A new method for deriving Padé approximants for some hypergeometric functions

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ABSTRACT

In this work a unified method for obtaining the Padé approximants for some hypergeometric functions is given. This method is based on a set of linear equations that are obtained from the determinant expressions for Padé approximants and their solution by using two simple theorems developed in the text.

1. INTRODUCTION AND THEORY

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

be a formal power series. The (m/n) Padé approximant to (1.1), if it exists, is defined to be the rational function

$$R_{m,n}(z) = \frac{P_{m,n}(z)}{Q_{m,n}(z)} \tag{1.2}$$

whose numerator $P_{m,n}(z)$ is a polynomial of degree at most m and whose denominator $Q_{m,n}(z)$ is a polynomial of degree at most n, such that

$$f(z) - R_{m,n}(z) = 0(z^{m+n+1}) \tag{1.3}$$

$R_{m,n}(z)$ can be expressed in the form (see Baker [1], p. 9):

$$R_{m,n}(z) = \frac{\det M}{\det N}, \tag{1.4}$$

where

$$M = \begin{bmatrix} z^n F_{m-n} & z^{n-1} F_{m-n+1} & \cdots & z^0 F_m \\ a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ a_{m-n+2} & a_{m-n+3} & \cdots & a_{m+2} \\ \vdots & \vdots & & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+n} \end{bmatrix} \tag{1.5}$$

where $F_j = \sum_{k=0}^{j} a_k z^k$, $j = 0, 1, \ldots$, and $F_j = 0$ and $a_j = 0$ for $j < 0$. N is obtained from M by replacing the first row of M by the vector $(z^n, z^{n-1} \ldots z^0)$. Of course, we assume in (1.4) that $\det N \neq 0$. Judging from above we have the following :

Lemma

$R_{m,n}(z)$, if it exists, is the solution to the set of linear equations

$$z^{m-r} F_r = z^{m-r} R_{m,n}(z) + \sum_{i=1}^{n} \delta_i a_{r+i}, \tag{1.6}$$

$r = m-n, m-n+1, \ldots, m,$

where the δ_i are additional unknowns, and the determinant of coefficients is non-zero.

Proof

Solve for $R_{m,n}(z)$ by using Cramer's rule.

It turns out that for some hypergeometric functions the equations in (1.6) for $m \geq n-1$ can be solved for $R_{m,n}(z)$ analytically in a simple form. In order to be able to do this we need the following result :

Theorem 1

Let T and γ_i, $i = 0, \ldots, n-1$, be defined by the following set of equations :

$$A_r = b_r T + c_r [\gamma_0 + \sum_{i=1}^{n-1} \frac{\gamma_i}{a + r + i - 1}], \tag{1.7}$$

$r = k, k+1, \ldots, k+n,$

or

$$A_r = b_r T + c_r \sum_{i=0}^{n-1} \frac{\gamma_i}{(a+r)_i}, \tag{1.8}$$

$r = k, k+1, \ldots, k+n,$

where

$a + r + i - 1 \neq 0$, $i = 1, \ldots, n-1$, $r = k, \ldots, k+n$.

Then

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$$T = \frac{\sum_{j=0}^{n} (-1)^j \binom{n}{j} (\alpha+k+j)_{n-1} \frac{A_{k+j}}{c_{k+j}}}{\sum_{j=0}^{n} (-1)^j \binom{n}{j} (\alpha+k+j)_{n-1} \frac{b_{k+j}}{c_{k+j}}} . \tag{1.9}$$

The Pochhammer symbol $(a)_j$ is defined by

$$(a)_j = \begin{cases} 1 & \text{if } j=0 \\ \prod_{i=1}^{j} (a+i-1) & \text{if } j = 1, 2, \ldots, \end{cases} \tag{1.10}$$

Proof

Multiplying (1.7) by $(\alpha+r)_{n-1}/c_r$ we obtain :

$$(\alpha+r)_{n-1} \frac{A_r}{c_r} = (\alpha+r)_{n-1} \frac{b_r}{c_r} T + [\gamma_0 (\alpha+r)_{n-1} + \sum_{i=1}^{n-1} \gamma_i \frac{(\alpha+r)_{n-1}}{\alpha+r+i-1}], \tag{1.11}$$

$r = k, \ldots, k+n.$

Now the term inside the square brackets on the right hand side of (1.11) is a polynomial in r of degree $\leqslant n-1$, since $(\alpha+r)_{n-1} = \prod_{j=1}^{n-1} (\alpha+r+j-1)$.

Similarly, if we multiply (1.8) by $(\alpha+r)_{n-1}/c_r$, we obtain :

$$(\alpha+r)_{n-1} \frac{A_r}{c_r} = (\alpha+r)_{n-1} \frac{b_r}{c_r} T + \left[\sum_{i=0}^{n-1} \gamma_i \frac{(\alpha+r)_{n-1}}{(\alpha+r)_i} \right],$$

$r = k, \ldots, k+n.$ (1.12)

The summation in the square brackets on the right hand side of (1.12) too is a polynomial in r of degree $\leqslant n-1$. Now if we define $\Delta p(x) = p(x+1) - p(x)$, and $\Delta^2 p(x) = \Delta[\Delta p(x)]$, etc., then $\Delta^n p(x) \equiv 0$ whenever $p(x)$ is a polynomial in x of degree $\leqslant n-1$. Let us now apply Δ^n to (1.11) or (1.12) with $r = k$. This is equivalent to multiplying each of the equations in (1.11) or (1.12) by a constant and then forming their sum. The new equation that we obtain from both (1.11) and (1.12) is

$$\Delta^n [(\alpha+k)_{n-1} \frac{A_k}{c_k}] = \Delta^n [(\alpha+k)_{n-1} \frac{b_k}{c_k} T], \tag{1.13}$$

where Δ^n operating on the summations on the right hand side of (1.11) and (1.12) gives zero. Solving (1.13) for T we obtain :

$$T = \frac{\Delta^n [(\alpha+k)_{n-1} \frac{A_k}{c_k}]}{\Delta^n [(\alpha+k)_{n-1} \frac{b_k}{c_k}]}, \tag{1.14}$$

which, upon using the relation

$$\Delta^n q(k) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} q(k+j), \tag{1.15}$$

is easily seen to be (1.9).

Another version of theorem 1 which is also useful is as follows :

Theorem 2

Let T and γ_i, $i = 0, \ldots, n-1$, be defined by the following set of equations :

$$A_r = b_r T + c_r \sum_{i=0}^{n-1} \gamma_i p_i(r), \quad r = k, k+1, \ldots, k+n \tag{1.16}$$

such that $p_i(x)$ are polynomials in x of degree $\leqslant n-1$. Then T is given by :

$$T = \frac{\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{A_{k+j}}{c_{k+j}}}{\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{b_{k+j}}{c_{k+j}}} . \tag{1.17}$$

Proof

Divide (1.16) by c_r and apply to the resulting equation with $r = k$ the operator Δ^n, remembering that $\Delta^n p_i(x) \equiv 0$ for all i. Finally, use (1.15) to obtain (1.17).

2. APPLICATIONS

Example 1

$$f(z) = {}_1F_1(1;\beta;z) = \sum_{j=0}^{\infty} \frac{z^j}{(\beta)_j}, \quad \beta \neq 0, -1, -2, \ldots$$

For this case $a_j = 1/(\beta)_j$, $j = 0, 1, \ldots$ Hence equations (1.6) for $m \geqslant n-1$ become

$$z^{m-r} F_r = z^{m-r} R_{m,n}(z) + \sum_{i=1}^{n} \frac{\delta_i}{(\beta)_{r+i}},$$

$r = m-n, \ldots, m.$ (2.1)

Using the relation

$$(a)_{p+q} = (a)_p (a+p)_q, \tag{2.2}$$

equations (2.1) can be put in the form

$$z^{m-r} F_r = z^{m-r} R_{m,n}(z) + \frac{1}{(\beta)_{r+1}} \sum_{i=0}^{n-1} \frac{\gamma_i}{(\beta+r+1)_i},$$

$r = m-n, \ldots, m,$ (2.3)

where $\gamma_i = \delta_{i+1}$, $i = 0, 1, \ldots, n-1$.

The linear system in (2.3) is of the form (1.8) with $A_r = z^{m-r} F_r$, $b_r = z^{m-r}$, $T = R_{m,n}(z)$, $c_r = 1/(\beta)_{r+1}$, $\alpha = \beta+1$, $k = m-n$. Hence using (1.9) of theorem 1 we obtain :

$$R_{m,n}(z) = \frac{\sum_{j=0}^{n} (-1)^j \binom{n}{j} (\beta+m-n+j+1)_{n-1} (\beta)_{m-n+j+1} z^{n-j} F_{m-n+j}}{\sum_{j=0}^{n} (-1)^j \binom{n}{j} (\beta+m-n+j-1)_{n-1} (\beta)_{m-n+j+1} z^{n-j}}, \tag{2.4}$$

which, upon making use of (2.2), can be put in the form

$$R_{m,n}(z)=\frac{\sum_{j=0}^{n}(-1)^j\binom{n}{j}(\beta)_{m+j}z^{n-j}F_{m-n+j}}{\sum_{j=0}^{n}(-1)^j\binom{n}{j}(\beta)_{m+j}z^{n-j}}, \tag{2.5}$$

For $\beta=1$, $f(z)=\exp(z)$ and for this case (2.5) becomes

$$R_{m,n}(z)=\frac{\sum_{j=0}^{n}(-1)^j\frac{(m+j)!}{(n-j)!j!}z^{n-j}\sum_{i=0}^{m-n+j}\frac{z^i}{i!}}{\sum_{j=0}^{n}(-1)^j\frac{(m+j)!}{(n-j)!j!}z^{n-j}}. \tag{2.6}$$

Example 2

$$f(z)={}_2F_1(1,\mu;\beta;z)=\sum_{j=0}^{\infty}\frac{(\mu)_j}{(\beta)_j}z^j,$$

$\mu,\beta\neq 0,-1,-2,\ldots$

For this case $a_j=(\mu)_j/(\beta)_j$, $j=0,1,\ldots$ Therefore, equations (1.6) for $m\geqslant n-1$ become

$$z^{m-r}F_r=z^{m-r}R_{m,n}(z)+\sum_{i=1}^{n}\delta_i\frac{(\mu)_{r+i}}{(\beta)_{r+i}}, \tag{2.7}$$

$r=m-n,\ldots,m.$

Using (2.2), the equations in (2.7) can be reexpressed in the form

$$z^{m-r}F_r=z^{m-r}R_{m,n}(z)+\frac{(\mu)_{r+1}}{(\beta)_{r+1}}\sum_{i=1}^{n}\delta_i\frac{(\mu+r+1)_{i-1}}{(\beta+r+1)_{i-1}},$$

$r=m-n,\ldots,m.$ (2.8)

Now, for $p=1,2,\ldots,$

$$\frac{(\mu+r+1)_p}{(\beta+r+1)_p}=\frac{(r+\mu+1)(r+\mu+2)\ldots(r+\mu+p)}{(r+\beta+1)(r+\beta+2)\ldots(r+\beta+p)}$$

$$=1+\sum_{q=1}^{p}\frac{A_{p,q}}{\beta+r+q}, \tag{2.9}$$

where $A_{p,q}$ are constants. Therefore,

$$\sum_{i=1}^{n}\delta_i\frac{(\mu+r+1)_{i-1}}{(\beta+r+1)_{i-1}}=\gamma_0+\sum_{i=1}^{n-1}\frac{\gamma_i}{\beta+r+i}, \tag{2.10}$$

where $\gamma_0=\sum_{i=1}^{n}\delta_i$ and γ_i, for $i=1,\ldots,n-1$, are linear combinations of products of the δ_j and the $A_{p,q}$, whose explicit form is not of interest to us in this work. Therefore, equations (2.8) can be written in the form (1.7) with A_r, b_r, and T as in example 1, and $c_r=(\mu)_{r+1}/(\beta)_{r+1}$, $\alpha=\beta+1$, $k=m-n$.

Using theorem 1 again we have :

$R_{m,n}(z)$

$$=\frac{\sum_{j=0}^{n}(-1)^j\binom{n}{j}(\beta+m-n+j+1)_{n-1}\frac{(\beta)_{m-n+j+1}}{(\mu)_{m-n+j+1}}z^{n-j}F_{m-n+j}}{\sum_{j=0}^{n}(-1)^j\binom{n}{j}(\beta+m-n+j+1)_{n-1}\frac{(\beta)_{m-n+j+1}}{(\mu)_{m-n+j+1}}z^{n-j}}$$

Making use of (2.2) we obtain :

$$R_{m,n}(z)=\frac{\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(\beta)_{m+j}}{(\mu)_{m-n+j+1}}z^{n-j}F_{m-n+j}}{\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(\beta)_{m+j}}{(\mu)_{m-n+j+1}}z^{n-j}}. \tag{2.12}$$

For $\beta=1$ $f(z)={}_1F_0(\mu;z)=(1-z)^{-\mu}$ and $R_{m,n}(z)$ for this case becomes :

$R_{m,n}(z)$

$$=\frac{\sum_{j=0}^{n}(-1)^j\frac{(m+j)!}{j!(n-j)!}\frac{z^{n-j}}{(\mu)_{m-n+j+1}}F_{m-n+j}}{\sum_{j=0}^{n}(-1)^j\frac{(m+j)!}{j!(n-j)!}\frac{z^{n-j}}{(\mu)_{m-n+j+1}}}. \tag{2.13}$$

For $\beta=\mu+1$, $f(z)={}_2F_1(1,\mu;\mu+1;z)$

$=\mu\sum_{j=0}^{\infty}\frac{z^j}{\mu+j}$, $\mu\neq 0,-1,-2,\ldots,$ hence (2.12) for this case becomes :

$$R_{m,n}(z)=\frac{\sum_{j=0}^{n}(-1)^j\binom{n}{j}(\mu+m-n+j+1)_n z^{n-j}F_{m-n+j}}{\sum_{j=0}^{n}(-1)^j\binom{n}{j}(\mu+m-n+j+1)_n z^{n-j}}. \tag{2.14}$$

For $\mu=1$ $f(z)={}_2F_1(1,1;2;z)=-z^{-1}\log(1-z)$ and hence (2.14) becomes :

$R_{m,n}(z)$

$$=\frac{\sum_{j=0}^{n}(-1)^j\frac{(m+j+1)!}{j!(n-j)!(m-n+j+1)!}z^{n-j}F_{m-n+j}}{\sum_{j=0}^{n}(-1)^j\frac{(m+j+1)!}{j!(n-j)!(m-n+j+1)!}z^{n-j}}. \tag{2.15}$$

For $\mu=1/2$ $f(z)={}_2F_1(1,1/2;3/2;z)=z^{-1/2}\operatorname{arctanh}z^{1/2}$, and $R_{m,n}(z)$ is similarly obtained from (2.14).

Example 3

$$f(z)={}_2F_0(1,\mu;z)=\sum_{j=0}^{\infty}(\mu)_j z^j,\quad \mu\neq 0,-1,-2,\ldots$$

Equations (1.6) for $m\geqslant n-1$ become

$$z^{m-r}F_r=z^{m-r}R_{m,n}(z)+\sum_{i=1}^{n}\delta_i(\mu)_{r+i},\quad r=m-n,\ldots,m, \tag{2.16}$$

which, with the help of (2.2), can be reexpressed in the form

$$z^{m-r}F_r=z^{m-r}R_{m,n}(z)+(\mu)_{r+1}\sum_{i=1}^{n}\delta_i(\mu+r+1)_{i-1},$$

$r=m-n,\ldots,m.$ (2.17)

Since each of the $(\mu+r+1)_{i-1}$, $i=1,\ldots,n$, is a polynomial in r of degree $\leqslant n-1$, theorem 2 applies and we obtain, with A_r, b_r and T as in the previous examples, and $c_r = (\mu)_{r+1}$,

$$R_{m,n}(z) = \frac{\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{z^{n-j}}{(\mu)_{m-n+j+1}} F_{m-n+j}}{\sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{z^{n-j}}{(\mu)_{m-n+j+1}}} . \qquad (2.18)$$

The case $\mu = 1$ is the Euler series which is related to the exponential integral $E_1(-1/z)$, and for this case $f(z) = {}_2F_0(1, 1; z) = \sum_{j=0}^{\infty} j!\, z^j$.

Example 4

$f(z) = \sum_{j=0}^{\infty} p(j) z^j$, p(j) is a polynomial in j of degree $s - 1 \geq 0$.

It can easily be verified that f(z) is a rational function whose numerator is of degree $\leq s-1$ and whose denominator is of degree exactly s.

Actually the denominator is just $(1-z)^s$. Letting $m \geq s-1$ and $n \geq m+1 \geq s$, equations (1.6) become :

$$z^{m-r} F_r = z^{m-r} R_{m,n}(z) + \sum_{i=1}^{n} \delta_i p(r+1),$$

$$r = m-n, \ldots, m. \qquad (2.19)$$

Since the sum on the right hand side of (2.19) is a polynomial in r of degree $\leq s-1 \leq n-1$, we can apply theorem 2 to this case. The result is

$$f(z) = R_{m,n}(z) = \frac{\sum_{j=0}^{n} (-1)^j \binom{n}{j} z^{n-j} F_{m-n+j}}{\sum_{j=0}^{n} (-1)^j \binom{n}{j} z^{n-j}} = R_{s-1,s}(z),$$

$$m \geq n-1 \geq s-1. \qquad (2.20)$$

Hence, we have actually constructed f(z) analytically from its Maclaurin series in a simple manner.

We note that $R_{m,n}(z)$ for example 1 and example 2, in particular for $f(z) = \exp(z)$, and $f(z) = (1-z)^{-\mu}$, have been given in the literature previously and can be found in Luke [3, p. 174, p. 192]. The cases $\exp(z)$ and $(1-z)^{-\mu}$ have also been reconsidered recently by Iserles [2] through Kummer's first identity for the confluent hypergeometric functions and Euler's theorem for the Gaussian hypergeometric functions.

CONCLUDING REMARKS

In this work we have presented a new method by which Padé approximants to some hypergeometric series can be obtained in closed analytic form with very little effort. This has been accomplished by defining the Padé approximants by the linear set of equations in (1.6) and then invoking the two theorems of section 1. It is worth noting that, unlike the previous methods, the method of the present work makes direct use of the series themselves.

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