On the approximation of square-integrable functions by exponential series

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ABSTRACT

The expansion of a real square-integrable function in a Legendre series is considered. Existence of best approximations from different sets of exponential functions and their mean convergence to the function in question are proved. As an extension of this result existence and mean convergence of some non-linear best approximations that have been developed by Longman are also demonstrated.

1. INTRODUCTION

Recently Longman [1, 2] has considered the problem of finding a best approximation to a real function f (t) that satisfies

$$K(w) = \int_0^\infty e^{-wt}[f(t)]^2\,dt < \infty \qquad (1)$$

for some $w \geqslant 0$, in terms of exponential functions, in the least-squares norm. Specifically, one looks for that set of parameters $\{A_r, \alpha_r\}_{r=1}^{n}$ (A_r, α_r may be restricted to be real or may be allowed to become complex) that will minimize the integral

$$I_n(w) = \int_0^\infty e^{-wt}[f(t) - \sum_{r=1}^{n} A_r e^{-\alpha_r t}]^2\,dt, \qquad (2)$$

where $\sum_{r=1}^{n} A_r e^{-\alpha_r t}$ is a real function of t, and the α_r are all distinct.

It turns out that all that one needs to know in order to minimize $I_n(w)$ in (2) is the Laplace transform $\bar{f}(p)$ of f(t) as defined by

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\,dt, \qquad (3)$$

along the real p-axis if the α_r are real, and in the complex p-plane if the α_r are allowed to become complex. $K(w) < \infty$ implies that $f(t) = o(e^{wt/2})$ as $t \to \infty$ which implies that the Laplace transform of $|f(t)|$ and hence that of f (t) exist for $\mathrm{Re}\, p > w/2$. As a matter of fact, the aim of this procedure is the approximate inversion of the Laplace transform.

It may happen, however, that in some cases it is impossible to minimize $I_n(w)$ in (2) with distinct α_r. An example of this is given in section 3 and there it is also shown that one can always minimize $I_n(w)$ with $\sum_{r=1}^{n} A_r e^{-\alpha_r t}$ replaced by $\sum_{r=1}^{m} \sum_{k=0}^{n_r - 1} A_{rk} t^k e^{-\alpha_r t}$ where $\sum_{r=1}^{m} n_r = n$. Of course, the n_r are to be found by trial and error.

As is clear from (2), one is considering an uncountably infinite set of exponential functions and from this set one is selecting those n members (if all $n_r = 1$) that will minimize $I_n(w)$. As will be shown in the next section $I_n(w)$ can be minimized on a fixed finite set of exponential functions; in fact, f(t) can be expanded in Legendre series of exponential functions in infinitely many forms and the partial sums of these expansions form sequences of best approximations which converge to f (t) in the mean. As an extension of this result it will also be shown that the sequence of best approximations of Longman, which should be modified as explained in the previous paragraph if necessary, converges to f (t) in the mean. Finally, the expansion of f(t) in a Legendre series can be regarded as a Laplace transform inversion method, and all that has to be known is the values of $\bar{f}(p)$ at a countably infinite number of points spaced equidistantly along the positive real axis in the p-plane. This fact has been noted by Lanczos [3].

2. EXPANSION OF f(t) IN A LEGENDRE SERIES

Let f(t) satisfy (1) for some $w \geqslant 0$. Then (1) can also be expressed as

$$K(w) = \int_0^\infty e^{-(w+b)t}[e^{bt/2} f(t)]^2\,dt < \infty \qquad (4)$$

for any real $b \geqslant 0$ such that $w + b > 0$. Now (4) implies that the function $e^{bt/2} f(t)$ is square-integrable

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with weight function $e^{-(w+b)t}$.

Theorem 1 : $f(t)$ can be expanded in the shifted Legendre polynomials $P_k^*(x)$ as

$$f(t) = e^{-bt/2} \sum_{k=o}^{\infty} a_k P_k^*(e^{-(w+b)t}) \qquad (5)$$

where $P_k^*(x)$ are related to the Legendre polynomials $P_k(x)$ through

$$P_k^*(x) = P_k(1-2x) = \sum_{m=o}^{k} C_{km} x^m \qquad (6)$$

and a_k are given as

$$a_k = (2k+1)(w+b) \sum_{m=o}^{k} C_{km} \bar{f}[(m+1)(w+b)-b/2] \qquad (7)$$

with $\bar{f}(p)$ as defined in (3); and this Legendre series converges to $f(t)$ in the mean on (o, ∞).

Proof : Let us map the interval (o, ∞) along the t-axis to $(o, 1)$ by the transformation $x = e^{-(w+b)t}$. Then (3) becomes

$$\bar{f}(p) = \frac{1}{w+b} \int_o^1 x^{\frac{p}{w+b}-1} F(x)\, dx \qquad (8)$$

where $F(x) \equiv f(t)$ and $x^{\frac{p}{w+b}-1}$ is defined to have its principal value. Now (4) implies that

$$\int_o^1 [x^{-\frac{b/2}{w+b}} F(x)]^2 dx < \infty \qquad (9)$$

and hence $x^{-\frac{b/2}{w+b}} F(x)$ can be expanded in a Legendre series on $(o, 1)$ as

$$x^{-\frac{b/2}{w+b}} F(x) = \sum_{k=o}^{\infty} a_k P_k^*(x). \qquad (10)$$

Using the orthogonality relation

$$\int_o^1 P_k^*(x) P_\ell^*(x)\, dx = \delta_{k\ell}/(2k+1)$$

and (6) one obtains

$$a_k = (2k+1) \sum_{m=o}^{k} C_{km} \int_o^1 x^m x^{-\frac{b/2}{w+b}} F(x)\, dx,$$

and upon using (8) in the last equation one gets (7). And finally, by writing (10) in the variable t one obtains (5).

Now (9) implies that the infinite series in (10) will converge to $x^{-\frac{b/2}{w+b}} F(x)$ in the mean on $(o, 1)$ [4], i. e., that

$$\lim_{n\to\infty} \int_o^1 [x^{-\frac{b/2}{w+b}} F(x) - \sum_{k=o}^{n} a_k P_k^*(x)]^2 dx = o,$$

and consequently

$$\lim_{n\to\infty} \int_o^1 e^{-wt} [f(t) - e^{-bt/2} \sum_{k=o}^{n} a_k P_k^*(e^{-(w+b)t})]^2 dt = 0$$

thus proving the theorem.

We note that expressions similar to (5) and (7) for the special case $w = 1$ and $b = o$ have been given by Lanczos [3].

Corollary :

$$\int_o^\infty e^{-wt}[f(t) - e^{-bt/2} \sum_{k=o}^{n} a_k P_k^*(e^{-(w+b)t})]^2 dt =$$

$$\min_{\{B_k\}_{k=o}^n} \int_o^\infty e^{-wt}[f(t) - \sum_{k=o}^{n} B_k e^{-k(w+b)t-bt/2}]^2 dt,$$

i. e., the n^{th} partial sum

$$s_n(t) = e^{-bt/2} \sum_{k=o}^{n} a_k P_k^*(e^{-(w+b)t})$$

of the infinite series in (5) is the best approximation to $f(t)$ from the set of functions $\{e^{-k(w+b)t-bt/2}\}_{k=o}^n$ on (o, ∞) in the least-squares norm

$$||f|| = \{ \int_o^\infty e^{-wt} [f(t)]^2 dx \}^{1/2}.$$

Proof : Since $\sum_{k=o}^{n} a_k P_k^*(x)$ minimizes the integral

$$\int_o^1 [x^{-\frac{b/2}{w+b}} F(x) - \sum_{k=o}^{n} a_k P_k^*(x)]^2 dx,$$

the result follows by writing this integral in the variable t.

3. EXISTENCE AND MEAN CONVERGENCE OF LONGMAN'S BEST APPROXIMATION

Let us start by assuming that $f(t)$, in addition to (1), satisfies also

$$\int_o^\infty e^{-wt/2} |f(t)|\, dt < \infty. \qquad (11)$$

This is not too drastic a restriction since all functions which are $0(e^{\beta t})$ with $\beta < w/2$ as $t \to \infty$ satisfy (11). Then (1) and (11) are sufficient for $f(t)$ to satisfy the Parseval's equality for Laplace transforms [5].

$$\int_o^\infty e^{-wt}[f(t)]^2 dt = \frac{1}{2\pi} \int_{w/2-i\infty}^{w/2+i\infty} |\bar{f}(p)|^2 \; |dp|. \qquad (12)$$

The integral $I_n(w)$ in (2) diverges if $\mathrm{Re}\,\alpha_r \leqslant -w/2$ for some r. One can see this if one writes $I_n(w)$ explicitly as a function of $\bar{f}(p)$. Then one has

$$I_n(w) = K(w) - 2\sum_{r=1}^{n} A_r \bar{f}(\alpha_r + w) + \sum_{r=1}^{n}\sum_{s=1}^{n} \frac{A_r A_s}{\alpha_r + \alpha_s + w} \qquad (13)$$

and if one substitutes $\alpha_r = -w/2 + iy$ for some r, then the term $\dfrac{A_r A_s}{\alpha_r + \alpha_s + w}$ becomes infinite for $\alpha_r = \bar{\alpha}_s$ and $A_r = \bar{A}_s$. Therefore, one must minimize $I_n(w)$ on the set of functions

$$G'_n = \{g_n(t) = \sum_{r=1}^{n} A_r e^{-\alpha_r t} \mid \alpha_r \text{ distinct},$$

$$\mathrm{Re}\,\alpha_r > -w/2,\ g_n(t) \text{ real}\}.$$

If $g_n(t)$ is in G'_n, then it satisfies both

$$\int_0^\infty e^{-wt}[g_n(t)]^2 dt < \infty \text{ and } \int_0^\infty e^{-wt/2}|g_n(t)|\,dt < \infty \qquad (14)$$

Now it is possible to apply the Parseval's equality to $f(t) - g_n(t)$. One obtains

$$\int_0^\infty e^{-wt}[f(t) - g_n(t)]^2 dt = \frac{1}{2\pi}\int_{w/2-i\infty}^{w/2+i\infty} |\bar{f}(p) - \bar{g}_n(p)|^2 |dp| \qquad (15)$$

where $\bar{g}_n(p) = \sum_{r=1}^{n} \dfrac{A_r}{p+\alpha_r}$ is the Laplace transform of $g_n(t)$. Define

$$\bar{G}'_n = \{\bar{g}_n(p) = \sum_{r=1}^{n} \frac{A_r}{p+\alpha_r} \mid \alpha_r \text{ distinct},$$

$$\mathrm{Re}\,\alpha_r > -w/2,\ \bar{g}_n(p) \text{ real for p real}\}.$$

Then it follows from (15) that finding a best approximation to $f(t)$ from G'_n in the norm

$$\|f\| = \{\int_0^\infty e^{-wt}[f(t)]^2 dt\}^{1/2} \qquad (16)$$

is equivalent to finding a best approximation to $\bar{f}(p)$ from $\bar{G}'_n$ in the norm

$$\|\bar{f}\|_L = \{\frac{1}{2\pi}\int_{w/2-i\infty}^{w/2+i\infty} |\bar{f}(p)|^2 |dp|\}^{1/2} \qquad (17)$$

Below the existence of a best $g_n(t)$ will be shown indirectly by the existence of a best $\bar{g}_n(p)$.

Before proceeding to the proof of the existence theorem let us make the observation that $\bar{G}'_n$ is not complete, i. e., a sequence $\{\bar{g}_{nk}(p)\}_{k=1}^{\infty}$ in $\bar{G}'_n$ does not have to have a limit in $\bar{G}'_n$. This one can see by considering the sequence

$$\{\bar{g}_{2k}(p) = \frac{k/2}{p+1-1/k} - \frac{k/2}{p+1+1/k}\}_{k=1}^{\infty}.$$

Although $\alpha_{1k} = 1 - 1/k$ and $\alpha_{2k} = 1 + 1/k$ are different, in the limit as $k \to \infty$ both α_{1k} and α_{2k} tend to 1 and $\bar{g}_{2k}(p) \to \dfrac{1}{(p+1)^2}$ which is not in $\bar{G}'_n$. Or one can express this by saying that $\bar{f}(p) = \dfrac{1}{(p+1)^2}$ which is the Laplace transform of $f(t) = te^{-t}$ does not have a best approximation in $\bar{G}'_2$ in the norm (17). However, $\dfrac{1}{(p+1)^2}$ does have a best rational approximation which is itself. This implies then that the best approximation to te^{-t} is itself and this suggests that one should extend G'_n and $\bar{G}'_n$ and consider the sets

$$G_n = \{g_n(t) = \sum_{r=1}^{m}\sum_{k=0}^{n_r-1} A_{rk} t^k e^{-\alpha_r t} \mid \alpha_r \text{ distinct},$$

$$\mathrm{Re}\,\alpha_r > -w/2,\ \sum_{r=1}^{m} n_r = n,\ g_n(t) \text{ real}\}$$

and the set of the Laplace transforms of the elements of G_n

$$\bar{G}_n = \{\bar{g}_n(p) = \sum_{r=1}^{m}\sum_{k=0}^{n_r-1} \frac{B_{rk}}{(p+\alpha_r)^{k+1}} \mid \alpha_r \text{ distinct},$$

$$\mathrm{Re}\,\alpha_r > -w/2,\ \sum_{r=1}^{m} n_r = n,\ \bar{g}_n(p) \text{ real for p real}\}.$$

It is clear that $\bar{G}_n$ is the set of all the rational functions at the form :

$$\sum_{i=0}^{n-1} \beta_i p^i / \sum_{j=0}^{n} \gamma_j p^j \text{ with } \beta_i \text{ and } \gamma_j \text{ real, degree}$$

of numerator strictly less than degree of denominator, such that the denominator does not vanish for p with $\mathrm{Re}\ p \geqslant w/2$. Again (14) is satisfied by all the members of G_n.

For future reference define the set of rational functions

$$H_n = \{ h_n(z) = \frac{\sum_{i=o}^{n-1} \beta_i z^i}{\sum_{j=o}^{n} \gamma_j z^j} \mid \sum_{j=o}^{n} \gamma_j z^j \neq o \text{ on } |z| \leq 1 \}.$$

Lemma : The set of polynomials

$\{[c + (c-w)z]^i (1-z)^{n-i}\}_{i=o}^{n}$, $c > w/2$, is linearly independent.

Proof : For $n = o$ and $n = 1$ the assertion is true. Suppose it is true for all $n \leq k$. Then one has to show that

$$\sum_{i=o}^{k+1} d_i [c + (c-w)z]^i (1-z)^{k+1-i} = o \tag{18}$$

if and only if $d_i = o$, $i = o, 1, \ldots, k+1$. Let $z = 1$ in (18). Then $d_{k+1} = o$, and now by cancelling a factor of $(1-z)$ in (18) one obtains

$$\sum_{i=o}^{k} d_i [c + (c-w)z]^i (1-z)^{k-i} = o$$

which by the induction hypothesis holds if and only if $d_i = o, 1.., k$.

Theorem 2 : Let f(t) satisfy (1) and (11). Then $\min_{g_n \epsilon G_n} \| f - g_n \|$ exists.

Proof : Since f(t) satisfies (1) and (11) and any $g_n(t)$ in G_n satisfies (14), Parseval's equality for $f(t) - g_n(t)$ holds; i. e.,

$$\| f - g_n \| = \| \bar{f} - \bar{g}_n \|_L$$

Using the Möbius transformation

$$p = \frac{c + (c-w)z}{1-z}, \quad c > w/2, \tag{19}$$

one maps the half-plane Re $p \geq w/2$ onto the unit disc $|z| \leq 1$. The straight line Re $p = w/2$ is mapped onto the unit circle $|z| = 1$. The point at infinity has as its image $z = 1$. In terms of the new variable z one can write

$$\| \bar{f} - \bar{g}_n \|_L = \{ \frac{c - w/2}{\pi} \int_{|z|=1} | \frac{\bar{f}(p(z))}{1-z} - \frac{\bar{g}_n(p(z))}{1-z} |^2 |dz| \}^{1/2}.$$

The function $\bar{g}_n(p)$, being a member of $\bar{G}_n$, is of the form $\sum_{i=o}^{n-1} \beta_i p^i / \sum_{j=o}^{n} \gamma_j p^j$ and, therefore, $h_n(z) = \bar{g}_n(p(z))/(1-z)$ is of the form

$$h_n(z) = \frac{\sum_{i=o}^{n-1} \beta_i [c + (c-w)z]^i (1-z)^{n-1-i}}{\sum_{j=o}^{n} \gamma_i [c + (c-w)z]^j (1-z)^{n-j}} \tag{20a}$$

$$= \frac{\sum_{i=o}^{n-1} \beta_i' z^i}{\sum_{j=o}^{n} \gamma_j' z^j}. \tag{20b}$$

Since $\bar{g}_n(p)$ is in $\bar{G}_n$, $h_n(z)$ is in H_n.

Now the problem is reduced to finding a best approximation from H_n to $h(z) = \bar{f}(p(z))/(1-z)$ in the norm

$$\|h\|_M = \{ \int_{|z|=1} |h(z)|^2 |dz| \}^{1/2}. \tag{21}$$

A theorem due to Walsh [6] guarantees the existence of a best approximation $h_n^*(z)$ of the form (20b). This theorem, for the purpose of the specific problem, which is least-squares approximation on the unit circle, can be stated as follows :

Theorem [Walsh] : Let the function h(z) be defined on a rectifiable curve C and let n(z) be a non-negative function on C such that

$$\int_C n(z) |dz| < \infty .$$

Suppose there exists at least one rational function r(z) in H_n such that

$$I[r] = \int_C n(z) |h(z) - r(z)|^2 |dz| < \infty.$$

Then there exists a rational function R(z) of the form (20b) for which

$$\int_C n(z) |h(z) - R(z)|^2 |dz|$$

is not greater than I[r] for any r(z) in H_n.
Now in our problem the unit circle is a rectifiable curve and $n(z) = 1$. Furthermore, since f(t) satisfies (1) and (11), $h(z) = \bar{f}(p(z))/(1-z)$ is defined on the unit circle. Also I[r] exists for any r(z) in H_n. Hence all the conditions of Walsh's theorem are satisfied and a best approximation $R(z) = h_n^*(z)$ exists.
Now $h_n^*(z)$, being the limit point of a sequence $\{h_{nk}(z)\}_{k=1}^{\infty}$ in H_n, may have poles in the region

$|z| \geqslant 1$. However, there cannot be any poles on the unit circle $|z| = 1$ since if there were, then $\|h - h_n^*\|_M$ would not exist. Therefore, all the poles of $h_n^*(z)$ are restricted to the exterior of the unit circle. Hence $h_n^*(z)$ is in H_n. Once $h_n^*(z)$ is known, the best approximation $\bar{g}_n^*(p)$ can be determined from $h_n^*(z)$ using (20a) as guaranteed by the lemma, and $\bar{g}_n^*(p)$ is in $\bar{G}_n$ since $h_n^*(z)$ is in H_n. This completes the proof of the existence theorem.

Theorem 3 : The sequence of the best approximations to $f(t)$ from G_n, $n = 1, 2, \ldots$, converges the $f(t)$ in the mean; i. e. $\lim\limits_{n\to\infty} \min\limits_{g_n \epsilon G_n} \| f - g_n \| = o$.

Proof : In the Corollary to Theorem 1 it was proved that the partial sums of the Legendre series in (5),

$$s_n^*(t) = e^{-bt/2} \sum_{k=o}^{n-1} a_k P_k^* (e^{-(w+b)t}),$$

with $b \geqslant o$ and a_k as given in (7), form a sequence of best approximations from the sets of functions

$$S_n = \{ s_n(t) = e^{-bt/2} \sum_{k=o}^{n-1} B_k e^{-k(w+b)t} | B_k \text{ real} \}, \quad n = 1, 2, 3, \ldots .$$

Now S_n is a proper n-dimensional subspace of G_n. Therefore,

$$\min_{g_n \epsilon G_n} \| f - g_n \| \leqslant \min_{s_n \epsilon S_n} \| f - s_n \|, \quad n = 1, 2, 3, \ldots . \tag{22}$$

Since from Theorem 1 the right hand side of (22) tends to zero as n tends to infinity, so does the left hand side, thus proving the theorem.

In some instances it may be difficult to compute $\bar{f}(p)$ for complex values of p or $f(t)$ behaves like a real exponential function as t becomes large. Then one can also consider best approximations to $f(t)$ from a more restricted class of functions, namely, real exponential functions, i. e., G_n, $\bar{G}_n$, and H_n can be replaced by

$$G_n^R = \{ g_n^R(t) = \sum_{r=1}^{m} \sum_{k=o}^{n_r - 1} A_{rk} t^k e^{\alpha_r t} | \alpha_r \text{ real, distinct}, \alpha_r > -w/2, \; g_n^R(t) \text{ real} \},$$

$$\bar{G}_n^R = \{ \bar{g}_n^R(p) = \sum_{r=1}^{m} \sum_{k=o}^{n_r - 1} \frac{B_{rk}}{(p + \alpha_r)^k} | \alpha_r \text{ real, distinct}, \alpha_r > -w/2, \; \bar{g}_n^R(p) \text{ real for p real} \},$$

$$H_n^R = \{ h_n^R(z) = \frac{\sum_{i=o}^{n-1} \beta_i z^i}{\sum_{j=o}^{n} \gamma_j z^j} | \sum_{j=o}^{n} \gamma_j z^j \neq o \; |z| \leqslant 1 \text{ and has real zeros only} \},$$

respectively. Then $\|f - g_n^R\|$ has a minimum on G_n^R which again is a consequence of Walsh's theorem [6]. The sequence of the best approximations to $f(t)$ from G_n^R, $n = 1, 2, \ldots$, converges to $f(t)$ in the mean, since the set S_n of the previous section is a proper n-dimensional subspace of G_n^R also, and G_n^R is an uncountably infinite dimensional subspace of G_n. Then one has

$$\min_{g_n \epsilon G_n} \| f - g_n \| \leqslant \min_{g_n^R \epsilon G_n^R} \| f - g_n^R \| \leqslant \min_{s_n \epsilon S_n} \| f - s_n \|.$$

4. CONCLUDING REMARKS

It has been shown that a square-integrable function $f(t)$ can be expanded in Legendre series in infinitely many forms, the coefficients of this series being simple expressions involving the Laplace transform of the function in question. As they stand the partial sums of there series cannot be used to obtain good approximations to $f(t)$ for t large. The reason for this is that the function $f(t)$ when expressed in the variable $x = e^{-(w+b)t}$ is not a smooth function near $x = o$ which corresponds to $t = \infty$. Therefore, the Legendre series suffers form convergence problems as t becomes large. In order to overcome this problem to some extent some acceleration convergence method should be applied to the Legendre series in question.

The existence of the best approximations of Longman, with some modification, has been shown by reducing the problem to one of approximation by rational functions on the unit circle. Convergence of the sequence of the best approximations to $f(t)$ in the mean has been shown by observing the fact that these best approximations are better (in the norm (16)) than the partial sums of the Legendre series which converge to $f(t)$ in the mean.

It has also been shown that the best approximation to $f(t)$ may be restricted to be a sum of real exponential functions and in this case too existence and convergence in the mean to $f(t)$ are guaranteed.

ACKNOWLEDGEMENT

The author wishes to thank Professor I. M. Longman for suggesting the problem and continued support and encouragement of this work. Thanks are due also to D. Levin for useful discussions and comments.

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