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THE PADÉ TABLE AND ITS CONNECTION WITH SOME WEAK EXPONENTIAL FUNCTION APPROXIMATIONS TO LAPLACE TRANSFORM INVERSION

by

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ABSTRACT

The Padé table to the Laplace transform is considered, the equivalence of the approximate Laplace transform inversion by the use of Padé approximants and some weak exponential function approximations to the inverse transform, is shown, and a characterization theorem is proved. A generalization to cover the case of multi-point Padé approximants and ordinary rational interpolation to the Laplace transform is also suggested. Prony's method of solving some non-linear equations is generalized.

1. INTRODUCTION

Let $f(t)$ be a real valued integrable* function on any finite subinterval of the semi-infinite interval $0 \leq t < \infty$. $\bar{f}(p)$, the Laplace transform of $f(t)$, is defined by the integral

$$(1.1) \qquad \bar{f}(p) = \int_0^\infty e^{-pt} f(t)dt ,$$

whenever, this integral converges.

One way of obtaining approximations to the inverse $f(t)$ of $\bar{f}(p)$ is by approximating $\bar{f}(p)$ by a sequence of rational functions $\bar{f}_n(p)$, $n = 1,2,\ldots$, and then inverting the $\bar{f}_n(p)$ exactly to obtain the sequence $f_n(t)$, $n = 1,2,\ldots$. The hope is that if the sequence $\{\bar{f}_n(p)\}$ converges to $\bar{f}(p)$ quickly, then so will the sequence $\{f_n(t)\}$ converge to $f(t)$ quickly. There are several ways of obtaining rational approximations to a given function, one of them being by expanding this function in a Taylor series and forming the Padé table associated with this Taylor series. For a detailed discussion of this subject and references to various applications, the reader is referred to Longman (1973).

In Section 2 of this work we review briefly some algebraic properties of the Padé approximants. In Section 3 we show that the approximate Laplace transform inversion by the use of the Padé approximants is equivalent to the approximation of the inverse transform by a linear combi-

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* In the sequel of this work, by "an integral" and "an integrable function" we shall mean an improper Riemann integral and an improperly Riemann integrable function, respectively.

nation of exponential functions in some weak sense and also give a characterization theorem for the approximation to the inverse transform. In Section 4 we extend the results of Section 3 to cover the case of multi-point Padé approximants and ordinary rational interpolation to Laplace transform. In Section 5 we deal with the problem of interpolation by a sum of exponential functions and generalize Prony's method of solution and Weiss and Mc Donough's result concerning this problem.

2. SOME ALGEBRAIC ASPECTS OF THE PADÉ APPROXIMANTS

Let the function $h(z)$ have a formal power series expansion of the form

$$(2.1)\qquad h(z) = \sum_{i=0}^{\infty} c_i z^i .$$

The (m,n) entry in the Padé table of (2.1), if it exists, is defined as the rational function

$$(2.2)\qquad h_{m,n}(z) = \frac{P_m(z)}{Q_n(z)} = \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{j=0}^{n} b_j z^j} , \quad b_o = 1 ,$$

such that the Maclaurin series expansion of $h_{m,n}(z)$ in (2.2) agrees with the formal power series in (2.1) up to and including the term z^{m+n}; i.e.,

$$(2.3)\qquad h(z) - h_{m,n}(z) = 0(z^{m+n+1}) .$$

It is possible to express (2.3) also in the form

(2.4) $$\sum_{i=0}^{m} a_i z^i - \left(\sum_{j=0}^{n} b_j z^j\right)\left(\sum_{i=0}^{\infty} c_i z^i\right) = O(z^{m+n+1}),$$

from which, by setting the coefficients of the powers z^i, $i = 0,1,\ldots,$ $m+n$, on the left hand side, equal to zero, we obtain the two sets of linear equations

(2.5a) $$\sum_{j=0}^{\min(i,n)} c_{i-j} b_j = a_i, \quad i = 0,1,\ldots,m$$

(2.5b) $$\sum_{j=0}^{\min(i,n)} c_{i-j} b_j = 0, \quad i = m+1,\ldots,m+n,$$

which, together with the condition $b_0 = 1$, completely determine $h_{m,n}(z)$. As is clear from equations (2.5), if the c_i are real, then the a_i, b_i and $h_{m,n}(z)$ (for real z) are all real. For the subject of the Padé table as defined above, see Baker (1975, Chapters 1 and 2).

For the purpose of the approximate inversion of the Laplace transform, the Padé approximants $h_{N+n-1,n}(z)$, where N is a non-negative integer, i.e., those $h_{m,n}(z)$ for which $m \geq n-1$, are of interest; therefore, we shall concentrate on these approximations.

Let us assume that in $h_{N+n-1,n}(z)$, all the common factors in the numerator $P_{N+n-1}(z)$ and in the denominator $Q_n(z)$, if there are any, have been cancelled out and that the numerator is of degree less than or equal to $N+n'-1$, and the denominator is of degree exactly n' $(n' \leq n)$; i.e., $h_{N+n-1,n}(z) \equiv h_{N+n'-1,n'}(z) = P_{N+n'-1}(z)/Q_{n'}(z)$.

Dividing the numerator $P_{N+n'-1}(z)$ by the denominator $Q_{n'}(z)$, we can express $h_{N+n-1,n}(z)$ as

$$(2.6) \qquad h_{N+n-1,n}(z) = R_{N-1}(z) + {}_Nh_n(z) ,$$

where

$$(2.7) \qquad {}_Nh_n(z) = \frac{\bar{P}_{n'-1}(z)}{Q_{n'}(z)} ,$$

such that $R_{N-1}(z)$ and $\bar{P}_{n'-1}(z)$ are polynomials of degree at most $N-1$ and $n'-1$ respectively, which are uniquely defined. Needless to say ${}_Nh_{n'}(z) \equiv {}_Nh_n(z)$.

<u>Lemma 2.1</u> The Maclaurin series expansion of the rational function ${}_Nh_n(z)$ agrees with the formal power series expansion in (2.1) through the terms z^i, $i = N, N+1, \ldots, N+2n-1$.

<u>Proof</u>. Let ${}_Nh_n(z)$ have the Maclaurin series expansion $\sum_{i=0}^{\infty} d_i z^i$ and let $R_{N-1}(z) = \sum_{i=0}^{N-1} e_i z^i$. Using these in (2.6) and substituting (2.6) in (2.3) we obtain:

$$(2.8a) \qquad c_i = e_i + d_i \quad , \qquad i = 0,1,\ldots,N-1$$

$$(2.8b) \qquad c_i = d_i \quad , \qquad i = N,N+1,\ldots,N+2n-1.$$

which is the desired result.

From this the following can easily be obtained.

<u>Corollary</u>. The Maclaurin series expansion of the N-th derivative of ${}_Nh_n(z)$, agrees with the formal N-th derivative of the series in (2.1),

obtained by differentiating (2.1) term by term N times, up to and including the term z^{2n-1}.

We note that the rational function ${}_Nh_n(z)$ is determined by the $2n$ equations given in (2.8b).

Since we assume that the denominator $Q_{n'}(z)$ of ${}_Nh_n(z)$ is exactly of degree n', we let $z_1, z_2, \ldots, z_s$, be all the zeros of $Q_{n'}(z)$, of multiplicities $\mu_1, \mu_2, \ldots, \mu_s$, respectively, such that $\sum_{j=1}^{s} \mu_j = n'$. Then it is possible to express ${}_Nh_n(z)$ in partial fractions as

$$(2.9) \qquad {}_Nh_n(z) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} \frac{A_{j,k}}{(z-z_j)^k} .$$

From (2.9), the Maclaurin series expansion of ${}_Nh_n(z)$ is

$$(2.10) \qquad {}_Nh_n(z) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} \frac{A_{j,k}}{(-z_j)^k} \sum_{i=0}^{\infty} \binom{-k}{i} \left(-\frac{z}{z_j}\right)^i ,$$

which, upon using the fact that $\binom{-k}{i} = (-1)^i \binom{k+i-1}{i}$, becomes

$$(2.11) \qquad {}_Nh_n(z) = \sum_{i=0}^{\infty} \left\{ \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} (-1)^k \binom{k+i-1}{i} \frac{A_{j,k}}{z_j^{k+i}} \right\} z^i .$$

If we now use the result of Lemma 2.1; i.e., equations (2.8b), we obtain the following result:

<u>Lemma 2.2</u> The parameters $A_{j,k}$ and z_j of the partial fraction decomposition of ${}_Nh_n(z)$ satisfy the $2n$ non-linear equations

$$(2.12)\qquad c_{N+i} = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} (-1)^k \binom{N+k+i-1}{N+i} \frac{A_{j,k}}{z_j^{N+k+i}}\,,\quad i = 0,1,\ldots,2n-1.$$

Consider now the power series expansion $\bar{h}(z) = \sum_{i=0}^{\infty} \bar{c}_i z^i$, $\bar{c}_i = c_{N+i}$, $i = 0,1,\ldots$, and let $\bar{h}_{m,n}$ be its Padé approximants. As can be seen by equations (2.5b), the coefficients of the denominator of $\bar{h}_{n-1,n}(z)$ and those of $h_{N+n-1,n}(z)$ satisfy the same equations. Assuming that the Padé table of $h(z)$ and hence that of $\bar{h}(z)$ are normal (see Baker (1975), Chapter 2), we know that the $h_{N+n-1,n}(z)$ and $\bar{h}_{n-1,n}(z)$ are irreducible and have denominators of degree exactly n and furthermore these denominators are identical. Being identical, the denominators of $h_{N+n-1,n}(z)$ and $\bar{h}_{n-1,n}(z)$ have the same zeros which we now assume are simple. Then ${}_Nh_n(z) = \sum_{j=1}^{n} A_j/(z-z_j)$ and $\bar{h}_{n-1,n}(z) = \sum_{j=1}^{n} \bar{A}_j/(z-z_j)$. Using these expansions, for ${}_Nh_n(z)$ equations (2.12) become

$$(2.13)\qquad c_{N+i} = \sum_{j=1}^{n} -\frac{A_j}{z_j^{N+i}}\,,\quad i = 0,1,\ldots,2n-1\,,$$

and recalling that $\bar{h}_{n-1,n}(z) \equiv {}_0\bar{h}_n(z)$, for $\bar{h}_{n-1,n}(z)$ equations (2.12) become

$$(2.14)\qquad \bar{c}_i = \sum_{j=1}^{n} -\frac{\bar{A}_j}{z_j^{i}}\,,\quad i = 0,1,\ldots,2n-1.$$

Remembering that $\bar{c}_i = c_{N+i}$, $i = 0,1,\ldots,$ and comparing equations (2.13) and (2.14) we see that

$$(2.15)\qquad A_j = \bar{A}_j z_j^N\,,\quad j = 1,2,\ldots,n\,.$$

Thus, we have shown that if $h(z)$ has a normal Padé table, then the partial fraction expansion of ${}_N h_n(z)$ can be obtained very easily from that of $\bar{h}_{n-1,n}(z)$, provided $\bar{h}_{n-1,n}(z)$ has simple poles. This way we also avoid the problem of the division of $P_{N+n-1}(z)$ by $Q_n(z)$.

3. APPROXIMATE LAPLACE TRANSFORM INVERSION BY THE USE OF THE PADÉ TABLE.

Let $f(t)$ be as described in Section 1 and let $\bar{f}(p)$ be its Laplace transform as defined in (1.1). One property of $\bar{f}(p)$ is that it is an analytic function of p whenever $\mathrm{Re}\, p > \gamma$, for some γ. Then $\bar{f}(p)$ is analytic at $p = w$ for w real and $w > \gamma$, and hence can be expanded in a Taylor series as

$$(3.1) \qquad F(z) \equiv \bar{f}(p) = \sum_{i=0}^{\infty} \frac{\bar{f}^{(i)}(w)}{i!} z^i, \quad z = p-w .$$

Another important property of the Laplace transform $\bar{f}(p)$ is that it goes to zero as $\mathrm{Re}\, p \to \infty$; and this implies that not every analytic function is a Laplace transform. This property, therefore, puts a restriction on the functions that can be used as approximations to $\bar{f}(p)$ for the purpose of inversion. For example, among the Padé approximations $F_{m,n}(z)$ to $F(z)$ (or equivalently $\bar{f}(p)$), only those with $m < n$ can be used for obtaining approximations to the inverse transform $f(t)$, whereas the others can not. In particular, among the $F_{N+n-1,n}(z)$, only those with $N = 0$, i.e. the $F_{n-1,n}(z)$, can be used for this purpose, and they have indeed been used with success. A question then is: Is it possible, somehow, to make use of the

$F_{N+n-1,n}(z)$, for $N > 0$, for the purpose of obtaining approximations to $f(t)$? Now we know the mathematical relationship between $\bar{f}(p)$ and the Padé approximations to it. Another even more interesting question then is: What is the mathematical relationship between $f(t)$ and the approximations to it obtained by inverting the Padé approximations $F_{n-1,n}(z)$, and what is the relationship between $f(t)$ and the approximations to it obtained by using the $F_{N+n-1,n}(z)$, for $N > 0$, if the answer to the first question is in the affirmative? The following theorem answers both of these questions simultaneously.

Theorem 3.1. Define the sets G_n as follows:

$$(3.2) \qquad G_n = \{g(t) = \sum_{j=1}^{r} \sum_{k=1}^{\sigma_j} B_{j,k} t^{k-1} e^{\alpha_j t} \mid \alpha_j \text{ distinct}, \sum_{j=1}^{r} \sigma_j = n' \leq n\} .$$

(It is clear from (3.2) that $G_1 \subset G_2 \subset G_3 \ldots$.)

Let now $g_n(t)$ be that function, if it exists, belonging to G_n, which approximates $f(t)$ on $[0,\infty)$ in the following weak sense:

$$(3.3) \qquad \int_0^\infty t^N e^{-wt} [f(t) - g_n(t)] t^i dt = 0, \quad i = 0,1,\ldots,2n-1.$$

Then $\bar{g}_n(p)$, the Laplace transform of $g_n(t)$ is simply ${}_N F_n(p-w)$, furthermore $g_n(t)$ is a real function of t.

Proof. If the function $g_n(t)$ exists, it is then of the form

$$(3.4) \qquad g_n(t) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} \frac{A_{j,k}}{(k-1)!} t^{k-1} e^{\alpha_j t}, \quad \sum_{j=1}^{s} \mu_j = n' \leq n.$$

Substituting (3.4) in (3.3) and using the relations

(3.5a) $$\int_0^{\infty} t^{\ell} e^{-pt} f(t)dt = (-1)^{\ell} \bar{f}^{(\ell)}(p),$$

(3.5b) $$\int_0^{\infty} t^{\nu} e^{-pt} dt = \frac{\nu!}{p^{\nu+1}}, \qquad \nu > -1 ,$$

we obtain

(3.6) $$(-1)^{N+i} \bar{f}^{(N+i)}(w) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} \frac{(N+i+k-1)!}{(k-1)!} \frac{A_{j,k}}{(w-\alpha_j)^{N+i+k}} ,$$

$$i = 0,1,\ldots,2n-1.$$

These equations can be rewritten as

(3.7) $$\frac{\bar{f}^{(N+i)}(w)}{(N+i)!} = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} (-1)^k \binom{N+i+k-1}{N+i} \frac{A_{j,k}}{(\alpha_j-w)^{N+i+k}} ,$$

$$i = 0,1,\ldots,2n-1.$$

Recalling from (3.1) that $\bar{f}^{(\ell)}(w)/\ell!$ is the coefficient of the power z^{ℓ} in the Maclaurin series expansion of $F(z)$, and using Lemma 2.2, we obtain the result

(3.8) $${}_N F_n(z) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} \frac{A_{j,k}}{(z-\alpha_j+w)^k} .$$

By using the definition $p = z+w$ we can express (3.8) as

(3.9) $${}_N F_n(p-w) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} \frac{A_{j,k}}{(p-\alpha_j)^k} .$$

Now, since the right hand side of equation (3.9) is nothing but $\bar{g}_n(p)$, we have $\bar{g}_n(p) = {}_N F_n(p-w)$. Since $\bar{f}(p)$ is real for real p, the Padé approximants $F_{m,n}(z)$ and hence ${}_N F_n(z)$ and equivalently $\bar{g}_n(p)$ are real for real p, therefore $g_n(t)$ is a real function of t too. This completes the proof.

Theorem 3.1 tells us then that $F_{N+n-1,n}(z)$ can be used for approximating the inverse transform $f(t)$ provided ${}_N F_n(z)$, i.e., that part of $F_{N+n-1,n}(z)$ which goes to zero as $p\to\infty$, is used as an approximation to $\bar{f}(p)$.

Now, by defining $\psi(t) = t^N e^{-wt}$ with N fixed, and $\varphi_i(t) = t^i$, $i = 0,1,\ldots,$ equations (3.3) can be written as

$$(3.3)' \qquad \int_0^\infty \psi(t)\,[\,f(t) - g_n(t)\,]\,\varphi_i(t)dt = 0, \quad i = 0,1,\ldots,2n-1,$$

which looks very much like a Galerkin-type approximation procedure. Therefore, by analogy with Galerkin approximation methods, we would expect the sequence $g_n(t)$, $n = 1,2,\ldots$, ignoring those $g_n(t)$ which do not exist, to converge to $f(t)$. Another justification for this expectation is the following: The sequences of Padé approximants along the diagonals usually converge very quickly, at least numerically. Now the $g_n(t)$ are obtained from the ${}_N F_n(p-w)$ which in turn are obtained from the Padé approximants $F_{N+n-1,n}(z)$, and these, for N fixed, form a diagonal of the Padé table.

For future reference, we state the following theorem:

Theorem 3.2. Let $u_r(x)$, $r = 0,1,2,\ldots$, be a set of polynomials which are orthogonal on an interval $[a,b]$, finite, semi-infinite, or infinite, with weight function $q(x)$, whose integral over any subinterval of $[a,b]$ is positive. If $A(x)$ is any real continuous function on (a,b) and $\int_a^b q(x)A(x)dx$ exists as an improper Riemann integral and if $\int_a^b q(x)A(x)u_r(x)dx = 0$, $r = 0,1,\ldots,k-1$, then $A(x)$ either changes sign at least k times in the interval (a,b) or is identically zero.

The proof of this theorem for $A(x)$ continuous on $[a,b]$ can be found in Cheney (1966, p.110) and carries over to the case in which $A(x)$ is as described above without any modification.

We now prove a characterization theorem for the approximations $g_n(t)$.

<u>Theorem 3.3</u>. Let $f(t)$ be as described in Section 1 and be continuous on $(0,\infty)$ and let $\bar{f}(p)$, its Laplace transform, be analytic for $\operatorname{Re} p > \gamma$. Let $w > \gamma$ and let $F(z)$ be defined as in (3.1).

Let $\bar{g}_n(p) = {}_NF_n(p-w)$, if it exists, and assume $\bar{g}_n(p)$ has no poles for $\operatorname{Re} p \geqslant w$. Then $D(t) = f(t) - g_n(t)$, where $g_n(t)$ is the inverse of $\bar{g}_n(p)$, changes its sign at least $2n$ times in the interval $(0,\infty)$ if $f \notin G_n$. If $f \in G_n$, then $D(t) \equiv 0$.

<u>Proof.</u> From Theorem 3.1, $g_n(t)$ is real and satisfies the equations

$$(3.10) \qquad \int_0^\infty t^N e^{-wt} D(t) t^i dt = 0, \quad i = 0,1,\ldots,2n-1.$$

Choosing v such that $\beta = w+v > 0$, we can write equations (3.10) in the form

$$(3.11) \qquad \int_0^\infty t^N e^{-\beta t}\bar{D}(t)t^i dt = 0, \quad i = 0,1,\ldots,2n-1,$$

where $\bar{D}(t) = e^{\nu t}D(t)$. By taking appropriate linear combinations, equations (3.11) can be expressed as

$$(3.12) \qquad \int_0^\infty t^N e^{-\beta t}\bar{D}(t)L_i^{(N)}(\beta t)dt = 0, \quad i = 0,1,\ldots,2n-1,$$

where $L_i^{(\alpha)}(x)$ are the Laguerre polynomials which are orthogonal on $[0,\infty)$ with weight function $x^\alpha e^{-x}$. It is easy to see that the $L_i^{(N)}(\beta t)$ are orthogonal on $[0,\infty)$ with weight function $t^N e^{-\beta t}$. Now, using Theorem 3.2, we conclude that $\bar{D}(t)$ and hence $D(t)$ change sign at least $2n$ times on $(0,\infty)$ or that they are identically zero. But $D(t) \equiv 0$ only when $f(t) \equiv g_n(t)$, and this proves the theorem.

4. GENERALIZATION TO MULTI-POINT PADE APPROXIMANTS AND RATIONAL INTERPOLATION

The result of Theorem 3.1 can be carried further as follows:

Theorem 4.1: Let $f(t)$ be as in Section 3 and let $g_n(t)$ be that function, if it exists, belonging to G_n, which approximates $f(t)$ on $[0,\infty)$ in the weak sense

$$(4.1) \qquad \int_0^\infty e^{-w_k t}[f(t) - g_n(t)]\, t^i dt = 0, \quad \begin{array}{l} i = 0,1,\ldots,n_k, \\ k = 1,2,\ldots,\ell, \end{array}$$

where the w_k are distinct and $\mathrm{Re}\, w_k > \gamma$, and $\sum_{k=1}^{\ell} (n_k + 1) = 2n$.

Then $\bar{g}_n(p)$, the Laplace transform of $g_n(t)$, is the ℓ-point Padé approximation to $\bar{f}(p)$, whose numerator is of degree at most $n-1$ and whose denominator is of degree at most n, and whose Taylor series expansions about the points $p = w_k$ agree with the Taylor series expansions of $\bar{f}(p)$ about the same points up to and including the terms $(p-w_k)^{n_k}$, $k = 1,2,\ldots,\ell$. (For the subject of multi-point Padé approximants see Baker (1975, Chapter 8).). Furthermore, if the w_k are real, then $g_n(t)$ is a real function of t.

<u>Proof</u>. The proof of the first part follows from the fact that equations (4.1), together with the help of equation (3.5a), can be written as

$$\bar{f}^{(i)}(w_k) = \bar{g}^{(i)}(w_k), \quad i = 0,1,\ldots,n_k, \quad k = 1,2,\ldots,\ell, \tag{4.2}$$

and the fact that $\bar{g}_n(p)$ is a rational function with numerator of degree at most $n-1$ and denominator of degree at most n. The proof of the second part follows from the fact that $\bar{g}_n(p)$ is real for real p, when the w_k are real. This can be seen easily by observing that Eqs. (4.2) when expressed in terms of the coefficients of the numerator and denominator of $\bar{g}_n(p)$, form a linear system of real equations. This, then completes the proof.

Setting $n_k = 0$ in Theorem 4.1, we can now show that the rational interpolation problem to $\bar{f}(p)$ too is simply related with an exponential function approximation to $f(t)$ in some weak sense.

Corollary. Let $g_n(t)$ be that function, if it exists, belonging to G_n, which approximates $f(t)$ on $[0,\infty)$, in the weak sense

$$(4.3)\qquad \int_0^\infty e^{-w_k t}\,[f(t) - g_n(t)]\,dt = 0, \quad k = 1,2,\ldots,2n$$

where the w_k are distinct and $\operatorname{Re} w_k > \gamma$. Then $\bar{g}_n(p)$, the Laplace transform of $g_n(t)$, is the rational function with numerator of degree at most $n-1$ and denominator of degree at most n, which interpolates $\bar{f}(p)$ at the points $p = w_k$, $k = 1,2,\ldots,2n$. As before, if the w_k are real, then $g_n(t)$ is a real function of t.

When the w_k are chosen to be real and equidistant, we can also prove a characterization theorem for $g_n(t)$, in the case when $\bar{g}_n(p)$ interpolates $\bar{f}(p)$ at the points $p = w_k$.

Theorem 4.2. Let $f(t)$ and $\bar{f}(p)$ be as in Theorem 3.3 and let $\bar{g}_n(p)$ be that rational function, if it exists, with numerator of degree at most $n-1$ and denominator of degree at most n, which interpolates $\bar{f}(p)$ at the $2n$ distinct real points $p = w_0 + k\delta$, $k = 0,1,\ldots,2n-1$, where $w_0 > \gamma$, $\delta > 0$, and assume that $\bar{g}_n(p)$ has no poles for $\operatorname{Re} p \geq w_0$. Then, $D(t) = f(t) - g_n(t)$, changes sign at least $2n$ times in the interval $(0,\infty)$, if $f \notin G_n$. If $f \in G_n$, then $D(t) \equiv 0$.

Proof. From the corollary to Theorem 4.1, $g_n(t)$ satisfies the equations

$$(4.4)\qquad \int_0^\infty e^{-w_0 t}\,D(t)\,e^{-k\delta t}\,dt = 0, \quad k = 0,1,\ldots,2n-1,$$

which can also be written as

$$(4.5)\qquad \int_0^\infty e^{-\delta t}\,\bar{D}(t)\,e^{-k\delta t}\,dt = 0, \quad k = 0,1,\ldots,2n-1\ ;$$

where $\bar{D}(t) = e^{(\delta - w_o)t}\,D(t)$. Now taking appropriate linear combinations, equations (4.5) can be written as

$$(4.6)\qquad \int_0^\infty e^{-\delta t}\,\bar{D}(t)\,P_k^*(e^{-\delta t})dt = 0, \quad k = 0,1,\ldots,2n-1$$

where $P_k^*(x)$ are the shifted Legendre polynomials which are orthogonal on the interval $[0,1]$ with weight function unity. Making the change of variable $x = e^{-\delta t}$ and defining $E(x) \equiv \bar{D}(t)$, we can express equations (4.6), in the new variable x, as

$$(4.7)\qquad \int_0^1 E(x)\,P_k^*(x)dx = 0, \quad k = 0,1,\ldots,2n-1.$$

Using now Theorem 3.2, we conclude that $E(x)$ either changes sign at least $2n$ times on $(0,1)$ or $E(x) \equiv 0$. Going back to the variable t, we see that $\bar{D}(t)$ and hence $D(t)$ either change sign at least $2n$ times on $(0,\infty)$ or are identically zero. But $D(t) \equiv 0$ only when $f(t) \equiv g_n(t)$ and this proves the theorem

5. PRONY's METHOD AND THE PADÉ TABLE

Suppose the function $c(x)$ is to be approximated by a sum of exponential functions

$$(5.1)\qquad u(x) = \sum_{j=1}^{n} \alpha_j e^{\sigma_j x},$$

where the α_j and σ_j are to be determined by the interpolation equations $c_i = c(i) = u(i)$, $i = 0,1,\ldots,2n-1$, which, on defining $e^{\sigma_j} = \zeta_j$, $j = 1,\ldots,n$, become

$$(5.2)\qquad c_i = \sum_{j=1}^{n} \alpha_j \zeta_j^i, \quad i = 0,1,\ldots,2n-1.$$

The non-linear equations have been solved by Prony (1795) and the relation of Prony's method of solution with the $(n-1,n)$ Padé approximant to the power series expansion $V(z) = \sum_{i=0}^{\infty} c_i z^i$ has been shown by Weiss and McDonough (1963). It turns out that the ζ_j are the inverses of the zeros of the denominator of the $(n-1,n)$ Padé approximant to $V(z)$, whenever this approximant exists.

Now $u(x)$ as given in (5.1) exists if the ζ_j are distinct. But whenever some of the ζ_j are equal, there is no such $\hat{u}(x)$. This implies that the function $u(x)$ in (5.1) must be modified. The following theorem shows how this modification is to be made and also generalizes the method of Prony and the result of Weiss and McDonough.

<u>Theorem 5.1</u> Let $c(x)$ be a given function and denote $c_i = c(i)$, $i = 0,1,2,\ldots$. Suppose furthermore that the Padé approximant

$V_{N+n-1,n}(z)$ to $V(z) = \sum_{i=0}^{\infty} c_i z^i$ exists. Then there exists a function $u(x)$ in G_n which interpolates $c(x)$ at the points $x = N, N+1, \ldots, N+2n-1$, and this $u(x)$ is related to ${}_NV_n(z)$.

<u>Proof.</u> If $u(x)$ exists, it is of the form

$$(5.3) \qquad u(x) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} (-1)^k \binom{k+x-1}{k-1} A_{j,k} \zeta_j^{k+x},$$

such that ζ_j are distinct and $\sum_{j=1}^{s} \mu_j = n' \leq n$. (It can be shown that any function in G_n can also be written as in (5.3).) Using now the conditions

$$(5.4) \qquad c_i = c(i) = u(i), \quad i = N, N+1, \ldots, N+2n-1,$$

we obtain the equations

$$(5.5) \qquad c_{N+i} = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} (-1)^k \binom{N+i+k-1}{k-1} A_{j,k} \zeta_j^{N+i+k}, \quad i = 0, 1, \ldots, 2n-1.$$

Upon setting $z_j = 1/\zeta_j$ and comparing equations (5.5) with equations (2.12), and using Lemma 2.2, we see that the $A_{j,k}$ and z_j are the parameters of the partial fraction decomposition of ${}_NV_n(z)$ provided ${}_NV_n(z)$ exists. But ${}_NV_n(z)$ exists, if $V_{N+n-1,n}(z)$ exists, and this proves the theorem.

As can be seen from the proof of Theorem 5.1, the interpolant $u(x)$ to $c(x)$ can easily be found by determining the parameters in the partial fraction decomposition of ${}_NV_n(z)$.

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