Best rational function approximation to Laplace transform inversion using a window function (*)

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ABSTRACT

The best rational function approximation for Laplace transform inversion due to Longman is modified by the introduction of an appropriate "window" function. This window function enables one to approximate the inverse transform f(t) by a linear combination $g_n(t)$ of n exponential functions accurately in a given interval about a given point along the t-axis. It is proved that the sequence of approximants $\{g_n(t)\}_{n=1}^{\infty}$ converges to f(t) in the mean. The method is illustrated by some numerical examples.

1. INTRODUCTION

Recently Longman [1, 2] has developed a technique, based on the least-squares method, for the approximate inversion of the Laplace transform $\bar{f}(p)$ of a square-integrable function f(t), where $\bar{f}(p)$ is defined as

$$\bar{f}(p) = L[f(t); p] = \int_0^{\infty} e^{-pt} f(t)\, dt \tag{1}$$

and f(t) is square-integrable in the sense that

$$\int_0^{\infty} e^{-wt}[f(t)]^2 dt < \infty \tag{2}$$

for some $w > o$. One looks for that rational approximation

$$\bar{g}_n(p) = \sum_{r=1}^{n} A_r/(p+\alpha_r) \tag{3}$$

to $\bar{f}(p)$ which will minimize the integral

$$I_n(w) = \int_0^{\infty} e^{-wt}[f(t) - g_n(t)]^2 dt \tag{4}$$

with respect to the parameters A_r and α_r, $r = 1,\ldots,n$, where $g_n(t)$ is the inverse transform of $\bar{g}_n(p)$; i.e.,

$$g_n(t) = \sum_{r=1}^{n} A_r e^{-\alpha_r t}\,. \tag{5}$$

What makes this method an approximate Laplace transform inversion method is that the integral $I_n(w)$ in (4), apart from a constant term which is independent of A_r and α_r, $r = 1,\ldots,n$, is expressible in terms of $\bar{f}(p)$, without needing to know f(t). The existence of $g_n(t)$ and the mean convergence of the sequence $\{g_n(t)\}_{n=1}^{\infty}$ to f(t) in the sense that

$$\lim_{n\to\infty} \min_{\{A_r,\alpha_r\}_{r=1}^{n}} I_n(w) = o \tag{6}$$

has been shown by the author [5].

Now let us consider some of the numerical aspects of this method. For $w > o$ the weight function e^{-wt} which appears in (4) is equal to 1 at $t = o$ and tends to zero quickly as t becomes large. This means that $g_n(t)$ will approximate f(t) closely for small values of t. However, for large values of t $g_n(t)$ may deviate from f(t) by large amounts and this would not increase the value of the integral $I_n(w)$ by a substantial amount, since for large t, e^{-wt} has a strong damping effect. If one wants to approximate f(t) also for values of t away from $t = o$ using the least-squares technique described above, one has to modify the weight function e^{-wt} in such a way that the new weight function would be appreciable in a certain interval away from $t = o$ and would tend to zero rapidly outside this interval, thus serving as a window. Naturally, the mathematical form of this window would be dictated by the requirement that $I_n(w)$ in (4), with e^{-wt} replaced by the window function, be a simple function of $\bar{f}(p)$, in the sense that one should be able to com-

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pute $I_n(w)$ easily on a computer, since one would like to use this technique for the approximate numerical inversion of the Laplace transform.

As for $w = o$, one can notice that equal weight is being attached to the square of the error $[f(t)-g_n(t)]$ for all values of t up to infinity, and it is clear that one cannot expect to approximate $f(t)$ accurately everywhere.

2. THE CHOICE OF THE WINDOW FUNCTION

Let $\Psi(t)$ be a non-negative function on (o, ∞), and let us seek to minimize the integral

$$I_n[\Psi] = \int_o^\infty \Psi(t)[f(t) - g_n(t)]^2 dt, \tag{7}$$

where $g_n(t)$ is as given in (5). Now $I_n[\Psi]$ can be written as

$$I_n[\Psi] = \int_o^\infty \Psi(t)[f(t)]^2 dt - 2\sum_{r=1}^{n} A_r L[\Psi(t)f(t);\alpha_r] + \sum_{r=1}^{n}\sum_{s=1}^{n} A_r A_s L[\Psi(t);\alpha_r + \alpha_s]. \tag{8}$$

This last expression would be useless for the purpose of Laplace transform inversion unless the term $L[\Psi(t) f(t); p]$ can be expressed as a simple function of $\bar{f}(p)$. This limits the choice of $\Psi(t)$ drastically since the only functions known to the author that satisfy this demand are polynomials t^N (N a positive integer) and e^{-wt} and linear combinations of their products. We recall that

$$L[t^N f(t); p] = (-1)^N \frac{d^N}{dp^N} \bar{f}(p), \tag{9a}$$

where N is a positive integer, and

$$L[e^{-wt} f(t); p] = \bar{f}(p + w). \tag{9b}$$

Now let us consider the "window" function defined as

$$\Psi(t) = t^N e^{-wt}, \tag{10}$$

where N is a positive integer, and $w > o$. t^N is very small for t but increases quite rapidly for larger values of t; on the other hand, e^{-wt} is a monotonically decreasing function of t, and tends to zero faster than any inverse power of t as t becomes large. This means that when t^N is multiplied by e^{-wt} the graph of t^N is bent towards the t-axis for large values of t, thus causing $\Psi(t)$ to have a maximum. This maximum occurs at $t_{max} = N/w$. The inflection points of $\Psi(t)$ are $t\pm = (N \pm \sqrt{N})/w$ and the distance between them is $\Delta = t_+ - t_- = 2\sqrt{N}/w$. Now Δ is a fairly good estimate of the width of $\Psi(t)$, in the sense that, $\Psi(t_\pm) \approx 1/2\ \Psi(t_{max})$. This means that $\Psi(t)$, practically, is appreciable in an interval of length Δ which is symmetric with respect to t_{max}, and furthermore this interval, as N becomes large, is away from $t = o$ since $\sqrt{N} \ll N$. Figure 1 shows two such windows. One expects to have better accuracy near t_{max} by making the width of the window smaller and leaving t_{max} unchanged. This can be achieved by exploiting the fact that both t_{max} and Δ depend on N and w. To illustrate this point suppose that one wants to approximate $f(t)$ in an interval about $t = N$. Suppose $w = 1$ initially. Then $\Psi_1(t) = t^N e^{-t}$ has $t_{max} = N$ and width $\Delta_1 = 2\sqrt{N}$. Now a narrower window can be obtained by choosing $\Psi_2(t) = [\Psi_1(t)]^2 = t^{2N}e^{-2t}$. Obviously t_{max} stays the same, but $\Delta_2 = \Delta_1/\sqrt{2}$. This example indicates that by increasing N and w simultaneously, one can reduce the width considerably and leave t_{max} unchanged, thus increasing the accuracy

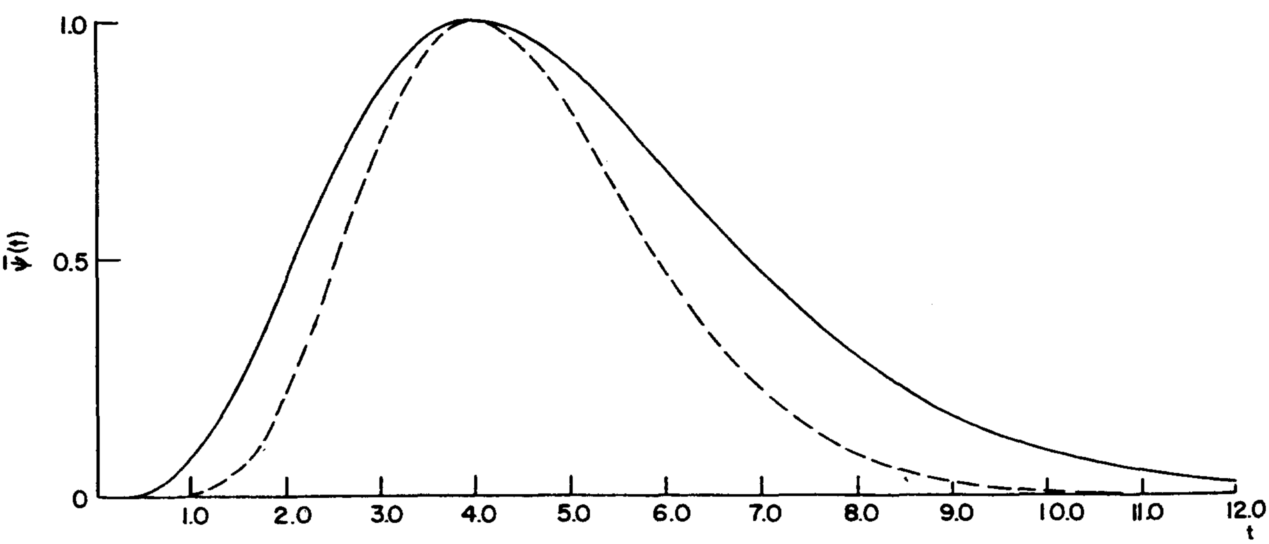

Fig. 1. Graphs of the normalized window functions $\bar{\Psi}_1(t) = \Psi_1(t)/\Psi_1(t_{max})$ (solid curve) and $\bar{\Psi}_2(t) = \Psi_2(t)/\Psi_2(t_{max})$ (dashed curve), where $\Psi_1(t) = t^4 e^{-t}$ and $\Psi_2(t) = t^8 e^{-2t}$

of the approximation near t_{max}. This is illustrated in Figure 1.

3. MINIMIZATION OF $I_n[\Psi]$

Consider the window function $\Psi(t)$ given in (10). Using (9a) and (9b) the integral $I_n[\Psi]$ given in (8) can be written as

$$I_n[\Psi] = \int_0^\infty t^N e^{-wt}[f(t)]^2 dt - 2\sum_{r=1}^n A_r(-1)^N \bar{f}^{(N)}(\alpha_r+w)$$

$$+\sum_{r=1}^n \sum_{s=1}^n A_r A_s N!/(\alpha_r+\alpha_s+w)^{N+1} \quad (11)$$

The first term is a constant independent of A_r and α_r, $r=1,\ldots,n$, which, although not known, does not affect the minimization of $I_n[\Psi]$ with respect to the parameters A_r, α_r, $r=1,\ldots,n$.

For a minimum (relative, not necessarily global) it is necessary (but not sufficient) that

$$\partial I_n/\partial A_r = 0 \quad \text{and} \quad \partial I_n/\partial \alpha_r = 0, \quad r=1,\ldots,n.$$

If one sets $\partial I_n/\partial A_r = 0$, one obtains

$$\bar{f}^{(N)}(\alpha_r+w) = \bar{g}_n^{(N)}(\alpha_r+w), \qquad r=1,\ldots,n, \quad (12)$$

and if one sets $\partial I_n/\partial \alpha_r = 0$, one obtains

$$\bar{f}^{(N+1)}(\alpha_r+w) = \bar{g}_n^{(N+1)}(\alpha_r+w), \qquad r=1,\ldots,n, \quad (13)$$

which for $N=0$ reduce to the equations given in [1]. Equations (12) and (13) can be solved by a minimization procedure as described in [2].

It is also interesting to note that the integral in (7) with the window $\Psi(t)$ as given in (10) can be written as a least-squares integral involving the difference $[\bar{f}^{(N)}(p) - \bar{g}_n^{(N)}(p)]$. Using

$$\int_0^\infty e^{-wt}[f(t)]^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\bar{f}(w/2+iy)|^2 dy \quad (14)$$

which is the Parseval equality for Laplace transforms [8], one can write

$$I_n[\Psi] = \int_0^\infty t^N e^{-wt}[f(t)-g_n(t)]^2 dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |L[t^{N/2}\{f(t)-g_n(t)\}; w/2+iy]|^2 dy. \quad (15)$$

For even N, say $N=2k$, (15) can be written as

$$I_n[\Psi] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\bar{f}^{(k)}(w/2+iy) - \bar{g}_n^{(k)}(w/2+iy)|^2 dy. \quad (16)$$

From this last equation one can see that the method of [1] with $N=0$ is equivalent to minimizing the modulus of the difference $[\bar{f}(p) - \bar{g}_n(p)]$ in the least-squares sense along the Bromwich contour which extends from $w/2 - i\infty$ to $w/2 + i\infty$. As has been remarked earlier, this method with $N=0$ can give good approximations to $f(t)$ for small t only. In order to obtain good accuracy for large t one should minimize $|\bar{f}^{(k)}(p) - \bar{g}_n^{(k)}(p)|$ in the least-squares sense along the Bromwich contour described above.

4. CONVERGENCE IN THE MEAN OF $\{g_n(t)\}_{n=1}^\infty$ TO $f(t)$

In this section we shall prove that the sequence $\{g_n(t)\}_{n=1}^\infty$ obtained by minimizing the integral $I_n[\Psi]$ in (15) converges to $f(t)$ in the mean; i.e., that

$$\lim_{n\to\infty} \min_{\{A_r,\alpha_r\}_{r=1}^n} I_n[\Psi] = 0. \quad (17)$$

The method of the proof is based on the method used in [5] in the proof of equation (6) of the present paper. Consider the transformation of variable $x = e^{-wt}$ which maps the interval $0 < t < \infty$ to the finite interval $0 < x < 1$. By defining $F(x) \equiv f(t)$ and $G_n(x) \equiv g_n(t)$ one can write the integral in $I_n[\Psi]$ in (15) as

$$I_n[\Psi] = 1/w^{N+1}\int_0^1 (-\log x)^N [F(x) - G_n(x)]^2 dx. \quad (18)$$

The weight function $\omega(x) = (-\log x)^N$ being a continuous positive integrable function on $(0, 1)$, there exists a complete set of polynomials $\{Q_m(x)\}_{n=0}^\infty$, where $Q_m(x)$ is a polynomial of degree m and $\int_0^1 \omega(x) Q_m(x) Q_k(x) dx = \delta_{mk}$.

If one defines

$$c_m \equiv \int_0^1 \omega(x) F(x) Q_m(x) dx,$$

then the linear combination $\sum_{m=0}^{n-1} c_m Q_m(x)$ minimizes $I_n[\Psi]$ in the space of functions $\{1, x, x^2, \ldots, x^{n-1}\}$; i.e.,

$$\int_0^1 \omega(x)[F(x) - \sum_{m=0}^{n-1} c_m Q_m(x)]^2 dx$$

$$\leqslant \int_0^1 \omega(x)[F(x) - \sum_{m=0}^{n-1} B_m x^m]^2 dx, \quad (19)$$

for any B_m, $m = 0, 1, \ldots, n-1$.

When written in terms of the variable t (19) becomes

$$\int_0^\infty t^N e^{-wt}[f(t) - \sum_{m=0}^{n-1} c_m Q_m(e^{-wt})]^2 dt$$

$$\leqslant \int_0^\infty t^N e^{-wt}[f(t) - \sum_{m=0}^{n-1} B_m e^{-wt}]^2 dt, \quad (20)$$

and the left-hand side of this last inequality tends to zero as n tends to infinity provided $f(t)$ satisfies

$$\int_0^{\infty} t^N e^{-wt}[f(t)]^2 dt < \infty$$

which is equivalent to the condition

$$\int_0^1 \omega(x)\,[F(x)]^2 dx < \infty,$$

which in turn is sufficient for the left-hand side of the inequality in (19) to tend to zero as n tends to infinity [9]. Once this has been established, the assertion in equation (17) can now be proved in the same manner as in [5].

5. APPLICATIONS

In this section the method that has been developed in the previous sections is applied to three Laplace transforms and the best approximations to their inverses are obtained.

Example 1

$f(t) = J_o(t)$, the Bessel function of the first kind of order zero. The Laplace transform of this function is $\bar{f}(p) = 1/(p^2+1)^{1/2}$.

One might think that the computation of high order derivatives of $\bar{f}(p)$ might become difficult as N gets large, but this is not so. Since in many problems $\bar{f}(p)$ is a function of elementary functions and/or special functions which satisfy a linear differential equation of some small order, the derivatives of $\bar{f}(p)$ can be computed recursively. For the example on hand one can start by writing

$$(p^2+1)^{1/2}\,\bar{f}(p) = 1$$

and differentiate this equation with respect to p as many times as is necessary. Then the recursion relation can be written as follows : Set

$$a_o = o,\ b_o = 1,\ \bar{f}(p) = 1/(p^2+1)^{1/2},$$

$$\bar{f}'(p) = -p\bar{f}(p)/(p^2+1),$$

then compute $a_k, b_k, \bar{f}^{(k)}(p)$ from

$$a_k = a_{k-1} + b_{k-1},\ b_k = b_{k-1} + 2,\quad k = 1, 2, \ldots$$

$$\bar{f}^{(k)}(p) = -[a_k\bar{f}^{(k-1)}(p) + b_k p\bar{f}^{(k-2)}(p)]/(p^2+1),\quad k = 1, 2, \ldots$$

Table 1 contains the parameters $\{A_r, \alpha_r\}_{r=1,2}$ of the approximants $g_2(t)$ for $w = 1, 2$ and $N = 2(2)8$. It is rather interesting to note that the parameters for the windows t^2e^{-t} and t^4e^{-2t} are close to each other. The reason for this is that for both of these windows $t_{max} = 2$ and hence $J_o(t)$ is approximated in an interval about t_{max}. The same is true for the windows t^4e^{-t} and t^8e^{-2t} for which $t_{max} = 4$. Now let us compare the accuracy of the approximations for the window $\Psi(t) = t^4e^{-t}$ and $\tilde{\Psi}(t) = t^8e^{-2t}$. For both windows $t_{max} = 4$ and the widths of $\Psi(t)$ and $\tilde{\Psi}(t)$ are $\Delta = 4$ and $\tilde{\Delta} = \sqrt{8}$ respectively. Let $g_2(t)$ and $\tilde{g}_2(t)$ be the best approximations to $J_o(t)$ obtained by using the windows $\Psi(t)$ and $\tilde{\Psi}(t)$ respectively. Figure 2 contains plots of the errors $J_o(t) - g_2(t)$ and $J_o(t) - \tilde{g}_2(t)$. As is seen from these plots, $\tilde{g}_2(t)$ ap-

Table 1

Parameters $\{A_r, \alpha_r\}$ for several window functions of the form given in (10) for the approximant $g_2(t)$. For all N and w $A_2 = \bar{A}_1$ and $\alpha_2 = \bar{\alpha}_1$ so that $g_2(t)$ is real.

	w = 1	w = 2
N = 2	A_1 = .38094145 + .32867462i α_1 = .23216051 + .96111519i	A_1 = .47407236 + .23494260i α_1 = .24025937 + .84660033i
N = 4	A_1 = .22022628 + .27126302i α_1 = .12765322 + 1.01426364i	A_1 = .39381427 + .31439020i α_1 = .23916952 + .93960917i
N = 6	A_1 = .18427276 + .20052531i α_1 = .08011072 + 1.00330344i	A_1 = .27274776 + .31612531i α_1 = .18251946 + 1.00604486i
N = 8	A_1 = .16362304 + .17448487i α_1 = .06249752 + 1.00184735i	A_1 = .20750643 + .25118995i α_1 = .11837302 + 1.01453567i

proximates $J_o(t)$ better inside the window $\tilde{\Psi}(t)$ than $g_2(t)$ does. Outside the window, however, the mean error increases in absolute value. Figure 3 contains the graph of the error $J_o(t) - g_2(t)$, where this time $g_2(t)$ is the best approximation to $J_o(t)$ obtained using the window $\Psi(t) = t^8 e^{-t}$. The error for this case also has the same features as those plotted in figure 2. The above-mentioned features have been observed to be present for all N and w for which best approximations have been computed.

Before treating examples 2 and 3 it would be appropriate to explain the use of different window functions and the interpretation of the numerical results for functions f(t) which tend to zero exponentially as t tends to infinity.

Suppose now that the best approximation $g_n(t)$ to f(t) is obtained by using the window $t^N e^{-wt}$. If t_{max} is large enough, then f(t) may, depending on the width of the window, decrease by several orders of magnitude inside the window as t becomes large. Since it is the square of the *absolute* error $|f(t) - g_n(t)|$ which is minimized, for large t inside the window this error may be small, but the approximant may be quite different from f(t); i.e., the relative error may be very large. This problem does not arise in the case of $J_o(t)$ since $J_o(t)$ is of the order of $t^{-1/2}$ as t becomes large.

Let $g_{n1}(t)$ and $g_{n2}(t)$ be the best approximations to f(t) inside the windows $\Psi_1(t) = t^N e^{-w_1 t}$ and $\Psi_2(t) = t^N e^{-w_2 t}$ respectively, where $w_1 > w_2$. Now $r_n(t) = e^{(w_1 - w_2)t/2} g_{n2}(t)$ is the best approximation to $h(t) = e^{(w_1 - w_2)t/2} f(t)$ inside the window $\Psi_1(t)$. But h(t) tends to zero less rapidly than f(t). Therefore, one would expect $r_n(t)$ to be a better approximation to h(t) than $g_{n1}(t)$ is to f(t), and consequently $g_{n2}(t)$ to be a better approximation to f(t) than $g_{n1}(t)$. That is, one could increase the accuracy of the best approximation by decreasing w. This reasoning can now be used even more efficiently as follows [7].

Let f(t) tend to zero like e^{-bt}, $b > o$. Then one can write the integral

$$I_n[\Psi] = \int_o^\infty t^N e^{-wt} [f(t) - g_n(t)]^2 dt \tag{21}$$

as

$$I_n[\Psi] = \int_o^\infty t^N e^{-(w+2b)t} \left[\frac{f(t) - g_n(t)}{e^{-bt}}\right]^2 dt \tag{22}$$

The expression $[f(t) - g_n(t)]/e^{-bt}$ can be viewed as the relative error of $g_n(t)$ at least for large t. Hence it is easy to see from (21) and (22) that the best approximation $g_n(t)$ to f(t) obtained by using the window $t^N e^{-wt}$ minimizes the "relative" error inside the window $t^N e^{-(w+2b)t}$.

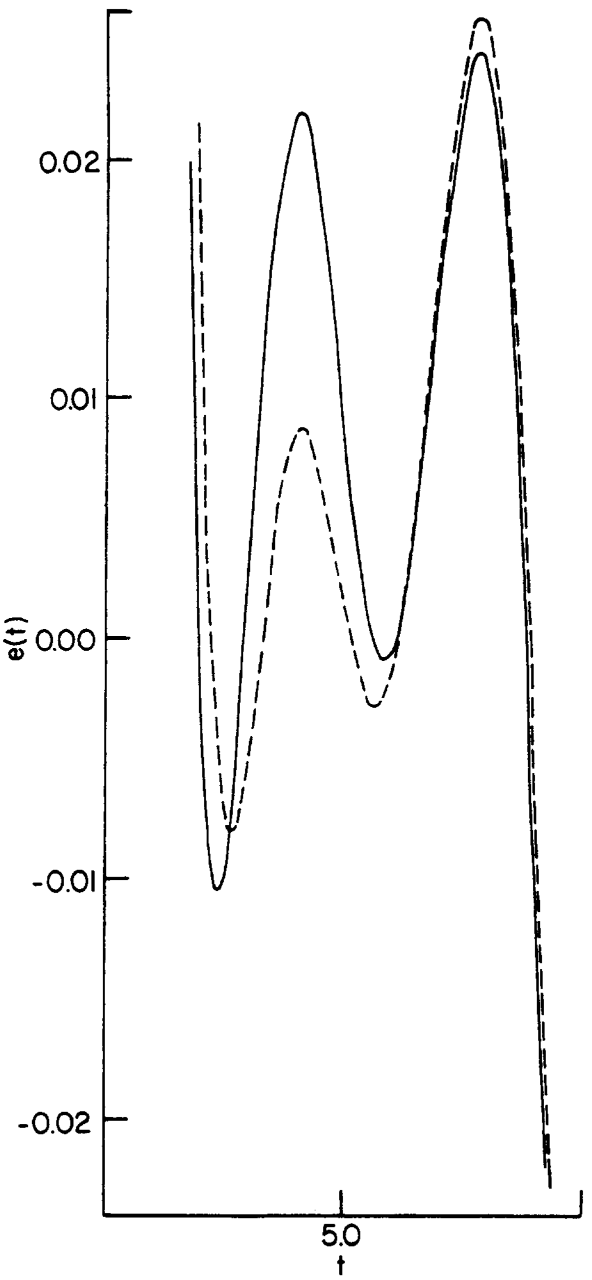

Fig. 2. Graphs of the errors $e(t) = J_o(t) - g_2(t)$ (solid curve) and $\tilde{e}(t) = J_o(t) - \tilde{g}_2(t)$ (dashed curve), where $g_2(t)$ and $\tilde{g}_2(t)$ are the best approximations to $J_o(t)$ obtained by minimizing $I_2[\Psi]$ using $\Psi(t) = t^4 e^{-t}$ and $\Psi(t) = t^8 e^{-2t}$ respectively.

Example 2

$f(t) = E_1(t) = \int_t^\infty e^{-x}/x\,dx$. The Laplace transform of this function is $\bar{f}(p) = \log(1+p)/p$. As t becomes large $E_1(t) \sim e^{-t}/t$ asymtotically, hence for large t it seems to be more convenient to approximate $tE_1(t)$

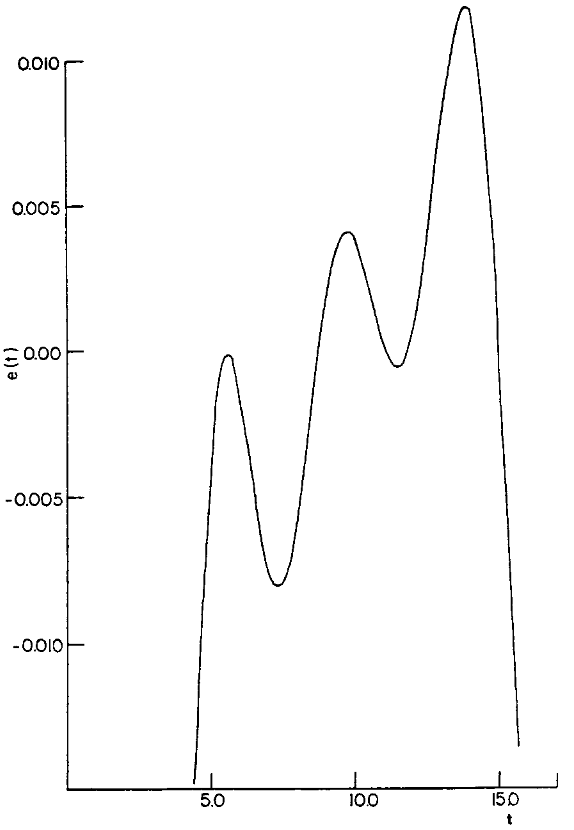

Fig. 3. Graph of the error $e(t) = J_o(t) - g_2(t)$, where $g_2(t)$ is the best approximation to $J_o(t)$ obtained by minimizing $I_2[\Psi]$ using $\Psi(t) = t^8 e^{-t}$.

by exponential functions rather than $E_1(t)$. Table 2 contains the parameters $\{A_r, \alpha_r\}$ for the approximation $g_2(t)$ to $E_1(t)$ obtained using the window $t^{10}e^{-t}$ and the approximations $h_2(t)$, $\tilde{h}_2(t)$, and $\tilde{\tilde{h}}_2(t)$ to $tE_1(t)$ obtained using the windows $t^{10}e^{-t}$, $t^{10}e^{-0.5t}$, and $t^{10}e^{-0.25t}$ respectively. Table 3 contains the values of $E_1(t)$ and the relative errors for the approximations $g_2(t)$, $h_2(t)/t$, $\tilde{h}_2(t)/t$, and $\tilde{\tilde{h}}_2(t)/t$. The assertion that for large t, $tE_1(t)$ is approximable by exponential functions better than $E_1(t)$ is confirmed by the 3rd and 4th columns of Table 3.

As for the relative error, as has been mentioned before, minimizing the square of the absolute error inside the window $t^{10}e^{-t}$ is equivalent to minimizing the "relative" error inside the window $t^{10}e^{-3t}$ since $E_1(t) \sim e^{-t}/t$ as t becomes large. For the window $t^{10}e^{-3t}$, $t_{max} = 10/3$ and $\Delta = \frac{2\sqrt{10}}{3}$ and $g_2(t)$ and $h_2(t)/t$ both approximate $E_1(t)$ very well in this window as confirmed by Table 3.

As has been stated previously, if one reduces w in $\Psi(t) = t^N e^{-wt}$ (keeping N fixed), then one obtains better accuracy in approximation. This is confirmed by the 5th and 6th columns of Table 3 which contain the relative errors for $\tilde{h}_2(t)/t$ and $\tilde{\tilde{h}}_2(t)/t$ respectively.

Example 3

$\bar{f}(p) = \frac{1}{p} \exp[-p/\sqrt{1+\sigma p}]$, $\sigma > o$. This function arises in problems of pulse propagation in viscoelastic media. The exact inverse of $\bar{f}(p)$ is not known analytically. The numerical inversion of this transform has been achieved by Longman [3, 4] by use of the Padé table and the Levin transformations [6].

$\bar{f}(p)$ has a simple pole at $p = o$ with residue 1 and a branch point at $p = -1/\sigma$ in the complex p-plane. Therefore, the inverse f(t) tends to 1 as t tends to infinity. If we subtract this, then $\bar{f}(p) - 1/p$ is the Laplace transform of $f(t) - 1$ and has only one singularity, namely, the branch point at $p = -1/\sigma$. This implies that $f(t) - 1$ tends to zero like $e^{-t/\sigma}$ times a power of t as t tends to infinity [10]. This piece of information now tells us that if one finds the best approximation to $f(t) - 1$ using the window $t^N e^{-wt}$, then one should look at those values of t which fall inside the window $t^N e^{-(w+2/\sigma)t}$ for maximum number of significant figures. This point is demonstrated in Table 5. But let us first look at Table 4 which contains the parameters $\{A_r, \alpha_r\}$ for $g_2(t)$, the best approximation to $f(t)-1$ with $\sigma = 1$, obtained using the window $t^{10}e^{-2t}$.

Table 2

The parameters $\{A_r, \alpha_r\}$ for $g_2(t)$, best approximation to $E_1(t)$, obtained using the window $t^{10}e^{-t}$, and $h_2(t)$, $\tilde{h}_2(t)$, and $\tilde{\tilde{h}}_2(t)$, best approximations to $tE_1(t)$, obtained using the windows $t^{10}e^{-t}$, $t^{10}e^{-0.5t}$, and $t^{10}e^{-0.25t}$ respectively.

	$g_2(t)$	$h_2(t)$	$\tilde{h}_2(t)$	$\tilde{\tilde{h}}_2(t)$
A_1	.35213648	.80495379	.82711670	.83865406
A_2	.76658326	- .41391736	- .39789717	- .38741964
α_1	1.15016959	.98546487	.98912073	.99075585
α_2	2.00997984	1.67695103	1.58623601	1.54059142

Table 3

Values of $E_1(t)$ and the relative errors for the approximations $g_2(t)$, $h_2(t)/t$, $\tilde{h}_2(t)/t$, and $\tilde{\tilde{h}}_2(t)/t$, where $g_2(t)$, $h_2(t)$, $\tilde{h}_2(t)$, and $\tilde{\tilde{h}}_2(t)$ are as described in Table 2.

		Relative errors for			
t	$E_1(t)$	$g_2(t)$	$h_2(t)/t$	$\tilde{h}_2(t)/t$	$\tilde{\tilde{h}}_2(t)/t$
2.0	4.8901×10^{-2}	3.2×10^{-3}	1.3×10^{-3}	7.4×10^{-4}	3.3×10^{-4}
2.4	2.8440×10^{-2}	8.8×10^{-5}	5.4×10^{-4}	1.1×10^{-3}	1.1×10^{-3}
2.8	1.6855×10^{-2}	2.1×10^{-3}	1.3×10^{-4}	6.1×10^{-4}	9.4×10^{-4}
3.2	1.0133×10^{-2}	2.2×10^{-3}	4.5×10^{-4}	3.0×10^{-5}	4.3×10^{-4}
3.6	6.1604×10^{-3}	7.6×10^{-4}	4.4×10^{-4}	3.4×10^{-4}	5.2×10^{-5}
4.0	3.7794×10^{-3}	1.3×10^{-3}	1.9×10^{-4}	4.6×10^{-4}	3.5×10^{-4}
4.4	2.3360×10^{-3}	3.2×10^{-3}	1.5×10^{-4}	3.8×10^{-4}	4.5×10^{-4}
4.8	1.4530×10^{-3}	4.1×10^{-3}	4.7×10^{-4}	1.6×10^{-4}	3.8×10^{-4}
5.2	9.0862×10^{-4}	3.6×10^{-3}	6.8×10^{-4}	1.1×10^{-4}	2.0×10^{-4}
5.6	5.7084×10^{-4}	1.2×10^{-3}	7.0×10^{-4}	3.8×10^{-4}	3.5×10^{-5}
6.0	3.6008×10^{-4}	3.1×10^{-3}	4.9×10^{-4}	5.6×10^{-4}	2.8×10^{-4}
6.4	2.2795×10^{-4}	9.5×10^{-3}	2.4×10^{-5}	6.8×10^{-4}	4.8×10^{-4}
6.8	1.4476×10^{-4}	1.8×10^{-2}	7.2×10^{-4}	6.3×10^{-4}	6.2×10^{-4}
7.2	9.2188×10^{-5}	2.8×10^{-2}	1.7×10^{-3}	4.3×10^{-4}	6.6×10^{-4}
7.6	5.8859×10^{-5}	4.1×10^{-2}	3.0×10^{-3}	6.3×10^{-5}	5.8×10^{-4}
8.0	3.7666×10^{-5}	5.5×10^{-2}	4.6×10^{-3}	4.9×10^{-4}	3.8×10^{-4}
8.4	2.4154×10^{-5}	7.0×10^{-2}	6.4×10^{-3}	1.2×10^{-3}	3.8×10^{-5}
8.8	1.5519×10^{-5}	8.7×10^{-2}	8.5×10^{-3}	2.1×10^{-3}	4.4×10^{-4}
9.2	9.9881×10^{-6}	1.0×10^{-1}	1.1×10^{-2}	3.2×10^{-3}	1.1×10^{-3}
9.6	6.4388×10^{-6}	1.2×10^{-1}	1.3×10^{-2}	4.5×10^{-3}	1.8×10^{-3}
10.0	4.1570×10^{-6}	1.4×10^{-1}	1.6×10^{-2}	5.9×10^{-3}	2.7×10^{-3}

Table 4

The parameters $\{A_r, \alpha_r\}$ for $g_2(t)$, the best approximation to $f(t) - 1$ with $\sigma = 1$, obtained using the window $t^{10}e^{-2t}$.

A_1	=	- .84978590
A_2	=	- .28910804
α_1	=	.93993479
α_2	=	2.81569326

Table 5

Exact values of the solution to the viscoelastic problem with $\sigma = 1$, and values of the approximant $g_2(t) + 1$, where $g_2(t)$ is as in Table 4. (The exact values have been taken from Longman [4]).

t	Exact	$g_2(t) + 1$
1.0	.65063	.65064
1.2	.71491	.71481
1.4	.766354	.766284
1.6	.807917	.807895
1.8	.841728	.841734
2.0	.869367	.869381
2.2	.892040	.892053
2.4	.910691	.910698
2.6	.926063	.926065
2.8	.938753	.938751
3.0	.949242	.949238

6. CONCLUSION

A Laplace transform inversion method developed by Longman and based on the least-squares method has been modified, and this modification allows one to approximate the inverse transform f(t) by a linear combination of exponential functions for values of t away from zero. A convergence proof has been given and the method is illustrated by three numerical examples. In all these examples the inverse transform f(t) has been approximated by sums of two exponential functions only and very good accuracy is obtained. This method has also been applied to functions of the form $f(t) = \sum_{k=1}^{M} a_k e^{-\beta_k t}$ where a_k and β_k are constants and M is large. The best approximations $g_2(t)$ for such functions are very good both for small *and* large t. This suggest that the method might be useful in electrical network theory where one tries to approximate a complicated circuit by a simpler one.

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