Some aspects of two-point Padé approximants

Avram Sidi (*)

ABSTRACT

Recently McCabe and Murphy have considered the two-point Padé approximants to a function for which (formal) power series expansions at the origin and at infinity are given. In this paper these approximations are slightly modified and determinant representations for them are given. The existence of various three-term recursion relations for the numerators and denominators of these approximants is shown. Based on these, a new continued fraction representation for these approximants is obtained and also an efficient recursive method is proposed for the determination of the coefficients of *all* the approximants that obtain from a given number of terms of the power series.

1. INTRODUCTION

Recently McCabe and Murphy [4] have considered the problem of determination of continued fractions for functions $f(z)$, which have formal power series expansions at $z = 0$ and $z = \infty$. Their problem, with a slight modification, is described below.
It is assumed that for small $|z|$

$$f(z) = \frac{a_0}{2} + a_1 z + a_2 z^2 + \ldots, \tag{1.1}$$

while for large $|z|$

$$f(z) = -\left(\frac{a_0}{2} + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \ldots\right), \tag{1.2}$$

and that $a_0 \neq 0$. We seek to approximate $f(z)$ by rational functions of the form

$$f_{i,j}(z) = \frac{p(z)}{q(z)} = \frac{\alpha_0 + \alpha_1 z + \ldots + \alpha_m z^m}{1 + \beta_1 z + \ldots + \beta_m z^m}, \quad m = 1, 2, \ldots \tag{1.3}$$

The $2m+1$ coefficients α_r and β_s are determined by the requirement that the expansions of $f_{i,j}(z)$ for small $|z|$ and large $|z|$ agree with (1.1) and (1.2) up to and including the terms $a_{i-1}z^{i-1}$ and $a_{-j}z^{-j}$ respectively such that $i + j = 2m$. Symbolically, we write this as

$$f(z) - f_{i,j}(z) = O(z^i, z^{-(j+1)}). \tag{1.4}$$

For future reference we shall express (1.4) in the following explicit form :

$$p(z) - q(z) f(z) = O(z^i) \text{ as } |z| \to 0 \tag{1.5}$$

$$\frac{p(z)}{z^m} - \frac{q(z)}{z^m} f(z) = O(z^{-(j+1)}) \text{ as } |z| \to \infty. \tag{1.6}$$

We note that by adding $\frac{a_0}{2}$ to $f(z)$ in (1.1) and (1.2) and also to $f_{i,j}(z)$ in (1.3) we obtain the problem treated by McCabe and Murphy, in which the rational approximations $f_{i,j}(z) - \frac{a_0}{2}$ have numerators that are polynomials of degree $m-1$ and not m.
It is worth noting that the notation $f_{i,j}(z)$ differs a great deal from that commonly used with ordinary (one-point) Padé approximants for which $f_{i,j}(z)$ usually means a rational function whose numerator and denominator are polynomials of degree at most i and j respectively.
Finally we have assumed that the series (1.1) and (1.2) are formal (convergent or asymptotic divergent) expansions of one and same function $f(z)$. Actually (1.1) and (1.2) can be considered to be two arbitrary formal expansions which are not necessarily related to the same function. But, as approximations, $f_{i,j}(z)$ can have significance only when they are related to the same function both at $z = 0$ and $z = \infty$. For examples of this, see McCabe and Murphy [4].
We note also that the rational functions $f_{i,j}(z)$ may not always exist; it is possible to construct series for which this is so. In this work, however, we assume that all $f_{i,j}(z)$ exist.
McCabe and Murphy have given a method, based on the q-d algorithm, for the computation of the coefficients of the continued fractions derived from (1.1)-(1.4).

(*) A. Sidi, Computer Science Dept., Technion, Israel Institute of Technology, Haifa, Israel

However, their algorithm necessitates the computation of many quantities which bear no relevance to $f_{i,j}(z)$. Besides using their algorithm only part of the approximants that derive from a given number of terms of (1.1) and (1.2) can be computed.
The purpose of this paper is,

1) to give determinant expressions for $f_{i,j}(z)$, similar to those for ordinary Padé approximants,
2) to develop some three-term recursion relations which are very useful in the construction of some continued fractions, namely, those of McCabe and Murphy and a new continued fraction which, upon contraction, gives one of the continued fractions of McCabe and Murphy, and
3) to give an efficient recursive algorithm for the computation of the α's and β's of *all* the $f_{i,j}(z)$ that derive from a given number of terms of the series (1.1) and (1.2).

2. SOME PROPERTIES

Assume that all the common factors of the numerator $p(z)$ and denominator $q(z)$ of the approximant $f_{i,j}(z)$ have been cancelled and that the reduced rational function is $\tilde{p}(z)/\tilde{q}(z)$, where $\tilde{p}(z)$ and $\tilde{q}(z)$ are polynomials of degree exactly $n \leq m$. The fact that $\tilde{p}(z)$ and $\tilde{q}(z)$ should be of the same degree follows from the fact that as $|z| \to \infty$

$f_{i,j}(z) = -a_0/2 + 0(z^{-1})$ and the assumption that $a_0 \neq 0$. Also $\tilde{q}(0) \neq 0$ since $f_{i,j}(z) = 0(1)$ as $|z| \to 0$. Hence we can set $\tilde{q}(0) = 1$. Therefore, we can define $f_{i,j}(z)$ by (1.3) with degree of $p(z)$ and degree of $q(z)$ exactly equal to $n \leq m$, such that $p(z)$ and $q(z)$ have no common factors apart from a multiplicative constant.
We shall define the approximant $f_{i,j}(z)$ to be normal if $n = m$ and the two-point Padé table to be normal if all $f_{i,j}(z)$ are normal.
In the remainder of this paper we shall assume that the two-point Padé table is normal.

Theorem 1

If $f_{i,j}(z)$ exists, it is unique.

Proof

If $i = 0$, then $j = 2m$, hence $f(z) - f_{i,j}(z) = 0(z^{-(2m+1)})$ only, i.e. $f_{0,j}(z)$ is the (m/m) ordinary Padé approximant derived from the series (1.2). From the uniqueness theorem for ordinary Padé approximants, see Baker([1], p. 8), it follows that $f_{0,j}(z)$ is unique.
If $i > 0$, assume that $f_{i,j}(z) = p(z)/q(z)$ is not unique. Then there is at least one more two-point Padé approximant, say $\bar{f}_{i,j}(z) = \bar{p}(z)/\bar{q}(z)$ different than $f_{i,j}(z)$.
Suppose that $f_{i,j}(z)$ and $\bar{f}_{i,j}(z)$ are irreducible and that $p(z)$ and $q(z)$ are of degree exactly $n \leq m$, and $\bar{p}(z)$ and $\bar{q}(z)$ are of degree exactly $\bar{n} \leq m$.
It follows from (1.4) that

$$f_{i,j}(z) - \bar{f}_{i,j}(z) = \frac{p(z)}{q(z)} - \frac{\bar{p}(z)}{\bar{q}(z)} = 0(z^i, z^{-(j+1)}), \tag{2.1}$$

which can be written as

$$\frac{p(z)\bar{q}(z) - \bar{p}(z)q(z)}{q(z)\bar{q}(z)} = 0(z^i, z^{-(j+1)}). \tag{2.2}$$

Now the numerator of the left hand side of (2.2) is a polynomial of degree $n + \bar{n} \leq 2m$, say $\sum_{k=0}^{n+\bar{n}} c_k z^k$, and the denominator is $0(1)$ as $|z| \to 0$. Therefore, it follows from (2.2) that $c_k = 0$, $0 \leq k \leq r$, where $r = \min(i-1, n+\bar{n})$
If we divide the numerator and denominator of the left hand side of (2.2) by $z^{n+\bar{n}}$, then the numerator becomes $\sum_{k=0}^{n+\bar{n}} c_{n+\bar{n}-k} z^{-k}$ and the denominator $q(z)\bar{q}(z)/z^{n+\bar{n}}$ is $0(1)$ as $|z| \to \infty$. Again, it follows from (2.2) that $c_k = 0$, $r \leq k \leq n + \bar{n}$, where $r = \max(0, n + \bar{n} - j)$. Hence $p(z)\bar{q}(z) - \bar{p}(z)q(z) \equiv 0$ from which we obtain

$$\frac{p(z)}{q(z)} \equiv \frac{\bar{p}(z)}{\bar{q}(z)}, \tag{2.3}$$

thus proving the theorem.

Theorem 2

Let $g(z) = 1/f(z)$. Then $g(z)$ has two formal power series expansions at $z = 0$ and $z = \infty$ obtained by inverting (1.1) and (1.2) in the usual sense. If $f_{i,j}(z)$ exists, then so does $g_{i,j}(z)$ and

$$g_{i,j}(z) = \frac{1}{f_{i,j}(z)}. \tag{2.4}$$

Proof

If $f_{i,j}(z)$ exists, then

$$\frac{1}{f(z)} - \frac{1}{f_{i,j}(z)} = \frac{f_{i,j}(z) - f(z)}{f(z)\, f_{i,j}(z)}. \tag{2.5}$$

Now both as $|z| \to 0$ and $|z| \to \infty$ the denominator of (2.5) is $0(1)$, hence by using (1.4) in the numerator of (2.5), we have

$$\frac{1}{f(z)} - \frac{1}{f_{i,j}(z)} = 0(z^i, z^{-(j+1)}). \tag{2.6}$$

By the uniqueness theorem (Theorem 1), (2.4) now follows.

We note that a result like (2.4) cannot be obtained by using the definition of McCabe and Murphy.

3. DETERMINANT REPRESENTATIONS

Theorem 3

If $f_{i,j}(z)$ exists, it can be expressed as the quotient of two $(m+1) \times (m+1)$ determinants as

$$f_{i,j}(z) = \frac{\det P(z)}{\det Q(z)} \equiv U(z), \tag{3.1}$$

where

$$Q(z) = \begin{bmatrix} 1 & z & z^2 & \cdots\cdots & z^m \\ a_{i-1} & a_{i-2} & a_{i-3} & \cdots\cdots & a_{i-m-1} \\ a_{i-2} & a_{i-3} & a_{i-4} & \cdots\cdots & a_{i-m-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i-m} & a_{i-m-1} & a_{i-m-2} & \cdots\cdots & a_{-j} \end{bmatrix} \tag{3.2}$$

and $P(z)$ is obtained from $Q(z)$ by replacing the first row of $Q(z)$ by the vector

$$(S_{m-1}, zS_{m-2}, z^2S_{m-3}, \ldots, z^{m-1}S_0, z^mT_0) \tag{3.3}$$

when $i \geqslant j$, i.e., $i \geqslant m$ and $j \leqslant m$, and

$$(T_0, zT_1, z^2T_2, \ldots, z^mT_m) \tag{3.4}$$

when $i \leqslant j$, i.e., $i \leqslant m$, $j \geqslant m$, where we have defined

$$S_k = \frac{a_0}{2} + a_1 z + \ldots + a_k z^k, \quad k = 0, 1, 2, \ldots \tag{3.5}$$

and

$$T_k = -\left(\frac{a_0}{2} + \frac{a_{-1}}{z} + \ldots + \frac{a_{-k}}{z^k}\right), \; k = 0, 1, 2, \ldots . \tag{3.6}$$

Proof

We shall give the proof for $i \geqslant j$ in detail and outline the proof for $i \leqslant j$ since it is similar to that for $i \geqslant j$. First of all, as can be seen from (3.2), $\det Q(z)$ is a polynomial of degree m at most. Similarly, from (3.3) it can be seen that $\det P(z)$ is also a polynomial of degree m at most. We now define the following transformations :

1) R_1 is that transformation which multiplies the first row of an $(m+1) \times (m+1)$ matrix by z^{-m}.
2) $C_\pm$ is that transformation which multiplies the k-th column of an $(m+1) \times (m+1)$ matrix by $z^{\pm(k-1)}$, $k = 1, 2, \ldots, m+1$.
3) $R_\pm$ is that transformation which multiplies the k-th row of an $(m+1) \times (m+1)$ matrix by $z^{\pm(i-k+1)}$, $k = 2, 3, \ldots, m+1$.

Let us now apply to $P(z)$ and $Q(z)$ first C_- and next R_+, and denote the new matrices by $P_1(z)$ and $Q_1(z)$, respectively. $Q_1(z)$ is of the form

$$Q_1(z) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_{i-1}z^{i-1} & a_{i-2}z^{i-2} & \cdots & a_{i-m-1}z^{i-m-1} \\ a_{i-2}z^{i-2} & a_{i-3}z^{i-3} & \cdots & a_{i-m-2}z^{i-m-2} \\ \vdots & \vdots & & \vdots \\ a_{i-m}z^{i-m} & a_{i-m-1}z^{i-m-1} & \cdots & a_{-j}z^{-j} \end{bmatrix} \tag{3.7}$$

and $P_1(z)$ is obtained from $Q_1(z)$ by replacing the first row of $Q_1(z)$ by the vector

$$(S_{m-1}, S_{m-2}, \ldots, S_0, T_0). \tag{3.8}$$

If $i = m$ set $P_2(z) \equiv P_1(z)$, otherwise (i.e., if $i > m$), add the 2nd,..., $(i-m+1)$ the rows to the first row of $P_1(z)$ and call the new matrix $P_2(z)$. In both cases the first row of $P_2(z)$ is

$$(S_{i-1}, S_{i-2}, \ldots, S_{i-m}, S^*_{i-m-1}), \tag{3.9}$$

where $S^*_{i-m-1} = S_{i-m-1}$ if $i > m$ and $S^*_{i-m-1} = T_0$ if $i = m$.

Now we have

$$U(z) = \frac{\det P_2(z)}{\det Q_1(z)}. \tag{3.10}$$

Subtracting $U(z)$ as given in (3.10) from $f(z)$ we obtain

$$f(z) - U(z) = \frac{\det P_3(z)}{\det Q_1(z)}, \tag{3.11}$$

where $P_3(z)$ is the matrix obtained from $P_2(z)$ by replacing the first row of $P_2(z)$ by the vector

$$(f - S_{i-1}, f - S_{i-2}, \ldots, f - S^*_{i-m-1}). \tag{3.12}$$

Applying now to $P_3(z)$ and $Q_1(z)$ first R_- and next C_+ we obtain the matrices $\overline{P}(z)$ and $Q(z)$ respectively, where $\overline{P}(z)$ is obtained from $Q(z)$ by replacing the first row of $Q(z)$ by the vector

$$[f - S_{i-1}, z(f - S_{i-2}), z^2(f - S_{i-3}), \ldots, z^m(f - S^*_{i-m-1})]. \tag{3.13}$$

As can be seen from (1.1) and (3.13), the elements of the first row of $\overline{P}(z)$ are power series which have z^i as their common factor, hence $\det \overline{P}(z) = 0(z^i)$ as $|z| \to 0$. Similarly $\det Q(z) = 0(1)$ as $|z| \to 0$. Therefore

$$f(z) - U(z) = \frac{\det \overline{P}(z)}{\det Q(z)} = 0(z^i) \quad \text{as} \quad |z| \to 0. \tag{3.14}$$

Let us now start with $U(z) = \det P_1(z) / \det Q_1(z)$ and, in $P_1(z)$, subtract from the first row the sum of the last j rows and call the new matrix $P_2'(z)$. If $i = m$, then the first row of $P_2'(z)$ is the vector

$$(T_0, T_1, \ldots, T_j), \tag{3.15}$$

and for $i > m$, the first row of $P_2'(z)$ becomes

$$(S_{i-m-1}, \ldots, S_0, T_0, T_1, \ldots, T_{-j}). \tag{3.16}$$

Subtracting now $U(z)$ as given by

$$U(z) = \frac{\det P_2'(z)}{\det Q_1(z)}, \tag{3.17}$$

from $f(z)$ we obtain

$$f(z) - U(z) = \frac{\det P_3'(z)}{\det Q_1(z)}, \tag{3.18}$$

where $P_3'(z)$ is obtained from $Q_1(z)$ by replacing the first row of $Q_1(z)$ by the vector $(f-A_0, f-A_1, \ldots, f-A_m)$, where $(A_0, A_1, \ldots, A_m)$ denotes either (3.15) or (3.16). Applying to $P_3'(z)$ and $Q_1(z)$ the transformations R_-, C_+ and R_1, in this order, we obtain the matrices $\hat{P}(z)$ and $\hat{Q}(z)$ respectively, where $\hat{P}(z)$ is obtained from $Q(z)$ by replacing the first row of $Q(z)$ by the vector

$$[z^{-m}(f-A_0), z^{-m+1}(f-A_1), \ldots, z^0(f-A_m)] \tag{3.19}$$

and $\hat{Q}(z)$ is obtained from $Q(z)$ by replacing the first row of $Q(z)$ by the row vector

$$(z^{-m}, z^{-m+1}, \ldots, z^{-1}, 1). \tag{3.20}$$

As can be seen from (1.2), (3.15), (3.16), and (3.19), $\det \hat{P}(z) = 0(z^{-(j+1)})$ as $|z| \to \infty$ and from (3.20) $\det \hat{Q}(z) = 0(1)$ as $|z| \to \infty$. Therefore,

$$f(z) - U(z) = \frac{\det \hat{P}(z)}{\det \hat{Q}(z)} = 0(z^{-(j+1)}) \text{ as } |z| \to \infty. \tag{3.21}$$

This completes the proof for the case $i \geqslant j$.

For the case $i \leqslant j$, it can be shown by similar means, that

$$f(z) - U(z) = \frac{\det \bar{R}(z)}{\det Q(z)} \tag{3.22}$$

where $\bar{R}(z)$ is obtained from $Q(z)$ by replacing the first row of $Q(z)$ by the vector

$$[f-S_{i-1}, \ldots, z^{i-1}(f-S_0), z^i(f-T_0), z^{i+1}(f-T_1), \ldots, z^m(f-T_{m-i})] \tag{3.23}$$

if $j > m$, and

$$[f-S_{i-1}, z(f-S_{i-2}), \ldots, z^{m-1}(f-S_0), z^m(f-T_0)] \tag{3.24}$$

if $j = m$. It can also be shown that

$$f(z) - U(z) = \frac{\det \hat{R}(z)}{\det \hat{Q}(z)}, \tag{3.25}$$

where $\hat{R}(z)$ is obtained from $\hat{Q}(z)$ by replacing the first row of $\hat{Q}(z)$ by the vector

$$[z^{-m}(f-T_{m-i}), z^{-m+1}(f-T_{m-i+1}), \ldots, z^0(f-T_j)]. \tag{3.26}$$

Using (3.22)-(3.26), it is now easy to see that (3.14) and (3.21) are satisfied also for the case $i \leqslant j$.

4. ASYMPTOTIC ERROR ESTIMATES

Using the determinant representations for $f(z) - f_{i,j}(z)$ which have been given in the previous section, we can now derive the asymptotic error bounds for $|z| \to 0$ and $|z| \to \infty$.

Following McCabe and Murphy let us define

$$D_{r,s} = \begin{vmatrix} a_r & a_{r+1} & \cdots & a_{r+s} \\ a_{r-1} & a_r & \cdots & a_{r+s-1} \\ \vdots & \vdots & & \vdots \\ a_{r-s} & a_{r-s+1} & \cdots & a_r \end{vmatrix},$$

$$r = 0, \pm 1, \pm 2, \ldots \quad s = 0, \pm 1, \pm 2, \ldots \tag{4.1}$$

Theorem 4

As $|z| \to 0$

$$|f(z) - f_{i,j}(z)| = \left| \frac{D_{i-m,m}}{D_{i-m-1,m-1}} \right| |z|^i [1 + 0(z)] \tag{4.2}$$

and as $|z| \to \infty$

$$|f(z) - f_{i,j}(z)| = \left| \frac{D_{i-m-1,m}}{D_{i-m,m-1}} \right| |z|^{-j-1} [1 + 0(z^{-1})] \tag{4.3}$$

both for $i \geqslant j$ and $i \leqslant j$.

Proof

The proof of (4.2) follows by examining the determinant representations $\det \bar{P}(z) / \det Q(z)$ $(i \geqslant j)$ and $\det \bar{R}(z) / \det Q(z)$ $(i \leqslant j)$ in the limit $|z| \to 0$, and the proof of (4.3) follows by examining $\det \hat{P}(z) / \det \hat{Q}(z)$ $(i \geqslant j)$ and $\det \hat{R}(z) / \det \hat{Q}(z)$ $(i \leqslant j)$ in the limit $|z| \to \infty$.

The results in (4.2) and (4.3) when $i \geqslant j$ check with those given by McCabe and Murphy who use the continued fraction formulation.

5. RECURSION RELATIONS

Using the determinant representation of section 3 it is possible to derive three-term recursion relations for the numerators and denominators of two-point Padé approximants. In order to derive one such relation we shall make use of the Sylvester determinant identity (see Gragg [6]).

Theorem 5 (Sylvester determinant identity)

Let

$$A' = \begin{bmatrix} a_{11} & a_1^T & a_{12} \\ a_1' & A & a_2' \\ a_{21} & a_2^T & a_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & * \\ * & a_{22} \end{bmatrix} = \begin{bmatrix} * & A_{12} \\ a_{21} & * \end{bmatrix}$$

$$= \begin{bmatrix} * & a_{12} \\ A_{21} & * \end{bmatrix} = \begin{bmatrix} a_{11} & * \\ * & A_{22} \end{bmatrix}, \tag{5.1}$$

where A' is an $(m+1) \times (m+1)$ matrix, A is an $(m-1) \times (m-1)$ matrix, and $A_{i,j}$, $i,j = 1,2$, are $m \times m$ matrices, a_{ij}, $i, j = 1, 2$ are scalars, a_i, a_i', $i = 1, 2$ are $(m-1)$-dimensional column vectors, and $*$ denotes m-dimensional row or column vectors. Then

$$\det A' \det A = \det A_{11} \det A_{22} - \det A_{12} \det A_{21}. \tag{5.2}$$

Let us denote the matrices $P(z)$ and $Q(z)$ in (3.1)-(3.3) by $P_{i,m}(z)$ and $Q_{i,m}(z)$ respectively. Let us also partition $P_{i,m}(z)$ and $Q_{i,m}(z)$ as in (5.1) with the notation therein.

Theorem 6

$P_{i,m}(z)$ and $Q_{i,m}(z)$ satisfy the recursion relations

$$\det P_{i,m}(z) \det A = \det P_{i,m-1}(z) \det A_{22} - z \det P_{i-1,m-1}(z) \det A_{21} \tag{5.3}$$

$$\det Q_{i,m}(z) \det A = \det Q_{i,m-1}(z) \det A_{22} - z \det Q_{i-1,m-1}(z) \det A_{21}, \tag{5.4}$$

with the same A, A_{21}, and A_{22} in both (5.3) and (5.4).

Proof

The proof of (5.4) follows from the easily verifiable fact that $\det A_{11} = Q_{i,m-1}(z)$ and $\det A_{12} = z \det Q_{i-1,m-1}(z)$.

The proof of (5.3) is based on the fact that $\det A_{11} = \det P_{i,m-1}(z)$ and $\det A_{12} = z \det P_{i-1,m-1}(z)$ both for $i \geq m$ and $i \leq m$, and this requires some attention. For $i \geq m$ it is easy to see that $\det A_{12} = z \det P_{i-1,m-1}(z)$. That $\det A_{11} = \det P_{i,m-1}(z)$ can be shown by subtracting from the first row of A_{11} the product of the row $(a_{m-1}, a_{m-2}, \ldots, a_0)$ with z^{m-1}. For $i \leq m$ it is easy to see that $\det A_{11} = \det P_{i,m-1}(z)$. That $\det A_{12} = z \det P_{i-1,m-1}(z)$ can be shown by adding to the first row of A_{12} the row $(a_{-1}, a_{-2}, \ldots, a_{-m})$.

Corollary

Let $p_{i,m}(z) = \det P_{i,m}(z) / \det Q_{i,m}(0)$ and $q_{i,m}(z) = \det Q_{i,m}(z) / \det Q_{i,m}(0)$. Then

$$p_{i,m}(z) = p_{i,m-1}(z) + \omega z\, p_{i-1,m-1}(z), \tag{5.5}$$

$$q_{i,m}(z) = q_{i,m-1}(z) + \omega z\, q_{i-1,m-1}(z), \tag{5.6}$$

where ω is the same constant both for (5.5) and (5.6).

Proof

It is easy to see from (3.2) that $\det Q_{i,m}(0) = \det A_{22}$ and $\det Q_{i,m-1}(0) = \det A$. Dividing (5.3) and (5.4) by $\det A \det A_{22}$ and defining $\omega = -\det Q_{i-1,m-1}(0) \det A_{21} / (\det A \det A_{22})$, (5.5) and (5.6) follow.

We note that since $q_{i,m}(0) = 1$, we see that $q_{i,m}(z) = 1 + \beta_1 z + \ldots + \beta_m z^m$ and $p_{i,m}(z) = \alpha_0 + \alpha_1 z + \ldots + \alpha_m z^m$ as in (1.3).

The next theorem gives another recursion relation which is based solely on the definition (1.1)-(1.4) of $f_{i,j}(z)$.

Theorem 7

Let $f_{k,\ell}(z) = p_{k,n}(z) / q_{k,n}(z)$, $q_{k,n}(0) = 1$, $k + \ell = 2n$, all k,n. Then

$$p_{i+1,m+1}(z) = \lambda p_{i,m}(z) + \mu p_{i,m+1}(z), \tag{5.7}$$

$$q_{i+1,m+1}(z) = \lambda q_{i,m}(z) + \mu q_{i,m+1}(z), \tag{5.8}$$

where λ and μ are the same constants for both (5.7) and (5.8) and $\lambda + \mu = 1$.

Proof

Let us, for the sake of simplicity, denote the coefficients of $p(z) \equiv p_{i,m+1}(z)$, $p^*(z) \equiv p_{i+1,m+1}(z)$, and $p'(z) \equiv p_{i,m}(z)$ by α_r, α_r^*, and α_r' respectively and define β_r, β_r^*, β_r' similarly. From the relation in (1.5) we have

$$p(z) - q(z) f(z) = 0(z^i) \quad \text{as} \quad |z| \to 0 \tag{5.9}$$

and

$$p^*(z) - q^*(z) f(z) = 0(z^{i+1}) \text{ as } |z| \to 0. \tag{5.10}$$

Let us devide (5.9) and (5.10) by β_{m+1} and β^*_{m+1} respectively and subtract (5.10) from (5.9). We obtain

$$\left[\frac{p(z)}{\beta_{m+1}} - \frac{p^*(z)}{\beta^*_{m+1}}\right] - \left[\frac{q(z)}{\beta_{m+1}} - \frac{q^*(z)}{\beta^*_{m+1}}\right] f(z) = 0(z^i) \text{ as}$$

$$|z| \to 0. \tag{5.11}$$

Now using the fact that

$a_{m+1}/\beta_{m+1} = a^*_{m+1}/\beta^*_{m+1} = -a_0/2$, we can see that in (5.11) $\bar{p}(z) = p(z)/\beta_{m+1} - p^*(z)/\beta^*_{m+1}$ and $\bar{q}(z) = q(z)/\beta_{m+1} - q^*(z)/\beta^*_{m+1}$ are both polynomials of degree m at most. Let us now divide $\bar{p}(z)$ and $\bar{q}(z)$ by z^m. Then

$$\frac{\bar{p}(z)}{z^m} - \frac{\bar{q}(z)}{z^m} f(z) = \frac{z}{\beta_{m+1}}\left[\frac{p(z)}{z^{m+1}} - \frac{q(z)}{z^{m+1}} f(z)\right]$$

$$- \frac{z}{\beta^*_{m+1}}\left[\frac{p^*(z)}{z^{m+1}} - \frac{q^*(z)}{z^{m+1}}\right] f(z) \tag{5.12}$$

Making use of (1.6), (5.12) becomes

$$\frac{\bar{p}(z)}{z^m} - \frac{\bar{q}(z)}{z^m} f(z) = 0(z^{-(2m-i+1)}) \text{ as } |z| \to \infty \tag{5.13}$$

Hence

$$f(z) - \frac{\bar{p}(z)}{\bar{q}(z)} = 0(z^i, z^{-(j+1)}), \quad i+j = 2m, \tag{5.14}$$

and by the uniqueness theorem $\bar{p}(z)/\bar{q}(z) = f_{i,j}(z)$.

Hence we have proved the existence of (5.7) and (5.8). We now have to find λ and μ. It can be seen easily that

$$\bar{q}(0) = \frac{1}{\beta_{m+1}} - \frac{1}{\beta^*_{m+1}}. \tag{5.15}$$

Therefore, $\mu = \beta^*_{m+1}/\beta_{m+1}$ and

$\lambda = (\beta_{m+1} - \beta^*_{m+1})/\beta_{m+1}$. This completes the proof.

We note that the method of proof of theorem 7 is similar to the one used by Longman [3] for ordinary Padé approximants.

The recursion relations given in (5.5-6) and (5.7-8) are quite fundamental in that they can be used to obtain other recursion relations as is shown below.

From the corollary to theorem 6 we have

$$t_{i,m}(z) = t_{i,m-1}(z) + \omega z t_{i-1,m-1}(z), \tag{5.16}$$

$$t_{i+1,m+1}(z) = t_{i+1,m}(z) + \omega' z t_{i,m}(z)\ ; \tag{5.17}$$

and from theorem 7 we have

$$t_{i+1,m}(z) = \lambda t_{i,m-1}(z) + \mu t_{i,m}(z), \tag{5.18}$$

where $t_{k,\ell}(z)$ stands for $p_{k,\ell}(z)$ or $q_{k,\ell}(z)$. Eliminating $t_{i,m-1}(z)$ and $t_{i+1,m}(z)$ from (5.16) - (5.18) and using the fact that $\lambda + \mu = 1$, we obtain :

Theorem 8

The $p_{k,\ell}$ and $q_{k,\ell}$ satisfy a three-term recursion relation of the form

$$t_{i+1,m+1}(z) = (Az+1)\, t_{i,m}(z) + Bz t_{i-1,m-1}(z),$$

where (5.19)

$A = \omega'$, $B = -\lambda\omega$.

Eliminating $t_{i+1,m+1}$ between (5.7) or (5.8) and (5.17) we obtain :

Theorem 9

The $p_{k,\ell}$ and $q_{k,\ell}$ satisfy a three-term recursion relation of the form

$$t_{i,m+1}(z) = (Dz+E)t_{i,m} + F t_{i+1,m}(z), \tag{5.20}$$

where $D = \omega'/\mu$, $E = -\lambda/\mu$, $F = 1/\mu$.

In order to understand the meaning of the various recursion relations obtained we first organize the two-point Padé approximants in a two-dimensional array as in figure 1.

	m=1	m=2	m=3	m=4	m=5	m=6
i=0	0/1	0/2	0/3	0/4	0/5	0/6
i=1	1/1	1/2	1/3	1/4	1/5	1/6
i=2	2/1	2/2	2/3	2/4	2/5	2/6
i=3	3/1*	3/2	3/3	3/4	3/5	3/6
i=4	4/1*	4/2	4/3	4/4	4/5	4/6
i=5	5/1*	5/2*	5/3	5/4	5/5	5/6
i=6	6/1*	6/2*	6/3	6/4	6/5	6/6

Fig. 1.

The starred positions have no meaning since for them $j < 0$. The entries between the two thick lines are those that can be computed when a_k, $-6 \leq k \leq 5$, are given. The first row (i = 0) consists of the entries on the main diagonal of the ordinary Padé table of (1.2), as was mentioned in the proof of theorem 1.

The recursion relations given in (5.5) and (5.6) together with (5.7) and (5.8) show that to a staircase in figure 1, e.g., (0/3), (1/3), (1/4), (2/4),..., there corresponds a continued fraction which, from a certain point on, is of the type

$$\ldots + \frac{\omega z}{1} + \frac{\lambda}{\mu} + \frac{\omega' z}{1} + \frac{\lambda'}{\mu'} + \ldots \; .$$

$$\lambda + \mu = \lambda' + \mu' = \ldots = 1. \tag{5.21}$$

It is not difficult to see that the continued fraction whose convergents are (1/1), (2/1), (2/2), (3/2),..., has the structure

$$c + \frac{dz}{1+ez} + \frac{\lambda_1}{\mu_1} + \frac{\omega_1 z}{1} + \frac{\lambda_2}{\mu_2} + \frac{\omega_2 z}{1} + \ldots, \quad \lambda_i + \mu_i = 1,$$

with $f_{1,1}(z) = c + dz/(1+ez)$, etc.

The recursion relations in (5.19), on the other hand, show that to a diagonal sequence in figure 1, e.g., (0/3), (1/4), (2/5),..., there corresponds a continued fraction which, from a certain point on, is of the type

$$\ldots + \frac{Bz}{Az+1} + \frac{B'z}{A'z+1} + \ldots \qquad (5.22)$$

It is easy to see that (5.22) is actually one of the continued fraction representations given in McCabe and Murphy, and can be obtained from (5.21) by contraction.

We note that McCabe and Murphy have applied a q-d algorithm to (5.22) for the determination of the $f_{i,j}(z)$. However, their algorithm necessitates the computation of quantities which have no connection with the $f_{i,j}(z)$, and one application of their algorithm produces only part of the $f_{i,j}(z)$ that derive from a given number of terms of (1.1) and (1.2).

It is worth mentioning that the continued fractions (5.21) and (5.22) are equivalent (in form) to those given in Perron ([7], p. 176).

Two other recursion relations can be obtained as follows: Apply the corollary of theorem 6 to $t_{i,m}$, $t_{i+1,m}$ and $t_{i+1,m+1}$, and theorem 7 to $t_{i+1,m}$, $t_{i+1,m+1}$, and $t_{i+2,m+1}$, and eliminate $t_{i+1,m+1}$.

Similarly, apply the corollary of theorem 6 to $t_{i+1,m}$, $t_{i+2,m}$, and $t_{i+2,m+1}$, and theorem 7 to $t_{i+1,m}$, $t_{i+2,m+1}$, and $t_{i+3,m+1}$, and eliminate $t_{i+2,m}$. The recursion relations that are obtained are summarized below :

Theorem 10

Between the $t_{k,\ell}(z)$ the following recursion relations hold :

$$t_{i+2,m+1}(z) = t_{i+1,m}(z) + \gamma z t_{i,m}(z) \qquad (5.23)$$

$$t_{i+3,m+1}(z) = t_{i+2,m+1}(z) + \delta z t_{i+1,m}(z) \qquad (5.24)$$

From theorem 10 we can see that to a sequence of the form (i/m), (i+1/m), (i+2/m+1), (i+3/m+1),..., e.g. (0/3), (1/3), (2/4), (3/4), (4/5),..., there corresponds a continued fraction, which, from a certain point on is of the type

$$\ldots + \frac{\gamma z}{1} + \frac{\delta z}{1} + \ldots \qquad (5.25)$$

This continued fraction has also been given by McCabe and Murphy.

6. COMPUTATIONAL ASPECTS

In this section we shall make use of some of the recursion relations that were established in the previous section in order to compute the α's and β's in the two-point Padé approximants.

From the determinant representation given in (3.1) in theorem 3, the denominator of $f_{i,j}(z)$ is

$\det Q(z) = \sum_{s=0}^{m} \gamma_s z^s$, where γ_s is the cofactor of z^s in the first row. From (1.3), however, the denominator of $f_{i,j}(z)$ is $q(z) = \sum_{s=0}^{m} \beta_s z^s$ with $\beta_0 = 1$. Therefore $q(z)$ is equal to $\det Q(z)$ up to a multiplicative constant. Consequently,

$$\gamma_s = d\beta_s, \quad s = 0, 1, \ldots, m, \qquad (6.1)$$

where $d = \det Q(0)$. From the theory of determinants we know that the inner products of the vector $(\gamma_0, \gamma_1, \ldots, \gamma_m)$ with the 2nd, 3rd,..., (m+1)th rows of $Q(z)$ are all zero. This together with (6.1) gives

$$\sum_{s=0}^{m} \beta_s a_{r-s} = 0, \quad r = i-m, i-m+1, \ldots, i-1, \qquad (6.2)$$

for *all* i and j.

Equations (6.2) with $\beta_0 = 1$ are enough for determining the β's. As is clear from (6.2), however, as m increases the number of equations to be solved increases too. Therefore, it is desirable to have a method which will obviate the need for solving a large system of equations. We now give one such method.

Let us first denote $f_{i,j}(z) = p_{i,m}(z) / q_{i,m}(z)$, where $i+j = 2m$,

$$p_{i,m}(z) = \sum_{s=0}^{m} \alpha_s^{(i,m)} z^s, \quad q_{i,m}(z) = \sum_{s=0}^{m} \beta_s^{(i,m)} z^s.$$

Theorem 11

If

$$\sum_{s=0}^{m} \beta_s^{(i,m)} a_{i-m-1-s} \neq 0, \qquad (6.3)$$

then

$$\beta_s^{(i+1,m+1)} = \beta_s^{(i+1,m)} + \omega \beta_{s-1}^{(i,m)}, \quad s = 0, 1, \ldots, m+1, \qquad (6.4)$$

where we have defined $\beta_{-1}^{(k,\ell)} = \beta_{\ell+1}^{(k,\ell)} = 0$, and ω is given by

$$\omega = -\frac{\sum_{s=0}^{m} \beta_s^{(i+1,m)} a_{i-m-s}}{\sum_{s=0}^{m} \beta_s^{(i,m)} a_{i-m-1-s}}. \qquad (6.5)$$

Proof

That a relation like that in (6.4) exists follows from the corollary to theorem 6. Also for $s = 0$ $\beta_0^{(i+1,m+1)} = \beta_0^{(i+1,m)} = 1$. Only (6.5) remains to be verified. Let us multiply equations (6.4) by a_{r-s} and sum from $s = 0$ to $s = m+1$. The left hand side, using (6.2), satisfies

$$I_r = \sum_{s=0}^{m+1} \beta_s^{(i+1,m+1)} a_{r-s} = 0, \quad r = i-m,\ i-m+1,\dots,i. \tag{6.6}$$

We now have to show that the right hand side

$$J_r = \sum_{s=0}^{m+1} [\beta_s^{(i+1,m)} + \omega\beta_{s-1}^{(i,m)}]\, a_{r-s}, \tag{6.7}$$

satisfies $J_r = 0$, $i-m \leq r \leq i$, too. But using (6.2) again

$$J_r = 0, \quad r = i-m+1,\dots,i. \tag{6.8}$$

Finally, using (6.3) and (6.5) it is easy to verify that $J_{i-m} = 0$ too. This proves the theorem.

In exactly the same way, starting from the relation (5.8) in theorem 7, we can prove the following :

Theorem 12

If

$$\sum_{s=0}^{m} \beta_s^{(i,m)} a_{i-m-1-s} \neq 0 \tag{6.9}$$

then

$$\mu\beta_s^{(i,m+1)} = \beta_s^{(i+1,m+1)} - \lambda\beta_s^{(i,m)} \tag{6.10}$$

where $\lambda + \mu = 1$, and

$$\lambda = \frac{\sum_{s=0}^{m+1} \beta_s^{(i+1,m+1)} a_{i-m-1-s}}{\sum_{s=0}^{m} \beta_s^{(i,m)} a_{i-m-1-s}}. \tag{6.11}$$

Theorems 11 and 12 enable us to compute recursively all of the two-point Padé approximants which derive from a given number of the coefficients a_s. For example, when a_k, $-6 \leq k \leq 5$ are given, the recursive computation of the β's can be performed as is shown in fig. 2.

0/1	0/2	0/3	0/4	0/5	0/6
1/1	1/2	1/3	1/4	1/5	1/6
2/1	2/2	2/3	2/4	2/5	2/6
*3/1	3/2	3/3	3/4	3/5	3/6
*4/1	4/2	4/3	4/4	4/5	4/6
*5/1	*5/2	5/3	5/4	5/5	5/6
*6/1	*6/2	6/3	6/4	6/5	6/6

Fig. 2.

As can be seen from figure 2 the starred positions which have no meaning enter the computation like the rest of the positions. The algorithm is started by letting $\beta_0^{(i,1)} = 1$, $\beta_1^{(i,1)} = -a_{i-1}/a_{i-2}$ in the first column. The second column, starting with (1/2), is computed from the first column using theorem 11. The position (0/2) is computed from (0/1) and (1/2) by using theorem 12. The third column is computed similarly. The 4th, 5th and 6th columns are now computed by making use of theorem 11 only.

The computation of the α's can be carried out in exactly the same way by using theorem 2. We compute the expansions for $g(z) = 1/f(z)$ by solving two triangular systems of linear equations and determine the β's for $g_{i,j}(z)$ using the algorithm above. From theorem 2 we know that the β's for $g_{i,j}(z)$ are very simply related to the α's of $f_{i,j}(z)$ through (2.4), and this solves the problem.

Once the β's have been computed it is also possible to compute the numerator of $f_{i,j}(z)$ without having to know the α's explicitly. For this let $(w_0(z), w_1(z),\dots,w_m(z))$ denote the first row of P(z) in the determinant representation given in theorem 3. Expanding both det P(z) and det Q(z) with respect to their first rows and making use of (6.1) we obtain

$$f_{i,j}(z) = \frac{\sum_{s=0}^{m} \beta_s w_s(z)}{\sum_{s=0}^{m} \beta_s z^s}. \tag{6.12}$$

We note that the recursion relations given in theorems 11 and 12 are similar in nature to those given by Clenshaw and Lord [2] and Sidi [5] in the computation of the Chebyshev-Padé table.

CONCLUSION

Two-point Padé approximants have been considered using the determinant approach. Asymptotic error estimates have been given. Two fundamental recursion relations have been derived and through them the existence of three different continued fraction representations, two of which are those given by Murphy and McCabe, has been shown. A recursive method for computing the parameters of all the approximants which derive from a given number of terms has been proposed. It is expected that as in the case of the ordinary Padé approximants, the determinant approach given in this paper will be useful in deriving other properties of two-point Padé approximants.

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