Two New Classes of Nonlinear Transformations for Accelerating the Convergence of Infinite Integrals and Series

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ABSTRACT

Two new classes of nonlinear transformations, the D-transformation to accelerate the convergence of infinite integrals and the d-transformation to accelerate the convergence of infinite series, are presented. In the course of the development of these transformations two interesting asymptotic expansions, one for infinite integrals and the other for infinite series, are derived. The transformations D and d can easily be applied to infinite integrals $\int_0^\infty f(t)\,dt$ whose integrands $f(t)$ satisfy linear differential equations of the form $f(t)=\sum_{k=1}^m p_k(t)f^{(k)}(t)$ and to infinite series $\sum_{r=1}^\infty f(r)$ whose terms $f(r)$ satisfy a linear difference equation of the form $f(r)=\sum_{k=1}^m p_k(r)\Delta^k f(r)$, such that in both cases the p_k have asymptotic expansions in inverse powers of their arguments. In order to be able to apply these transformations successfully one need not know explicitly the differential equation that the integrand satisfies or the difference equation that the terms of the series satisfy; mere knowledge of the existence of such a differential or difference equation and its order m is enough. This broadens the areas to which these methods can be applied. The connection between the D- and d-transformations with some known transformations in shown. The use and the remarkable efficiency of the D- and d-transformations are demonstrated through several numerical examples. The computational aspects of these transformations are described in detail.

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1. INTRODUCTION

In this work we present some nonlinear transformations to accelerate the convergence of slowly converging infinite integrals and infinite series. These transformations, in a sense, combine the G-transformation of Gray, Atchison, and McWilliams [1] and the confluent ϵ-algorithm of Wynn [4] on the one hand, and some transformations that were obtained by the first author [5–7] for accelerating the convergence of infinite integrals and series on the other.

The G-transformation and the confluent ϵ-algorithm have in common the property that they integrate exactly from zero to infinity functions $f(t)$ which satisfy linear differential equations with constant coefficients on $(0,\infty)$; i.e., (in the notation of Gray, Atchison, and McWilliams [1]) the quantity $G_m[F(t);k]$ of the G-transformation and (in the notation of Wynn [4]) the quantity $\epsilon_{2m}(t)$ of the confluent ϵ-algorithm are exactly equal to $\int_0^\infty f(t)\,dt$. For more details on both the G-transformation and the confluent ϵ-algorithm the reader is referred to [1].

For future reference, when dealing with infinite series, we define $A^{(\gamma)}$ to be the set of functions $a(x)$, which, as $x\to\infty$, have asymptotic expansions in inverse powers of x, of the form

$$a(x)\sim x^{\gamma}\left(\alpha_0+\frac{\alpha_1}{x}+\frac{\alpha_2}{x^2}+\cdots\right), \tag{1.1}$$

and when dealing with infinite integrals, we define $A^{(\gamma)}$ to be the set of infinitely differentiable functions $a(x)$, satisfying (1.1), and such that their derivatives of any order have asymptotic expansions, which can be obtained by differentiating that in (1.1) formally term by term.

It turns out numerically that the G-transformation and the confluent ϵ-algorithm work efficiently on functions of the form $f=ag$, where $a\in A^{(\gamma)}$ for some γ and where g satisfies a linear differential equation with constant coefficients, provided $F(x)=\int_0^x f(t)\,dt$ is not monotonic as $x\to\infty$—and therefore, on sums of functions of this form.

In the work of Levin [6] too we find that there is a strong connection between the differential equation that the integrand satisfies and the method of accelerating the convergence of the infinite integral, or between the recursion relation that the terms of the infinite series satisfy and the method of accelerating the convergence of the infinite series. In this work of Levin methods are given for accelerating the convergence of infinite integrals of the form $\int_0^\infty a(t)\phi(t)\,dt$ and infinite series of the form $\sum_{r=1}^\infty a(r)\phi(r)$, where $a\in A^{(\gamma)}$ for some γ, and where $\phi(t)$ satisfies a *specific* differential equation in the case of the infinite integrals, and $\phi(r)$, $r=1,2,\ldots$, satisfy a *specific* recursion relation in the case of the infinite series. That is, for each different function $\phi(t)$ or different sequence $\phi(r)$, $r=1,2,\ldots$, one has a *different* transformation. This point will be further clarified below.

So far nonlinear transformations have been developed for a limited class of infinite integrals and infinite series. The main purpose of this work is to develop a transformation or a class of transformations that will work efficiently on a large class of infinite series and infinite integrals that arise in many problems of applied mathematics and physics. A property common to most of these problems is the fact that many of the functions of applied mathematics and physics satisfy linear differential equations and/or linear recursion relations. This fact will be the starting point in the development of our nonlinear transformations.

We shall see that it is possible to obtain a class of nonlinear transformations $D^{(m)}$, that will accelerate the convergence of infinite integrals $\int_0^\infty f(t)\,dt$ where $f(t)$ satisfies *any* linear differential equation of order m with *any* coefficients in $A^{(\gamma)}$ for some values of γ. Similarly we shall see that it is possible to obtain another class of transformations $d^{(m)}$ for accelerating the convergence of infinite series $\sum_{r=1}^\infty f(r)$ where $f(r)$, $r=1,2,\ldots$, satisfy *any* linear $(m+1)$-term recursion relation with coefficients in $A^{(\gamma)}$ for some values of γ. This recursion relation can be written as a linear mth-order difference equation with coefficients again in $A^{(\gamma)}$ for some values of γ, this difference equation being the discrete analogue of the differential equation mentioned above.

Levin [6], in the development of his transformations for infinite integrals of the form $S=\int_0^\infty a(t)\phi(t)\,dt$, where $a\in A^{(\gamma)}$ for some γ and where $\phi(t)$ satisfies a second-order linear differential equation, made use of an asymptotic expansion of the "remainder" $\int_x^\infty a(t)\phi(t)\,dt$, of the form

$$\int_x^\infty a(t)\phi(t)\,dt \sim \sum_{k=0}^{\infty} \tau_k\theta_k(x), \tag{1.2}$$

where τ_k are constants and $\theta_k(x)$ are functions which depend on $a(x)$, $a'(x)$, $\phi(x)$, $\phi'(x)$, and *explicitly* on the coefficients of the differential equation that $\phi(x)$ satisfies. For infinite series of the form $\tilde{S}=\sum_{r=1}^\infty a(r)\phi(r)$, where $a(x)$ considered as a function of the continuous variable x is in $A^{(\gamma)}$ for some γ and where $\phi(r)$, $r=1,2,\ldots$, satisfy a linear 3-term recursion relation, Levin [6] derived for the "remainder" $\sum_{r=R}^\infty a(r)\phi(r)$ an asymptotic expansion of the form

$$\sum_{r=R}^{\infty} a(r)\phi(r) \sim \sum_{k=0}^{\infty} \tau_k\theta_k(R), \tag{1.3}$$

where τ_k are constants and $\theta_k(R)$ are quantities that depend on the elements of the series, and *explicitly* on the coefficients of the recursion relation that the $\phi(r)$ satisfy. Since the $\theta_k(x)$ in (1.2) and the $\theta_k(R)$ in (1.3) enter Levin's transformations, these transformations cannot be used unless one knows fully the differential equation that $\phi(t)$ satisfies or the recurrence relation that the $\phi(r)$ satisfy.

In Sections 2 and 5 of this work, in the development of the $D^{(m)}$- and the $d^{(m)}$-transformations, we make use of some asymptotic expansions which are interesting by themselves. For an infinite integral $\int_0^\infty f(t)\,dt$, where $f(t)$ satisfies a linear differential equation of order m of the form

$$f(t)=\sum_{k=1}^{m} p_k(t)f^{(k)}(t) \tag{1.4}$$

with $p_k \in A^{(k)}$, $k=1,2,\ldots,m$, under certain mild conditions to be described in Section 2, we obtain the asymptotic expansion

$$\int_x^\infty f(t)\,dt \sim \sum_{k=0}^{m-1} f^{(k)}(x)\sum_{i=0}^{\infty}\beta_{k,i}x^{j_k-i} \quad \text{as} \quad x\to\infty, \tag{1.5}$$

where the j_k are integers with the property $j_k \leq k+1$, $k=0,1,\ldots,m-1$. For an infinite series $\sum_{r=1}^{\infty} f(r)$, where $f(r)$, $r=1,2,\ldots$, satisfy a linear difference equation of order m of the form

$$f(r)=\sum_{k=1}^{m} p_k(r)\Delta^k f(r) \tag{1.6}$$

with $p_k \in A^{(k)}$, $k=1,2,\ldots,m$, when the $p_k(x)$ are considered as functions of the continuous variable x, again under certain mild conditions to be described in Section 5, we obtain the asymptotic expansion

$$\sum_{r=R}^{\infty} f(r) \sim \sum_{k=0}^{m-1} \Delta^k f(R)\sum_{i=0}^{\infty}\beta_{k,i}R^{j_k-i} \quad \text{as} \quad R\to\infty, \tag{1.7}$$

again with the j_k being integers with the property $j_k \leq k+1$, $k=0,1,\ldots,m-1$. Both in (1.5) and in (1.7) the $\beta_{k,i}$ are constants. Since the quantities $f^{(k)}(x)$, $k=0,1,\ldots,m-1$, and $\Delta^k f(r)$, $k=0,1,\ldots,m-1$, enter the $D^{(m)}$- and the $d^{(m)}$-transformations respectively, we see that full knowledge of the differential equation (1.4) or the difference equation (1.6) is not required.

To the best of our knowledge, the existence of such asymptotic expansions has not been known in full generality until now except for a few special cases like

$$-\mathrm{si}(x)=\int_x^\infty \frac{\sin t}{t}\,dt \sim \frac{\sin x}{x}\left(\frac{0!+1!}{x}-\frac{2!+3!}{x^3}+\frac{4!+5!}{x^5}-\cdots\right)$$
$$+\left(\frac{\sin x}{x}\right)'\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\cdots\right),$$

where $(\sin t)/t$ satisfies a linear second-order differential equation (see Example 4.1 in Section 4),

$$\int_x^\infty J_0(t)\,dt \sim J_0(x)\left(\frac{1}{x}-\frac{1^2\times 3}{x^3}+\frac{1^2\times 3^2\times 5}{x^5}-\frac{1^2\times 3^2\times 5^2\times 7}{x^7}+\cdots\right)$$
$$+J_0'(x)\left(1-\frac{1^2}{x^2}+\frac{1^2\times 3^2}{x^4}-\frac{1^2\times 3^2\times 5^2}{x^6}+\cdots\right),$$

where $J_0(t)$ also satisfies a linear second-order differential equation (see Example 4.2 in Section 4), and

$$\sum_{r=R}^{\infty}\frac{1}{r^s} \sim R\frac{1}{R^s}\left[\frac{1}{s-1}+\frac{1}{2}\frac{1}{R}+\frac{B_2}{2}\binom{s}{1}\frac{1}{R^2}+\frac{B_4}{4}\binom{s+2}{3}\frac{1}{R^4}+\cdots\right],$$

where $s>1$ B_i are the Bernoulli numbers, and $\binom{s}{k}$, $k=0,1,\ldots,$ are the binomial coefficients. This last expansion can easily be obtained from

$$\sum_{r=1}^{R}\frac{1}{r^s} \sim \left(\zeta(s)-\frac{1}{s-1}\frac{1}{R^{s-1}}\right)$$
$$+\frac{1}{R^s}\left[\frac{1}{2}-\frac{B_2}{2}\binom{s}{1}\frac{1}{R}-\frac{B_4}{4}\binom{s+2}{3}\frac{1}{R^3}-\cdots\right],$$

where $\zeta(s)=\sum_{r=1}^{\infty}1/r^s$ is the Riemann ζ-function (see [17, p. 538]). Here the terms $f(r)=1/r^s$, $r=1,2,\ldots,$ satisfy the 2-term recursion relation $f(r+1)=(1+1/r)^s f(r)$.

In Section 3 we shall show that the D-transformation, in a sense, generalizes the G-transformation, the confluent ϵ-algorithm, and the P-transformation of Levin [7], and in Section 6 we shall show that the d-transformation generalizes the ϵ-algorithm of Wynn [3] and the t- and u-transformations of Levin [5].

In Sections 4 and 7 we shall illustrate the use of the D- and d-transformations with several examples of infinite integrals and infinite series. It turns out that the D- and d-transformations work efficiently on all the integrals and series on which the known methods work efficiently, and in addition to that they work on infinite integrals and series such as $\int_0^\infty \sin(at^2+bt)\,dt$, $\int_0^\infty J_0(t)J_1(t)\,dt/t$, $\sum_{r=1}^\infty J_0(\lambda_r x)/[\lambda_r J_1(\lambda_r)]^2$ [λ_r is the rth positive zero of $J_0(x)$], and $\sum_{r=0}^\infty \cos(r+\frac{1}{2})\beta P_r(\cos\phi)$, on which the known transformations fail to work.

The computational aspects of these transformations are discussed in detail in Section 8.

2. THE D-TRANSFORMATION FOR INFINITE INTEGRALS.

Let us define $B^{(m)}$ to be the set of functions f which are integrable on $(0,\infty)$ and which satisfy linear mth-order differential equations of the form

$$f(x)=\sum_{k=1}^{m} p_k(x)f^{(k)}(x), \tag{2.1}$$

where $p_k \in A^{(k)}$, $k=1,2,\ldots,m$. We assume that this m is minimal.

We shall now develop the D-transformation which will accelerate the convergence of slowly converging infinite integrals whose integrands are in $B^{(m)}$. In the next two sections we shall deal with some special cases of the D-transformation and apply it to some infinite integrals whose integrands are in $B^{(2)}$ and $B^{(3)}$.

Let us start by integrating (2.1) from $x>0$ to infinity:

$$\int_x^\infty f(t)\,dt=\sum_{k=1}^{m}\int_x^\infty p_k(t)f^{(k)}(t)\,dt. \tag{2.2}$$

Assuming that $\lim_{x\to\infty} p_k(x)f^{(k-1)}(x)=0$, $k=1,2,\ldots,m$, and integrating by parts the right-hand side of (2.2), we obtain

$$\int_x^\infty f(t)\,dt=-\sum_{k=1}^{m} p_k(x)f^{(k-1)}(x)-\int_x^\infty p_1'(t)f(t)\,dt$$
$$-\sum_{k=2}^{m}\int_x^\infty p_k'(t)f^{(k-1)}(t)\,dt. \tag{2.3}$$

Assuming next that $\lim_{x\to\infty} p_k'(x)f^{(k-2)}(x)=0$, $k=2,3,\ldots,m$, and integrating by parts the last term on the right-hand side of (2.3) we obtain

$$\int_x^\infty f(t)\,dt = -\sum_{k=1}^m p_k(x)f^{(k-1)}(x) + \sum_{k=2}^m p_k'(x)f^{(k-2)}(x)$$
$$+\int_x^\infty [-p_1'(t)+p_2''(t)]f(t)\,dt + \sum_{k=3}^m \int_x^\infty p_k''(t)f^{(k-2)}(t)\,dt. \tag{2.4}$$

Assuming, in general, that $\lim_{x\to\infty} p_k^{(i-1)}(x)f^{(k-i)}(x)=0$, $k=i,\ i+1,\ldots,m$, $i=3,4,\ldots,m$, we keep integrating by parts until all derivatives of f disappear in the last term on the right-hand side of (2.4). The final result is

$$\int_x^\infty f(t)\,dt = \sum_{i=1}^m (-1)^i \sum_{k=i}^m p_k^{(i-1)}(x)f^{(k-i)}(x)$$
$$+\int_x^\infty \left[\sum_{k=1}^m (-1)^k p_k^{(k)}(t)\right] f(t)\,dt. \tag{2.5}$$

Rearranging the first term on the right-hand side of (2.5), we obtain

$$\int_x^\infty f(t)\,dt = \sum_{k=0}^{m-1} a_{1,k}(x)f^{(k)}(x) + \int_x^\infty a_1(t)f(t)\,dt, \tag{2.6}$$

where

$$a_{1,k}(x) = \sum_{j=k+1}^m (-1)^{j+k} p_j^{(j-k-1)}(x), \qquad k=0,1,\ldots,m-1,$$
$$a_1(x) = \sum_{k=1}^m (-1)^k p_k^{(k)}(x). \tag{2.7}$$

We now use the fact that if $h\in A^{(\gamma)}$, i.e., $h(x)=h_0x^\gamma + O(x^{\gamma-1})$ as $x\to\infty$, then $h'(x)=\gamma h_0 x^{\gamma-1}+O(x^{\gamma-2})$, i.e., $h'\in A^{(\gamma-1)}$, and if $h\in A^{(0)}$, then $h'\in$

$A^{(-2)}$. Since $p_j \in A^{(j)}$, $j = 1, 2, \ldots, m$, then $a_{1,k} \in A^{(k+1)}$, $k = 0, 1, \ldots, m-1$, and $a_1 \in A^{(0)}$. Now a_1, being the derivative of $a_{1,0}$, does not contain the power x^{-1} in its asymptotic expansion; hence it can be expressed as

$$a_1(x) = \alpha_1 + c_1(x) \tag{2.8}$$

where $\alpha_1 = \sum_{k=1}^{m} (-1)^k k! p_{k,0}$, $p_{k,0} = \lim_{x\to\infty} x^{-k} p_k(x)$, $k = 1, 2, \ldots, m$, and $c_1 \in A^{(-2)}$. Assuming that $\alpha_1 \neq 1$, (2.6) can be written as

$$\int_x^\infty f(t)\, dt = \sum_{k=0}^{m-1} b_{1,k}(x) f^{(k)}(x) + \int_x^\infty b_1(t) f(t)\, dt, \tag{2.9}$$

where

$$\begin{aligned} b_{1,k}(x) &= \frac{a_{1,k}(x)}{1-\alpha_1}, \qquad k = 0, 1, \ldots, m-1, \\ b_1(x) &= \frac{c_1(x)}{1-\alpha_1}. \end{aligned} \tag{2.10}$$

We now see that $b_{1,k} \in A^{(k+1)}$, $k = 0, 1, \ldots, m-1$, but $b_1 \in A^{(-2)}$; i.e., $b_1(x) = O(x^{-2})$ as $x \to \infty$. Therefore, the integral $\int_x^\infty b_1(t) f(t)\, dt$ converges to zero faster than $\int_x^\infty f(t)\, dt$ as $x \to \infty$.

In order to continue the above process we shall prove the following lemma.

LEMMA 1. *Let $b_l \in A^{(-l-1)}$, $l \geq 1$, and $\sum_{k=1}^{m} l(l-1)\cdots(l-k+1) p_{k,0} \neq 1$. Then*

$$\int_x^\infty b_l(t) f(t)\, dt = \sum_{k=0}^{m-1} b_{l+1,k}(x) f^{(k)}(x) + \int_x^\infty b_{l+1}(t) f(t)\, dt, \tag{2.11}$$

where $b_{l+1,k} \in A^{(k-l)}$, $k = 0, 1, \ldots, m-1$, and $b_{l+1} \in A^{(-l-2)}$.

PROOF. Substituting (2.1) in $\int_x^\infty b_l(t)f(t)\,dt$ and using the procedure that led to (2.6), we obtain

$$\int_x^\infty b_l(t)f(t)\,dt = \sum_{k=0}^{m-1} a_{l+1,k}(x)f^{(k)}(x) + \int_x^\infty a_{l+1}(t)f(t)\,dt, \quad (2.12)$$

where

$$\begin{aligned} a_{l+1,k}(x) &= \sum_{j=k+1}^{m} (-1)^{j+k}\big[b_l(x)p_j(x)\big]^{(j-k-1)}, \qquad k=0,1,\ldots,m-1, \\ a_{l+1}(x) &= \sum_{k=1}^{m} (-1)^k\big[b_l(x)p_k(x)\big]^{(k)}. \end{aligned} \quad (2.13)$$

The conditions $\lim_{x\to\infty} p_k^{(i-1)}(x)f^{(k-i)}(x)=0$, $k=i, i+1,\ldots,m$, $i=1,2,\ldots,m$, that were imposed previously are sufficient for (2.12) to be true for all $l\geqslant 1$, since they also imply $\lim_{x\to\infty}[b_l(x)p_k(x)]^{(i-1)}f^{(k-i)}(x)=0$, $k=i, i+1,\ldots,m$, $i=1,2,\ldots,m$, which are sufficient for (2.12) to hold.

Now $a_{l+1,k}\in A^{(k-l)}$, $k=0,1,\ldots,m-1$, and $a_{l+1}\in A^{(-l-1)}$. Since $a_{l+1}\in A^{(-l-1)}$ and $b_l\in A^{(-l-1)}$, we can write

$$a_{l+1}(x)=\alpha_{l+1}b_l(x)+c_{l+1}(x) \quad (2.14)$$

with $\alpha_{l+1}=\sum_{k=1}^{m} l(l-1)\cdots(l-k+1)p_{k,0}$ and $c_{l+1}\in A^{(-l-2)}$. Since by assumption $\alpha_{l+1}\neq 1$, we obtain (2.11) with

$$\begin{aligned} b_{l+1,k}(x) &= \frac{a_{l+1,k}(x)}{1-\alpha_{l+1}}, \qquad k=0,1,\ldots,m-1, \\ b_{l+1}(x) &= \frac{c_{l+1}(x)}{1-\alpha_{l+1}}, \end{aligned} \quad (2.15)$$

so that $b_{l+1,k}\in A^{(k-l)}$ and $b_{l+1}\in A^{(-l-2)}$, thus proving the lemma.

Starting now with Equation (2.9), with $a_{1,k}(x)$, $k=0,1,\ldots,m-1$, $a_1(x)$, α_1, $c_1(x)$, $b_{1,k}(x)$, $k=0,1,\ldots,m-1$, and $b_1(x)$ already defined in (2.7), (2.8), and (2.10), and assuming that $\sum_{k=1}^{m} l(l-1)\cdots(l-k+1)p_{k,0}\neq 1$ for $l\geqslant 1$, we use Lemma 1 to define recursively $a_{l+1,k}(x)$, $k=0,1,\ldots,m-1$, $a_{l+1}(x)$, α_{l+1}, $c_{l+1}(x)$, $b_{l+1,k}(x)$, $k=0,1,\ldots,m-1$, and $b_{l+1}(x)$, $l=1,2,\ldots,n-1$, by Equations (2.13), (2.14), and (2.15). We then sum Equation (2.9) and

$$\int_x^\infty b_l(t)f(t)\,dt=\sum_{k=0}^{m-1} b_{l+1,k}(x)f^{(k)}(x)+\int_x^\infty b_{l+1}(t)f(t)\,dt,$$

$$l=1,2,\ldots,n-1, \qquad (2.16)$$

which are obtained by repeated application of Lemma 1. The result is

$$\int_x^\infty f(t)\,dt=\sum_{k=0}^{m-1} \beta_k^n(x)f^{(k)}(x)+\int_x^\infty b_n(t)f(t)\,dt, \qquad (2.17)$$

where

$$\beta_k^n(x)=\sum_{l=1}^{n} b_{l,k}(x), \qquad k=0,1,\ldots,m-1. \qquad (2.18)$$

In (2.17) $\beta_k^n\in A^{(k+1)}$, $k=0,1,\ldots,m-1$, and $b_n\in A^{(-n-1)}$. Let the asymptotic series of $p_1(x)$ begin with the power x^{i_1}. Since $p_1\in A^{(1)}$, $i_1\leqslant 1$. The condition $\lim_{x\to\infty}p_1(x)f(x)=0$ which was previously imposed implies that $f(x)=o(x^{-i_1})$ as $x\to\infty$. Since $b_n(x)=O(x^{-n-1})$ and $f(x)=o(x^{-i_1})$ as $x\to\infty$, the integral $\int_x^\infty b_n(t)f(t)\,dt$ is $o(x^{-n-i_1})$ as $x\to\infty$. If we now choose n and x large enough, we can neglect $\int_x^\infty b_n(t)f(t)\,dt$ in (2.17) and still get a good approximation to $S=\int_0^\infty f(t)\,dt$; i.e., we can approximate S as

$$S\doteqdot\int_0^x f(t)\,dt+\sum_{k=0}^{m-1}\beta_k^n(x)f^{(k)}(x). \qquad (2.19)$$

Of course, in (2.19) one has to compute the functions $\beta_k^n(x)$, $k=0,1,\ldots,m-1$. But the computation of these functions becomes difficult, as is seen from Equations (2.18), (2.15), (2.14), and (2.13). However, as we shall show below, we can still make use of (2.19) to derive a transformation that will produce a good approximation to S.

Since

$$\beta_k^n(x)=\beta_k^{n-1}(x)+b_{n,k}(x), \qquad k=0,1,\dots,m-1, \tag{2.20}$$

and $b_{n,k}\in A^{(k-n+1)}$, we can write

$$\beta_k^n(x)=x^{k+1}\left[\beta'_{k,0}+\frac{\beta'_{k,1}}{x}+\frac{\beta'_{k,2}}{x^2}+\cdots+\frac{\beta'_{k,n}}{x^n}+O(x^{-n-1})\right]$$
$$\text{as } x\to\infty, \qquad k=0,1,\dots,m-1, \tag{2.21}$$

where the coefficients $\beta'_{k,j}$, $j=0,1,\dots,n$, are the same for all $\beta_k^l(x)$, $k=0,1,\dots,m-1$, with $l\geq n$. Therefore, we deduce from (2.17), (2.21), and the fact that $\int_x^\infty b_{n+1}(t)f(t)\,dt=o(x^{-n-1-i_1})$ as $x\to\infty$ that $\int_x^\infty f(t)\,dt$ has a true asymptotic expansion which is given as

$$\int_x^\infty f(t)\,dt\sim\sum_{k=0}^{m-1}f^{(k)}(x)x^{k+1}\left(\beta'_{k,0}+\frac{\beta'_{k,1}}{x}+\frac{\beta'_{k,2}}{x^2}+\cdots\right). \tag{2.22}$$

In many cases it occurs that the asymptotic series of $p_k(x)$, $k=1,2,\dots,m$, do not all start with the power x^k; i.e., some may start with the lower power of x. We then assume that, in general, $p_k\in A^{(i_k)}$, $i_k\leq k$, $k=1,2,\dots,m$. If one now follows the steps that led to (2.17), one can see that some of the β_k^n, $k=0,1,\dots,m-1$, may have asymptotic series which do not start with x^{k+1}, but with a lower integer power of x, say x^{j_k}. For example, if $p_k\in A^{(0)}$, $k=1,2,\dots,m$ (which for instance, with $m=2$, is the case for Bessel's equation), then from (2.7)–(2.17) one can see that $a_{l,k}\in A^{(-l+1)}$, $k=0,1,\dots,m-1$, $a_l\in A^{(-l-1)}$, and hence $\alpha_l=0$, $b_{l,k}\equiv a_{l,k}$, $k=0,1,\dots,m-1$, and $b_l\equiv a_l$, $l=1,2\dots$. Therefore, $\beta_k^n\in A^{(0)}$. In general, $\beta_k^n\in A^{(j_k)}$, with j_k integers and $j_k\leq k+1$, $k=0,1,\dots,m-1$. Actually, $\beta_k^n(x)=O(b_{1,k}(x))$ as $x\to\infty$; hence from (2.7) $j_k\leq\max(i_{k+1},i_{k+2}-1,\dots,i_m-m+k+1)$, $k=0,1,\dots,m-1$. Therefore (2.22) can be replaced by

$$\int_x^\infty f(t)\,dt\sim\sum_{k=0}^{m-1}f^{(k)}(x)x^{j_k}\left(\beta_{k,0}+\frac{\beta_{k,1}}{x}+\frac{\beta_{k,2}}{x^2}+\cdots\right). \tag{2.23}$$

We can summarize all that has been said so far in the following theorem.

THEOREM 1. *Let f be integrable on $[0,\infty)$ and satisfy the linear mth-order differential equation $f(x)=\sum_{k=1}^{m} p_k(x)f^{(k)}(x)$ with $p_k \in A^{(i_k)}$, $i_k \leq k$, $k=1,2,\ldots,m$. If $\lim_{x\to\infty} p_k^{(i-1)}(x)f^{(k-i)}(x)=0$, $k=i,i+1,\ldots,m$, $i=1,2,\ldots,m$, and if for any integer $l \geq -1$ we have $\sum_{k=1}^{m} l(l-1)\cdots(l-k+1)p_{k,0}\neq 1$, where $p_{k,0}=\lim_{x\to\infty} x^{-k}p_k(x)$, then, as $x\to\infty$, $\int_x^\infty f(t)\,dt$ has an asymptotic expansion of the form (2.23) with $j_k \leq \max(i_{k+1}, i_{k+2}-1,\ldots,i_m-m+k+1)$, $k=0,1,\ldots,m-1$.*

We note that the conditions stated in Theorem 1 are *sufficient*. In all the examples done by the authors all of these conditions were seen to hold simultaneously. Therefore, the authors feel that the result (2.23) of Theorem 1 might hold even with a smaller number of conditions.

We note in passing that if f satisfies (2.23) such that $\beta_{0,0}\neq -1$, then f is in $B^{(m)}$. This can be proved by differentiating both sides of (2.22).

Having established (2.23), we now define our D-transformation. Following Schucany, Gray, and Owen [2] and Levin [5–7], we demand that the approximation $D^{(m)}_{n_0,n_1,\ldots,n_{m-1}}$ to $S=\int_0^\infty f(t)\,dt$ satisfy the $N=1+\sum_{k=0}^{m-1} n_k$ equations

$$D^{(m)}_{n_0,n_1,\ldots,n_{m-1}}=\int_0^{x_l} f(t)\,dt+\sum_{k=0}^{m-1} f^{(k)}(x_l)x_l^{j_k}\sum_{i=0}^{n_k-1}\frac{\bar{\beta}_{k,i}}{x_l^i},$$

$$l=1,2,\ldots,N, \quad (2.24)$$

with $\bar{\beta}_{k,i}$ constants and x_l chosen to satisfy $0<x_1<x_2<\cdots<x_N$. The equations (2.24) form a linear set in N unknowns, namely, $D^{(m)}_{n_0,n_1,\ldots,n_{m-1}}$ and $\bar{\beta}_{k,i}$, $i=0,1,\ldots,n_k-1$, $k=0,1,\ldots,m-1$, and can, in general, be solved for the N unknowns. $D^{(m)}_{n_0,n_1,\ldots,n_{m-1}}$ is expected to be a good approximation to S; however, the coefficients $\bar{\beta}_{k,i}$ do not have to be identical to the $\beta_{k,i}$ in (2.23), since the asymptotic series in (2.23) are usually infinite. As it turns out, the choice of the x_l is important, and this point has been investigated by the second author; see Note Added in Proof. For the sake of simplicity, however, we choose

$$x_l=\xi+(l-1)\tau, \qquad l=1,2,\ldots,N, \quad \xi>0, \quad \tau>0.$$

Following Gray, Atchison, and McWilliams [1], we then denote $D^{(m)}_{n_0,n_1,\ldots,n_{m-1}}$ by $D^{(m)}_{n_0,n_1,\ldots,n_{m-1}}[F(\xi);\tau]$ where $F(\xi)=\int_0^\xi f(t)\,dt$. Usually it is more convenient to use the "diagonal" transformation $D^{(m)}_{n,n,\ldots,n}[F(\xi);\tau]\equiv D^{(m)}_n[F(\xi);\tau]$. For the case $m=2$ and $j_0=j_1=0$, which occurs frequently, $D^{(m)}_n[F(\xi);\tau]$ is given below:

$$D_n^{(2)}[F(\xi);\tau]=\frac{\begin{vmatrix} F(\xi) & F(\xi+\tau) & \cdots & F(\xi+2n\tau) \\ f(\xi) & f(\xi+\tau) & \cdots & f(\xi+2n\tau) \\ \dfrac{f(\xi)}{\xi} & \dfrac{f(\xi+\tau)}{\xi+\tau} & \cdots & \dfrac{f(\xi+2n\tau)}{\xi+2n\tau} \\ \vdots & \vdots & & \vdots \\ \dfrac{f(\xi)}{\xi^{n-1}} & \dfrac{f(\xi+\tau)}{(\xi+\tau)^{n-1}} & \cdots & \dfrac{f(\xi+2n\tau)}{(\xi+2n\tau)^{n-1}} \\ f'(\xi) & f'(\xi+\tau) & \cdots & f'(\xi+2n\tau) \\ \dfrac{f'(\xi)}{\xi} & \dfrac{f'(\xi+\tau)}{\xi+\tau} & \cdots & \dfrac{f'(\xi+2n\tau)}{\xi+2n\tau} \\ \vdots & \vdots & & \vdots \\ \dfrac{f'(\xi)}{\xi^{n-1}} & \dfrac{f'(\xi+\tau)}{(\xi+\tau)^{n-1}} & \cdots & \dfrac{f'(\xi+2n\tau)}{(\xi+2n\tau)^{n-1}} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ f(\xi) & f(\xi+\tau) & \cdots & f(\xi+2n\tau) \\ \dfrac{f(\xi)}{\xi} & \dfrac{f(\xi+\tau)}{\xi+\tau} & \cdots & \dfrac{f(\xi+2n\tau)}{\xi+2n\tau} \\ \vdots & \vdots & & \vdots \\ \dfrac{f(\xi)}{\xi^{n-1}} & \dfrac{f(\xi+\tau)}{(\xi+\tau)^{n-1}} & \cdots & \dfrac{f(\xi+2n\tau)}{(\xi+2n\tau)^{n-1}} \\ f'(\xi) & f'(\xi+\tau) & \cdots & f'(\xi+2n\tau) \\ \dfrac{f'(\xi)}{\xi} & \dfrac{f'(\xi+\tau)}{\xi+\tau} & \cdots & \dfrac{f'(\xi+2n\tau)}{\xi+2n\tau} \\ \vdots & \vdots & & \vdots \\ \dfrac{f'(\xi)}{\xi^{n-1}} & \dfrac{f'(\xi+\tau)}{(\xi+\tau)^{n-1}} & \cdots & \dfrac{f'(\xi+2n\tau)}{(\xi+2n\tau)^{n-1}} \end{vmatrix}} \tag{2.25}$$

In general, we can write $D_n^{(m)}[F(\xi);\tau]$ as the quotient of two determinants using Cramer's rule as in (2.25). It also turns out that $D_n^{(m)}[F(\xi);\tau]$ is an "average" of $F(\xi), F(\xi+\tau),\dots,F(\xi+mn\tau)$; i.e.,

$$D_n^{(m)}[F(\xi);\tau]=\frac{\sum_{j=0}^{mn}\gamma_j F(\xi+j\tau)}{\sum_{j=0}^{mn}\gamma_j}, \tag{2.26}$$

where γ_j are the cofactors of $F(\xi+j\tau)$ and are dependent on $f(x)$, $f'(x),\ldots,f^{(m-1)}(x)$. Therefore, the D-transformation can be viewed as a nonlinear summability method.

3. SOME SPECIAL CASES OF THE D-TRANSFORMATION AND AN EXTENSION OF THE CONFLUENT ϵ-ALGORITHM

For $m=1$ and $j_0=0$ the system of equations (2.24) reduces to that given by Levin [7] in his development of the P-transformation. Hence the $D_n^{(1)}$-transformation is identical to the P-transformation of Levin. The P-transformation has been developed for functions of the form $f(x)=e^{i\omega x}h(x)$ with ω constant and $h\in A^{(\gamma)}$, $\gamma<0$. It is not difficult to show that these functions satisfy first-order linear differential equations of the form $f(x)=p(x)f'(x)$ with $p(x)=[h'(x)/h(x)+i\omega]^{-1}$, so that $p\in A^{(0)}$, which implies that $j_0=0$ in (2.23). It is worth mentioning that the P-transformation has worked very efficiently on the Bromwich integral, which is used in inverting Laplace transforms. This fact may indicate that the D-transformation would also produce very good results.

If the asymptotic expansion (2.23) turns out to be finite, i.e.,

$$\int_x^\infty f(t)\,dt=\sum_{k=0}^{m-1} f^{(k)}(x)x^{j_k}\sum_{i=0}^{n_k-1}\frac{\beta_{k,i}}{x^i}, \tag{3.1}$$

then $S=\int_0^\infty f(t)\,dt\equiv D^{(m)}_{n'_0,n'_1,\ldots,n'_{m-1}}[F(\xi);\tau]$ for all $n'_k\geqslant n_k$, $k=0,1,\ldots,m-1$, $\xi>0$, and $\tau>0$. As an example of (3.1) we can consider the Bessel functions of the 1st kind of odd order, namely, $J_{2k+1}(x)$, $k=0,1,2,\ldots$. Using the relation [16]

$$\int_x^\infty J_{2k+1}(t)\,dt=J_0(x)+2\sum_{l=1}^{k}J_{2l}(x),$$

together with the different recursion relations between the J_l, we finally arrive at (3.1). For $k=1$ (3.1) becomes

$$\int_x^\infty J_3(t)\,dt=J_3(x)\left(\frac{1}{x}+\frac{24}{x^3}\right)+J_3'(x)\left(1+\frac{8}{x^2}\right).$$

As another example we can consider the functions which satisfy linear differential equations of the form

$$f(x) = \sum_{k=1}^{m} \left(\sum_{j=0}^{k} c_{k,j} x^j \right) f^{(k)}(x). \tag{3.2}$$

Then we can see from (2.7) that $a_1(x)$ is a constant; hence $b_1 \equiv 0$, and $b_{1,k}(x)$ are polynomials of degree $k+1$, $k=0,1,\ldots,m-1$, i.e.,

$$\int_x^\infty f(t)\,dt = \sum_{k=0}^{m-1} \left(\sum_{j=0}^{k+1} \beta_{k,j} x^{k+1-j} \right) f^{(k)}(x). \tag{3.3}$$

If $p_k(x)$, $k=1,2,\ldots,m$, in (2.1) are constants, then $b_{1,k}(x)$, $k=0,1,\ldots,m-1$, in (2.3) are constants too, and hence $D_1^{(m)}$ with $j_k=0$, $k=0,1,\ldots,m-1$, like the G_m-transformation and the ϵ_{2m} in the confluent ϵ-algorithm, integrates exactly functions which satisfy linear mth-order differential equations with constant coefficients.

We can make use of (3.3) to obtain another transformation which will integrate exactly functions which satisfy equations of the type (3.2). We start by writing (3.3) as

$$S - F(x) = \sum_{k=0}^{m-1} \left(\sum_{j=0}^{k+1} \beta_{k,j} x^{k+1-j} \right) f^{(k)}(x) \tag{3.4}$$

and differentiate this equation $M = m(m+3)/2$ times to obtain a total of $M+1$ equations for the $M+1$ constants $S = \int_0^\infty f(t)\,dt$ and $\beta_{k,j}$, $j=0,1,\ldots,k+1$, $k=0,1,\ldots,m-1$. We now define the C-transformation for any integrable function f. We demand that the approximation $C^{(m)}$ to S, satisfy the $M+1$ equations

$$\frac{d^j}{dx^j}\left[C^{(m)} - F(x) - \sum_{k=0}^{m-1} \left(\sum_{j=0}^{k+1} \bar{\beta}_{k,j} x^{k+1-j} \right) f^{(k)}(x) \right] = 0,$$

$$j=0,1,\ldots,M, \tag{3.5}$$

where $\bar{\beta}_{k,j}$ are constants. Equations (3.5) can, in general, be solved for the $M+1$ unknowns $C^{(m)} \equiv C^{(m)}[F(x)]$ and $\bar{\beta}_{k,j}$, $j=0,1,\ldots,k+1$, $k=0,1,\ldots,m-1$. Obviously $C^{(m)}[F(x)] \equiv S$ if f satisfies (3.2). It is easy to see that the

G-transformation is an extension of the confluent ϵ-algorithm of Wynn, since this algorithm can be obtained by solving the set of equations

$$\frac{d^j}{dx^j}\left[\epsilon_{2m}-F(x)-\sum_{k=0}^{m-1}\bar{\beta}_k f^{(k)}(x)\right]=0, \qquad j=0,1,\ldots,m+1, \quad (3.6)$$

for ϵ_{2m}.

4. APPLICATIONS OF THE D-TRANSFORMATION

In this section we shall illustrate the use of the D-transformation that was developed in Section 2 for several functions which are in $B^{(2)}$ and $B^{(3)}$. The integrals of all those functions considered in this section converge very slowly. The high-order G-transformations work efficiently on some of these functions but fail to work on most others.

It is worthwhile to make a few comments on the practical application of the D-transformation. In order to be able to apply the D-transformation efficiently one has to know (1) the smallest possible positive integer m such that the integrand f is in $B^{(m)}$, and (2) the parameters j_k, $k=0,1,\ldots,m-1$, in the asymptotic expansion of $\int_x^\infty f(t)\,dt$. If the upper bounds for the j_k are known, then we can substitute these upper bounds for the j_k in the equations (2.24). If the differential equation that f satisfies is not readily obtained, then we proceed as follows: Since integrability of f at infinity means that $\lim_{x\to\infty}\int_x^\infty f(t)\,dt=0$, we must have

$$\lim_{x\to\infty} x^{j_k} f^{(k)}(x)=0, \qquad k=0,1,\ldots,m-1, \quad (4.1)$$

in the asymptotic expansion in (2.23). Therefore, we replace j_k in (2.24) by the minimum σ_k of $k+1$ and s_k, where s_k is the largest of the integers s for which $\lim_{x\to\infty} x^s f^{(k)}(x)=0$. If, with j_k replaced by σ_k, the first few coefficients, say $\bar{\beta}_{k,0}, \bar{\beta}_{k,1},\ldots,\bar{\beta}_{k,r_k}$, in (2.24), turn out to be very small compared with the rest of the coefficients, then we can assume that $j_k=\sigma_k-r_k$ in (2.23) and replace j_k by σ_k-r_k in (2.24); and if indeed $j_k=\sigma_k-r_k$, then we are likely to obtain better accuracy for the approximations to $S=\int_0^\infty f(t)\,dt$ using the same number of the $F(x_j)$.

We shall now state a lemma that will be useful in determining the order of the differential equations that the function f whose integral is to be evaluated satisfies.

Lemma 2. *If the functions f and g satisfy linear differential equations of order m and n respectively, then their product fg and their sum $f+g$, in general, satisfy linear differential equations of orders less than or equal to mn and $m+n$ respectively.*

Proof. Let f and g satisfy

$$f=\sum_{k=1}^{m} p_k f^{(k)}, \tag{4.2a}$$

$$g=\sum_{l=1}^{n} q_l g^{(l)}. \tag{4.2b}$$

Multiplying (4.2a) and (4.2b), we get

$$fg=\sum_{k=1}^{m}\sum_{l=1}^{n} p_k q_l f^{(k)} g^{(l)}. \tag{4.3}$$

In (4.3) we have to be able to express the mn products $f^{(k)}g^{(l)}$, $k=1,2,\ldots,m$, $l=1,2,\ldots,n$, as linear combinations of $(fg)^{(r)}$, $r=1,2,\ldots,mn$. Using Leibnitz's rule for differentiating the product of two functions, we have

$$(fg)^{(r)}=\sum_{s=0}^{r}\binom{r}{s} f^{(s)} g^{(r-s)}, \qquad r=1,2,\ldots,mn. \tag{4.4}$$

In (4.4) if $s=0$, $f^{(s)}$ is replaced by $\sum_{k=1}^{m} p_k f^{(k)}$ using (4.2a), and if $s>m$, then $f^{(s)}$ is expressed as a combination of $f', f'',\ldots,f^{(m)}$, by differentiating (4.2a) $s-m$ times. The same can be done for $g^{(r-s)}$ when $r-s=0$ and $r-s>n$. Then we will have expressed $(fg)^{(r)}$, $r=1,2,\ldots,mn$, as combinations of the mn products $f^{(k)}g^{(l)}$, $k=1,2,\ldots,m$, $l=1,2,\ldots,n$; i.e.,

$$(fg)^{(r)}=\sum_{k=1}^{m}\sum_{l=1}^{n} A_{kl} f^{(k)} g^{(l)}, \qquad r=1,2,\ldots,mn. \tag{4.5}$$

Now (4.5) is a set of mn linear equations in the mn unknowns $f^{(k)}g^{(l)}$, $k=1,2,\ldots,m$, $l=1,2,\ldots,n$, and can, in general, be solved to give $f^{(k)}g^{(l)}$ as linear combinations of $(fg)^{(r)}$, $r=1,2,\ldots,mn$. Hence (4.3) becomes

$$fg=\sum_{r=1}^{mn} A_r(fg)^{(r)}. \tag{4.6}$$

For the case of $f+g$ we add (4.2a) and (4.2b) to obtain

$$f+g=\sum_{k=1}^{m} p_k f^{(k)}+\sum_{l=1}^{n} q_l g^{(l)}. \tag{4.7}$$

In (4.7) we have to express $f^{(k)}$, $k=1,2,\ldots,m$, and $g^{(l)}$, $l=1,2,\ldots,n$, as combinations of $(f+g)^{(r)}$, $r=1,2,\ldots,m+n$. Using $(f+g)^{(r)}=f^{(r)}+g^{(r)}$ and the fact that for $r>m$ $f^{(r)}$ can be written as a combination of $f',f'',\ldots,f^{(m)}$, and for $r>n$ $g^{(r)}$ can be written as a combination of $g',g'',\ldots,g^{(n)}$, the result follows as in the previous case; i.e.,

$$f+g=\sum_{r=1}^{m+n} B_r(f+g)^{(r)}. \tag{4.8}$$

COROLLARY 1. *If f satisfies a linear differential equation of order m, then, in general, f^2 satisfies a linear differential equation of order $m(m+1)/2$ or less.*

PROOF. The proof follows from the fact that the number of unknowns $f^{(k)}g^{(l)}$ in (4.3) is $m(m+1)/2$ when $f=g$.

COROLLARY 2. *If the coefficients p_k, $k=1,2,\ldots,m$, and q_l, $l=1,2,\ldots,n$, have asymptotic expansions in inverse powers of x as $x\to\infty$, then so do A_r, $r=1,2,\ldots,mn$, in* (4.6) *and B_r, $r=1,2,\ldots,m+n$, in* (4.8).

COROLLARY 3. *If $f\in B^{(m)}$ and $g\in A^{(\gamma)}$ or $g(x)=e^{\alpha x}$, then fg satisfies a linear differential equation of order m or less with coefficients that have asymptotic expansions in inverse powers of x as $x\to\infty$.*

The proofs of these corollaries are easy and we omit them.

From the experience gained in the use of the P- and high-order G-transformations, we expect that as n tends to infinity, $D_n^{(m)}[F(\xi;\tau)]$ should tend to

$\int_0^\infty f(t)\,dt$ quickly if $f \in B^{(m)}$ and satisfies all the conditions of Theorem 1. The numerical results in the following examples indeed confirm this. Convergence properties of the D-transformation have been taken up by the second author in a separate paper, see Note Added in Proof.

EXAMPLE 4.1.

$$I=\int_0^\infty \frac{\sin t}{t}\,dt=\frac{\pi}{2}=1.5707963267 95\ldots.$$

The integrand $f(t)=(\sin t)/t$ is integrable at infinity and satisfies the differential equation $f=-(2/t)f'-f''$; hence it is in $B^{(2)}$. Therefore, we expect $D_n^{(2)}$ to work efficiently. Also since $\sin t$ satisfies a second-order linear differential equation with constant coefficients, we expect G_n to work efficiently too. This point has been mentioned in the introduction. From the differential equation that f satisfies we see that $j_0 \leqslant -1$ and $j_1 \leqslant 0$. However, we used the prescription proposed for the case in which the differential equation is not known explicitly, and in (2.24) we replaced j_0 by $\sigma_0=0$ and j_1 by $\sigma_1=0$, where $\sigma_k=\min(k+1, s_k)$, $k=0,1$. Accordingly $D_n^{(2)}[f(\xi);\tau]$ was computed by solving the $2n+1$ linear equations

$$D_n^{(2)}[F(\xi);\tau]=F(\xi+j\tau)+\frac{\sin(\xi+j\tau)}{\xi+j\tau}\sum_{i=0}^{n-1}\frac{\bar{\beta}_{0,i}}{(\xi+j\tau)^i}$$
$$+\left[\frac{\sin(\xi+j\tau)}{\xi+j\tau}\right]'\sum_{i=0}^{n-1}\frac{\bar{\beta}_{1,i}}{(\xi+j\tau)^i},\qquad j=0,1,\ldots,2n, \quad (4.9)$$

with $\xi=1$ and $\tau=1$, and $F(x)=\int_0^x[(\sin t)/t]\,dt$. The finite integrals $F(\xi+j\tau)$ were computed correctly to 14 decimal points using a Gauss-Legendre quadrature formula. We also computed $G_{2n}[F(\xi);\tau)]$. The results of the computations with $D_n^{(2)}$ and G_{2n} are given in Table 1. $D_n^{(2)}[F(\xi);\tau]$ has been compared with $G_{2n}[F(\xi);\tau]$, since they both use the same finite integrals, namely, $F(\xi+j\tau)$, $j=0,1,\ldots,2n$.

For the integral $\int_x^\infty[(\sin t)/t]\,dt$ we have the asymptotic expansion

$$\int_x^\infty \frac{\sin t}{t}\,dt \sim \frac{\sin x}{x}\left(\frac{1!}{x}-\frac{3!}{x^3}+\frac{5!}{x^5}-\cdots\right)+\frac{\cos x}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\cdots\right), \quad (4.10)$$

TABLE 1
VALUES OF THE APPROXIMATIONS $D_n^{(2)}[F(1);1]$ AND $G_{2n}[F(1);1]$ TO THE INTEGRAL $I=\int_0^\infty [(\sin t)/t]\,dt=\pi/2$[a]

n	$D_n^{(2)}[F(1);1]$	$G_{2n}[F(1);1]$
2	1.63	1.56
4	1.5716	1.572
6	1.5707943	1.57080
8	1.57079606	1.5707955
10	1.5707963323	1.5707963326

[a] $n=2(2)10$

which can easily be obtained by integrating $(\sin t)/t$ by parts a sufficient number of times. We note that since $[(\sin x)/x]'=(\cos x)/x-(\sin x)/x^2$, we can rearrange (4.10) and obtain

$$\int_x^\infty \frac{\sin t}{t}dt \sim \frac{\sin x}{x}\left(\frac{0!+1!}{x}-\frac{2!+3!}{x^3}+\frac{4!+5!}{x^5}-\cdots\right)$$
$$+\left(\frac{\sin x}{x}\right)'\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\cdots\right), \tag{4.11}$$

which is in the form (2.23). We see that $j_0=-1$ and $j_1=0$; i.e., j_0 and j_1 are actually equal to their upper bounds given above.

We now expect at least the first few $\bar{\beta}_{k,i}$, $k=0,1$, in the solution of the equations (4.9) to be similar to the corresponding coefficients in the asymptotic expansion given in (4.11). This indeed turns out to be the case. For example, for $n=10$ we obtained $\bar{\beta}_{0,0}=-0.000068$, $\bar{\beta}_{0,1}=2.0076$, $\bar{\beta}_{0,2}=-0.32$, $\bar{\beta}_{0,3}=-1.41$ corresponding to 0, 2, 0, -8 respectively, and $\bar{\beta}_{1,0}=0.999982$, $\bar{\beta}_{1,1}=-0.00071$, $\bar{\beta}_{1,2}=-1.84$ corresponding to 1, 0, -2 respectively. The rest of the coefficients become large as quickly as those in the asymptotic expansion (4.11).

We also note that if one truncates both of the asymptotic series in (4.11) at the power x^{-9} and uses these truncated series in the computation of $\int_x^\infty [(\sin t)/t]\,dt$ with $x=20$, the error in this computation is of the order of 10^{-8}, whereas the $D_{10}^{(2)}[F(1);1]$ approximation which is obtained by using $\int_0^{21}[(\sin t)/t]\,dt$ has an error of the order of 10^{-9}.

EXAMPLE 4.2.

$$I_a=\int_0^\infty J_0(t)\,dt=1, \qquad I_b=\int_0^\infty \frac{tJ_0(t)}{1+t^2}dt=K_0(1)=0.42102\,44382\,401\ldots.$$

Since $f(t)=J_0(t)$ is integrable at infinity and satisfies the differential equation $f=-(1/t)f'-f''$, we have $f\in B^{(2)}$. The function $g(t)=tJ_0(t)/(1+t^2)$ also is in $B^{(2)}$, and this is suggested by Corollary 3 of Lemma 2, since $h(t)=t/(1+t^2)$ is in $A^{(-1)}$ and $f\in B^{(2)}$.

Therefore, we expect the $D_n^{(2)}$-transformation to work efficiently on both I_a and I_b, and this is confirmed by the results in Table 2. The results in this table were obtained by replacing j_0 and j_1 with $\sigma_0=0$ and $\sigma_1=0$ as in Example 1.

For the integral $\int_x^\infty J_0(t)\,dt$ there is also a known asymptotic expansion [9] which is given by

$$\int_x^\infty J_0(t)\,dt \sim J_0(x)\left(\frac{1}{x}-\frac{1^2\times 3}{x^3}+\frac{1^2\times 3^2\times 5}{x^5}-\frac{1^2\times 3^2\times 5^2\times 7}{x^7}+\cdots\right)$$
$$-J_1(x)\left(1-\frac{1^2}{x^2}+\frac{1^2\times 3^2}{x^4}-\frac{1^2\times 3^2\times 5^2}{x^6}+\cdots\right). \qquad (4.12)$$

Since $[J_0(x)]'=-J_1(x)$, (4.12) is exactly of the form (2.23).

In this case too it turns out that the first few $\bar{\beta}_{k,i}$, $k=0,1$, in (2.24) are similar to the corresponding coefficients of the asymptotic expansion in (4.12). For example, for $n=10$ we obtained $\bar{\beta}_{0,0}=-0.00001$, $\bar{\beta}_{0,1}=1.001$, $\bar{\beta}_{0,2}=-0.049$, $\bar{\beta}_{0,3}=-1.88$, corresponding to 0, 1, 0, -3 respectively, and $\bar{\beta}_{1,0}=0.99998$, $\bar{\beta}_{1,1}=0.0001$, $\bar{\beta}_{1,2}=-0.981$, corresponding to 1, 0, -1 respectively. The rest of the coefficients become large as quickly as those in the asymptotic expansion in (4.12).

If we truncate the asymptotic series in (4.12) at the power x^{-9} and use the truncated series in the computation of $\int_x^\infty J_0(t)\,dt$ with $x=20$, the error in this computation is of the order of 10^{-6}. However, the $D_{10}^{(2)}[F_a(1);1]$-approximation has an error of order 10^{-9}.

The G_n-transformation was tried on I_a and I_b, and the results that were obtained for $G_{2n}[F(\xi);\tau]$ were as good as those for $D_n^{(2)}[F(\xi);\tau]$ in Table 2. The reason that G_n works on I_a and I_b so well is that as t becomes large $J_0(t)=\cos t\,h_1(t)+\sin t\,h_2(t)$ with $h_1,h_2\in A^{(-1/2)}$, and that $\cos t$ and $\sin t$ satisfy a linear differential equation with constant coefficients.

TABLE 2

VALUES OF THE APPROXIMATIONS $D_n^{(2)}[F_a(1);1]$ AND $D_n^{(2)}[F_b(1);1]$ TO $I_a=\int_0^\infty J_0(t)\,dt$ AND $I_b=\int_0^\infty tJ_0(t)/(1+t^2)\,dt$ RESPECTIVELY[a]

n	$D_n^{(2)}[F_a(1);1]$	$D_n^{(2)}[F_b(1);1]$
2	1.04	0.43
4	1.003	0.4212
6	0.999994	0.421027
8	0.99999998	0.4210244433
10	0.9999999986	0.4210244434
12	0.999999999984	0.42102443824 07

[a]For $n=2(2)12$, where $F_a(x)=\int_0^x J_0(t)\,dt$ and $F_b(x)=\int_0^x tJ_0(t)/(1+t^2)\,dt$.

The integrals I_a and I_b were computed also by Levin [6] using a transformation designed to work exclusively on integrals of the form

$$\int_0^\infty h(t)J_\nu(t)\,dt, \qquad h\in A^{(\gamma)},$$

and very good results were obtained.

EXAMPLE 4.3.

$$I(a,b)=\int_0^\infty \sin(at^2+bt)\,dt$$
$$=\sqrt{\frac{\pi}{2a}}\left\{\sin\left(\frac{b^2}{4a}\right)\left[C\left(\frac{b}{\sqrt{2\pi a}}\right)-\frac{1}{2}\right]\right.$$
$$\left.-\cos\left(\frac{b^2}{4a}\right)\left[S\left(\frac{b}{\sqrt{2\pi a}}\right)-\frac{1}{2}\right]\right\}, \qquad a>0,$$

where

$$C(x)=\int_0^x\cos\left(\frac{\pi}{2}t^2\right)dt \quad\text{and}\quad S(x)=\int_0^x\sin\left(\frac{\pi}{2}t^2\right)dt.$$

In this example the zeros of the integrand $f(t) = \sin(at^2 + bt)$ get closer to each other as $t \to \infty$; hence it is obvious that $f(t)$ does not have a behavior which would enable the G_n-transformation to work efficiently. Indeed, the G_n-transformation failed to work in this case. However, f is in $B^{(2)}$, since it is integrable at infinity and satisfies the differential equation $f = [2a/(2at + b)^3]f' - [1/(2at+b)^2]f''$, therefore, we expect the $D_n^{(2)}$-transformation to work efficiently. As we can see from the differential equation, $j_0 \leqslant -3$ and $j_1 \leqslant -2$. In this example we computed $D_n^{(2)}[F(\xi);\tau]$ by replacing j_0 and j_1 in (2.24) with -3 and -2 respectively and by letting $\xi = 0.2$, $\tau = 0.2$. We considered the two cases $a = \pi/2$, $b = 0$, and $a = \pi/2$, $b = \pi/2$. For these cases we have $I(\pi/2, 0) = \frac{1}{2}$ and $I(\pi/2, \pi/2) = 0.39920505585256\ldots$. The results are given in Table 3.

EXAMPLE 4.4.

$$I = \int_0^\infty \frac{\log(1+t)}{1+t^2}\,dt = \frac{\pi}{4}\log 2 + G = 1.46036211675 3\ldots,$$

where $G = 0.91596559417 7219\ldots$ is Catalan's constant.

The function $f(t) = [\log(1+t)]/(1+t^2)$ is integrable at infinity and satisfies the differential equation

$$f = -\frac{5t^2+4t+1}{2(2t+1)}f' - \frac{(t^2+1)(t+1)}{2(2t+1)}f'';$$

therefore, $f \in B^{(2)}$. We also note that $j_0 \leqslant 1$ and $j_1 \leqslant 2$. Numerical results indicate that $j_0 = 1$ and $j_1 = 2$ exactly. For this case, and for many others

TABLE 3

VALUES OF THE APPROXIMATIONS $D_n^{(2)}[F(0.2); 0.2]$ TO $I(\pi/2, 0)$ AND $I(\pi/2, \pi/2)$, WHERE $I(a,b) = \int_0^\infty \sin(at^2 + bt)\,dt$[a]

n	$D_n^{(2)}[F(0.2);0.2](\pi/2,0)$	$D_n^{(2)}[F(0.2);0.2](\pi/2,\pi/2)$
2	0.12	0.46
4	0.495	0.397
6	0.4993	0.399212
8	0.5000001	0.399205044
10	0.499999999989	0.3992050585518

[a]For $n = 2(2)10$, where $F(x) = \int_0^x \sin(at^2 + bt)\,dt$

which contain logarithmic terms multiplied by functions in $A^{(\gamma)}$, the choice of the x_j, $j=1,2,\ldots,N$, in the equations (2.24) becomes very important. If we choose the x_j equidistantly as in the previous examples, the approximations $D_n^{(2)}[F(\xi);\tau]$ are poor for large n. If we choose $x_j=\xi e^{(j-1)\tau}$, $j=1,2,\ldots,N$, then the convergence of the approximations $D_n^{(2)}$ obtained by solving the equations (2.24) improves considerably. In our computations we chose $\xi=1$ and $\tau=0.2$. The results of the computations are given in Table 4.

We note that the G_n-transformation failed to work in this case.

EXAMPLE 4.5.

$$I=\int_0^\infty\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}=1.5707963267795\ldots.$$

The integrand $f(t)=[(\sin t)/t]^2$ is integrable at infinity, and since $(\sin t)/t$ is in $B^{(2)}$ from Example 4.1, $[(\sin t)/t]^2$ satisfies a differential equation of order 3 or less with coefficients in $A^{(\gamma)}$ for some values of γ, according to Corollaries 1 and 2 of Lemma 2. Indeed, f satisfies the differential equation

$$f=-\frac{2x^2+3}{4x}f'-\frac{3}{4}f''-\frac{x}{8}f''';$$

hence $f\in B^{(3)}$. Therefore, we except the $D_n^{(3)}$-transformation to produce good results. The G_n-transformation does not work efficiently on this function, although $(\sin t)^2$ satisfies a linear differential equation of order 3 with constant coefficients. The parameters j_k, $k=0,1,2$, in (2.24) satisfy $j_0\leqslant 1$, $j_1\leqslant 0$, $j_2\leqslant 1$, and from numerical results it turns out that $j_0=1$, $j_1=0$, $j_2=1$. In the

TABLE 4

VALUES OF THE APPROXIMATIONS $D_n^{(2)}$ TO

$$I=\int_0^\infty\frac{\log(1+t)}{1+t^2}dt^{a}$$

n	$D_n^{(2)}$
2	1.14
4	1.46085
6	1.46042
8	1.46036208
10	1.4603621191

[a]Obtained using $x_j=e^{(j-1)0.2}$, $j=1,2,\ldots,N$, for $n=2(2)10$

TABLE 5
THE APPROXIMATIONS $D_n^{(3)}[F(1);1]$ TO $I=\int_0^\infty [(\sin t)/t]^2\,dt$ [a]

n	$D_n^{(3)}[F(1);1]$
2	1.61
4	1.5709
6	1.570793
8	1.57079635
10	1.570796326888

[a] For $n=2(2)10$.

computation of $D_n^{(3)}$ we assumed that the equation was not known and replaced j_k by $\sigma_k=\min(k+1, s_k)$, $k=0,1,2$, which in this case turned out to be equal to 1. The approximations $D_n^{(3)}[F(\xi);\tau]$ to I were obtained by using $\xi=1$ and $\tau=1$. The results of the computations are given in Table 5.

EXAMPLE 4.6.

$$I=\int_0^\infty J_0(t)J_1(t)\frac{dt}{t}=\frac{2}{\pi}=0.63661977236758\ldots.$$

Since the functions $J_0(t)$ and $J_1(t)$ are in $B^{(2)}$, according to Lemma 2 and Corollary 2 their product satisfies a linear differential equation of order 4 or less; in fact, $J_0(t)J_1(t)/t$ turns out to be in $B^{(3)}$, the differential equation being rather complicated. We assumed that the differential equation was not known and replaced the j_k, $k=0,1,2$, in (2.24) by the σ_k. Since $f^{(k)}(x)=O(x^{-2})$ as $x\to\infty$, $k=0,1,2,\ldots$, it turns out that $\sigma_k=1$, $k=0,1,2$. In the computation of $D_n^{(3)}[F(\xi);\tau]$ we chose $\xi=1$ and $\tau=1$. The results of the computations are given in Table 6.

TABLE 6
VALUES OF THE APPROXIMATIONS
$D_n^{(3)}[F(1);1]$ TO $I=\int_0^\infty [J_0(t)J_1(t)/t]\,dt$ [a]

n	$D_n^{(3)}[F(1);1]$
2	0.6341
4	0.6366097
6	0.63661991
8	0.636619772204
10	0.63661977234 0

[a] For $n=2(2)10$

We note that the G_n-transformation failed to work in this case too. $D_n^{(3)}[F(\xi);1]$ was compared with $G_{3n}[F(\xi);\tau]$. $G_{30}[F(1);1]$ was seen to be correct to 5 decimal places, and $F(31)$ was seen to be correct to 3 decimal places.

5. THE d-TRANSFORMATION FOR INFINITE SERIES

Let $\tilde{B}^{(m)}$ be the set of infinite sequences $\{f(r)\}$ whose elements $f(r)$, $r=1,2,3,\ldots$, satisfy linear mth-order difference equations of the form

$$f(r)=\sum_{k=1}^{m} p_k(r)\Delta^k f(r), \tag{5.1}$$

where $\Delta^0 f(r)=f(r)$, $\Delta f(r)=f(r+1)-f(r)$, $\Delta^2 f(r)=\Delta[\Delta f(r)]$, etc., and $p_k(x)$, considered as functions of the continuous variable x, are in $A^{(k)}$, $k=1,2,\ldots,m$.

In this section we are going to make use of the ideas that were developed in Section 2 to derive the d-transformation that will give good approximations to the sums of infinite series of the form $\sum_{r=1}^{\infty} f(r)$, where $\{f(r)\}\in\tilde{B}^{(m)}$. In essence the d-transformation is the discrete analogue of the D-transformation for infinite integrals.

In analogy to Theorem 1 for infinite integrals whose integrands are in $B^{(m)}$, we now state Theorem 2 for infinite series whose associated sequences are in $\tilde{B}^{(m)}$.

THEOREM 2. *Let* $|\sum_{r=1}^{\infty} f(r)|<\infty$, *and let* $f(r)$, $r=1,2,\ldots$, *satisfy the linear* m *th-order difference equation* $f(r)=\sum_{k=1}^{m} p_k(r)\Delta^k f(r)$ *with* $p_k\in A^{(i_k)}$, $i_k\leqslant k$, $k=1,2,\ldots,m$. *If*

$$\lim_{r\to\infty}\left[\Delta^{i-1}p_k(r)\right]\left[\Delta^{k-i}f(r)\right]=0, \qquad k=i,i+1,\ldots,m, \quad i=1,2,\ldots,m, \tag{5.2}$$

and

$$\sum_{k=1}^{m} l(l-1)\cdots(l-k+1)p_{k,0}\neq 1, \qquad l\geqslant -1, \quad l \textit{ integer}, \tag{5.3}$$

where $p_{k,0}=\lim_{r\to\infty} r^{-k}p_k(r)$, *then* $\sum_{r=R}^{\infty} f(r)$, *for* $R\to\infty$, *has an asymptotic* *expansion of the form*

$$\sum_{r=R}^{\infty} f(r) \sim \sum_{k=0}^{m-1} \Delta^k f(R) R^{j_k} \left(\beta_{k,0} + \frac{\beta_{k,1}}{R} + \frac{\beta_{k,2}}{R^2} + \cdots \right) \tag{5.4}$$

with $j_k \leqslant \max(i_{k+1}, i_{k+2}-1, \ldots, i_m - m + k + 1)$, $k = 0, 1, \ldots, m-1$.

The proof of this theorem is analogous to that of Theorem 1 of Section 2 with $D \equiv d/dx$ replaced by Δ, $\int_x^\infty$ replaced by $\sum_{r=R}^\infty$, and integration by parts replaced by "summation by parts" using the formula

$$\sum_{r=R}^{R'} g(r)\Delta h(r) = -g(R-1)h(R) + g(R')h(R'+1) - \sum_{r=R}^{R'} [\Delta g(r-1)] h(r). \tag{5.5}$$

In order to show the similarity of the proof of Theorem 2 to that of Theorem 1, we give the first step of it in detail, the rest being analogous.

Summing $f(r) = \sum_{k=1}^m p_k(r)\Delta^k f(r)$ from $r = R$ to infinity and using (5.5) repeatedly together with the conditions in (5.2), we obtain analogously to (2.6)

$$\sum_{r=R}^{\infty} f(r) = \sum_{k=0}^{m-1} a_{1,k}(R)\Delta^k f(R) + \sum_{r=R}^{\infty} a_1(r) f(r), \tag{5.6}$$

where

$$\begin{aligned} a_{1,k}(R) &= \sum_{j=k+1}^{m} (-1)^{j+k}\Delta^{j-k-1} p_j(R-j+k), \qquad k = 0, 1, \ldots, m-1, \\ a_1(r) &= \sum_{k=1}^{m} (-1)^k \Delta^k p_k(r-k), \qquad r \geqslant R. \end{aligned} \tag{5.7}$$

We now use the fact that if $h \in A^{(\gamma)}$ [i.e., $h(r) = h_0 r^\gamma + O(r^{\gamma-1})$ as $r \to \infty$], then $\Delta h(r) = \gamma h_0 r^{\gamma-1} + O(r^{\gamma-2})$ as $r \to \infty$, i.e., $\Delta h \in A^{(\gamma-1)}$; and if $h \in A^{(0)}$, then $\Delta h \in A^{(-2)}$. This property of the difference operator Δ is similar to that of the differential operator D. We then see that $a_{1,k} \in A^{(k+1)}$, $k = 0, 1, \ldots, m-1$, and $a_1 \in A^{(0)}$ and is of the form

$$a_1(r) = \alpha_1 + c_1(r), \tag{5.8}$$

where $\alpha_1=\sum_{k=1}^{m}(-1)^k k! p_{k,0}$ and $c_1\in A^{(-2)}$. Since $\alpha_1\neq 1$ according to the hypothesis stated in the theorem, (5.6) can be written as

$$\sum_{r=R}^{\infty} f(r)=\sum_{k=0}^{m-1} b_{1,k}(R)\Delta^k f(R)+\sum_{r=R}^{\infty} b_1(r)f(r), \tag{5.9}$$

where

$$\begin{aligned} b_{1,k}(R)&=\frac{a_{1,k}(R)}{1-\alpha_1}, \qquad k=0,1,\ldots,m-1,\\ b_1(r)&=\frac{c_1(r)}{1-\alpha_1}, \qquad r\geqslant R, \end{aligned} \tag{5.10}$$

and where $b_{1,k}\in A^{(k+1)}$, $k=0,1,\ldots,m-1$, and $b_1\in A^{(-2)}$. Since $f(r)=o(1)$ and $b_1(r)=O(r^{-2})$ as $r\to\infty$, the sum $\sum_{r=R}^{\infty} b_1(r)f(r)$ converges to zero faster than the sum $\sum_{r=R}^{\infty} f(r)$ as $R\to\infty$.

We now make use of (5.4) to derive the d-transformation for infinite series. We demand that the approximation $d^{(m)}_{n_0,n_1,\ldots,n_{m-1}}$ to $S=\sum_{r=1}^{\infty} f(r)$ satisfy the $N=1+\sum_{k=0}^{m-1} n_k$ equations

$$d^{(m)}_{n_0,n_1,\ldots,n_{m-1}}=\sum_{r=1}^{R_l} f(r)+\sum_{k=0}^{m-1}\left[\Delta^k f(R_l+1)\right](R_l+1)^{i_k}\sum_{i=0}^{n_k-1}\frac{\bar\beta_{k,i}}{(R_l+1)^i}, \qquad l=1,2,\ldots,N, \tag{5.11}$$

with $\bar\beta_{k,i}$ constants and R_l chosen to satisfy $0\leqslant R_1<R_2<\cdots<R_N$, and with $\sum_{r=1}^{R_1} f(r)=0$ when $R_1=0$. The equations (5.11) form a linear set in N unknowns, namely, $d^{(m)}_{n_0,n_1,\ldots,n_{m-1}}$ and $\bar\beta_{k,i}$, $i=0,1,\ldots,n_k-1$, $k=0,1,\ldots,m-1$, and can, in general, be solved for $d^{(m)}_{n_0,n_1,\ldots,n_{m-1}}$. Again for the sake of simplicity we choose

$$R_l=\xi+(l-1)\tau, \qquad l=1,2,\ldots,N,$$

where $\xi\geqslant 0$, $\tau\geqslant 1$ are integers. We denote $d^{(m)}_{n_0,n_1,\ldots,n_{m-1}}$ by $d^{(m)}_{n_0,n_1,\ldots,n_{m-1}}[F(\xi);\tau]$, where $F(\xi)$ is the sum of the first ξ terms of $\sum_{r=1}^{\infty} f(r)$, i.e., $F(\xi)=\sum_{r=1}^{\xi} f(r)$. We also define the "diagonal" d-transformation as $d^{(m)}_{n,n,\ldots,n}[F(\xi);\tau]\equiv d^{(m)}_n[F(\xi);\tau]$.

$d_n^{(m)}[F(\xi);1]$ for the case $m=2$ and $j_1=j_2=0$, which occurs frequently, is given below:

$$d_n^{(2)}[F(\xi);1]=\frac{\begin{vmatrix} F(\xi) & F(\xi+1) & \cdots & F(\xi+2n) \\ f(\xi+1) & f(\xi+2) & \cdots & f(\xi+2n+1) \\ \dfrac{f(\xi+1)}{\xi+1} & \dfrac{f(\xi+2)}{\xi+2} & \cdots & \dfrac{f(\xi+2n+1)}{\xi+2n+1} \\ \vdots & \vdots & & \vdots \\ \dfrac{f(\xi+1)}{(\xi+1)^{n-1}} & \dfrac{f(\xi+2)}{(\xi+2)^{n-1}} & \cdots & \dfrac{f(\xi+2n+1)}{(\xi+2n+1)^{n-1}} \\ \Delta f(\xi+1) & \Delta f(\xi+2) & \cdots & \Delta f(\xi+2n+1) \\ \dfrac{\Delta f(\xi+1)}{\xi+1} & \dfrac{\Delta f(\xi+2)}{\xi+2} & \cdots & \dfrac{\Delta f(\xi+2n+1)}{\xi+2n+1} \\ \vdots & \vdots & & \vdots \\ \dfrac{\Delta f(\xi+1)}{(\xi+1)^{n-1}} & \dfrac{\Delta f(\xi+2)}{(\xi+2)^{n-1}} & \cdots & \dfrac{\Delta f(\xi+2n+1)}{(\xi+2n+1)^{n-1}} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ f(\xi+1) & f(\xi+2) & \cdots & f(\xi+2n+1) \\ \dfrac{f(\xi+1)}{\xi+1} & \dfrac{f(\xi+2)}{\xi+2} & \cdots & \dfrac{f(\xi+2n+1)}{\xi+2n+1} \\ \vdots & \vdots & & \vdots \\ \dfrac{f(\xi+1)}{(\xi+1)^{n-1}} & \dfrac{f(\xi+2)}{(\xi+2)^{n-1}} & \cdots & \dfrac{f(\xi+2n+1)}{(\xi+2n+1)^{n-1}} \\ \Delta f(\xi+1) & \Delta f(\xi+2) & \cdots & \Delta f(\xi+2n+1) \\ \dfrac{\Delta f(\xi+1)}{\xi+1} & \dfrac{\Delta f(\xi+2)}{\xi+2} & \cdots & \dfrac{\Delta f(\xi+2n+1)}{\xi+2n+1} \\ \vdots & \vdots & & \vdots \\ \dfrac{\Delta f(\xi+1)}{(\xi+1)^{n-1}} & \dfrac{\Delta f(\xi+2)}{(\xi+2)^{n-1}} & \cdots & \dfrac{\Delta f(\xi+2n+1)}{(\xi+2n+1)^{n-1}} \end{vmatrix}} \tag{5.12}$$

Like the *D*-transformation, the *d*-transformation is also a nonlinear summability method.

6. SOME SPECIAL CASES OF THE *d*-TRANSFORMATION

For $m=1$ and $j_0=0$ the system of equations (5.11) reduces to that given by Levin [5] in his development of the *t*-transformation. Hence the $d_n^{(1)}$-transformation is identical to the *t*-transformation of Levin [5], namely,

$d_n^{(1)}[F(\xi);1] \equiv t_n[F(\xi+1)]$. For $m=1$ and $j_0=1$ the $d_n^{(1)}$-transformation turns out to be identical to the u-transformations of Levin [5], namely $d_n^{(1)}[F(\xi);1] \equiv u_n[F(\xi+1)]$ with $n>1$. The fact that the t- and u-transformations accelerate the convergence of slowly converging series whose associated sequences are in $\tilde{B}^{(1)}$, in the sense that as $n \to \infty$ t_n and u_n converge to the sums of the series quickly, indicates that the $d_n^{(m)}$-transformation will be as efficient as the t- and u-transformations when applied to series whose associated sequences are in $\tilde{B}^{(m)}$ for any $m \geqslant 1$; i.e., as $n \to \infty$, $d_n^{(m)}$ will tend to the sums of these series quickly. This was indeed the case in all the examples that were considered. (See Section 7.)

It also turns out that $d_1^{(m)}[F(\xi);1]$ with $j_k=0$, $k=0,1,\ldots,m-1$, is identical to the $e_m[F(\xi+m)]$-transformation of Shanks [8], which in turn is identical to $\epsilon_{2m}[F(\xi)]$ of the ϵ-algorithm of Wynn [3].

When the d-transformation is applied to power series, rational approximations are obtained. From what has been said above we see that the rational approximations obtained using $d_1^{(m)}$ with $j_k=0$, $k=0,1,\ldots,m-1$, are just the Padé approximants, and those obtained using $d_n^{(1)}$ with $j_0=1$ are the u-approximants of Levin [6], see also Longman [10]. The Padé and u-approximants have proved to be very efficient for power series whose coefficients satisfy simple recurrence relations. However, for power series whose coefficients satisfy complicated recurrence relations of the form (5.1) we expect that the rational approximations obtained using the $d^{(m)}$-transformation will be more appropriate. These rational approximations can be cast into a form which is convenient to use. For example, the $d_n^{(2)}[F(\xi);1]$ approximation to a power series $\sum_{r=1}^{\infty} a_r x^{r-1}$, with $j_0=j_1=0$, after some elementary row and column transformations on (5.12), can be expressed as $d_n^{(2)}[F(\xi);1]=N/D$, where

$$N=\begin{vmatrix} x^{2n}\sum\limits_{r=1}^{\xi} a_r x^{r-1} & x^{2n-1}\sum\limits_{r=1}^{\xi+1} a_r x^{r-1} & \cdots & \sum\limits_{r=1}^{\xi+2n} a_r x^{r-1} \\ a_{\xi+1} & a_{\xi+2} & \cdots & a_{\xi+2n+1} \\ \dfrac{a_{\xi+1}}{\xi+1} & \dfrac{a_{\xi+2}}{\xi+2} & \cdots & \dfrac{a_{\xi+2n+1}}{\xi+2n+1} \\ \vdots & \vdots & & \vdots \\ \dfrac{a_{\xi+1}}{(\xi+1)^{n-1}} & \dfrac{a_{\xi+2}}{(\xi+2)^{n-1}} & \cdots & \dfrac{a_{\xi+2n+1}}{(\xi+2n+1)^{n-1}} \\ a_{\xi+2} & a_{\xi+3} & \cdots & a_{\xi+2n+2} \\ \dfrac{a_{\xi+2}}{\xi+1} & \dfrac{a_{\xi+3}}{\xi+2} & \cdots & \dfrac{a_{\xi+2n+2}}{\xi+2n+1} \\ \vdots & \vdots & & \vdots \\ \dfrac{a_{\xi+2}}{(\xi+1)^{n-1}} & \dfrac{a_{\xi+3}}{(\xi+2)^{n-1}} & \cdots & \dfrac{a_{\xi+2n+2}}{(\xi+2n+1)^{n-1}} \end{vmatrix}, \qquad (6.1)$$

and D is obtained from N by replacing the first row in the determinant expression in (6.1) with the row vector $(x^{2n}, x^{2n-1},\ldots,x,1)$.

As is seen from (6.1), the rational approximation obtained using $d_n^{(2)}[F(\xi);1]$ has numerator of degree $2n+\xi-1$ and denominator of degree $2n$. In general, the rational approximations obtained using $d_n^{(m)}[F(\xi);1]$ with $j_k=0$, $k=0,1,\ldots,m-1$, will have numerators of degree $mn+\xi-1$ and denominators of degree mn.

7. APPLICATIONS OF THE d-TRANSFORMATION

In many problems of applied mathematics the solution is obtained in the form of an infinite series $\sum_{r=1}^{\infty}a(r)\phi_r(x)$, where $\phi_r(x)$ are orthogonal polynomials, or elementary functions, or special functions, or products of them. Hence the functions $\phi_r(x)$ satisfy a linear recursion relation of some finite order hence the sequences $\{\phi_r(x)\}$ are usually in $\tilde{B}^{(m)}$ for some m. As we shall see below, if $a(r)=t^r$ or $a\in A^{(\gamma)}$ for some γ, then in general, $\{a(r)\phi_r(x)\}\in \tilde{B}^{(m)}$ too.

Several methods for accelerating the convergence of series of orthogonal functions have been developed in the past. Mention can be made of the methods of Maehley [11] and Clenshaw and Lord [14] for Chebyshev series, of Holdeman [12] for series of orthogonal polynomials in general, and of Fleischer [13] for Legendre series. A recursive method for the computation of the approximations of Clenshaw and Lord has been given by the second author [15]. All these methods are of the Padé type. We finally mention the methods of Levin [6] for accelerating the convergence of series of orthogonal polynomials, which are like the d-transformation, but unlike the d-transformation require full knowledge of the recursion relations that these polynomials satisfy. This point has been explained in detail in Section 1. One drawback of all these methods is that one needs different transformations for different kinds of series, whereas the same d-transformation can be used for all of these series. If the function represented by the infinite series in question is analytic, then the d-transformation, like the methods given in [11, 12, 13, 14], can be used to analytically continue the series to regions in which the series diverges. This point is briefly illustrated in Example 7.1.

In this section we shall illustrate the use of the d-transformation on different infinite series of the form mentioned in the previous paragraph; in particular, we shall deal with series whose associated sequences are in $\tilde{B}^{(2)}$ and $\tilde{B}^{(4)}$.

The choice of the parameters j_k in (5.11) is exactly as explained in Section 4 for the D-transformation with the derivative operator replaced by the forward difference operator Δ. In analogy to Lemma 2 and its Corollaries 1, 2, and 3 of Section 4 we state

LEMMA 3. *If $\{f(r)\}$ and $\{g(r)\}$ satisfy linear difference equations of orders m and n respectively, i.e.,*

$$f(r)=\sum_{k=1}^{m} p_k(r)\Delta^k f(r), \qquad g(r)=\sum_{l=1}^{n} q_l(r)\Delta^l g(r), \tag{7.1}$$

then their product $\{f(r)g(r)\}$ and their sum $\{f(r)+g(r)\}$, in general, satisfy linear difference equations of orders less than or equal to mn and $m+n$ respectively, i.e.,

$$\begin{aligned} f(r)g(r)&=\sum_{k=1}^{mn} A_k(r)\Delta^k[f(r)g(r)],\\ f(r)+g(r)&=\sum_{k=1}^{m+n} B_k(r)\Delta^k[f(r)+g(r)]. \end{aligned} \tag{7.2}$$

COROLLARY 1. *If $\{f(r)\}$ satisfies a linear difference equation of order m, then, in general, $\{[f(r)]^2\}$ satisfies a linear difference equation of order $m(m+1)/2$ or less.*

COROLLARY 2. *If the coefficients $p_k(r)$, $k=1,2,\dots,m$, and $q_l(r)$, $l=1,2,\dots,n$, in (7.1) have asymptotic expansions in inverse powers of r as $r\to\infty$, then so do $A_k(r)$, $k=1,2,\dots,mn$, and $B_k(r)$, $k=1,2,\dots,m+n$, in (7.2).*

COROLLARY 3. *If $\{f(r)\}\in\tilde{B}^{(m)}$ and $g(r)=t^r$ or $g\in A^{(\gamma)}$ for some γ, then the terms of the sequence $\{g(r)f(r)\}$ satisfy a linear difference equation of order m or less with coefficients that have asymptotic expansions in inverse powers of r as $r\to\infty$.*

The proofs of Lemma 3 and its Corollaries 1, 2, and 3 are identical to those of Lemma 2 and its corollaries if one uses

$$\Delta^N[f(r)g(r)]=\sum_{k=0}^{N}\binom{N}{k}\sum_{j=0}^{N-k}\binom{N-k}{j}[\Delta^{N-j}f(r)][\Delta^{k+j}g(r)], \tag{7.3}$$

in the same manner that Leibnitz's rule for differentiating the product of two functions was used in Section 4. The formula given in (7.3) can be proved by induction on N.

EXAMPLE 7.1.

$$g(x)=\frac{1}{2}\sqrt{\frac{1-x}{2}}=\sum_{r=0}^{\infty}\frac{P_r(x)}{(1-2r)(2r+3)}, \qquad -1\leqslant x\leqslant 1,$$

where $P_r(x)$ is the Legendre polynomial of degree r, $r=0,1,2,\ldots$. This series converges slowly for $|x|\leqslant 1$ and diverges for $|x|>1$.

Since the Legendre polynomials satisfy the linear three-term recursion relation

$$P_{r+1}(x)=\frac{2r+1}{r+1}xP_r(x)-\frac{r}{r+1}P_{r-1}(x),$$

the terms $C_r=P_r(x)/[(1-2r)(2r+3)]$ of the infinite series above satisfy the second-order difference equation

$$C_r=p_1(r)\,\Delta C_r+p_2(r)\,\Delta^2 C_r,$$

where

$$p_1(r)=\frac{(1-x)4r^2+(11-6x)2r+(28-5x)}{(x-1)(4r^2+12r)+(5x-13)},$$

$$p_2(r)=\frac{2r^2+11r+14}{(x-1)(4r^2+12r)+(5x-13)}.$$

It is seen that for $x\neq 1$, $p_1\in A^{(0)}$ and $p_2\in A^{(0)}$ when p_1 and p_2 are considered as functions of r for fixed x. However, for $x=1$, $p_1\in A^{(1)}$ and $p_2\in A^{(2)}$. This sudden transition of p_1 and p_2 from $A^{(0)}$ to $A^{(1)}$ and $A^{(2)}$ respectively indicates that $x=1$ is a point of singularity of $g(x)$, and indeed $x=1$ is a branch point of $g(x)$. We therefore expect the $d_n^{(2)}$-transformation to produce good approximations to $g(x)$ for $-1\leqslant x<1$; i.e., where the series converges to $g(x)$, and for x not too close to 1. This is verified by computations done for $-1\leqslant x<1$. It also turns out that the $d_n^{(2)}$-transformation produces good approximations to $g(x)$ for $x<-1$, i.e., even when the series diverges. In Table 7 we give the values of $d_n^{(2)}[F(0,x);1]$ [with j_0 and j_1 in (5.11) replaced

TABLE 7
VALUES OF THE APPROXIMATIONS
$d_n^{(2)}[F(0,x);1]$ TO $g(x)=\sqrt{(1-x)/2}/2$[a]

n	$d_n^{(2)}[F(0,-1.5);1]$	$d_n^{(2)}[F(0,0.5);1]$	$d_n^{(2)}[F(0,0.9);1]$
2	0.559015	0.2505	0.116
4	0.5590166998	0.249998	0.1114
6	0.5590169943 72	0.24999989	0.11177
8	0.5590169943 7493	0.24999999997 8	0.111800
10	0.5590169943 7485	0.2500000000 27	0.1118039
Exact	0.5590169943 74947	0.2500000000 00	0.1118033988 74

[a]For $x=-1.5, 0.5, 0.9$, and $n=2(2)10$. Exact values of $g(x)$ are given in the bottom row.

by their upper bound, which is zero] for $x=-1.5, 0.5, 0.9$, where $F(\xi, x)$ is the ξth partial sum of the infinite series evaluated at x; i.e., $F(\xi, x)=\sum_{r=0}^{\xi-1} P_r(x)/[(1-2r)(2r+3)]$, and $F(0,x)=0$.

We see from Table 7 that for $x=0.9$, which is close to $x=1$ [the branch point of $g(x)$], the convergence of $d_n^{(2)}[F(0,0.9);1]$ to $g(x)$ as n increases is much slower than that of $d_n^{(2)}[F(0,x);1]$ for $x=-1.5$ and $x=0.5$, which are far from $x=1$. In order to accelerate the convergence of the approximations $d_n^{(2)}[F(\xi,x);\tau]$ to $g(x)$ one can use higher values of ξ and τ in the equations (5.11). In our computations we took $\xi=1$ and $\tau=2$ and computed the approximations $d_n^{(2)}[F(1,x);2]$. In Table 8 we given the values of $d_n^{(2)}[F(1,x);2]$ for $x=0.9$, which clearly show a great improvement in the convergence over those exhibited in Table 7.

TABLE 8
VALUES OF THE APPROXIMATIONS
$d_n^{(2)}[F(1,0.9);2]$ TO $g(x)=\sqrt{(1-x)/2}/2$[a]

n	$d_n^{(2)}[F(1,0.9);2]$
2	0.112
4	0.111805
6	0.1118032
8	0.1118033393
10	0.1118033988 5
Exact	0.1118033988 74

[a]For $n=2(2)10$. The exact value of $g(x)$ for $x=0.9$ is given at the bottom.

EXAMPLE 7.2.

$$g(x) = \operatorname{sgn} x = \frac{4}{\pi} \sum_{r=1}^{\infty} \frac{\sin(2r-1)x}{2r-1}, \qquad -\pi < x < \pi.$$

The terms $f(r) = \sin(2r-1)x$, $r = 1, 2, \ldots$, satisfy the three-term recursion relation

$$f(r+1) = 2\cos 2x\, f(r) - f(r-1);$$

hence the terms $C_r = [4\sin(2r-1)x]/[\pi(2r-1)]$ of the infinite series satisfy the second-order difference equation

$$C_r = p_1(r)\,\Delta C_r + p_2(r)\,\Delta^2 C_r,$$

where

$$p_1(r) = -1 + \frac{\cos 2x - 3}{(1-\cos 2x)(2r+1)},$$

$$p_2(r) = -\frac{2r+3}{(1-\cos 2x)(4r+2)}.$$

We see that for $\cos 2x \neq 1$, i.e., $x \neq 0, \pm\pi$, $\{C_r\} \in \tilde{B}^{(2)}$, and $p_1 \in A^{(0)}$, $p_2 \in A^{(0)}$. For $x = 0, \pm\pi$ the difference equation is singular, which indicates that at $x = 0, \pm\pi$ the function $g(x)$ has signularities. Indeed, $g(x)$ has jump discontinuities at $x = 0, \pm\pi$. We therefore expect the $d_n^{(2)}$-transformation to produce good approximations to $g(x)$ for x not too close to $0, \pm\pi$, where discontinuities occur, and this is verified by Table 9, which gives values of $d_n^{(2)}[F(0,x);1]$ with j_0 and j_1 in (5.11) replaced by their upper bound, which is zero, for $x = \pi/6$ and $x = \pi/2$, where $F(\xi, x) = (4/\pi)\sum_{r=1}^{\xi}[\sin(2r-1)x]/(2r-1)$ and $F(0,x) = 0$.

It is worth mentioning that $d_n^{(2)}[F(0,\pi/2);1]$ has turned out to be very accurate despite the fact that the coefficients $\bar{\beta}_{k,i}$ in the equations (5.11) came out very large. The reason for the $\bar{\beta}_{k,i}$ to be large is that for $x = \pi/2$ the elements of the series are $C_r = (-1)^{r+1}4/[\pi(2r-1)]$, which puts $\{C_r\}$ in $\tilde{B}^{(1)}$, and this causes the instability in the computation of the coefficients $\bar{\beta}_{k,i}$.

We note that Fourier sine and cosine series and series of Chebyshev polynomials of the first and second kinds can be summed with equal efficiency by using the d-transformation, since $\{\sin rx\}$, $\{\cos rx\}$, $\{T_r(x)\}$, and

TABLE 9

VALUES OF THE APPROXIMATIONS $d_n^{(2)}[F(0,x);1]$ TO $g(x)=\operatorname{sgn} x$ [a]

n	$d_n^{(2)}[F(0,\pi/6);1]$	$d_n^{(2)}[F(0,\pi/2);1]$
2	1.032	1.00010
4	1.00031	0.99999983
6	0.9999979	1.0000000036
8	0.9999999908	0.999999999980
10	0.999999999932	0.99999999999993

[a] For $x=\pi/6$ and $x=\pi/2$, and $n=2(2)10$.

$\{U_r(x)\}$, where $T_r(x)$ and $U_r(x)$ are the Chebyshev polynomials of the first and second kinds respectively, all satisfy similar recursion relations, namely,

$$\sin(r+1)x=2\cos x\sin rx-\sin(r-1)x,$$

$$\cos(r+1)x=2\cos x\cos rx-\cos(r-1)x,$$

$$T_{r+1}(x)=2xT_r(x)-T_{r-1}(x),$$

$$U_{r+1}(x)=2xU_r(x)-U_{r-1}(x),$$

for $r\geq 1$.

EXAMPLE 7.3.

$$g(x)=\log\frac{1}{x}=2\sum_{r=1}^{\infty}\frac{J_0(\lambda_r x)}{[\lambda_r J_1(\lambda_r)]^2},\qquad 0<x\leq 1,$$

where λ_r is the rth positive zero of $J_0(x)$.

We have not been able to find a recursion relation that the terms $C_r=2J_0(\lambda_r x)/[\lambda_r J_1(\lambda_r)]^2$ of this infinite series satisfy; however, we can prove that $\{C_r\}\in\tilde{B}^{(2)}$. For this we shall make use of the following results [16]:

$$\lambda_r=(r-\tfrac{1}{4})\pi+a(r), \tag{7.4}$$

$$H_\nu^{(1)}(x)=e^{i(x-\nu\pi/2-\pi/4)}b_\nu(x), \tag{7.5}$$

$$J_\nu(x)=\operatorname{Re}H_\nu^{(1)}(x), \tag{7.6}$$

where $a(r)$, considered as a function of r, is in $A^{(-1)}$; $b_\nu(x)$, considered as a function of x, is in $A^{(-1/2)}$; and $H_\nu^{(1)}(x)$ is the Hankel function of the first kind of order ν.

First of all, we see from (7.4) that $1/\lambda_r^2$ considered as a function of r is in $A^{(-2)}$. Secondly, from (7.5) and (7.6)

$$J_1(\lambda_r) = \operatorname{Re} H_1^{(1)}(\lambda_r) = \operatorname{Re}\{e^{i[r\pi-\pi+a(r)]}b_1[r\pi-\pi/4+a(r)]\}$$

$$=(-1)^{r+1}\operatorname{Re}\{e^{ia(r)}b_1[r\pi-\pi/4+a(r)]\}.$$

Now since $a\in A^{(-1)}$, i.e., $a(r)=O(r^{-1})$ as $r\to\infty$, we have $e^{ia(r)}\in A^{(0)}$. Using the fact that $b_1[r\pi-\pi/4+a(r)]\in A^{(-1/2)}$ [since $b_1(r)\in A^{(-1/2)}$], we have that $J_1(\lambda_r)\in A^{(-1/2)}$, and hence $1/[J_1(\lambda_r)]^2\in A^{(1)}$.

Finally, we show that $\{J_0(\lambda_r x)\}$ for fixed x is in $\tilde{B}^{(2)}$. Using (7.4) and (7.5), we have

$$H_0^{(1)}(\lambda_r x) = e^{ir\pi x}K(r),$$

where $K(r)=e^{ia(r)x}\{e^{-i(\pi x/4-\pi/4)}b_0[r\pi x-\pi x/4+a(r)x]\}$. Since $e^{ia(r)x}\in A^{(0)}$ and $b_0[r\pi x-\pi x/4+a(r)x]\in A^{(-1/2)}$, we have $K(r)\in A^{(-1/2)}$. Since $\{e^{ir\pi x}\}\in\tilde{B}^{(1)}$ for x fixed, we have $\{H_0^{(1)}(\lambda_r x)\}\in\tilde{B}^{(1)}$ using Corollary 3 to Lemma 3. Also since $J_0(\lambda_r x) = [H_0^{(1)}(\lambda_r x)+\bar{H}_0^{(1)}(\lambda_r x)]/2$, we have $\{J_0(\lambda_r, x)\}\in\tilde{B}^{(2)}$ in accordance with Corollary 2 of Lemma 3. And finally $\{J_0(\lambda_r x)/[\lambda_r J_1(\lambda_r)]^2\}\in\tilde{B}^{(2)}$, again by Corollary 2 of Lemma 3. Therefore, we expect the $d_n^{(2)}$-transformation to work efficiently for $0<x\leqslant 1$ and x not too close to zero, where the function $g(x)=\log(1/x)$ has a singularity. It turns out that the $d_n^{(2)}$-transformation produces very good approximations to $g(x)$ even when $x>1$. In Table 10 we give values of $d_n^{(2)}[F(0,x);1]$ with j_0 and j_1 replaced by $\sigma_0=1$ and $\sigma_1=1$ for $x=0.6$ and $x=1.4$, where $F(\xi,x)=$

TABLE 10

VALUES OF THE APPROXIMATIONS $d_n^{(2)}[F(0,x);1]$ TO $g(x)=\log(1/x)$[a]

n	$d_n^{(2)}[F(0,0.6);1]$	$d_n^{(2)}[F(0,1.4);1]$
2	0.51034	−0.336
4	0.51077	−0.36437
6	0.51082556	−0.36472198
8	0.510825623725	−0.3647223659
10	0.51082562376559	−0.364722366620986
Exact	0.51082562376 65599	−0.364722366621212

[a] For $x=0.6$ and $x=1.4$ and $n=2(2)10$. Exact values of $g(x)$ are given in the bottom row.

$\sum_{r=1}^{\xi} J_0(\lambda_r x)/[\lambda_r J_1(\lambda_r)]^2$ and $F(0,x)=0$. Numerical results indicate that $j_0=j_1=0$.

EXAMPLE 7.4.

$$g(\beta,\phi)=\begin{cases}1/\sqrt{2(\cos\beta-\cos\phi)}, & 0\leq\beta<\phi<\pi,\\ 0, & 0<\phi<\beta<\pi,\end{cases}$$

$$g(\beta,\phi)=\sum_{r=0}^{\infty}\cos\left(r+\tfrac{1}{2}\right)\beta P_r(\cos\phi),\qquad 0\leq\beta<\pi,\quad 0<\phi<\pi,\quad \beta\neq\phi.$$

Since both $\cos(r+\frac{1}{2})\beta$, $r=0,1,2,\ldots$, and $P_r(\cos\phi)$, $r=0,1,2,\ldots$, satisfy three-term recursion relations and hence second-order difference equations, the terms $C_r=\cos(r+\frac{1}{2})\beta P_r(\cos\phi)$, $r=0,1,2,\ldots$, of the infinite series satisfy a fourth-order difference equation by Corollary 1 of Lemma 3. This difference equation is quite complicated and will not be given here. We expect the $d_n^{(4)}$-transformation to produce good approximations to $g(\beta,\phi)$ for ϕ not too close to β, since $g(\beta,\phi)$ goes to infinity as ϕ approaches β. Table 11 contains values of $d_n^{(4)}[F(0,\beta,\phi);1]$ with $j_k=0$, $k=0,1,2,3$, for $\beta=2\pi/3$, $\phi=\pi/6$ and for $\beta=\pi/6$, $\phi=2\pi/3$, where $F(\xi,\beta,\phi)=\sum_{r=0}^{\xi-1}\cos(r+\frac{1}{2})\beta P_r(\cos\phi)$ and $F(0,\beta,\phi)=0$.

We note that the partial sums $F(\xi,2\pi/3,\pi/6)$ for $\xi\leq 50$ are of order 10^{-2} at best, and the $d_6^{(2)}$-transformation improves the accuracy by about 12 significant figures.

TABLE 11

VALUES OF THE APPROXIMATIONS $d_n^{(4)}[F(0,\beta,\phi),1]$ TO $g(\beta,\phi)$[a]

2	$\approx 4\times10^{-6}$	0.604998
3	$\approx -2\times10^{-8}$	0.60500026
4	$\approx -2\times10^{-10}$	0.605000333358
5	$\approx -2\times10^{-13}$	0.6050003337080
6	$\approx -2\times10^{-14}$	0.605000333706045
Exact	0	0.605000333706055

[a]For $\beta=2\pi/3$, $\phi=\pi/6$ and for $\beta=\pi/6$, $\phi=2\pi/3$, with $n=2(1)6$. Exact values of $g(\beta,\phi)$ are given in the bottom row.

8. COMPUTATIONAL ASPECTS

In this section we shall describe briefly the computational aspects of the D-transformation, those of the d-transformation being similar.

When one is given a function f to integrate between zero and infinity, one should find out whether this function is in $B^{(m)}$ for some m and what this m is. Then, as was described in Section 4, the parameters j_k in (2.24) should be replaced either by their upper bounds or by the σ_k whose determination was also described in Section 4. Once this is done, one should pick up the x_j in (2.24) in such a way that the function $f(t)$ has a smooth behavior between two consecutive x_j's and hence can be integrated accurately there without much effort being wasted. An example of poor behavior was given in Example 4.3, where $f(t)=\sin(at^2+bt)$ with $a=\pi/2$, $b=0$ and with $a=\pi/2$, $b=\pi/2$. This function oscillates with increasing frequency as t becomes large, and if the distance between x_j and x_{j+1} is so large as to enable the function to oscillate a large number of times there, then the accurate computation of the integrals $u_j=\int_{x_{j-1}}^{x_j} f(t)\,dt$ and hence of $F(x_j)=\sum_{i=1}^{j} u_i$ becomes a hard task. This is the reason why we took $x_j=\xi+(j-1)\tau$ with ξ and τ small (actually we took $\xi=0.2$ and $\tau=0.2$). With this choice of ξ and τ, at least the first few u_j which are needed for the computation of $D_n^{(2)}[F(\xi);\tau]$, say up to $n=15$, can be computed accurately without any difficulty. We have already pointed out in Example 4.4 that sometimes the choice $x_j=\xi+(j-1)\tau$ causes the approximations $D_n^{(m)}[F(\xi);\tau]$ to be poor even for large n, and this problem can be dealt with by taking, for example, $x_j=\xi+(j-1)^l\tau$, $l>1$, or $x_j=\xi e^{(j-1)\tau}$; in Example 4.4 we chose the latter. Now that we have chosen the x_j appropriately, we compute the u_j, the $F(x_j)$, and the matrix of the equations (2.24) and solve the linear system in (2.24). We used the LINSYST subroutine subprogram [18] to solve the linear system in (2.24). This subroutine solves linear systems by LU decomposition and iterative improvement. The use of Cramer's rule, as in (2.25) for example, is not advised, since as n and hence the number of equations become large, the determinant of the linear system (2.24) decreases rapidly and thus errors are introduced in the computation of the approximations $D_n^{(m)}$. Also it is worth mentioning that the system (2.24) becomes very ill conditioned as n increases. However, this does not seem to affect the approximations $D_n^{(m)}$.

In order to give proper meaning to the numerical results, attention must be paid to the coefficients $\bar{\beta}_{k,i}$, as well as $D_n^{(m)}$, in the solution of (2.24). In most cases the D-transformation produces approximations $D_n^{(m)}$ which, as n becomes large, converge very quickly to the right value. For instance, in most of the examples in Section 4, $D_n^{(m)}$ is correct to ten significant figures or more. As the $D_n^{(m)}$, $n=1,2,\ldots$, converge, the first few $\bar{\beta}_{k,i}$, for each k, approach the corresponding $\beta_{k,i}$ in the asymptotic expansion in (2.23), and this can be recognized very easily even for small n.

It may happen, though very rarely, that the convergence pattern of the approximations $D_n^{(m)}$ is disrupted by some $D_n^{(m)}$, say for $n = r$; i.e., it may happen that $D_r^{(m)}$ is a worse approximation than $D_{r-1}^{(m)}$, $D_{r+1}^{(m)}$ being better than $D_{r-1}^{(m)}$. This is accompanied by a disruption in the convergence pattern of the first few $\bar{\beta}_{k,i}$ for each k to $\beta_{k,i}$ in (2.23), the rest of the coefficients becoming much larger than those for $n = r \pm 1$. The reason for this is that the matrix of the equations (2.24) for $n = r$, for computational purposes, is singular. Needless to say, $D_r^{(m)}$ should be discarded.

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Note Added in Proof

The convergence analysis of the D- and d-transformations has been taken up by the second author within a more general framework in the paper

A. Sidi, Some properties of a generalization of the Richardson extrapolation process, *J. Inst. Maths. Applics.* 24:327–346 (1979).

In this paper the question of the significance of the x_l that go into the definition of the D-transformation is also considered.

It turns out that by selecting the x_l in an appropriate manner, the D-transformation can be modified and made more efficient for integrands $f(x)$ that oscillate an infinite number of times as $x \to \infty$. This subject is taken up in the papers

A. Sidi, Extrapolation methods for oscillatory infinite integrals, *J. Inst. Maths. Applics.* 26:1–20 (1980).

A. Sidi, The numerical evaluation of very oscillatory infinite integrals by extrapolation, *Math. Comp.* 38 (1982) (in press).

Convergence properties of the T-transformation of Levin [5] have been taken up in the papers

A. Sidi, Convergence properties of some non-linear sequence transformations, *Math. Comp.* 33:315–326 (1979).

A. Sidi, Analysis of convergence of the T-transformation for power series, *Math. Comp.* 35:833–850 (1980).

The t- and u-transformations, which are special cases of the T-transformation, are simply $d^{(1)}$-transformations as mentioned in Section 6 of the present work.