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Some Properties of a Generalization of the Richardson Extrapolation Process

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The Richardson extrapolation process is generalized to cover a large class of sequences. Error bounds for the approximations are obtained and some convergence theorems for two different limiting processes are given. The results are illustrated by an oscillatory infinite integral.

1. Introduction

THE PURPOSE of this paper is to generalize the well-known extrapolation process due to Richardson and to analyse, in some detail, the convergence properties of this generalization. In view of this analysis we shall also give some simple criteria for the efficient implementation of this "generalized Richardson extrapolation process" (GREP). An illustrative numerical example will also be appended.

Definition 1.1. We shall say that a function $A(y)$, defined for $0 < y \leqslant b$, for some $b > 0$, where y can be a discrete or continuous variable, belongs to the set $F^{(m)}$, for some integer $m > 0$, if there exist functions $\phi_k(y)$, $\beta_k(y)$, $0 \leqslant k \leqslant m-1$, and a constant A, such that

$$A = A(y) + \sum_{k=0}^{m-1} \phi_k(y)\beta_k(y), \tag{1.1}$$

where $A = \lim_{y \to 0+} A(y)$ whenever this limit exists, in which case $\lim_{y \to 0+} \phi_k(y) = 0$, $0 \leqslant k \leqslant m-1$, and $\beta_k(\xi)$, as functions of the continuous variable ξ, are continuous for $0 \leqslant \xi \leqslant b$, and for some constants $r_k > 0$, as $\xi \to 0+$, have Poincaré-type asymptotic expansions of the form

$$\beta_k(\xi) \sim \sum_{i=0}^{\infty} \beta_{k,i}\xi^{ir_k}, \quad k = 0, 1, \ldots, m-1. \tag{1.2}$$

If, in addition, the functions $B_k(t) \equiv \beta_k(t^{1/r_k})$, as functions of the continuous variable t, are infinitely differentiable for $0 \leqslant t \leqslant b^{r_k}$, we shall say that $A(y)$ belongs to the set $F_\infty^{(m)}$.

Remark. If $\lim_{y \to 0+} A(y)$ does not exist, then in the nomenclature of Shanks (1955), A is

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said to be the anti-limit of $A(y)$. In this case, for at least one k, $\lim_{k\to 0+} \phi_k(y)$ does not exist as is obvious from (1.1) and (1.2).

The problem is to find (or approximate) A whether it is the limit or the anti-limit of $A(y)$ as $y \to 0+$.

Definition 1.2 (*GREP*). Let $A(y) \in F^{(m)}$, for some integer $m > 0$, with the same notation as in Definition 1.1. Denote the vector $(n_0, n_1, \ldots, n_{m-1})$ by n. Then $A_n^{(m,j)}$, the approximation to A, and the parameters $\bar{\beta}_{k,i}$, $0 \leqslant i \leqslant n_k$, $0 \leqslant k \leqslant m-1$, are defined as the solution of the set of linear equations

$$A_n^{(m,j)} = A(y_l) + \sum_{k=0}^{m-1} \phi_k(y_l) \sum_{i=0}^{n_k} \bar{\beta}_{k,i} y_l^{ir_k}, \quad j \leqslant l \leqslant j+N, \tag{1.3}$$

where

$$N = \sum_{k=0}^{m-1} (n_k+1) \quad \text{and} \quad b \geqslant y_0 > y_1 > y_2 > \ldots,$$

such that $y_l > 0$ for all $l \geqslant 0$ and $\lim_{l\to\infty} y_l = 0$, provided of course that the matrix of the coefficients of equations (1.3) is non-singular.

Remark. The origin of this definition is in the work of Levin & Sidi (1975), which deals with the approximation of some infinite integrals and series. A brief outline of the important results of this work will be given in the next section.

We note that, in general, equations (1.3) have to be solved numerically on a computer by using a linear equation solver. (Only in a few cases can $A_n^{(m,j)}$ be computed in a simple manner and these are the T-transformations of Levin (1973) and ordinary Romberg integration.) In particular, we write equations (1.3) in the form:

$$Qc = d, \tag{1.4}$$

where Q is the matrix of the linear system (1.3) whose first column is the $(N+1)$-dimensional vector $(1, 1, \ldots, 1)^T$ (T denotes transpose), c is the $(N+1)$-dimensional vector of unknowns whose first element is $A_n^{(m,j)}$, and d is the vector $(A(y_j), A(y_{j+1}), \ldots, A(y_{j+N}))^T$. Let the first row of Q^{-1}, the inverse of Q, be the vector $(\gamma_0, \gamma_1, \ldots, \gamma_N)$. Then $c = Q^{-1}d$ implies

$$A_n^{(m,j)} = \sum_{l=0}^{N} \gamma_l A(y_{j+l}), \tag{1.5}$$

and from $Q^{-1}Q = I$ it follows that

$$\sum_{l=0}^{N} \gamma_l = 1. \tag{1.6}$$

In view of (1.6) and (1.5), $A_n^{(m,j)}$ seems to be some kind of an average of the $A(y_i)$. But the weights γ_l of this average depend on the $\phi_k(y_i)$ in a very complicated manner. In some cases (see Levin, 1973; Levin, 1975; Levin & Sidi, 1975) the $\phi_k(y)$ depend on $A(y)$, hence the GREP can, in general, be viewed as a "non-linear summability method" (see Section 4).

In the next section we shall give examples of functions belonging to $F^{(m)}$ and we shall also summarize the basic points of the work of Levin & Sidi (1975) for later use. In Section 3 we shall derive some useful bounds on $A - A_n^{(m,j)}$ for two different limiting processes and give some convergence theorems which are based in part on Sidi (1977, Chapter 5). In Section 4 we shall comment on the two limiting processes of Section 3 in the light of the Silverman–Toeplitz theorem on summability. In Section 5 we shall illustrate the results of Sections 3 and 4 with a numerical example of the use of the results of Levin & Sidi (1975) on infinite integrals.

2. Examples of Functions in $F^{(m)}$

Functions belonging to $F^{(m)}$ come up in a natural way in numerical integration through the Euler–MacLaurin sum formula and generalizations of it. In what follows we assume that the function $g(x)$ is infinitely differentiable on $[0, 1]$ and define the "generalized trapezoidal rule" approximations to

$$I = \int_0^1 G(x)\,dx,$$

where $G(x) = w(x)g(x)\,dx$ by the formula:

$$T(h) = h \sum_{j=1}^{n} G\left(\frac{2j-1+\alpha}{2n}\right), \quad |\alpha| < 1, \quad nh = 1.$$

The following generalizations of the Euler–Maclaurin sum formula are due to Navot (1961, 1962)

(a) If $w(x) = x^\beta$, $-1 < \beta < 0$, then for $\alpha = 0$ for example (midpoint rule)

$$I \sim T(h) + h^2 \sum_{k=0}^{\infty} a_k h^{2k} + h^{1+\beta} \sum_{k=0}^{\infty} b_k h^k.$$

(b) If $w(x) = x^\beta \log x$, $-1 < \beta < 0$, then again for $\alpha = 0$

$$I \sim T(h) + h^2 \sum_{k=0}^{\infty} a_k h^{2k} + h^{1+\beta} \sum_{k=0}^{\infty} b_k h^k + h^{1+\beta} \log h \sum_{k=0}^{\infty} c_k h^k.$$

Similar results for the case $\beta > 0$ for ordinary trapezoidal and Simpson rules have been given by Fox (1967). Fox has also used GREP (of low order) for approximating the singular integrals in (a) and (b) but has not gone as far as developing the method as generally as in Definition 1.2.

The results of Navot (1961, 1962) have been extended by Lyness & Ninham (1967) as follows:

(c) If $w(x) = x^\beta(1-x)^\delta$, $-1 < \beta < 0$ and $-1 < \delta < 0$, then

$$I \sim T(h) + h^{1+\delta} \sum_{k=0}^{\infty} a_k h^k + h^{1+\beta} \sum_{k=0}^{\infty} b_k h^k.$$

(d) $w(x) = x^\beta(1-x)^\delta \log x$, $-1 < \beta < 0$ and $-1 < \delta < 0$, then

$$I \sim T(h) + h^{1+\delta} \sum_{k=0}^{\infty} a_k h^k + h^{1+\beta} \sum_{k=0}^{\infty} b_k h^k + h^{1+\beta} \log h \sum_{k=0}^{\infty} c_k h^k.$$

Generalization of these results to multiple integrals on hypercubes and hyperspheres have been given by Lyness & McHugh (1970) and lately by Lyness (1976).

Recently, two other important examples connected with infinite integrals and series have been given by Levin & Sidi (1975). For future reference their results are summarized below:

Definition 2.1. We shall say that a function $\alpha(x)$, defined for $x > a \geqslant 0$, belongs to the set $A^{(\gamma)}$ if it is infinitely differentiable for all $x > a$ and if, as $x \to \infty$, it has a Poincaré-type asymptotic expansion of the form

$$\alpha(x) \sim x^{\gamma} \sum_{i=0}^{m} \alpha_i/x^i, \tag{2.1}$$

and all its derivatives, as $x \to \infty$, have Poincaré-type asymptotic expansions which are obtained by differentiating the right-hand side of (2.1) term by term.

From this definition it follows that $A^{(\gamma)} \supset A^{(\gamma-1)} \supset A^{(\gamma-2)} \supset \ldots$.

Remark. It also follows that if $\alpha(x)$ is in $A^{(0)}$, then it is infinitely differentiable for all $x > a$ including $x = \infty$ (but not necessarily analytic at $x = \infty$).

THEOREM 2.1. *Let $f(x)$ be defined for $x > a \geqslant 0$, and satisfy a homogeneous linear differential equation of order m of the form*

$$f(x) = \sum_{k=1}^{\infty} p_k(x) f^{(k)}(x), \tag{2.2}$$

where $p_k \in A^{(i_k)}$ but $p_k \notin A^{(i_k-1)}$, such that i_k are integers satisfying $i_k \leqslant k$, $1 \leqslant k \leqslant m$. Let also

$$\lim_{x \to \infty} p_k^{(i-1)}(x) f^{(k-i)}(x) = 0, \quad i \leqslant k \leqslant m, \quad 1 \leqslant i \leqslant m. \tag{2.3}$$

If for any integer $l = -1, 1, 2, 3, \ldots$,

$$\sum_{k=1}^{m} l(l-1)\ldots(l-k+1)\bar{p}_k \neq 1, \tag{2.4}$$

where

$$\bar{p}_k = \lim_{x \to \infty} x^{-k} p_k(x), \quad 1 \leqslant k \leqslant m, \tag{2.5}$$

then

$$\int_a^{\infty} f(t)\, dt = \int_a^{x} f(t)\, dt + \sum_{k=0}^{m-1} f^{(k)}(x) x^{\rho_k} \theta_k(x), \tag{2.6}$$

where $\theta_k \in A^{(0)}$ and ρ_k are integers satisfying

$$\rho_k \leqslant \max(i_{k+1}, i_{k+2}-1, \ldots, i_m - m+k+1), \quad 0 \leqslant k \leqslant m-1. \tag{2.7}$$

It also follows that $\lim_{x \to \infty} f^{(k)}(x) x^{\rho_k} = 0$, $0 \leqslant k \leqslant m-1$.

THEOREM 2.2. *Let the elements of the sequence $\{f_r\}_{r=1}^{\infty}$ satisfy a homogeneous linear* *difference equation of order m of the form*

$$f_r = \sum_{k=1}^{m} p_k(r)\Delta^k f_r, \tag{2.8}$$

where Δ *is the forward difference operator operating on the index* r, *and* $p_k(x)$, *as functions of the continuous variable* x, *are in* $A^{(i_k)}$ *but not in* $A^{(i_k-1)}$, *such that* i_k *are integers satisfying* $i_k \leqslant k$, $1 \leqslant k \leqslant m$. *Let also*

$$\lim_{r\to\infty} [\Delta^{i-1}p_k(r)][\Delta^{k-i}f_r] = 0, \quad i \leqslant k \leqslant m, \quad 1 \leqslant i \leqslant m. \tag{2.9}$$

If for every integer $l = -1, 1, 2, 3, \ldots$ (2.4) *holds together with* (2.5) *then*

$$\sum_{r=1}^{\infty} f_r = \sum_{r=1}^{R-1} f_r + \sum_{k=0}^{m-1} (\Delta^k f_R)R^{\rho_k}\psi_k(R), \tag{2.10}$$

where $\psi_k \in A^{(0)}$ *and* ρ_k *are integers satisfying* (2.7). *It also follows that* $\lim_{R\to\infty} (\Delta^k f_R)R^{\rho_k} = 0$, $0 \leqslant k \leqslant m-1$.

The proofs of both theorems are by construction and can be found in Levin & Sidi (1975), see also Sidi (1978) for the case $m = 1$ of Theorem 2.2. Using these theorems the D- and d-approximations are defined as in Definition 1.2.

Definition 2.2. Let $f(t)$ be as in Theorem 2.1 with the same notation. The approximation $D_n^{(m,j)}$ to

$$\int_a^\infty f(t)\,dt,$$

where n denotes the vector $(n_0, n_1, \ldots, n_{m-1})$, and the constants $\bar{\theta}_{k,i}$, $0 \leqslant i \leqslant n_k$, $0 \leqslant k \leqslant m-1$, are defined as the solution of the linear equations

$$D_n^{(m,j)} = \int_a^{x_l} f(t)\,dt + \sum_{k=0}^{m-1} f^{(k)}(x_l)x_l^{\rho_k} \sum_{i=0}^{n_k} \bar{\theta}_{k,i}/x_l^i, \quad j \leqslant l \leqslant j+N, \tag{2.11}$$

where $N = \sum_{k=0}^{m-1} (n_k+1)$ and $a < x_0 < x_1 < \ldots$, such that $\lim_{l\to\infty} x_l = \infty$, provided the matrix of equations (2.11) is non-singular. (If the ρ_k are not known exactly, then they can be replaced in (2.11) by $\sigma_k = \min\,(k+1, s_k)$, where

$$s_k = \max\,\{s|s \text{ integer}, \lim_{x\to\infty} x^s f^{(k)}(x) = 0\}.$$

Then $\rho_k \leqslant \sigma_k \leqslant k+1$ and $\lim_{x\to\infty} f^{(k)}(x)x^{\sigma_k} = 0$, $0 \leqslant k \leqslant m-1$.) The finite integrals

$$\int_0^x f(t)\,dt$$

can be computed very accurately by using a low order Gaussian rule.

Definition 2.3. Let the sequence $\{f_r\}_{r=1}^{\infty}$ be as in Theorem 2.2 with the same notation.

The approximation $d_n^{(m,j)}$ to $\sum_{r=1}^{\infty} f_r$, where n denotes the vector $(n_0, n_1, \ldots, n_{m-1})$, and the constants $\bar{\psi}_{k,i}$, $0 \leqslant i \leqslant n_k$, $0 \leqslant i \leqslant m-1$, are defined as the solution of the linear equations

$$d_n^{(m,j)} = \sum_{r=1}^{R_l-1} f_r + \sum_{k=0}^{m-1} (\Delta^k f_{R_l}) R_l^{\rho_k} \sum_{i=0}^{n_k} \bar{\psi}_{k,i}/R_l^i, \quad j \leqslant l \leqslant j+N, \tag{2.12}$$

where $N = \sum_{k=0}^{m-1} (n_k+1)$ and $1 \leqslant R_0 < R_1 < \ldots$, provided the matrix of equations (2.12) is non-singular. (If the ρ_k are not known exactly, then they can be replaced in (2.12) by $\sigma_k = \min(k+1, s_k)$, where

$$s_k = \max\{s | s \text{ integer}, \lim_{R\to\infty} R^s(\Delta^k f_R) = 0\}.$$

Then $\rho_k \leqslant \sigma_k \leqslant k+1$ and $\lim_{R\to\infty} (\Delta^k f_R) R^{\sigma_k} = 0$, $0 \leqslant k \leqslant m-1$).

It is obvious that the processes described by the approximations $D_n^{(m,j)}$ and $d_n^{(m,j)}$ are exactly the GREP defined in Definition 1.2, provided one lets

$$y = 1/x, \qquad A(y) \equiv \int_a^x f(t)\,dt \quad \text{and} \quad \phi_k(y) \equiv f^{(k)}(x) x^{\rho_k}$$

in Definition 2.2 and

$$y = 1/R, \qquad A(y) \equiv \sum_{r=1}^{R-1} f_r \quad \text{and} \quad \phi_k(y) \equiv (\Delta^k f_R) R^{\rho_k}$$

in Definition 2.3.

It is worth noting that the D- and d-approximations have proved to be extremely efficient for accelerating the convergence of infinite integrals and series of different kinds which could not be handled by the well-known methods of Euler (see Bromwich, 1942, p. 62; Shanks, 1955), the G-transformations of Gray, Atchison & McWilliams (1971). For numerical examples of varying degree of complexity, see Levin & Sidi (1975). The d-approximations for the case $m = 1$, are originally due to Levin (1973) and some aspects of their convergence theory have been analysed in Sidi (1978, 1979). Also the case $m = 1$ of the D-approximations for Fourier integrals is due to Levin (1975).

3. Error Bounds and Convergence Theorems

In this section we shall analyse the convergence properties of $A_n^{(m,j)}$ for two kinds of limiting processes:

(a) Process 1: n fixed, $j \to \infty$.
(b) Process 2: j fixed, $n \to \infty$, i.e. $n_k \to \infty$, $k = 0, \ldots, m-1$.

We shall be using the notation of Definitions 1.1 and 1.2, and for convenience we shall denote $\mu = m-1$.

If the equations in (1.3) are solved using Cramer's rule, then for $A_n^{(m,j)}$ we obtain

$A_n^{(m,j)} = \det M/\det K$ where M and K are $(N+1)\times(N+1)$ matrices. The $(l+1)$th column of M is the $(N+1)$-dimensional vector

$$(A(y_{j+l}), \phi_0(y_{j+l})v_{n_0}^0(y_{j+l}), \phi_1(y_{j+l})v_{n_1}^1(y_{j+l}), \ldots, \phi_\mu(y_{j+l})v_{n_\mu}^\mu(y_{j+l}))^T, \quad l = 0, 1, \ldots, N, \tag{3.1}$$

where T denotes transpose and $v_s^k(y)$ are the $(s+1)$-dimensional row vectors given by

$$v_s^k(y) = (1, y^{r_k}, y^{2r_k}, \ldots, y^{sr_k}), \quad k = 0, \ldots, \mu. \tag{3.2}$$

For example, for $m = 2$ ($\mu = 1$), $n_0 = 1$, $n_1 = 2$, the matrix M takes the form

$$M = \begin{bmatrix} A(y_j) & A(y_{j+1}) \ldots & A(y_{j+5}) \\ \phi_0(y_j) & \ldots\ldots\ldots & \phi_0(y_{j+5}) \\ \phi_0(y_j)y_j^{r_0} & \ldots\ldots\ldots & \phi_0(y_{j+5})y_{j+5}^{r_0} \\ \phi_1(y_j) & \ldots\ldots\ldots & \phi_1(y_{j+5}) \\ \phi_1(y_j)y_j^{r_1} & \ldots\ldots\ldots & \phi_1(y_{j+5})y_{j+5}^{r_1} \\ \phi_1(y_j)y_j^{2r_1} & \ldots\ldots\ldots & \phi_1(y_{j+5})y_{j+5}^{2r_1} \end{bmatrix}.$$

The matrix K is obtained from M by replacing the first row of M by the $(N+1)$-dimensional vector $(1, 1, \ldots, 1)$.

If we now denote the cofactor of $A(y_{j+l})$ in the first row of M by δ_l and expand $\det M$ and $\det K$ with respect to their first rows, we can write

$$A_n^{(m,j)} = \frac{\sum_{l=0}^{N} \delta_l A(y_{j+l})}{\sum_{l=0}^{N} \delta_l}. \tag{3.3}$$

From (3.3) and (1.5) it is clear that

$$\gamma_l = \delta_l \Big/ \sum_{i=0}^{N} \delta_i, \quad l = 0, 1, \ldots, N, \tag{3.4}$$

and (1.6) is again seen to be trivially satisfied.

LEMMA 3.1. *The error in the approximation $A_n^{(m,j)}$ satisfies the equality*

$$A - A_n^{(m,j)} = \sum_{l=0}^{N} \gamma_l \sum_{k=0}^{\mu} \phi_k(y_{j+l})\beta_k(y_{j+l}). \tag{3.5}$$

Proof. The result follows by substituting (1.1) in (1.5) and using (1.6).

Corollary. With the help of (3.4), (3.5) can be re-expressed in the form

$$A - A_n^{(m,j)} = \frac{\sum_{l=0}^{N} \delta_l \sum_{k=0}^{\mu} \phi_k(y_{j+l})\beta_k(y_{j+l})}{\sum_{l=0}^{N} \delta_l} = \frac{\det M_1}{\det K}, \tag{3.6}$$

where M_1 is the matrix obtained from M by replacing the first row of M by the row vector

$$\left(\sum_{k=0}^{\mu} \phi_k(y_j)\beta_k(y_j), \ldots, \sum_{k=0}^{\mu} \phi_k(y_{j+N})\beta_k(y_{j+N})\right). \tag{3.7}$$

For future reference we shall number the 2nd, ..., $(N+1)$th rows of the matrix M_1 (and/or K) with two indices as follows: we shall give the 2nd row the indices (0, 0), the 3rd row, the indices (0, 1), ..., the (n_0+2)th row the indices $(0, n_0)$. In the same manner we shall give the next (n_1+1) rows the indices $(1, 0)$, $(1, 1)$, ..., $(1, n_1)$, etc. Then the last $(n_\mu+1)$ rows will have the indices $(\mu, 0)$, ..., (μ, n_μ). Thus the row $(\phi_k(y_j)y_j^{ir_k}, \ldots, \phi_k(y_{j+N})y_{j+N}^{ir_k})$ has the indices (k, i).

Starting with (3.6) we shall now analyse the convergence properties of the two limiting processes defined in the beginning of this section.

(*a*) *Process* 1.

THEOREM 3.1. *The approximation* $A_n^{(m,j)}$ *satisfies*

$$A - A_n^{(m,j)} = \sum_{l=0}^{N} \gamma_l \sum_{k=0}^{\mu} \phi_k(y_{j+l}) w_{n_k}^k(y_{j+l}), \tag{3.8}$$

where

$$w_s^k(y) = \beta_k(y) - \sum_{i=0}^{s} \beta_{k,i} y^{ir_k}, \quad k = 0, \ldots, \mu, \tag{3.9}$$

with $\beta_{k,i}$ *as defined in* (1.2).

Proof. Let us subtract from the first row of M_1 the sum of the products of the rows (k, i) by $\beta_{k,i}$, $i = 0, 1, \ldots, n_k$, $k = 0, 1, \ldots, \mu$, and leave the 2nd, ..., $(N+1)$th rows unchanged. Let us denote the new matrix by M'. The first row of M', by (3.7) and (3.9), is

$$\left(\sum_{k=0}^{\mu} \phi_k(y_j) w_{n_k}^k(y_j), \ldots, \sum_{k=0}^{\mu} \phi_k(y_{j+N}) w_{n_k}^k(y_{j+N})\right) \tag{3.10}$$

and furthermore $\det M' = \det M_1$, hence $A - A_n^{(m,j)} = \det M'/\det K$. If we now use the fact that the cofactors of the first row of M' are still the δ_l and expand $\det M'$ and $\det K$ with respect to their first rows, (3.8) follows.

Remark. The assumption that $\beta_k(y)$ have Poincaré-type asymptotic expansions implies that

$$w_s^k(y) = 0(y^{(s+1)r_k}) \quad \text{as } y \to 0+.$$

This together with (3.8) implies that

$$A - A_n^{(m,j)} = \sum_{l=0}^{N} \gamma_l \sum_{k=0}^{\mu} \phi_k(y_{j+l}) 0(y_{j+l}^{(n_k+1)r_k}) \quad \text{as } j \to \infty, \tag{3.11}$$

which shows that $A_n^{(m,j)}$ is indeed the generalization of the Richardson extrapolation process.

The following result can now be easily obtained from (3.8).

Corollary 1. The approximation $A_n^{(m,j)}$ satisfies the inequality

$$|A - A_n^{(m,j)}| \leqslant \left(\sum_{l=0}^{N} |\gamma_l| \right) \sum_{k=0}^{\mu} \max_{j \leqslant i \leqslant j+N} |\phi_k(y_i)| \max_{j \leqslant i \leqslant j+N} |w_{n_k}^k(y_i)|. \tag{3.12}$$

Corollary 2. As $j \to \infty$, hence as $y_j \to 0+$, $A_n^{(m,j)}$ satisfies the inequality

$$|A - A_n^{(m,j)}| \leqslant \left(\sum_{l=0}^{N} |\gamma_l| \right) \sum_{k=0}^{\mu} \varepsilon_k^{(j)} 0(y_j^{(n_k+1)r_k}), \tag{3.13}$$

where $\varepsilon_k^{(j)} = \max_{j \leqslant i \leqslant j+N} |\phi_k(y_i)|$, and if $\lim_{y \to 0+} A(y) = A$, then

$$|A - A_n^{(m,j)}| \leqslant \left(\sum_{l=0}^{N} |\gamma_l| \right) o(y_j^{\alpha}) \quad \text{as } j \to \infty, \tag{3.14}$$

where

$$\alpha = \min_{0 \leqslant k \leqslant \mu} \{(n_k + 1) r_k\}. \tag{3.15}$$

Proof. (3.13) follows from (3.12) easily if we recall that

$$w_s^k(y) = \beta_{k,s+1} y^{(s+1)r_k} + 0(y^{(s+2)r_k}) \quad \text{as } y \to 0+,$$

since (1.2) is a Poincaré-type asymptotic expansion, and also that $y_j > y_{j+1} > \dots$. From Definition 1.1, if $\lim_{y \to 0+} A(y) = A$, then $\lim_{y \to 0+} \phi_k(y) = 0$, in which case $\varepsilon_k^{(j)} = o(1)$ as $j \to \infty$. Using this together with (3.15), (3.14) now follows.

As an immediate consequence of Corollary 2, we obtain the following:

Corollary 3. If $\lim_{y \to 0+} A(y) = A$ and

$$\sup_j \left(\sum_{l=0}^{N} |\gamma_l| \right) \leqslant L < \infty, \tag{3.16}$$

then $|A - A_n^{(m,j)}| \to 0$ as $j \to \infty$ and the rate of convergence is given by

$$|A - A_n^{(m,j)}| = o(y_j^{\alpha}), \tag{3.17}$$

at least.

Remark. If $\phi_k(y) = 0(y^{\bar{r}_k})$ as $y \to 0+$ for some constants $\bar{r}_k$ and if (3.16) holds, then (3.17) holds with $\alpha = \min_{0 \leqslant k \leqslant \mu} \{\tau_k\}$, where $\tau_k = \bar{r}_k + (n_k + 1) r_k$, $0 \leqslant k \leqslant \mu$, as can be seen from (3.13). This means that if n_k are sufficiently large such that $\tau_k > 0$ for all k, then as $j \to \infty$ $A - A_n^{(m,j)} \to 0$ provided (3.16) is satisfied, whether $\lim_{y \to 0+} A(y)$ exists or not. (For an example of this situation see Sidi, 1978.)

(*b*) *Process* 2.

THEOREM 3.2. *The approximation* $A_n^{(m,j)}$ *satisfies*

$$A-A_n^{(m,j)}=\sum_{l=0}^{N}\gamma_l\sum_{k=0}^{\mu}\phi_k(y_{j+l})u_{n_k}^k(y_{j+l}), \tag{3.18}$$

where

$$u_s^k(y)=\beta_k(y)-\pi_s^k(y),\quad k=0,\ldots,\mu. \tag{3.19}$$

such that

$$\pi_s^k(y)=\sum_{i=0}^{s}\alpha_{s,i}^k y^{ir_k} \tag{3.20}$$

is the best polynomial approximation of degree s to $\beta_k(y)$ *in powers of* y^{r_k}, $0\leqslant k\leqslant\mu$, *on the interval* $[0,y_j]$.

Proof. Let us subtract from the first row of M_1 the sum of the products of the rows (k,i) by $\alpha_{n_k,i}^k$, $i=0,1,\ldots,n_k$, $k=0,1,\ldots,\mu$, and leave 2nd, . . ., $(N+1)$th rows unchanged. Let us denote the new matrix by M''. The first row of M'', by (3.19) and (3.20), is

$$\left(\sum_{k=0}^{\mu}\phi_k(y_j)u_{n_k}^k(y_j),\ldots,\sum_{k=0}^{\mu}\phi_k(y_{j+N})u_{n_k}^k(y_{j+N})\right), \tag{3.21}$$

and furthermore, $\det M''=\det M_1$. (3.18) is now obtained by expanding $\det M''$ and $\det K$ with respect to their first rows.

The following results can easily be derived from (3.18).

Corollary 1. The approximation $A_n^{(m,j)}$ satisfies the inequality

$$|A-A_n^{(m,j)}|\leqslant\left(\sum_{l=0}^{N}|\gamma_l|\right)\sum_{k=0}^{\mu}\max_{j\leqslant i\leqslant j+N}|\phi_k(y_i)|\max_{j\leqslant i\leqslant\infty}|u_{n_k}^k(y_i)|. \tag{3.22}$$

Corollary 2. If $A(y)\in F_\infty^{(m)}$ (see Definition 1.1), then as $n_k\to\infty$, $0\leqslant k\leqslant\mu$,

$$|A-A_n^{(m,j)}|\leqslant\left(\sum_{l=0}^{N}|\gamma_l|\right)\sum_{k=0}^{\mu}\eta_k^{(j)}o(n_k^{-\lambda_k}),\quad\text{any }\lambda_k>0, \tag{3.23}$$

where $\eta_k^{(j)}=\max_{j\leqslant i\leqslant j+N}|\phi_k(y_i)|$, $0\leqslant k\leqslant\mu$, and if $\lim_{y\to0+}A(y)=A$, then

$$|A-A_n^{(m,j)}|\leqslant\left(\sum_{l=0}^{N}|\gamma_l|\right)o(\nu^{-\lambda})\quad\text{any }\lambda>0, \tag{3.24}$$

where

$$\nu=\min(n_0,n_1,\ldots,n_\mu).$$

Proof. The proof of (3.23) follows from (3.22) and the fact that $\max_{0\leqslant y\leqslant y_j}|u_s^k(y)|=o(s^{-\lambda})$ as $s\to\infty$, for any $\lambda>0$, which is a standard result of approximation theory. The proof of (3.24) follows from (3.23) and the fact that $\lim_{y\to0+}\phi_k(y)=0$, hence $\phi_k(y)=0(1)$ for $0\leqslant y\leqslant y_j$, $0\leqslant k\leqslant\mu$.

Corollary 3. If $A(y) \in F_\infty^{(m)}$ and $\lim_{y \to 0+} A(y) = A$, and

$$\sup_{n_0, n_1, \ldots, n_\mu} \left(\sum_{l=0}^{N} |\gamma_l| \right) \leqslant L < \infty, \tag{3.25}$$

then, as $n_k \to \infty$, $0 \leqslant k \leqslant \mu$,

$$|A - A_n^{(m,j)}| = o(v^{-\lambda}), \quad \text{for any } \lambda > 0. \tag{3.26}$$

Remark. If $A(y) \in F_\infty^{(m)}$, $\phi_k(y) = 0(y^{\bar{r}_k})$ as $y \to 0+$, $n_k = 0(v)$ as $n \to \infty$, $0 \leqslant k \leqslant \mu$ and if $\lim_{y \to 0+} A(y)$ does not exist, i.e. $\lim_{y \to 0+} |\phi_k(y)| = \infty$, for at least one k, then (3.24) still holds provided y_i are chosen such that $y_i = 0(i^p)$ for some $p < 0$. If, furthermore, (3.25) is satisfied, then (3.26) holds too.

There are two immediate practical conclusions that one can draw from Theorems 3.1 and 3.2 and their corollaries.

1. The smaller $\sum_{l=0}^{N} |\gamma_l|$ the smaller the error bounds are expected to be in (3.12) and (3.22). Now $\sum_{l=0}^{N} |\gamma_l| \geqslant 1$, therefore one should adjust the y_i in (1.3) such that $\sum_{l=0}^{N} |\gamma_l|$ will be small and as close to 1 as possible. The implications of this from the numerical point of view will be taken up in Section 6.
2. As can be seen from the corollaries to Theorems 3.1 and 3.2, the fact that $u_s^k(y) \to 0$ as $s \to \infty$ much faster than $w_s^k(y) \to 0$ as $y \to 0+$, suggests that Process 2 would have much better convergence properties than Process 1.

Both of these conclusions seem to be correct as a large number of numerical examples of various kinds have shown. One such example will be given in Section 5. For a theoretical verification of the last conclusion for Levin's transformation see Sidi (1978, 1979).

Finally, we note that it is difficult to check rigorously under what circumstances conditions (3.16) and (3.25) hold. (As a matter of fact, in general, no simple expression for the γ_l is available.) However, in some special cases the behaviour of $\sum_{l=0}^{N} |\gamma_l|$ can be analysed quite simply (see Sidi, 1979). We shall say more on this point in the next section.

Before we close this section, we shall give another closed expression for the error $A - A_n^{(m,j)}$.

THEOREM 3.3. *Let the functions $\beta_k(\xi)$ in Definition 1.2 be such that $\tilde{\beta}_k(\xi) \equiv \beta_k(\xi)\xi^{r_k}$ are of the form*

$$\tilde{\beta}_k(\xi) = \mathscr{L}[\omega_k(t); \xi^{-r_k}] = \int_0^\infty \exp(-t/\xi^{r_k})\, \omega_k(t)\, dt, \quad k = 0, 1, \ldots, \mu. \tag{3.27}$$

(*i.e. they are Laplace transforms*), *where the functions $\omega_k(t)$ are infinitely differentiable* *on* $[0, \infty)$. *Then the error satisfies*

$$A - A_n^{(m,j)} = \sum_{l=0}^{N} \gamma_l \sum_{k=0}^{\mu} \phi_k(y_{j+l}) y_{j+l}^{n_k r_k} \mathscr{L}[\omega_k^{(n_k+1)}(t); y_{j+l}^{-r_k}]. \tag{3.28}$$

Proof. From Laplace transform theory we have

$$\beta_k(\xi) = \frac{\tilde{\beta}_k(\xi)}{\xi^{r_k}} = \xi^{n_k r_k} \mathscr{L}[\omega_k^{(n_k+1)}(t); \xi^{-r_k}] + \sum_{i=0}^{n_k} \omega_k^{(i)}(0) \xi^{i r_k}, \quad k = 0, 1, \ldots, \mu. \tag{3.29}$$

Now, subtracting from the first row of the matrix M_1 the sum of the products of the rows (k, i) by $\omega_k^{(i)}(0)$, $i = 0, 1, \ldots, n_k$, $k = 0, 1, \ldots, \mu$, and repeating the arguments which lead to (3.8) and (3.18) and using (3.29) the result follows.

Remark. Using Watson's lemma in (3.27), it is easy to identify the $\omega_k^{(i)}(0)$ as $\beta_{k,i}$, $i = 0, 1, \ldots$, $k = 0, 1, \ldots, \mu$.

We note that Theorem 3.3 is the generalization of Theorem 4.1 in Sidi (1978). The latter has enabled the author to prove convergence theorems on Process 2 for the $d_n^{(1,j)}$ approximations (or Levin's T-transformations), which are more powerful than Theorem 3.2. Applications of Theorem 3.3 in connection with the D-transformation for Fourier & Hankel transforms will be taken up in a future paper.

4. GREP as a Summability Method

Definition 4.1. The infinite matrix

$$\Lambda = \begin{bmatrix} \lambda_{00} & \lambda_{01} & \lambda_{02} & \cdots \\ \lambda_{10} & \lambda_{11} & \lambda_{12} & \cdots \\ \lambda_{20} & \lambda_{21} & \lambda_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{4.1}$$

is said to be regular if for every convergent sequence $\{z_k\}_{k=0}^{\infty}$ of numbers, the sequence $\{\zeta_n\}_{n=0}^{\infty}$, where

$$\zeta_n = \sum_{k=0}^{\infty} \lambda_{nk} z_k,$$

converges and to the same limit.

THEOREM 4.1 (Silverman–Toeplitz). *The infinite matrix* Λ *in* (3.1) *is regular if and only if*

(1) $$\lim_{n\to\infty} \lambda_{nk} = 0 \qquad \text{for all } k,$$

(2) $$\lim_{n\to\infty} \sum_{k=0}^{\infty} \lambda_{nk} = 1$$

(3) $$\sup_n \sum_{k=0}^{\infty} |\lambda_{nk}| \leqslant L < \infty \quad \text{for some } L > 0.$$

The proof of this theorem can be found in Powell & Shah (1972, pp. 23–27).

It turns out that for Process 1 and Process 2 we can define infinite matrices B and C of the form (4.1). For simplicity we take $n_0 = n_1 = \ldots = n_\mu = \nu$ and denote $A_n^{(m,j)}$ by $A_\nu^{(m,j)}$ and the corresponding γ_l by $\gamma_{\nu,l}^{(j)}$. Now $N = m(\nu+1)$.

For Process 1 we define the matrix B as follows:

$$b_{j,k} = \begin{cases} 0 & 0 \leqslant k < j \quad \text{if } j \neq 0 \\ \gamma_{\nu,k-j}^{(j)} & j \leqslant k \leqslant j+N \\ 0 & k > j+N. \end{cases} \tag{4.2}$$

As can be seen from (4.2) the matrix B is a band matrix since N is fixed and

$$A_\nu^{(m,j)} = \sum_{k=0}^{\infty} b_{j,k} A(y_k), \quad j = 0, 1, 2. \ldots.$$

For Process 2 the matrix C is defined as follows:

$$c_{\nu,l} = \begin{cases} 0 & 0 \leqslant l < j \quad \text{if } j \neq 0 \\ \gamma_{\nu,l-j}^{(j)} & j \leqslant l \leqslant j+N \\ 0 & l > j+N. \end{cases} \tag{4.3}$$

From (4.3) it follows that C is a "stair case" type matrix in that each row has m nonzero elements more than the previous row, since j is fixed, and N increases by m when ν increases by 1. Also, for this case

$$A_\nu^{(m,j)} = \sum_{l=0}^{\infty} c_{\nu,l} A(y_l), \quad \nu = 0, 1, 2, \ldots.$$

Now the three conditions of Theorem 4.1 are sufficient for the matrix Λ to be regular. They become necessary if we require that the sequence $\{\zeta_n\}_{n=0}^{\infty}$ converge to the limit of $\{z_k\}_{k=0}^{\infty}$ for *any* $\{z_k\}_{k=0}^{\infty}$. In our case the matrices B and C are applied only to very special sequences. This then raises the question whether the matrices B and C have to satisfy the conditions of Theorem 4.1 and under what circumstances they do. Now it can easily be verified that the second condition of Theorem 4.1 is automatically satisfied by both B and C since

$$\sum_{k=0}^{\infty} b_{j,k} = \sum_{l=0}^{N} \gamma_l = 1 \quad \text{and} \quad \sum_{l=0}^{\infty} c_{\nu,l} = \sum_{l=0}^{N} \gamma_l = 1.$$

The first condition is automatically satisfied by matrix B since $b_{j,k} = 0$ for j large enough and k fixed. Numerous computations for convergent infinite integrals and series, by using the D- and d-transformations of Levin & Sidi (1975) have shown that this condition is satisfied for Process 2 too, although no proof of this is available yet. The same computations have shown that the third condition of Theorem 4.1, which is just (3.16) for Process 1 and (3.25) for Process 2, is satisfied when the functions $\phi_k(y)$ are all oscillatory as $y \to 0+$ and it is not satisfied when some of the $\phi_k(y)$ are monotonic as $y \to 0+$ and $\phi_k(y_l)$ vary slowly as l increases. No proof of this observation is available yet either. For some simple cases with $m = 1$, like the ordinary Romberg integration (see Bauer, Rutishauser & Stiefel, 1963), and the t- and

u-transformations of Levin (1973) (see also Sidi, 1978) as applied to oscillatory sequences, all three conditions of Theorem 4.1 can be shown to hold.

Finally, we note that the numerical experience gained by the use of the D- and d-transformations and some theoretical results in Sidi (1978, 1979) suggest that whether the third condition of Theorem 4.1 is satisfied or not, convergence takes place in both Process 1 and Process 2, in some cases. The numerical rate of convergence, however, depends very strongly on the size of $\sum_{l=0}^{N} |\gamma_l|$ and/or on the rate at which $\gamma_l \to 0$ as $n_k \to \infty$, $k = 0, \ldots, \mu$, for fixed l.

Actually, the following have been observed to be satisfied simultaneously:

(1) $A_n^{(m,j)} \to A$ quickly (both Process 1 and Process 2).
(2) $\gamma_l \to 0$ quickly as $n_k \to \infty$, $k = 0, \ldots, \mu$, l fixed (Process 2).
(3) $\sum_{l=0}^{N} |\gamma_l|$ is small, and if it increases its increase is slow (Process 1 and Process 2).

The numerical example in the next section will clarify these points further.

5. A Numerical Example

We shall now apply the D-transformation of Levin & Sidi to the integral

$$\int_0^\infty J_0(t)\,dt = 1,$$

where $J_0(t)$ is the Bessel function of the first kind of order zero. Now $f(t) = J_0(t)$ satisfies all the conditions of Theorem 2.1 with $m = 2$, $i_0 = -1$, $i_1 = 0$ as can be seen from Bessel's equation of order zero, $f = -(1/t)f' - f''$. Therefore, a relation of the form (2.6) exists with $\rho_0 \leqslant -1$, $\rho_1 \leqslant 0$. See Longman (1959), as $x \to \infty$,

$$\int_x^\infty J_0(t)\,dt \sim J_0(x)\left(\frac{1}{x} - \frac{1^2.3}{x^3} + \frac{1^2.3^2.5}{x^5} - \ldots\right) + [J_0(x)]'\left(1 - \frac{1^2}{x^2} + \frac{1^2.3^2}{x^4} - \ldots\right).$$

Hence $\rho_0 = -1$, $\rho_1 = 0$ exactly. σ_0 and σ_1, by their definition, turn out to be equal to 0. Computing the finite integrals

$$\int_0^{x_l} f(t)\,dt$$

numerically (and accurately) and solving equations (2.11) we obtain the approximation $D_n^{(2,j)}$ to

$$\int_0^\infty J_0(t)\,dt = 1.$$

We consider, as in Section 4, the approximations with $n_0 = n_1 = \nu$ and use the notation therein. For further details the reader is referred to Levin & Sidi (1975).

TABLE 1

	$\nu = 2$		$\nu = 4$	
j	$\sum_{l=0}^{N} \lvert\gamma_{\nu,l}^{(j)}\rvert$	$\lvert 1-D_\nu^{(2,j)}\rvert$	$\sum_{l=0}^{N} \lvert\gamma_{\nu,l}^{(j)}\rvert$	$\lvert 1-D_\nu^{(2,j)}\rvert$
1	2·662	9×10^{-5}	4·048	1×10^{-7}
3	2·006	1×10^{-5}	2·668	3×10^{-8}
5	1·714	2×10^{-6}	2·147	4×10^{-9}
7	1·547	4×10^{-7}	1·876	7×10^{-10}
9	1·443	1×10^{-7}	1·723	2×10^{-10}

(*a*) *Process* 1

In Table 1 we exhibit some of the results obtained for $D_\nu^{(2,j)}$ and $\sum_{l=0}^{N} |\gamma_{\nu,l}^{(j)}|$ with $\nu = 2, 4$, using $x_l = 3(l+1)/2$, $l = 0, 1, \ldots$.

(*b*) *Process* 2

In Tables 2a, 2b and 2c we exhibit some of the results obtained for $\gamma_{\nu,0}^{(0)}$, $\sum_{l=0}^{N} |\gamma_{\nu,l}^{(0)}|$ and $D_\nu^{(2,0)}$ using $x_l = l+1$, $x_l = 3(l+1)/2$, $x_l = 2(l+1)$, respectively, $l = 0, 1, 2, \ldots$.

Table 3 exhibits part of the matrix C for Process 2 (see previous section) obtained by using $x_l = 3(l+1)/2$, $l = 0, 1, 2, \ldots$.

Let us now compare Process 1 and Process 2 with the help of Tables 1 and 2b, which have been computed by taking $x_l = 3(l+1)/2$, $l = 0, 1, \ldots$, hence by using the same sequence of finite integrals

$$\int_0^{x_l} J_0(t)\, dt.$$

As is seen the rate of convergence for Process 2 is much greater than that of Process 1. Also if we compare two approximants, one from each table, whose computations are done by using about the same number of finite integrals, like $D_2^{(2,9)}$ in Table 1 (16 finite integrals) and $D_7^{(2,0)}$ (17 finite integrals), we see that Process 2 is superior to Process 1 in this kind of comparison too. This second kind of comparison becomes especially meaningful for the d-transformation for infinite series, since it implies that given a finite number of terms of the series, Process 2 gives much better accuracy than Process 1. This observation is in agreement with one of the two conclusions of Section 3.

6. Remarks on Computational Aspects

It turns out that the matrix Q of equations (1.3), as the n_k become larger, becomes very ill-conditioned. This causes the computed values of the $\bar{\beta}_{k,i}$ to be very inaccurate.

TABLE 2a

$x_l = l+1, \quad l = 0, 1, 2, \ldots$

ν	$\gamma_{\nu,0}^{(0)}$	$\sum_{l=0}^{N} \lvert\gamma_{\nu,l}^{(0)}\rvert$	$\lvert 1-D_\nu^{(2,0)}\rvert$
1	$2{\cdot}5 \times 10^{-1}$	$3{\cdot}16 \times 10^{1}$	2×10^{-2}
3	$-4{\cdot}1 \times 10^{-4}$	$1{\cdot}09 \times 10^{1}$	2×10^{-4}
5	$7{\cdot}2 \times 10^{-7}$	$4{\cdot}11 \times 10^{2}$	2×10^{-6}
7	$6{\cdot}2 \times 10^{-7}$	$6{\cdot}93 \times 10^{5}$	3×10^{-7}
9	$-1{\cdot}0 \times 10^{-12}$	$4{\cdot}30 \times 10^{3}$	1×10^{-8}

TABLE 2b

$x_l = 3(l+1)/2, \quad l = 0, 1, 2, \ldots$

ν	$\gamma_{\nu,0}^{(0)}$	$\sum_{l=0}^{N} \lvert\gamma_{\nu,l}^{(0)}\rvert$	$\lvert 1-D_\nu^{(2,0)}\rvert$
1	$1{\cdot}5 \times 10^{-2}$	1	4×10^{-4}
3	$2{\cdot}9 \times 10^{-5}$	1·06	2×10^{-6}
5	$2{\cdot}0 \times 10^{-8}$	1·52	5×10^{-9}
7	$7{\cdot}4 \times 10^{-12}$	2·21	2×10^{-11}
9	$1{\cdot}6 \times 10^{-15}$	3·22	2×10^{-13}

TABLE 2c

$x_l = 2(l+1), \quad l = 0, 1, 2, \ldots$

ν	$\gamma_{\nu,0}^{(0)}$	$\sum_{l=0}^{N} \lvert\gamma_{\nu,l}^{(0)}\rvert$	$\lvert 1-D_\nu^{(2,0)}\rvert$
1	$2{\cdot}3 \times 10^{-2}$	1	3×10^{-4}
3	$1{\cdot}6 \times 10^{-5}$	1	2×10^{-7}
5	$-5{\cdot}6 \times 10^{-9}$	1·00000001	4×10^{-10}
7	$-3{\cdot}6 \times 10^{-12}$	1·005	2×10^{-12}
9	$1{\cdot}4 \times 10^{-16}$	1	6×10^{-14}

However, the accuracy of the computed value of $A_n^{(m,j)}$ seems to be unaffected by the ill-conditioning of Q. What does seem to have an effect on the accuracy of the computed value of $A_n^{(m,j)}$ is the size of $\sum_{l=0}^{N} |\gamma_l|$, the same quantity that affects the error $A - A_n^{(m,j)}$ in the true approximation. If we let $\bar{A}_n^{(m,j)}$ be the computed value of $A_n^{(m,j)}$, then $|A - A_n^{(m,j)}|$ and $|A_n^{(m,j)} - \bar{A}_n^{(m,j)}|$ increase (decrease) simultaneously as $\sum_{l=0}^{N} |\gamma_l|$ increases (decreases). The effect of $\sum_{l=0}^{N} |\gamma_l|$ on $|A_n^{(m,j)} - \bar{A}_n^{(m,j)}|$ can be explained to some

TABLE 3

v	$\gamma_{v,0}^{(0)}$	$\gamma_{v,1}^{(0)}$	$\gamma_{v,2}^{(0)}$	$\gamma_{v,3}^{(0)}$	$\gamma_{v,4}^{(0)}$
1	$1{\cdot}5\times10^{-2}$	$4{\cdot}3\times10^{-2}$	$3{\cdot}2\times10^{-1}$	$8{\cdot}2\times10^{-2}$	$5{\cdot}4\times10^{-1}$
3	$2{\cdot}9\times10^{-5}$	$4{\cdot}4\times10^{-4}$	$1{\cdot}1\times10^{-2}$	$7{\cdot}5\times10^{-3}$	$1{\cdot}6\times10^{-1}$
5	$2{\cdot}0\times10^{-8}$	$1{\cdot}0\times10^{-6}$	$1{\cdot}2\times10^{-4}$	$-1{\cdot}4\times10^{-5}$	$7{\cdot}8\times10^{-3}$
7	$7{\cdot}4\times10^{-12}$	$-1{\cdot}2\times10^{-10}$	$5{\cdot}9\times10^{-7}$	$-3{\cdot}3\times10^{-6}$	$1{\cdot}6\times10^{-4}$
9	$1{\cdot}6\times10^{-15}$	$-3{\cdot}2\times10^{-12}$	$1{\cdot}8\times10^{-9}$	$-4{\cdot}4\times10^{-8}$	$1{\cdot}9\times10^{-6}$

extent as follows: suppose that the $A(y_s)$ have been computed with an error of ε_s, $s = 0, 1, 2, \ldots$. Then

$$\bar{A}_n^{(m,j)} = \sum_{l=0}^{N} \gamma_l [A(y_{j+l}) + \varepsilon_{j+l}]$$

which together with (1.5) implies

$$|A_n^{(m,j)} - \bar{A}_n^{(m,j)}| \leqslant \left(\sum_{l=0}^{N} |\gamma_l| \right) \max_{j \leqslant s \leqslant j+N} |\varepsilon_s|.$$

Hence if $A(y_s)$ are of the same order of magnitude and have r correct significant decimal digits, then $\varepsilon_s/A(y_s) \sim 10^{-r+1}$, and if $A_n^{(m,j)}$ is of the same order of magnitude as the $A(y_s)$ and $\sum_{l=0}^{N} |\gamma_l| \sim 10^q$ for some integer $q \geqslant 0$ then

$$|A_n^{(m,j)} - \bar{A}_n^{(m,j)}|/A_n^{(m,j)} \sim 10^{-r+1+q},$$

i.e. $\bar{A}_n^{(m,j)}$ has $\sim r-q$ correct significant decimal digits. Now if $A_n^{(m,j)}$ agrees with A in the first q' significant decimal digits and $q' \leqslant r-q$, then $\bar{A}_n^{(m,j)}$ can be taken as $A_n^{(m,j)}$.

If r is large enough, that is the $A(y_s)$ have been computed sufficiently accurately, then even if $\sum_{l=0}^{N} |\gamma_l|$ may be large, $\bar{A}_n^{(m,j)}$ will be sufficiently accurate to be taken as $A_n^{(m,j)}$.

From what has been said above we can conclude that if the vector $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_N)^T$ is known, then we can simultaneously (1) compute $\bar{A}_n^{(m,j)}$, and (2) through $\sum_{l=0}^{N} |\gamma_l|$ obtain an estimate of the correct number of significant figures in $\bar{A}_n^{(m,j)}$, the computed value of $A_n^{(m,j)}$.

The vector γ can be found by solving the set of linear equations

$$Q^T \gamma = e_1,$$

where $e_1 = (1, 0, 0, \ldots, 0)^T$. Therefore, the amount of computing to be done for determining $A_n^{(m,j)}$ and γ is the same as that for $A_n^{(m,j)}$ and the $\bar{\beta}_{k,i}$.

7. Further Results on GREP

So far we have been concerned with $A_n^{(m,j)}$ and have given bounds on $|A - A_n^{(m,j)}|$. It turns out that the $\bar{\beta}_{k,i}$ are approximations to the $\beta_{k,i}$ and tend to them in the limit for both Process 1 and Process 2.

We start by solving equation (1.3) for $\bar{\beta}_{p,q}$ for $0 \leqslant p \leqslant \mu$ and $0 \leqslant q \leqslant n_p$. Using Cramer's rule again the result is $-\bar{\beta}_{p,q} = \det \bar{M}/\det K$, where K is as described in Section 3 and $\bar{M}$ is the matrix obtained from K by replacing the (p, q)th row of K by the row vector $(A(y_j), \ldots, A(y_{j+N}))$. Let us denote the cofactor of $A(y_{j+1})$ in the (p, q)th row of $\bar{M}$ by $\bar{\delta}_l$, $l = 0, 1, \ldots, N$. Then expanding $\det \bar{M}$ and $\det K$ with respect to their (p, q)th rows we obtain

$$-\bar{\beta}_{p,q} = \frac{\sum_{l=0}^{N} \bar{\delta}_l A(y_{j+l})}{\sum_{l=0}^{N} \bar{\delta}_l \phi_p(y_{j+l}) y_{j+l}^{qr_p}} \tag{7.1}$$

from which we immediately identify

$$-\bar{\gamma}_l = \bar{\delta}_l/\det K, \quad l = 0, 1, \ldots, N, \tag{7.2}$$

where $(\bar{\gamma}_0, \bar{\gamma}_1, \ldots, \bar{\gamma}_N)$ is the row of Q^{-1} which corresponds to $\bar{\beta}_{p,q}$ in the vector c in (1.4), i.e.

$$\bar{\beta}_{p,q} = \sum_{l=0}^{N} \bar{\gamma}_l A(y_{j+l}). \tag{7.2a}$$

LEMMA 7.1. *$\bar{\beta}_{p,q}$ satisfies*

$$\beta_{p,q} - \bar{\beta}_{p,q} = \sum_{l=0}^{N} \bar{\gamma}_l \left[\sum_{k=0}^{\mu} \phi_k(y_{j+l}) \beta_k(y_{j+l}) - \beta_{p,q} \phi_p(y_{j+l}) y_{j+l}^{qr_p} \right]. \tag{7.3}$$

Proof. Substituting (1.1) in (7.1) we obtain

$$-\bar{\beta}_{p,q} = \frac{A \sum_{l=0}^{N} \bar{\delta}_l - \sum_{l=0}^{N} \bar{\delta}_l \sum_{k=0}^{\mu} \phi_k(y_{j+l}) \beta_j(y_{j+l})}{\sum_{l=0}^{N} \bar{\delta}_l \phi_p(y_{j+l}) y_{j+l}^{qr_p}}. \tag{7.4}$$

Now $\sum_{l=0}^{N} \bar{\delta}_l = 0$ since it is just $\det \bar{K}$, where $\bar{K}$ is the matrix obtained from K by replacing its (p, q)th row by the vector $(1, 1, \ldots, 1)$, thus giving two identical rows in $\bar{K}$, namely the first at the (p, q)th. (7.3) then follows by adding $\beta_{p,q}$ to both sides of (7.4) and using (7.2).

Corollary.

$$\beta_{p,q} - \bar{\beta}_{p,q} = \det \bar{M}_1/\det K, \tag{7.5}$$

where $\bar{M}_1$ is obtained from $\bar{M}$ by replacing its (p, q)th row by the vector $(a_0, a_1, \ldots, a_N)$, where

$$a_l = \sum_{k=0}^{\mu} \phi_k(y_{j+l}) \beta_k(y_{j+l}) - \beta_{p,q} \phi_p(y_{j+l}) y_{j+l}^{qr_p}, \quad l = 0, 1, \ldots, N. \tag{7.6}$$

(*a*) *Process* 1

THEOREM 7.1. $\bar{\beta}_{p,q}$ *satisfies the equality*

$$\beta_{p,q} - \bar{\beta}_{p,q} = \sum_{l=0}^{N} \bar{\gamma}_l \sum_{k=0}^{\mu} \phi_k(y_{j+l}) w_{n_k}^k(y_{j+l}), \tag{7.7}$$

where $w_s^k(y)$ *are as defined in* (3.9).

Proof. Let us subtract from the (p, q)th row of $\bar{M}_1$ the products of the rows (k, i) by $\beta_{k,i}$, $0 \leqslant i \leqslant n_k$, $0 \leqslant k \leqslant \mu$, $(k, i) \neq (p, q)$, and leave the rest of the rows unchanged, and call the new matrix $\bar{M}'$. The (p, q)th row of $\bar{M}'$ is now given by (3.10). Expanding $\det \bar{M}'$ with respect to the (p, q)th row and using (7.2), (7.7) follows.

(*b*) *Process* 2

THEOREM 7.2. $\bar{\beta}_{p,q}$ *satisfies the equality*

$$\beta_{p,q} - \bar{\beta}_{p,q} = \sum_{l=0}^{N} \bar{\gamma}_l \left[\sum_{\substack{k=0 \\ k \neq p}}^{\mu} \phi_k(y_{j+l}) u_{n_k}^k(y_{j+l}) + \phi_p(y_{j+l}) y_{j+l}^{q r_p} u_{n_p}^{p,q}(y_{j+l}) \right], \tag{7.8}$$

where $u_s^k(y)$ *are as defined in* (3.19) *and* (3.20), *and* $u_{n_p}^{p,q}(y)$ *is the best polynomial approximation to* $w_q^p(y)/y^{(q+1)r_p}$ *in powers of* y^{r_p}, *of degree* $n_p - q - 1$, *in the interval* $[0, y_j]$.

Proof. Similar to those of Theorem 7.1 and Theorem 3.2.

Starting with (7.7) and (7.8) we can give upper bounds for $|\beta_{p,q} - \bar{\beta}_{p,q}|$ and prove convergence theorems under some special circumstances as we did for $|A - A_n^{(m,j)}|$ in Section 3.

We now give another result that corresponds to Theorem 3.3.

THEOREM 7.3. *Let* $A(y)$ *be as in Theorem* 3.3. *Then*

$$\beta_{p,q} - \bar{\beta}_{p,q} = \sum_{l=0}^{N} \bar{\gamma}_l \sum_{k=0}^{\mu} \phi_k(y_{j+l}) y_{j+l}^{n_k r_k} \mathscr{L}[w_k^{(n_k+1)}(t); y_{j+l}^{-r_k}]. \tag{7.9}$$

Proof. Like that of Theorem 7.1 and Theorem 3.3.

Special cases of Theorems 7.1, 7.2, and 7.3 for the case of the T-transformations of Levin have been used by the present author (see Sidi, 1978, 1979), to prove convergence of $\bar{\beta}_{p,q}$ to $\beta_{p,q}$ for both Process 1 and Process 2.

Note. Numerous computations with Process 2 have shown that $\bar{\beta}_{p,q} \to \beta_{p,q}$ as $n_k \to \infty$, $0 \leqslant k \leqslant \mu$, this convergence being very quick for $q = 0$, less quick for $q = 1$, etc. What happens is that as q increases the $\bar{\gamma}_l$ become very large in absolute value and do not have a fixed sign. This, through (7.2a), introduces very severe round-off error propagation in the computation of $\bar{\beta}_{p,q}$.

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