

Uniqueness of Padé Approximants From Series of Orthogonal Polynomials

By Avram Sidi

Abstract. It is proved that whenever a nonlinear Padé approximant, derived from a series of orthogonal polynomials, exists, it is unique.

Let $\phi_r(x)$, $r = 0, 1, 2, \dots$, be a set of polynomials which are orthogonal on an interval $[a, b]$, finite, semi-infinite, or infinite, with weight function $w(x)$, whose integral over any subinterval of $[a, b]$ is positive; i.e.,

$$(1) \quad \int_a^b w(x)\phi_r(x)\phi_s(x) dx = 0 \quad \text{if } r \neq s.$$

Then it is known that $\phi_r(x)$ is a polynomial of degree exactly r .

Suppose now $f(x)$ is a function which has a formal expansion of the form

$$(2) \quad f(x) = \sum_{r=0}^{\infty} a_r \phi_r(x)$$

on $[a, b]$. The (m, n) Padé approximant to $f(x)$ is defined to be the rational function

$$(3) \quad S_{m,n}(x) = \frac{P(x)}{Q(x)} = \frac{\sum_{r=0}^m p_r \phi_r(x)}{\sum_{s=0}^n q_s \phi_s(x)}$$

having an expansion in $\phi_r(x)$, $r = 0, 1, 2, \dots$, which agrees with that of $f(x)$ given in (2) up to and including the term $a_{m+n} \phi_{m+n}(x)$. It is assumed that the polynomials $P(x)$ and $Q(x)$ have no common factor, apart from a constant, and that $Q(x)$ does not vanish on $[a, b]$. It is worth mentioning that the approximations defined above are the ones called "nonlinear Padé approximants" in [2].

THEOREM 1. *If $g(x)$ is any continuous function on $[a, b]$ such that $\int_a^b w(x)g(x)\phi_r(x) dx = 0$, $r = 0, 1, \dots, k-1$, then $g(x)$ either changes sign at least k times in the interval (a, b) or is identically zero.*

The proof of this theorem can be found in [1, p. 110].

As a consequence of Theorem 1, it follows that if $Q(x)$ is nonzero on $[a, b]$, then $q_0 \neq 0$; hence one can normalize $Q(x)$ by taking $q_0 = 1$.

THEOREM 2. *If the (m, n) th nonlinear Padé approximant $P(x)/Q(x)$ to f exists, in the sense of (3), and, after dividing out common factors, if Q is of one sign on $[a, b]$, then it is unique.*

Proof. By the definition of $S_{m,n}(x) = P(x)/Q(x)$ one has

$$(4) \quad f(x) - S_{m,n}(x) = \sum_{r=m+n+1}^{\infty} A_r \phi_r(x).$$

Received July 27, 1976; revised November 1, 1976.

AMS (MOS) subject classifications (1970). Primary 41A20.

Copyright © 1977, American Mathematical Society

If $\bar{S}_{m,n}(x) = \bar{P}(x)/\bar{Q}(x)$ is another (m, n) Padé approximant to (1), then

$$(5) \quad f(x) - \bar{S}_{m,n}(x) = \sum_{r=m+n+1}^{\infty} \bar{A}_r \phi_r(x).$$

Subtracting (4) from (5) one obtains

$$(6) \quad S_{m,n}(x) - \bar{S}_{m,n}(x) = \sum_{r=m+n+1}^{\infty} (\bar{A}_r - A_r) \phi_r(x).$$

Now since $S_{m,n}(x)$ and $\bar{S}_{m,n}(x)$ are continuous on $[a, b]$ so is $D(x) \equiv S_{m,n}(x) - \bar{S}_{m,n}(x)$. Then from (6) it follows that $D(x)$ satisfies $\int_a^b w(x) D(x) \phi_r(x) dx = 0, r = 0, 1, \dots, m+n$. Hence by Theorem 1, $D(x)$ either changes sign at least $m+n+1$ times on (a, b) , or is identically zero there. But

$$(7) \quad D(x) = \frac{P(x)}{Q(x)} - \frac{\bar{P}(x)}{\bar{Q}(x)} = \frac{P(x)\bar{Q}(x) - \bar{P}(x)Q(x)}{Q(x)\bar{Q}(x)},$$

i.e., the numerator of $D(x)$ is a polynomial of degree at most $m+n$, therefore, can have at most $m+n$ zeros on (a, b) . Since $Q(x)$ and $\bar{Q}(x)$ are nonzero on $[a, b]$, $D(x)$ changes sign at most $m+n$ times on (a, b) . Therefore, $D(x) \equiv 0$; hence $S_{m,n}(x) \equiv \bar{S}_{m,n}(x)$. Q.E.D.

So far Padé approximants from Legendre series [2] and Chebyshev series have been considered [3], [4]. As is explained in [2], the determination of the $q_s, s = 1, 2, \dots, n$, in general, involves the solution of n nonlinear equations, the determination of the p_r being trivial then. However, these n equations may have several solutions. But, as is mentioned in [2], only one solution with $Q(x) \neq 0$ on $[a, b]$ has been found for the examples in [2]. By Theorem 2 there is no other solution, and it is at this point that the result of Theorem 2 becomes important.

The author wishes to thank Professor I. M. Longman for encouragement and help.

Department of Geophysics and Planetary Sciences
Tel-Aviv University
Ramat-Aviv, Israel

1. E. W. CHENEY, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966. MR 36 #5568.
2. J. FLEISCHER, "Nonlinear Padé approximants for Legendre series," *J. Mathematical Phys.*, v. 14, 1973, pp. 246-248. MR 48 #592.
3. C. W. CLENSHAW & K. LORD, "Rational approximations from Chebyshev series," *Studies in Numerical Analysis*, Academic Press, London, 1974, pp. 95-113. MR 50 #8914.
4. A. SIDI, "Computation of the Chebyshev-Padé table," *J. Comput. Appl. Math.*, v. 1, 1975, pp. 69-71. MR 52 #4580.