Computation of the Chebyshev-Padé table

Avram Sidi (*)

ABSTRACT

A recursive method is given for the computation of the coefficients in the Chebyshev-Padé table. This is a table, recently defined by Clenshaw and Lord for Chebyshev series, which is analogous to the Padé table for power series. The method enables one to compute the whole of the triangular part of the table which derives from the given number of terms in the original Chebyshev series taken into account. The recursive method given by Clenshaw and Lord only enables one to compute the coefficients in half of this table.

1. INTRODUCTION

Recently Clenshaw and Lord [1] have defined the Chebyshev-Padé (C P) table of a function f(x) which has a formal Chebyshev series expansion on [-1, 1] of the form

$$f(x) = \sum_{r=o}^{\infty}{}' a_r T_r(x) \qquad (1)$$

where $T_r(x) = \cos(r \cos^{-1} x)$ is the Chebyshev polynomial of the first kind, of order r, and $\sum_{r=o}^{\infty}{}' u_r$ denotes the sum $\frac{1}{2} u_o + u_1 + u_2 + \cdots$.

The (m, n) member of the CP table is defined as the rational function

$$S_{m,n}(x) = \frac{P_m(x)}{Q_n(x)} = \frac{\sum_{r=o}^{m}{}' p_r T_r(x)}{\sum_{r=o}^{n}{}' q_r T_r(x)} \qquad (2)$$

having a formal Chebyshev series expansion agreeing with (1) up to and including the term $a_{m+n} T_{m+n}(x)$. It is assumed that the polynomials $P_m(x)$ and $Q_n(x)$ have no common factor, apart from a constant, and that $Q_n(x)$ has no zeros on [-1, 1]. It then follows [1] that $q_o \neq o$, and we can normalize by taking $q_o = 2$. This is done in order to maintain the analogy with the normal Padé table. In order to compute the p_r and the q_r one first solves the linear set of homogeneous equations

$$\sum_{s=o}^{n} \gamma_s a_{|r-s|} = o, \quad r = m+1, m+2, \ldots, m+n \qquad (3)$$

for the γ_s by letting $\gamma_o = 1$. Since $|r-s| \leqslant m+n$ for $m+1 \leqslant r \leqslant m+n$ and $o \leqslant s \leqslant n$, the determination of the γ_s requires knowledge of $a_o, a_1, \ldots, a_{m+n}$ which are also needed for the determination of $S_{m,n}(x)$.

Once (3) is solved for the γ_s, the q_s may be computed from

$$q_s = \mu \sum_{i=o}^{n-s} \gamma_i \gamma_{s+i}, \quad s = 1, 2, \ldots, n$$

where

$$\mu^{-1} = \frac{1}{2} \sum_{i=o}^{n} \gamma_i^2$$

and the p_r may be computed from

$$p_r = \frac{1}{2} \sum_{s=o}^{n}{}' q_s (a_{r+s} + a_{|r-s|}), \quad r = o, 1, \ldots, m.$$

For the basis of the above formulae the reader is referred to the original paper of Clenshaw and Lord.

2. COMPUTATION OF THE γ_s

In the determination of $S_{m,n}(x)$ the solution of (3) for the γ_s is the most difficult part of the work, particularly if n is large. Now Clenshaw and Lord have shown in [1] how to obtain the γ_s for $m \geqslant n$ recursively. However, their method for the computation of the γ_s is not applicable to the case $m < n$, because the first row ($m = o$) of the CP table is not known. Below is given a recursive method for the computation of the complete CP table. This method is similar in nature to the one given in [1]. Essentially one computes the (m, n + 1) element assuming knowledge of the (m + 1, n) and (m, n) elements and this requires knowledge of the first column ($n = o$). However, the first column is known since

$$S_{m,o}(x) = \sum_{r=o}^{m}{}' a_r T_r(x),$$

and this means that given $a_o, a_1, \ldots, a_{m+n}$ one can

(*) Department of Environmental Sciences, Tel-Aviv University, Ramat Aviv, Israel

compute the coefficients of the $S_{i,j}(x)$ in the entire triangle $o \leqslant i+j \leqslant m+n$.

Let us denote by $\gamma_s^{(m,n)}$ the γ_s for the (m,n) element and set $\gamma_o^{(m,n)} = 1$ and $\gamma_{-1}^{(m,n)} = \gamma_{n+1}^{(m,n)} = o$.

Proposition :

If $\gamma_s^{(m+1,n)}$ and $\gamma_s^{(m,n)}$, $s = o, 1, \ldots, n$ are known, then $\gamma_s^{(m,n+1)}$ can be computed recursively from

$$\gamma_s^{(m,n+1)} = \gamma_s^{(m+1,n)} + \omega^{(m,n+1)} \gamma_{s-1}^{(m,n)}, \qquad s = o, 1, \ldots, n+1 \qquad (4)$$

where $\omega^{(m,n+1)}$ is given as

$$\omega^{(m,n+1)} = - \frac{\sum_{s=o}^{n} \gamma_s^{(m+1,n)} a_{|m+1-s|}}{\sum_{s=o}^{n} \gamma_s^{(m,n)} a_{|m-s|}} \qquad (5)$$

Proof :

If one multiplies (4) by $a_{|r-s|}$ and sums the resulting equation from $s = o$ to $s = n+1$, one obtains

$$\sum_{s=o}^{n+1} \gamma_s^{(m,n+1)} a_{|r-s|} = \sum_{s=o}^{n+1} \gamma_s^{(m+1,n)} a_{|r-s|} + \omega^{(m,n+1)} \sum_{s=o}^{n+1} \gamma_{s-1}^{(m,n)} a_{|r-s|}.$$

Now in the first sum on the right hand side of this equation if one inserts $\gamma_{n+1}^{(m+1,n)} = o$ and in the second sum $\gamma_{-1}^{(m,n)} = o$, then one has

$$\sum_{s=o}^{n+1} \gamma_s^{(m,n+1)} a_{|r-s|} = \sum_{s=o}^{n} \gamma_s^{(m+1,n)} a_{|r-s|} + \omega^{(m,n+1)} \sum_{s=o}^{n} \gamma_s^{(m,n)} a_{|r-s-1|}. \qquad (6)$$

The right hand side of (6) is zero for $r = m+2$, $m+3, \ldots, m+n+1$ by (3), and, therefore, so is the left hand side. One can make the left hand side zero for $r = m+1$ if one chooses $\omega^{(m,n+1)}$ to satisfy

$$\sum_{s=o}^{n} \gamma_s^{(m+1,n)} a_{|m+1-s|} + \omega^{(m,n+1)} \sum_{s=o}^{n} \gamma_s^{(m,n)} a_{|m-s|} = o, \qquad (7)$$

the solution of which gives (5).

Since $|m+1-s| \leqslant m+n+1$ and $|m-s| \leqslant m+n$ for $o \leqslant s \leqslant n$, the determination of $\omega^{(m,n+1)}$ and hence of $\gamma_s^{(m,n+1)}$ requires knowledge of $a_o, a_1, \ldots, a_{m+n+1}$ as required by the definition of $S_{m,n+1}(x)$. Hence we have shown that if the $\gamma_s^{(m,n+1)}$ satisfy (4) with (5), then they also satisfy (3).

It is worthy of note that the Padé table for power series can be used to compute elements of the CP table as suggested by Chisholm [2]. However, as will be shown below, this enables one to compute only the same half of the CP table that one is able to compute using the algorithm of Clenshaw and Lord. The procedure is as follows :

Start by writing $x = \cos\theta$, $t = e^{i\theta}$; then $T_r(x) = \frac{1}{2}(t^r + t^{-r})$ and the Chebyshev series in (1) becomes

$$f(x) = \frac{1}{2}\left(\sum_{r=o}^{\infty}{}' a_r t^r + \sum_{r=o}^{\infty}{}' a_r t^{-r} \right).$$

Let the (m,n) element of the Padé table for $\sum_{r=o}^{\infty}{}' a_r t^r$ be

$$E_{m,n}(t) = \frac{\sum_{r=o}^{m} \alpha_r t^r}{\sum_{s=o}^{n} \beta_s t^s}$$

Accordingly the (m,n) element of the Padé table for $\sum_{r=o}^{\infty}{}' a_r t^{-r}$ is $E_{m,n}(t^{-1})$. If one sums the corresponding elements of the two tables and divides the result by 2, one obtains

$$\bar{S}_{m,n}(x) = \frac{1}{2}[E_{m,n}(t) + E_{m,n}(t^{-1})] \qquad (8)$$

$$= \frac{\sum_{i=-n}^{m} \left(\sum_{r=\max(o,i)}^{\min(i+n,m)} \alpha_r \beta_{r-i} \right) T_{|i|}(x)}{\sum_{j=-n}^{n} \left(\sum_{r=\max(o,j)}^{\min(j+n,m)} \beta_r \beta_{r-j} \right) T_{|j|}(x)}$$

It is seen that $\bar{S}_{m,n}(x)$ is of the form

$$\bar{S}_{m,n}(x) = \frac{\sum_{i=o}^{\max(m,n)} \bar{\alpha}_i T_i(x)}{\sum_{j=o}^{n} \bar{\beta}_j T_j(x)} \qquad (9)$$

Now since $\sum_{r=o}^{\infty}{}' a_r t^r - E_{m,n}(t) = \sum_{r=m+n+1}^{\infty} A_r t^r$, one has

$$f(x) - \bar{S}_{m,n}(x) = \sum_{r=m+n+1}^{\infty} A_r T_r(x).$$

This last equality together with (9) shows that $\bar{S}_{m,n}(x) = S_{m,n}(x)$ for $m \geqslant n$. For $m < n$, however $\bar{S}_{m,n}(x)$ is of the form

$$\bar{S}_{m,n}(x) = \frac{\sum_{i=1}^{n} \bar{\alpha}_i T_i(x)}{\sum_{j=1}^{n} \bar{\beta}_j T_j(x)}$$

and, therefore, does not, in general, belong to the CP table of f(x).

ACKNOWLEDGMENT

The author wishes to thank Professor I. M. Longman for suggesting the problem and for the many valuable suggestions for the presentation of the material.

REFERENCES

1. C. W. Clenshaw and K. Lord, "Rational approximations from Chebyshev series", Studies in Numerical Analysis, Academic Press, London, 1974, pp. 95-113
2. J. S. R. Chisholm, private communication to Professor I. M. Longman.