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Extrapolation Methods for Oscillatory Infinite Integrals

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The non-linear transformations for accelerating the convergence of slowly convergent infinite integrals due to Levin & Sidi (1975) are modified in two ways. These modifications enable one to evaluate accurately some oscillatory infinite integrals with less work. Special emphasis is placed on the evaluation of Fourier and Hankel transforms and some simple algorithms for them are given. Convergence properties of these modifications are analysed in some detail and powerful convergence theorems are proved for certain cases including those of the Fourier and Hankel transforms treated here. Several numerical examples are also supplied.

1. Introduction

RECENTLY Levin & Sidi (1975) have given non-linear methods for the accurate evaluation of infinite integrals and series which have proved to be very efficient in accelerating the convergence of slowly converging infinite integrals and series of various kinds. The numerous examples given in their work show that these methods have very good convergence properties. Lately, convergence properties of these methods have been partially analysed by the present author, see Sidi (1979*a*, 1979*b*, 1980).

The purpose of the present work is to modify the methods of Levin & Sidi (1975) to deal with some oscillatory infinite integrals. These modifications enable one to evaluate accurately these integrals with less work. Convergence properties of the new methods are also analysed.

Before we go on we shall give a brief outline of the results of Levin & Sidi (1975) and Sidi (1979*b*) that bear relevance to the present work.

Definition 1. We shall say that a function $\alpha(x)$, defined for $x > a \geqslant 0$, belongs to the set $A^{(\gamma)}$ if it is infinitely differentiable for all $x > a$ and if, as $x \to \infty$, it has a Poincaré-type asymptotic expansion of the form

$$\alpha(x) \sim x^{\gamma} \sum_{i=0}^{\infty} \alpha_i/x^i, \tag{1.1}$$

and all derivatives, as $x \to \infty$, have Poincaré-type asymptotic expansions which are obtained by differentiating the right-hand side of (1.1) term by term.

From this definition it follows that $A^{(\gamma)} \supset A^{(\gamma-1)} \supset \dots$.

Remark. It also follows that if $\alpha(x)$ is in $A^{(0)}$, then it is infinitely differentiable for all $x > a$ including $x = \infty$ (but not necessarily analytic at $x = \infty$).

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THEOREM 1. *Let $f(x)$ be defined for $x > a \geqslant 0$, be integrable at infinity, and satisfy a homogeneous linear differential equation of order m of the form*

$$f(x) = \sum_{k=1}^{m} p_k(x) f^{(k)}(x), \tag{1.2}$$

where $p_k \in A^{(i_k)}$ but $p_k \notin A^{(i_k - 1)}$, such that i_k are integers satisfying $i_k \leqslant k$, $1 \leqslant k \leqslant m$. Let also

$$\lim_{x\to\infty} p_k^{(i-1)}(x) f^{(k-i)}(x) = 0, \quad i \leqslant k \leqslant m, \quad 1 \leqslant i \leqslant m. \tag{1.3}$$

If for every integer $l = -1, 1, 2, 3, \ldots,$

$$\sum_{k=1}^{m} l(l-1)\ldots(l-k+1)\bar{p}_k \neq 1, \tag{1.4}$$

where

$$\bar{p}_k = \lim_{x\to\infty} x^{-k} p_k(x), \quad 1 \leqslant k \leqslant m, \tag{1.5}$$

then

$$\int_a^\infty f(t)\,dt = \int_a^x f(t)\,dt + \sum_{k=0}^{m-1} x^{\rho_k} f^{(k)}(x)\beta_k(x) \tag{1.6a}$$

such that ρ_k are integers satisfying

$$\rho_k \leqslant \max(i_{k+1}, i_{k+2}-1, \ldots, i_m - m+k+1), \quad 0 \leqslant k \leqslant m-1, \tag{1.6b}$$

and $\beta_k \in A^{(0)}$, $0 \leqslant k \leqslant m-1$. It also follows that

$$\lim_{x\to\infty} x^{\rho_k} f^{(k)}(x) = 0, \quad 0 \leqslant k \leqslant m-1.$$

Remark. An interesting point about this result is that the functions $x^{\rho_k}\beta_k(x)$ depend only on the $p_k(x)$ and are independent of $f(x)$. An immediate consequence of this is that if there is another function $g(x)$ besides $f(x)$ satisfying equation (1.2) and the rest of the conditions of Theorem 1, then both $f(x)$ and $g(x)$ satisfy (1.6a) with the same ρ_k and $\beta_k(x)$, i.e.

$$\int_x^\infty \begin{Bmatrix} f(t) \\ g(t) \end{Bmatrix} dt = \sum_{k=0}^{m-1} x^{\rho_k} \begin{Bmatrix} f^{(k)}(x) \\ g^{(k)}(x) \end{Bmatrix} \beta_k(x). \tag{1.7}$$

We shall apply Theorem 1 to some Fourier and Hankel transforms in Section 2.

Definition 2. Let $f(x)$ be as in Theorem 1 with the same notation and let $n = \{n_0, n_1, \ldots, n_{m-1}\}$, where n_k are non-negative integers. The approximation $D_n^{(m,j)}$ to $\int_a^\infty f(t)\,dt$ together with the parameters $\bar{\beta}_{k,i}$, $0 \leqslant i \leqslant n_k$, $0 \leqslant k \leqslant m-1$, are defined by the set of linear equations

$$D_n^{(m,j)} = \int_a^{x_l} f(t)\,dt + \sum_{k=0}^{m-1} x_l^{\rho_k} f^{(k)}(x_l) \sum_{i=0}^{n_k} \bar{\beta}_{k,i}/x_l^i, \quad j \leqslant l \leqslant j+N, \tag{1.8}$$

where $N = \sum_{k=0}^{m-1} (n_k+1)$ and x_l are chosen such that $a < x_0 < x_1 < x_2 < \ldots,$ and

$\lim_{l\to\infty} x_l = \infty$, provided that the matrix of the equations in (1.8) is non-singular. If the ρ_k are not known exactly, then in (1.8), the ρ_k can be replaced by the integers σ_k defined by $\sigma_k = \min(s_k, k+1)$, where

$$s_k = \max\left\{s \mid \lim_{x\to\infty} x^s f^{(k)}(x) = 0,\ s \text{ integer}\right\}, \quad 0 \leqslant k \leqslant m-1.$$

It can be shown that $\rho_k \leqslant \sigma_k \leqslant k+1$ and that

$$\lim_{x\to\infty} x^{\sigma_k} f^{(k)}(x) = 0, \quad 0 \leqslant k \leqslant m-1.$$

(The integrals $\int_a^{x_l} f(t)\,dt$ can be evaluated very accurately by using a low order Gaussian rule.)

(The notation in the definition above is slightly different from that given in Levin & Sidi, 1975.)

Convergence properties of the approximations have been analysed for two kinds of limiting processes, see Sidi (1979*b*):

(1) Process 1: n fixed, $j \to \infty$,
(2) Process 2: j fixed, $n_k \to \infty$, $0 \leqslant k \leqslant m-1$;

and Process 2, under certain circumstances, has been shown to have better convergence properties. In particular (under certain circumstances),

$$\left|\int_a^{\infty} f(t)\,dt - D_n^{(m,j)}\right| \to 0,$$

(1) at least like x_j^{ν} for Process 1 and (2) faster than any inverse power of ν for Process 2, where $\nu = \min\{n_0, n_1, \ldots, n_{m-1}\}$. It also turns out that the $\bar{\beta}_{k,i}$ in equations (1.8) are approximations to the coefficients $\beta_{k,i}$ in the asymptotic expansions of the $\beta_k(x)$ in equation (1.6), where

$$\beta_k(x) \sim \sum_{i=0}^{\infty} \beta_{k,i}/x^i \quad \text{as } x \to \infty, \quad 0 \leqslant k \leqslant m-1. \tag{1.9}$$

The rates of convergence for $D_n^{(m,j)}$ given in Sidi (1979*b*) indicate that they depend mainly on j and n and very little on m. But the amount of computing that has to be done for obtaining $D_n^{(m,j)}$ depends very strongly on m. Firstly, the number of the finite integrals $\int_a^{x_l} f(t)\,dt$ that have to be computed is $j+N+1$. Secondly, the number of the equations that have to be solved to obtain $D_n^{(m,j)}$ is $N+1$. And N increases as m increases. In view of this observation we ask whether we can do something to obtain an approximation to $\int_a^{\infty} f(t)\,dt$ which is about as accurate as $D_n^{(m,j)}$ but requires less computing. It turns out that this is possible for certain oscillatory integrals provided the x_l are chosen in a suitable way, and the method is modified, as we show in Section 3. In Section 4 we will give another modification that will further simplify things. We shall apply the new methods of Sections 3 and 4 to Fourier and Hankel transforms. Some convergence theorems will also be supplied in Sections 5 and 6.

2. Applications of Theorem 1 to Fourier and Hankel Transforms

We now give two applications of Theorem 1 to Fourier and Hankel transforms. The results of this section will be of use in the remainder of this paper.

Fourier Transforms

We shall consider the integral $\int_x^\infty f(t)\,dt$, where

$$f(x) = g(x)\begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}, \tag{2.1}$$

where $g(x)$ is of the form

$$g(x) = h(x)\,e^{\phi(x)}, \tag{2.2}$$

such that $\phi(x)$ is a real polynomial in x of degree $k \geqslant 0$, for some integer k, and $h \in A^{(\gamma)}$ for some γ. If $k > 0$, then for $g(x)$ to be integrable at infinity $\lim_{x\to\infty} \phi(x) = -\infty$ is necessary. If $k = 0$, then $g \in A^{(\gamma)}$, hence $\gamma < 0$ in order for $f(x)$ to be integrable at infinity.

Now let

$$u(x) = \begin{Bmatrix}\cos x\\ \sin x\end{Bmatrix}.$$

Then $u''(x)+u(x) = 0$. Since $u(x) = f(x)/g(x)$ by (2.1), we have $(f/g)''+(f/g) = 0$. Substituting (2.2) in this differential equation we obtain $f = p_1 f' + p_2 f''$, where

$$\begin{aligned} p_1(x) &= \frac{2(\phi'+h'/h)}{1+(\phi'+h'/h)^2-(\phi'+h'/h)'} \\ p_2(x) &= \frac{-1}{1+(\phi'+h'/h)^2-(\phi'+h'/h)'}. \end{aligned} \tag{2.3}$$

Now if $k > 0$, then it can be shown using (2.3) and the properties of $\phi(x)$ and $h(x)$ that $p_j \in A^{(i_j)}$, with $i_1 = -k+1$ and $i_2 = -2k+2$. If $k = 0$, i.e. $\phi(x) \equiv$ constant, then $p_j \in A^{(i_j)}$, with $i_1 = -1$ and $i_2 = 0$. Since $i_j < j$, $j = 1, 2$ in all these cases, we see that $\bar{p}_1 = \bar{p}_2 = 0$ in Theorem 1. It can now be verified that all the conditions of Theorem 1 are satisfied with $m = 2$ and

$$\begin{aligned} \rho_0 &= \max(i_1, i_2-1) = \begin{cases} -k+1 & \text{if } k > 0 \\ -1 & \text{if } k = 0 \end{cases} \\ \rho_1 &= i_2 = \begin{cases} -2k+2 & \text{if } k > 0 \\ 0 & \text{if } k = 0. \end{cases} \end{aligned} \tag{2.4}$$

Hence

$$\int_x^\infty f(t)\,dt = x^{\rho_0} f(x)\beta_0(x) + x^{\rho_1} f'(x)\beta_1(x), \tag{2.5}$$

where $\beta_0, \beta_1 \in A^{(0)}$, and $\beta_0(x)$ and $\beta_1(x)$ are the same both for $u(x) = \cos x$ and $u(x) = \sin x$.

Hankel Transforms

We now consider the integral $\int_x^\infty f(t)\,dt$, where

$$f(x) = g(x)\begin{Bmatrix} J_\nu(x) \\ Y_\nu(x) \end{Bmatrix}, \tag{2.6}$$

where $g(x)$ is as described above in the sub-section on Fourier transforms, with the same notation, the only difference being that when $k = 0, \gamma < 1/2$, and $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of order ν of the first and second kind, respectively. If we let $u(x)$ denote $J_\nu(x)$ or $Y_\nu(x)$, then $u(x)$ satisfies the differential equation

$$x^2u''(x)+xu'(x)+(x^2-\nu^2)u(x) = 0,$$

i.e. Bessel's equation. Again using the fact that $f = gu$, we have $u = f/g$. Substituting this in the differential equation above we obtain $f = p_1 f' + p_2 f''$, where

$$\begin{aligned} p_1(x) &= \frac{2x^2(\phi'+h'/h)-x}{w(x)} \\ p_2(x) &= \frac{-x^2}{w(x)}, \end{aligned} \tag{2.7}$$

where

$$w(x) = x^2\left[\left(\phi'+\frac{h'}{h}\right)^2 - \left(\phi'+\frac{h'}{h}\right)'\right] - x\left(\phi'+\frac{h'}{h}\right) + x^2 - \nu^2. \tag{2.8}$$

From (2.7) and (2.8) it can be shown that $p_j \in A^{(i_j)}$ where i_j are the same as those obtained for the Fourier transforms above. Therefore, $\bar{p}_1 = \bar{p}_2 = 0$, and we can verify that all the conditions of Theorem 1 are satisfied with $m = 2$, and ρ_0 and ρ_1 are the ones given in (2.4). Hence (2.5) holds with $f(x)$ as given in (2.6), and $\beta_0, \beta_1 \in A^{(0)}$. Again $\beta_0(x)$ and $\beta_1(x)$ are the same both for $u(x) = J_\nu(x)$ and $u(x) = Y_\nu(x)$.

3. The $\bar{D}$-Transformation

Suppose that the function $f(x)$ is as in Theorem 1 with the same notation and that it and/or some of its derivatives vanish an infinite number of times at infinity, i.e. assume that there exist x_l, $l = 0, 1, 2, \ldots$, such that $a < x_0 < x_1 < x_2 < \ldots$, $\lim_{l\to\infty} x_l = \infty$ and that

$$f^{(k_i)}(x_l) = 0, \quad l = 0, 1, \ldots, \quad 0 \leqslant k_1 < k_2 < \ldots < k_p \leqslant m-1. \tag{3.1}$$

Obviously with this choice of the x_l, the matrix of Equations (1.8) is singular since it has columns all of whose elements are zero. This problem can be remedied by reducing the number of the equations as follows:

Definition 3. Let the function $f(x)$ be as in the first paragraph of this section. Denote $E = \{0, 1, \ldots, m-1\}$ and $E_p = \{k_1, \ldots, k_p\}$, $q = m-p$, and let $\bar{n} = \{n_i | \, i \in E \backslash E_p\}$.

Then $\bar{D}_{\bar{n}}^{(q,j)}$, the approximation to $\int_a^\infty f(t)\,dt$, together with the parameters $\bar{\beta}_{k,i}$, $0 \leqslant i \leqslant n_k$, $k \in E \backslash E_p$, are defined as the solution to the set of linear equations

$$\bar{D}_{\bar{n}}^{(q,j)} = \int_a^{x_l} f(t)\,dt + \sum_{\substack{k=0 \\ k \notin E_p}}^{m-1} x^{\rho_k} f^{(k)}(x) \sum_{i=0}^{n_k} \bar{\beta}_{k,i}/x_l^i, \quad j \leqslant l \leqslant j+\bar{N}, \tag{3.2}$$

where

$$\bar{N} = \sum_{\substack{k=0 \\ k \notin E_p}}^{m-1} (n_k+1).$$

If the ρ_k are not known exactly, then the σ_k of Definition 2 can be used instead of the ρ_k.

We now demonstrate the use of the $\bar{D}$-Transformation with a few examples.

Example 1—Fourier Transforms

Consider the integral $\int_a^\infty f(t)\,dt$ with $a \geqslant 0$, where $f(x)$ is as described in Section 2, sub-section on Fourier transforms. Let $u(x)$ denote cos x or sin x. By taking x_0 to be the smallest zero of $u(x)$ greater than a, and letting $x_0, x_1, x_2, \ldots$, be the consecutive zeros of $u(x)$, we have $p = 1$, $k_1 = 0$, $q = 1$ in (3.1). Equations (3.2) then become

$$\bar{D}_{\bar{n}}^{(1,j)} = F(x_l) + x^{\rho_1} g(x_l) u'(x_l) \sum_{i=0}^{n_1} \frac{\bar{\beta}_{1,i}}{x_l^i}, \quad j \leqslant l \leqslant j+n_1+1, \tag{3.3}$$

where

$$F(x) = \int_a^x f(t)\,dt.$$

Now $|u'(x_l)| = 1$ and $u'(x_l)u'(x_{l+1}) = -1$, $l = 0, 1, \ldots$. Since $x_l = x_0 + l\pi$, $l = 0, 1, \ldots$, Equations (3.3) have a simple solution (see Appendix A) given by

$$\bar{D}_{\bar{n}}^{(1,j)} = \frac{\sum_{r=0}^{n_1+1} \binom{n_1+1}{r} (x_0/\pi+j+r)^{n_1} F(x_{j+r})/[x_{j+r}^{\rho_1} g(x_{j+r})]}{\sum_{r=0}^{n_1+1} \binom{n_1+1}{r} (x_0/\pi+j+r)^{n_1}/[x_{j+r}^{\rho_1} g(x_{j+r})]}. \tag{3.4}$$

Numerical example. We have used (3.4) to evaluate the integral

$$\int_0^\infty t \sin t/(1+t^2)\,dt = \pi/(2e).$$

For this case, $a = 0$, $k = 0$, $\gamma = -1$. Some of the results are given in Table 1.

Example 2—Hankel Transforms

Consider the integral $\int_a^\infty f(t)\,dt$ with $a \geqslant 0$, where $f(x)$ is as described in Section 2,

TABLE 1

Approximations $\bar{D}_{\bar{n}}^{(1,0)}$ *for* $\int_0^{\infty} t \sin t/(1+t^2)\, dt = \pi/(2e)$

n_1	$\bar{D}_{\bar{n}}^{(1,0)}$
1	0·57792
3	0·5778616
5	0·57786368
7	0·577863674888
9	0·57786367489538
Exact	0·57786367489546

sub-section on Hankel transforms. Let $u(x)$ denote $J_\nu(x)$ or $Y_\nu(x)$. Again by taking x_0 to be the smallest zero of $u(x)$ greater than a, and letting $x_0, x_1, x_2, \ldots,$ be the consecutive zeros of $u(x)$ we have $p = 1$, $k_1 = 0$, $q = 1$ in (3.1). Equations (3.2) again become

$$\bar{D}_{\bar{n}}^{(1,j)} = F(x_l)+x^{\rho_1}g(x)u'(x_l)\sum_{i=0}^{n_1}\frac{\bar{\beta}_{1,i}}{x_l^i}, \quad j \leqslant l \leqslant j+n_1+1, \tag{3.5}$$

where $F(x) = \int_a^x f(t)\, dt$ and $u'(x_l)u'(x_{l+1}) < 0$, $l = 0, 1, \ldots$. But this time $u'(x_l)$ is not as simple as that of Example 1 and the x_l are not equidistant as those of Example 1. Research on a simple algorithm for solving (3.5) is under way although at the time of writing there does not seem to be a simple solution for $\bar{D}_{\bar{n}}^{(q,j)}$ as that given in (3.4). Therefore, the solution of Equations (3.5) has to be found by solving (3.5) numerically on a computer.

Numerical example. We have solved (3.5) for the case $\int_0^{\infty} J_0(t)\, dt = 1$, i.e. $a = 0$, $g(x) \equiv 1$ $(k = 0, \gamma = 0)$ and $\nu = 0$. Some of the results are given in Table 2.

TABLE 2

Approximations $\bar{D}_{\bar{n}}^{(1,0)}$ *for* $\int_0^{\infty} J_0(t)\, dt = 1$

n_1	$\bar{D}_{\bar{n}}^{(1,0)}$
1	0·9995
3	0·999997
5	1·00000001
7	0·999999999988
9	0·9999999999998
Exact	1

Example 3 – $\int_0^\infty (\sin t/t)^2\, dt = \pi/2$

The integrand $f(x) = (\sin x/x)^2$ satisfies all the conditions of Theorem 1 with $m = 3$ and (1.6) holds with $\rho_0 = 1, \rho_1 = 0, \rho_2 = 1$, see Levin & Sidi (1975; example 4.5). If we choose $x_l = (l+1)\pi$, $l = 0, 1, \ldots$, then $f(x_l) = f'(x_l) = 0$, $l = 0, 1, \ldots$, i.e. $p = 2$, $k_1 = 0$, $k_2 = 1$, $q = 1$. Table 3 shows some of the results obtained for $\bar{D}_{\bar{n}}^{(1,0)}$.

TABLE 3

Approximations $\bar{D}_{\bar{n}}^{(1,0)}$ *for* $\int_0^\infty (\sin t/t)^2\, dt = \pi/2$

n_2	$\bar{D}_{\bar{n}}^{(1,0)}$
1	1·572
3	1·570795
5	1·5707962
7	1·570796329
9	1·57079632673
Exact	1·570796326795 . . .

4. The $\tilde{D}$-Transformation

The $\bar{D}$-transformation that has been introduced in the previous section is especially useful when the x_l in Equation (3.1) are readily available, either from tables or can be computed by use of tables. Otherwise they have to be computed numerically by solving Equations (3.1) and this may be not so practical. We now give a modification of the D-transformation which is very simple to implement and has proved to be as efficient as the $\bar{D}$-transformation.

There are cases in which the integral $\int_x^\infty f(t)\, dt$ can be expressed in the form

$$\int_x^\infty f(t)\, dt = \sum_{k=0}^{m-1} v_k(x) b_k(x), \tag{4.1}$$

where the functions $v_k(x)$ satisfy $\lim_{x\to\infty} v_k(x) = 0$, are simpler than $f^{(k)}(x)$, and some of them have an infinite number of zeros at infinity that are readily available, and $b_k \in A^{(0)}$, $0 \leqslant k \leqslant m-1$. Let then these zeros be x_l, $l = 0, 1, \ldots$, such that $a < x_0 < x_1 < \ldots$, and $\lim_{l\to\infty} x_l = \infty$ and such that

$$v_{k_i}(x_l) = 0, \quad l = 0, 1, \ldots, \quad 0 \leqslant k_1 < k_2 < \ldots < k_p \leqslant m-1. \tag{4.2}$$

Definition 4. Let the function $f(x)$ be as in Theorem 1 and as in the previous paragraph and let E, E_p, q, $\bar{n}$ and $\bar{N}$ be as in Definition 3. Then $\tilde{D}_{\bar{n}}^{(q,j)}$, the approximation to

$\int_a^\infty f(t)\,dt$, and the parameters $\bar{\beta}_{k,i}$, are defined as the solution to the set of linear equations

$$\tilde{D}_{\bar{n}}^{(q,j)} = \int_a^{x_l} f(t)\,dt + \sum_{\substack{k=0\\k\notin E_p}}^{m-1} v_k(x_l) \sum_{i=0}^{n_k} \bar{\beta}_{k,i}/x_l^i, \quad j \leqslant l \leqslant j+\bar{N}. \tag{4.3}$$

We now apply the $\tilde{D}$-transformation to a few examples.

Example 4—Fourier Transforms

Consider the integral of Example 1 with the same notation. Using the properties of $h(x)$ and $\phi(x)$ in (2.2) we can re-express (2.5) in the form

$$\int_x^\infty f(t)\,dt = v_0(x)b_0(x)+v_1(x)b_1(x) \tag{4.4}$$

where $b_0, b_1 \in A^{(0)}$ and

$$\begin{aligned} v_0(x) &= x^{\rho_0+\gamma}\, e^{\phi(x)}\, u(x) \\ v_1(x) &= x^{\rho_1+\gamma}\, e^{\phi(x)}\, u'(x) \end{aligned} \tag{4.5}$$

where $u(x)$ denotes $\cos x$ or $\sin x$ as in Example 1.

Letting now $x_0, x_1, x_2, \ldots$, be as in Example 1 we have once more $p = 1$, $k_1 = 0$, $q = 1$, in (4.2). Hence Equations (4.3) become

$$\tilde{D}_{\bar{n}}^{(1,j)} = F(x_l)+v_1(x_l)\sum_{i=0}^{n_1} \bar{\beta}_{1,i}/x_l^i, \quad j \leqslant l \leqslant j+n_1+1. \tag{4.6}$$

The solution of these equations is as before (see Example 1 and Appendix A)

$$\tilde{D}_{\bar{n}}^{(1,j)} = \frac{\displaystyle\sum_{r=0}^{n_1+1} \binom{n_1+1}{r} (x_0/\pi+j+r)^{n_1} F(x_{j+r})/[x_{j+r}^{\rho_1+\gamma}\, e^{\phi(x_{j+r})}]}{\displaystyle\sum_{r=0}^{n_1+1} \binom{n_1+1}{r} (x_0/\pi+j+r)^{n_1}/[x_{j+r}^{\rho_1+\gamma}\, e^{\phi(x_{j+r})}]}. \tag{4.7}$$

TABLE 4

Approximations $\tilde{D}_{\bar{n}}^{(1,0)}$ *for* $\int_0^\infty (\sin t/\sqrt{4+t^2})\,dt = (\pi/2)[I_0(2)-L_0(2)]$

n_1	$\tilde{D}_{\bar{n}}^{(1,0)}$
1	0·5372
3	0·537447
5	0·53745040
7	0·53745038905
9	0·5374503890636
Exact	0·5374503890637326

Numerical example. We have used (4.7) to approximate the integral

$$\int_0^\infty (\sin t/\sqrt{4+t^2})\,dt = (\pi/2)[I_0(2)-L_0(2)],$$

where $I_0(x)$ and $L_0(x)$ are the modified Bessel and Struve functions of order zero, respectively, i.e. $g(x) = (4+x^2)^{-\frac{1}{2}} \in A^{(-1)}$, hence $k = 0$, $\gamma = -1$, $a = 0$. Some of the results are given in Table 4.

Example 5—Hankel Transforms

Consider the integral of Example 2 with the same notation. The Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ have the property that

$$u(x) = \begin{Bmatrix} J_\nu(x) \\ Y_\nu(x) \end{Bmatrix} = \cos x\,\alpha_0(x) + \sin x\,\alpha_1(x), \quad \text{for all } \nu \tag{4.8}$$

where $\alpha_0, \alpha_1 \in A^{(-\frac{1}{2})}$, which follows by manipulating the asymptotic expansions of $J_\nu(x)$ and $Y_\nu(x)$ as $x \to \infty$. Using (4.8) together with the properties of $h(x)$ and $\phi(x)$, we can re-express (2.5) in the form given by (4.4) such that $b_0, b_1 \in A^{(0)}$ and

$$\left.\begin{aligned} v_0(x) &= x^{\rho+\gamma-\frac{1}{2}}\,e^{\phi(x)}\cos x \\ v_1(x) &= x^{\rho+\gamma-\frac{1}{2}}\,e^{\phi(x)}\sin x \end{aligned}\right\} \rho = \max(\rho_0, \rho_1). \tag{4.9}$$

Letting now $x_0, x_1, \ldots$, be consecutive zeros of $\sin x$, greater than a, we have $p = 1$, $k_1 = 1$, $q = 1$ in (4.2). Hence Equations (4.3) become

$$\tilde{D}_{\bar{n}}^{(1,j)} = F(x_l) + v_0(x_l) \sum_{i=0}^{n_0} \bar{\beta}_{0,i}/x_l^i, \quad j \leqslant l \leqslant j+n_0+1. \tag{4.10}$$

The solution of these equations is again (see Examples 1, 4, and Appendix A).

$$\tilde{D}_{\bar{n}}^{(1,j)} = \frac{\displaystyle\sum_{r=0}^{n_0+1} \binom{n_0+1}{r} (x_0/\pi+j+r)^{n_0} F(x_{j+r})/[x_{j+r}^{\rho+\gamma-\frac{1}{2}}\,e^{\phi(x_{j+r})}]}{\displaystyle\sum_{r=0}^{n_0+1} \binom{n_0+1}{r} (x_0/\pi+j+r)^{n_0}/[x_{j+r}^{\rho+\gamma-\frac{1}{2}}\,e^{\phi(x_{j+r})}]}. \tag{4.11}$$

Numerical example. We have used (4.11) to approximate the integral $\int_0^\infty J_0(2t)[\theta_1(t)/\theta_2(t)]\,dt$, where

$$\theta_1(t) = t(t^2+1/3)^{\frac{1}{2}}\{2t^2 \exp -[1/5(t^2+1)^{\frac{1}{2}}] - (2t^2+1)\exp[-1/5(t^2+1/3)^{\frac{1}{2}}]\},$$

and

$$\theta_2(t) = (2t^2+1)^2 - 4t^2(t^2+1/3)^{\frac{1}{2}}(t^2+1)^{\frac{1}{2}}.$$

This integral has been computed to 4-figure accuracy by Longman (1956) using the Euler transformation.

First of all the integral above can be put in the form

$$\int_0^\infty J_0(t)[\theta_1(t/2)/\theta_2(t/2)]\,dt/2,$$

i.e. $g(x) = \theta_1(x/2)/[2\theta_2(x/2)]$, and $a = 0$. Although the function $g(x)$ has a complicated appearance, a careful analysis of the functions $\theta_1(t)$ and $\theta_2(t)$ shows that $g(x)$ is of the form given in (2.2) with $\phi(x) = -x/10$ ($k = 1$) and $h \in A^{(1)}$, hence $\rho_0 = \rho_1 = 0$, and therefore $\rho = 0$, see (4.9). The fact that $\phi(x) = -\alpha x$ may tempt one to believe that $e^{\phi(x)}$ makes the integral converge quickly at infinity, hence no extrapolation is necessary. However, $e^{\phi(x)}$ is not negligible until x becomes very large, this being due to the fact that α is very small. Numerical results also indicate that $F(x)$ converges very slowly as x increases. Table 5 shows some of the results obtained for this integral.

TABLE 5

The finite integrals $F(x_{n_0+1})$ and the approximations $\tilde{D}_{\bar{n}}^{(1,0)}$ to the integral

$$\int_0^\infty J_0(t)[\theta_1(t/2)/\theta_2(t/2)]\,dt/2$$

n_0	$F(x_{n_0+1})$	$\tilde{D}_{\bar{n}}^{(1,0)}$
1	−0·0724	−0·0252
3	−0·0509	−0·02661
5	−0·0399	−0·026608992
7	−0·0339	−0·026608998119
9	−0·0307	−0·02660899812797

In connection with the second column of Table 5, we note that only $F(x_l)$, $l \leqslant n_0+1$ enter the computation of $\tilde{D}_{\bar{n}}^{(1,0)}$.

Example 6
$$\int_0^\infty J_0(t)J_1(t)\,dt/t = 2/\pi$$

This example has been treated by Levin & Sidi (1975, example 4.6). It can be shown that $f(x) = J_0(x)J_1(x)/x$ satisfies all the conditions of Theorem 1 with $m = 3$. The approximations $D_{\bar{n}}^{(3,0)}$ (Definition 2) to this integral have been computed by replacing ρ_0, ρ_1, ρ_2 by $\sigma_0, \sigma_1, \sigma_2$ which all turn out to be 1, i.e.

$$\int_x^\infty f(t)\,dt = \sum_{k=0}^{2} xf^{(k)}\beta_k(x), \tag{4.12}$$

with $\beta_k \in A^{(0)}$, $k = 0, 1, 2$. If we substitute (4.8) into (4.12), then we can show after some manipulation that

$$\int_x^\infty f(t)\,dt = \sum_{k=0}^{2} v_k(x)b_k(x), \tag{4.13}$$

where $b_k \in A^{(0)}$, and

$$\begin{aligned} v_0(x) &= x^{-1} \\ v_1(x) &= x^{-1}\cos 2x \\ v_2(x) &= x^{-1}\sin 2x. \end{aligned} \tag{4.14}$$

Letting $x_l = (l+1)\pi/2$, $l = 0, 1, \ldots$, so that $v_2(x_l) = 0$, $l = 0, 1, \ldots$, we have $p = 1$,

TABLE 6

Approximations $\tilde{D}_{\bar{n}}^{(2,0)}$ *to the integral* $\int_0^\infty J_0(t)J_1(t)\,dt/t = 2/\pi$

$n_0 = n_1$	$D_{\bar{n}}^{(2,0)}$
1	0·6360
3	0·636616
5	0·6366199
7	0·636619770
9	0·6366197724
Exact	0·63661977236 . . .

$q = 2$, $k_1 = 2$, in (4.2). Therefore, $\tilde{D}_{\bar{n}}^{(2,j)}$ are given by solving Equations (4.3), with $\bar{n} = \{n_0, n_1\}$. Table 6 shows some of these results.

5. Convergence Theorems for $\bar{D}$ and $\tilde{D}$-Transformations

Convergence properties of the $\bar{D}$- and $\tilde{D}$-transformations are very similar to those of the D- and d-transformations of Levin & Sidi (1975) which have been partially analysed in Sidi (1979*b*).

Let Q be the $(\bar{N}+1)\times(\bar{N}+1)$ matrix of the equations in (3.2) (for the $\bar{D}$-transformation) or in (4.3) (for the $\tilde{D}$-transformation), such that the first column of Q is the vector $(1, 1, \ldots, 1)^T$, (T denotes transpose). Let also c be the vector of unknowns whose first element is $\bar{D}_{\bar{n}}^{(q,j)}$ or $\tilde{D}_{\bar{n}}^{(q,j)}$, and the rest are the parameters $\bar{\beta}_{k,i}$. Let also

$$d = (F(x_j), F(x_{j+1}), \ldots, F(x_{j+\bar{N}}))^T,$$

where

$$F(x) = \int_a^x f(t)\,dt.$$

Then Equations (3.2) or (4.3) can be expressed as $Qc = d$. If we denote the first row of the inverse matrix Q^{-1} by $(\gamma_0, \gamma_1, \ldots, \gamma_{\bar{N}})$, then we have

$$D = \sum_{l=0}^{\bar{N}} \gamma_l F(x_{j+l}), \tag{5.1}$$

where D denotes $\bar{D}_{\bar{n}}^{(q,j)}$ or $\tilde{D}_{\bar{n}}^{(q,j)}$. Using the fact that $Q^{-1}Q = I$, it follows that $\sum_{l=0}^{\bar{N}} \gamma_l = 1$, and therefore $\sum_{l=0}^{\bar{N}} |\gamma_l| \geq 1$.

Using

$$\int_a^\infty f(t)\,dt = F(x_l) + \sum_{\substack{k=0\\ k\notin E_p}}^{m-1} x_l^{\rho_k} f^{(k)}(x_l)\beta_k(x_l), \quad l = 0, 1, \ldots, \tag{5.2}$$

which follows from (1.6) and (3.1) and the fact that

$$\int_a^\infty f(t)\,dt = F(x_l) + \sum_{\substack{k=0\\ k\notin E_p}}^{m-1} v_k(x_l)b_k(x_l), \quad l = 0, 1, \ldots, \tag{5.3}$$

which follows from (4.1) and (4.2) and the techniques of Sidi (1979*b*), the following results can be proved.

Process 1 *($j \to \infty$, $\bar{n}$ fixed)*

THEOREM 2. *Let $f(x)$ be as in Theorem* 1 *and let the x_l be as explained in Section* 3 *(for the $\bar{D}$-transformation) or in Section* 4 *(for the $\tilde{D}$-transformation), with the same notation. Let $\psi_k(x)$ and $\lambda_k(x)$ denote $x^{\rho_k} f^{(k)}(x)$ and $\beta_k(x)$, respectively or $v_k(x)$ and $b_k(x)$, respectively, such that*

$$\lambda_k(x) \sim \sum_{i=0}^{\infty} \lambda_{k,i}/x^{i}, \quad \text{as } x \to \infty. \tag{5.4}$$

Define

$$w_s^k(x) = \lambda_k(x) - \sum_{i=0}^{s} \lambda_{k,i}/x^i. \tag{5.5}$$

Then the approximation D satisfies the equality

$$\int_a^\infty f(t)\,dt - D = \sum_{l=0}^{\bar{N}} \gamma_l \sum_{\substack{k=0\\ k\notin E_p}}^{m-1} \psi_k(x_{j+l}) w_{n_k}^k(x_{j+l}). \tag{5.6}$$

Proof. Similar to Theorem 3.1 in Sidi (1979*b*).

Corollary. If

$$\sup_j \left(\sum_{l=0}^{\bar{N}} |\gamma_l| \right) \leqslant L < \infty, \tag{5.7}$$

then

$$\left| \int_a^\infty f(t)\,dt - D \right| = o(x_j^{-\eta}), \quad \text{as } j \to \infty, \tag{5.8}$$

at least, where

$$\eta = \min \bar{n} + 1. \tag{5.9}$$

Proof. Similar to those of the corollaries to Theorem 3.1 in Sidi (1979*b*).

Process 2 *(j fixed, $n_k \to \infty$, $k \in E\backslash E_p$)*

THEOREM 3. *Let $f(x)$ be as in Theorem* 2 *with the same notation. Transform the interval $x_j \leqslant x \leqslant \infty$ to $0 \leqslant \xi \leqslant 1$ by using the transformation $\xi = x_j/x$. Let*

$$\pi_s^k(\xi) = \sum_{i=0}^{s} \alpha_{s,i}^k \xi^i, \tag{5.10}$$

be the best polynomial approximation of degree s to the function $\lambda_k(x_j/\xi)$, $k \in E\backslash E_p$.

Define

$$u_s^k(x) = \lambda_k(x) - \pi_s^k(x_j/x), \quad k \in E\backslash E_p. \tag{5.11}$$

Then D satisfies the equality

$$\int_a^\infty f(t)\,dt - D = \sum_{l=0}^{\bar{N}} \gamma_l \sum_{\substack{k=0 \\ k \notin E_p}}^{m-1} \psi_k(x_{j+l}) u_{n_k}^k(x_{j+l}). \tag{5.12}$$

Proof. Similar to Theorem 3.2 in Sidi (1979*b*).

Corollary. If

$$\sup_{\bar{n}} \left(\sum_{l=0}^{\bar{N}} |\gamma_l| \right) \leqslant L < \infty, \tag{5.13}$$

then

$$\left| \int_a^\infty f(t)\,dt - D \right| = o(\eta^{-r}), \quad \text{as } n_k \to \infty, \quad k \in E\backslash E_p, \tag{5.14}$$

at least, for every $r > 0$, where η is as defined in (5.9).

Proof. Similar to those of the corollaries to Theorem 3.2 in Sidi (1979*b*).

Similar results for the $\bar{\beta}_{k,i}$ in Equations (3.2) or (4.3) can be obtained as in Sidi (1979*b*). Essentially, the $\bar{\beta}_{k,i}$ turn out to be approximations to the $\lambda_{k,i}$, and under certain circumstances it can be shown that $\lambda_{k,i} - \bar{\beta}_{k,i} \to 0$. In this work, however, we shall not deal with the $\bar{\beta}_{k,i}$.

We note here that (5.7) and (5.13) are sufficient but not necessary for the convergence of D to $\int_a^\infty f(t)\,dt$. Usually it is very difficult to see theoretically whether this condition is satisfied or not. In one case though this can be done with relative ease and we turn to that case now.

6. Convergence Theorems for $\bar{D}$- and $\tilde{D}$-Transformations Applied to Fourier and Hankel Transforms

We now apply the results of the previous section to the Fourier and Hankel transforms that we dealt with in Sections 2, 3 and 4. Actually, we shall do this in more general terms and in greater detail. We shall let, as in the previous section, $\psi_k(x)$ denote either $x^{\rho_k} f^{(k)}(x)$ or $v_k(x)$, $\lambda_k(x)$ denote either $\beta_k(x)$ or $b_k(x)$, and be as in (5.4). We shall assume as in Sections 3 and 4 that there exist x_l, $l = 0, 1, 2, \ldots$, such that $a < x_0 < x_1 < \ldots$, $\lim_{l \to \infty} x_l = \infty$, and that

$$\psi_k(x_l) = 0, \quad l = 0, 1, \ldots, \quad k \in E_{m-1} = \{k_1, k_2, k_{m-1}\}, \tag{6.1}$$

and

$$\psi_k(x_l)\psi_k(x_{l+1}) < 0, \quad l = 0, 1, \ldots, \quad k \in E\backslash E_{m-1}. \tag{6.2}$$

Actually the set $E\backslash E_{m-1}$ contains only one value of k, say $k = r$. For the sake of simplicity we shall write $\psi(x)$ instead of $\psi_r(x)$ and $\lambda(x)$ instead of $\lambda_r(x)$. Then (6.2) becomes

$$\psi(x_l)\psi(x_{l+1}) < 0, \quad l = 0, 1, \ldots. \tag{6.3}$$

Similarly, (5.2) and/or (5.3) become

$$\int_a^\infty f(t)\,dt = F(x_l) + \psi(x_l)\lambda(x_l), \quad l = 0, 1, \ldots, \tag{6.4}$$

and Equations (3.2) and/or (4.3) become

$$D = F(x_l) + \psi(x_l) \sum_{i=0}^{n} \bar{\lambda}_i / x_l^i, \quad j \leqslant l \leqslant j+n+1, \tag{6.5}$$

where n denotes n_r for short.

For the Fourier and Hankel transforms dealt with in Sections 2, 3 and 4 we have the following: $m = 2$, $p = 1 = q$. For the $\bar{D}$-transformation (6.3) holds provided $g(x) > 0$ for $x > a$. For the $\tilde{D}$-transformation (6.3) always holds.

The solution for D of Equations (6.5), by using Cramer's rule can be expressed as $D = \det M/\det K$, where M and K are $(n+2)\times(n+2)$ matrices. The matrix M is given by

$$M = \begin{bmatrix} F(x_j)/\psi(x_j) & F(x_{j+1})/\psi(x_{j+1}) & \ldots & F(x_{j+n+1})/\psi(x_{j+n+1}) \\ 1 & 1 & & 1 \\ x_j^{-1} & x_{j+1}^{-1} & \ldots & x_{j+n+1}^{-1} \\ \vdots & \vdots & & \vdots \\ x_j^{-n} & x_{j+1}^{-n} & \ldots & x_{j+n+1}^{-n} \end{bmatrix} \tag{6.6}$$

and the matrix K is obtained from M by replacing the first row of M by the row vector $(1/\psi(x_j), 1/\psi(x_{j+1}), \ldots, 1/\psi(x_{j+n+1}))$.

If we now denote the minor of $F(x_{j+l})/\psi(x_{j+l})$ in M (or of $1/\psi(x_{j+l})$ in K) by V_l, and expand $\det M$ and $\det K$ with respect to their first rows we obtain

$$D = \frac{\sum_{l=0}^{n+1} (-1)^l [V_l/\psi(x_{j+l})] F(x_{j+l})}{\sum_{l=0}^{n+1} (-1)^l [V_l/\psi(x_{j+l})]}, \tag{6.7}$$

hence

$$\gamma_l = (-1)^l [V_l/\psi(x_{j+l})] \Big/ \sum_{i=0}^{n+1} (-1)^i [V_i/\psi(x_{j+i})], \quad l = 0, \ldots, n+1, \tag{6.8}$$

in Equation (5.1), where $(\gamma_0, \gamma_1, \ldots, \gamma_{n+1})$ is the first row of the inverse matrix Q^{-1}.

The minors V_l are given by

$$\begin{aligned} V_0 &= V(x_{j+1}^{-1}, \ldots, x_{j+n+1}^{-1}) \\ V_l &= V(x_j^{-1}, \ldots, x_{j+l-1}^{-1}, x_{j+l+1}^{-1}, \ldots, x_{j+n+1}^{-1}), \quad 0 < l < n, \\ V_{n+1} &= V(x_j^{-1}, \ldots, x_{j+n}^{-1}) \end{aligned} \tag{6.9}$$

where

$$V(\xi_0, \xi_1, \ldots, \xi_n) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \xi_0 & \xi_1 & \cdots & \xi_n \\ \xi_0^2 & \xi_1^2 & \cdots & \xi_n^2 \\ \vdots & \vdots & & \vdots \\ \xi_0^n & \xi_1^n & \cdots & \xi_n^n \end{bmatrix} \tag{6.10}$$

is the Vandermonde determinant. It is known that

$$V(\xi_0, \xi_1, \ldots, \xi_n) = \prod_{0 \leqslant i < j \leqslant n} (\xi_j - \xi_i) \tag{6.11}$$

and therefore if $\xi_0 < \xi_1 < \ldots < \xi_n$, then $V(\xi_0, \xi_1, \ldots, \xi_n) > 0$.

Since $x_0 < x_1 < \ldots$, all of the V_l have the same sign. Using this fact, together with (6.3), we can see that the quantities $(-1)^l V_l/\psi(x_{j+l})$, $0 \leqslant l \leqslant n+1$, all have the same sign. Therefore, $\gamma_l > 0$ for all l, which follows from (6.8). (We have also proved that if (6.3) is satisfied, then equations (6.5) always have a unique solution, since we have shown that the determinant of the matrix of coefficients is non-zero.) Consequently

$$\sum_{l=0}^{n+1} |\gamma_l| = \sum_{l=0}^{n+1} \gamma_l = 1. \tag{6.12}$$

Theorem 2 then becomes

THEOREM 4. *The approximation D to $\int_a^\infty f(t)\,dt$ satisfies*

$$\int_a^\infty f(t)\,dt - D = \sum_{l=0}^{n+1} \gamma_l \psi(x_{j+l}) w_n(x_{j+l}), \tag{6.13}$$

where

$$w_n(x) = \lambda(x) - \sum_{i=0}^{n} \lambda_i'/x^i \tag{6.14}$$

and $\lambda(x)$, which is in $A^{(0)}$, has the following asymptotic expansion:

$$\lambda(x) \sim \sum_{i=0}^{\infty} \lambda_i'/x^i \quad \text{as } x \to \infty. \tag{6.15}$$

Corollary. For n fixed and $j \to \infty$ (i.e. $x_j \to \infty$)

$$\left| \int_a^\infty f(t)\,dt - D \right| = o(x_j^{-n-1}). \tag{6.16}$$

Proof. From (6.13) it follows that

$$\left| \int_a^\infty f(t)\,dt - D \right| \leqslant \left(\sum_{l=0}^{n+1} |\gamma_l| \right) \max_{x \geqslant x_j} |\psi(x)| \max_{x \geqslant x_j} |w_n(x)|. \tag{6.17}$$

(6.16) now follows from (6.12) and by using the fact that $\psi(x) = o(1)$ and $w_n(x) = o(x^{-n-1})$ as $x \to \infty$.

Similarly, Theorem 3 becomes

THEOREM 5. *The approximation D to $\int_a^\infty f(t)\,dt$ satisfies*

$$\int_a^\infty f(t)\,dt - D = \sum_{l=0}^{n+1} \gamma_l \psi(x_{j+l}) u_n(x_{j+l}), \tag{6.18}$$

where

$$u_n(x) = \lambda(x) - \pi_n(x_j/x) \tag{6.19}$$

and

$$\pi_n(\xi) = \sum_{i=0}^{n} \alpha_{n,i} \xi^i \tag{6.20}$$

is the best polynomial approximation of degree n in ξ to $\lambda(x_j/\xi)$ in $[0, 1]$.

Corollary. Defining $\xi_l = x_j/x_l$, $l = 0, 1, \ldots$, we have

$$\left|\int_a^\infty f(t)\,dt - D\right| \leqslant \max_{j \leqslant l \leqslant j+n+1} |\psi(x_j/\xi_l)| \max_{0 \leqslant \xi \leqslant 1} |u_n(x_j/\xi)| \tag{6.21}$$

hence for j fixed and $n \to \infty$

$$\left|\int_a^\infty f(t)\,dt - D\right| = o(n^{-\mu}) \quad \text{for any } \mu > 0. \tag{6.22}$$

Proof. (6.21) follows from (6.18) and (6.12). (6.22) follows from the fact that $\psi(x) = o(1)$ for all $x > a$ and the fact that $\max_{0 \leqslant \xi \leqslant 1} |u_n(x_j/\xi)| = o(n^{-\mu})$ for any $\mu > 0$. This in turn is a consequence of the fact that $\lambda(x)$ is infinitely differentiable for all $x > a$ including $x = \infty$ (hence $\lambda(x_j/\xi)$ is infinitely differentiable for $0 \leqslant \xi \leqslant 1$).

Another proof of (6.21) will be given in Appendix B.

Note. The situation in which $\gamma_l > 0$ for all l is the most ideal situation from the numerical point of view. This suggests that the error in the computed value of D is of the order of the maximum of the errors in $F(x_l)$. For this point see Sidi (1979*b*).

7. A Further Application to Complex Fourier Transforms

In Section 6 we have seen that the oscillatory nature of $\psi(x_l)$, $l = 0, 1, \ldots$, forces the condition in (6.12) which in turn guarantees the rates of convergence in (6.16) and (6.22). It turns out for complex Fourier transforms too, the points x_l, $l = 0, 1, \ldots$, can be chosen so as to make $\psi(x_l)$ oscillatory.

Let us put $f(x) = e^{ix} g(x)$ where $g(x)$ is again as in Section 2, sub-section on Fourier transforms. Then $f(x)$ satisfies the linear first order homogeneous differential equation $f = p_1 f'$, where

$$p_1(x) = (i + \phi' + h'/h)^{-1}. \tag{7.1}$$

A simple analysis shows that if $k > 0$, then $p_1 \in A^{(-k+1)}$, and if $k = 0$, then $p_1 \in A^{(0)}$.

Hence in both cases $\bar{p}_1 = 0$ and all the conditions of Theorem 1 are satisfied and

$$\rho_0 = i_1 = \begin{cases} -k+1, & \text{if } k > 0 \\ 0, & \text{if } k = 0. \end{cases} \tag{7.2}$$

Hence

$$\int_x^\infty f(t)\,dt = x^{\rho_0} f(x) \beta_0(x), \tag{7.3}$$

where $\beta_0 \in A^{(0)}$. Using the properties of $g(x)$, we can also express (7.3) in the form

$$\int_x^\infty f(t)\,dt = x^{\rho_0+\gamma} e^{\phi(x)} e^{ix} b_0(x), \tag{7.4}$$

with $b_0 \in A^{(0)}$.

Consider now the integral $\int_a^\infty f(t)\,dt$ with $a \geqslant 0$ and $f(x)$ as above. By taking x_0 to be the smallest zero of $\sin x$ which is greater than a and letting $x_0, x_1, x_2, \ldots$, be the consecutive zeros of $\sin x$, i.e. $x_l = x_0 + l\pi$, $l = 0, 1, \ldots$, we have that $e^{ix_l} e^{ix_{l+1}} = -1$, $l = 0, 1, \ldots$. Hence from (7.3) we obtain

$$\int_a^\infty f(t)\,dt = \int_a^{x_l} f(t)\,dt + (-1)^l x_l^{\rho_0} g(x_l) \beta_0(x_l), \quad l = 0, 1, \ldots, \tag{7.5}$$

and from (7.4), we obtain

$$\int_a^\infty f(t)\,dt = \int_a^{x_l} f(t)\,dt + (-1)^l x_l^{\rho_0+\gamma} e^{\phi(x_l)} b_0(x_l), \quad l = 0, 1, \ldots. \tag{7.6}$$

This means that $\psi(x)$, in the notation of Section 6, satisfies (6.3), in the D-transformation of Levin & Sidi (1975) provided $g(x) > 0$ for $x > a$, and in the $\tilde{D}$-transformation, always. Hence the approximations $D_{n_0}^{(1,j)}$ and $\tilde{D}_{n_0}^{(1,j)}$ are computed by solving Equations (6.5), or equivalently, from

$$D = \frac{\sum_{r=0}^{n_0+1} (-1)^r \binom{n_0+1}{r} (x_0/\pi + j + r)^{n_0} F(x_{j+r})/\psi(x_{j+r})}{\sum_{r=0}^{n_0+1} (-1)^r \binom{n_0+1}{r} (x_0/\pi + j + r)^{n_0}/\psi(x_{j+r})}, \tag{7.7}$$

where D stands for $D_{n_0}^{(1,j)}$ or $\tilde{D}_{n_0}^{(1,j)}$ depending on the choice of $\psi(x)$.

As mentioned above, Theorems 4 and 5 and their corollaries apply directly to these approximations.

We note that the result given here for the D-transformation is a generalization of that due to Levin (1975), while the $\tilde{D}$-transformation is new. However, the special choice of the x_l, which has been guided by the results of Section 6, gives considerable improvement over the results of Levin (1975) who uses $x_{l+1} - x_l = 1$, $l = 0, 1, \ldots$.

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Appendix A

We shall now consider the solution of the linear equations

$$T = F(x_l)+v(x_l)\sum_{i=0}^{n} \delta_i/x_l^i, \quad j \leqslant l \leqslant j+n+1, \tag{A.1}$$

where T and the δ_i are unknowns and $x_0 > 0$ and $x_l = x_0+ls$, $l = 0, 1, \ldots$, such that $s > 0$ is a fixed constant. Defining $\delta_i' = \delta_i/s^i$, $i = 0, \ldots, n$, we can write equations (A.1) in the form

$$T = F(x_l)+v(x_l)\sum_{i=0}^{n} \delta_i'/(x_0/s+l)^i, \quad j \leqslant l \leqslant j+n+1. \tag{A.2}$$

Now these equations can be solved using Cramer's rule and the procedure described in Levin (1973), and the result is

$$T = \frac{\sum_{r=0}^{n+1} (-1)^r \binom{n+1}{r} (x_0/s+j+r)^n F(x_{j+r})/v(x_{j+r})}{\sum_{r=0}^{n+1} (-1)^r \binom{n+1}{r} (x_0/s+j+r)^n/v(x_{j+r})}. \tag{A.3}$$

Appendix B

The following proof of (6.21) has been given by Prof. Nira Richter-Dyn of Tel-Aviv University (Private communication).

Subtracting (6.5) from (6.4) and dividing by $\psi(x_l)$ we obtain

$$\frac{\int_a^\infty f(t)\,dt - D}{\psi(x_l)} = \lambda(x_l) - \sum_{i=0}^{n} \bar{\lambda}_i/x_l^i, \quad j \leqslant l \leqslant j+n+1. \tag{B.1}$$

Defining $\xi = x_j/x$ and $\xi_l = x_j/x_l$, $l = 0, 1, \ldots$, we write (B.1) as

$$\frac{\int_a^\infty f(t)\,dt - D}{\psi(x_l)} = \lambda(x_j/\xi_l) - p_n(\xi_l), \quad j \leqslant l \leqslant j+n+1, \tag{B.2}$$

where $p_n(\xi)$ is the nth degree polynomial

$$p_n(\xi) = \sum_{i=0}^{n} (\bar{\lambda}_i/x_j^i)\xi^i. \tag{B.3}$$

Using (6.3) in (B.2) we have that $\lambda(x_j/\xi) - p_n(\xi)$ assumes alternately positive and negative values on the $n+2$ consecutive points $\xi_j > \xi_{j+1} > \ldots > \xi_{j+n+1}$. Invoking now one of the theorems of de la Vallée Poussin, see Cheney (1966, p. 77), we have

$$\min_{j \leqslant l \leqslant j+n+1} |\lambda(x_j/\xi_l) - p_n(\xi_l)| \leqslant \max_{0 \leqslant \xi \leqslant 1} |u_n(x_j/\xi)|. \tag{B.4}$$

Substituting (B.4) into (B.2), (6.21) now follows.

Of course in order to be able to write down the steps above we need to show that Equations (6.5) have a unique solution. And this has been done prior to Theorem 4.