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Interpolation at Equidistant Points by a Sum of Exponential Functions

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A set of necessary and sufficient conditions for the existence and uniqueness of a solution to the problem of interpolation at equidistant points by a sum of exponential functions is given. Simultaneously a simple method for constructing the solution is developed. The confluent interpolation problem in which all the points of interpolation coincide is dealt with similarly. Integral representations for the solutions to both problems are given and a limit result is proved.

1. INTRODUCTION

Suppose the function $f(x)$ is to be approximated by a sum of exponential functions

$$u(x) = \sum_{j=1}^{n} \alpha_j e^{\sigma_j x}, \tag{1.1}$$

where the α_j and σ_j are unknown parameters to be determined by the interpolation conditions

$$c_i = f(x_0 + ih) = u(x_0 + ih), \qquad i = 0, 1,..., 2n - 1. \tag{1.2}$$

Substituting (1.1) in (1.2), and defining $\bar{\alpha}_j = \alpha_j e^{\sigma_j x_0}$ and $\zeta_j = e^{\sigma_j h}$, $j = 1,..., n$, we have

$$c_i = \sum_{j=1}^{n} \bar{\alpha}_j \zeta_j^i, \qquad i = 0, 1,..., 2n - 1. \tag{1.3}$$

These equations have been solved for the $\bar{\alpha}_j$ and ζ_j by Prony [7], and the relation of Prony's method of solution with the $(n - 1/n)$ Padé approximant $F_{n-1,n}(z)$ to the power series $F(z) = \sum_{i=0}^{2n-1} c_i z^i$ has been shown by Weiss and McDonough [8]. It turns out that the ζ_j are the reciprocals of the poles of $F_{n-1,n}(z)$ if $F_{n-1,n}(z)$ has simple poles.

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Prony's method is discussed in several books on numerical analysis, e.g. Hildebrand [3, pp. 378–386], Lanczos [6, pp. 272–280], Hamming [2, pp. 620–627]. Some theoretical aspects of the interpolation problem, when the function $f(x)$ is completely monotonic, have been dealt with by Kammler [4].

Prony's method cannot be applied if $F_{n-1,n}(z)$ has multiple poles. In this case it turns out that $u(x)$ as given in (1.1) does not exist. The interpolation problem, however, may have a solution provided $u(x)$ is modified in an appropriate manner, and this is the subject of the present work. With this modification we give a set of necessary and sufficient conditions for the existence and uniqueness of the solution to the interpolation problem, and simultaneously give a simple method for constructing it. Later we do the same for the confluent interpolation problem, in which all the points of interpolation coincide. Under certain cnditions, we prove that the solution to the confluent interpolation problem is the limit of that of interpolation at equidistant points when the distance between them tends to zero.

Since both problems are ultimately connected with Padé approximants, we start with them.

2. Padé Approximants

Let

$$g(z) = \sum_{i=0}^{\infty} c_i z^i \tag{2.1}$$

be a formal power series. The (m/n) entry in the Padé table of (2.1), if it exists, is defined as the rational function

$$g_{m,n}(z) = \frac{P_{m,n}(z)}{Q_{m,n}(z)} = \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{i=0}^{n} b_i z^i}, \qquad b_0 = 1, \tag{2.2}$$

such that the Maclaurin series expansion of $g_{m,n}(z)$ in (2.2) agrees with the formal power series in (2.1) up to and including the term $c_{m+n} z^{m+n}$, i.e.,

$$g(z) - g_{m,n}(z) = O(z^{m+n+1}) \qquad \text{as} \quad z \to 0. \tag{2.3}$$

For the subject of the Padé table as defined above, see Baker [1, Chaps. 1, 2].

It can be verified that for there to be a solution it is necessary and sufficient that the equations

$$\sum_{j=0}^{\min(i,n)} c_{i-j} b_j = 0, \qquad i = m+1, \ldots, m+n, \tag{2.4}$$

have a solution with $b_0 = 1$. Once the b_j have been determined, the a_i can be computed from

$$a_i = \sum_{j=0}^{\min(i,n)} c_{i-j} b_j, \qquad i = 0, 1,..., m. \tag{2.4a}$$

Although $P_{m,n}(z)$ and $Q_{m,n}(z)$ in (2.2) may be non-unique, the fraction $g_{m,n}(z)$ is unique, as stated in the following theorem.

THEOREM 2.1. *If $g_{m,n}(z)$ exists, it is unique.* ∎

For a proof of this see Baker [1, p. 8].

We shall now concentrate on the approximations $g_{n-1,n}(z)$ since these are relevant to the problem of interpolation described in the previous section.

DEFINITION 2.1. A rational function $v(z)$ is said to have property R if its numerator polynomial has degree strictly .less than that of its denominator polynomial, i.e., if $\lim_{z\to\infty} v(z) = 0$.

If $g_{n-1,n}(z)$ has property R, then after cancelling common factors from the numerator and denominator, we can express $g_{n-1,n}(z)$ as

$$g_{n-1,n}(z) = \frac{\bar{P}(z)}{\bar{Q}(z)}, \tag{2.5}$$

where the degree of $\bar{P}(z)$ is strictly less than that of $\bar{Q}(z)$, and the degree of $\bar{Q}(z)$ is n', for some $n' \leqslant n$. Let $z_1,..., z_s$ be the zeros of $\bar{Q}(z)$ of multiplicities $\mu_1,..., \mu_s$ respectively, so that $\sum_{j=1}^{s} \mu_j = n'$. Since $b_0 = 1$, $z_j \neq 0$ for all j. Then, for some constants $A_{j,k}$, $1 \leqslant k \leqslant \mu_j$, $1 \leqslant j \leqslant s$, $g_{n-1,n}(z)$ has the following unique partial fraction expansion:

$$g_{n-1,n}(z) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} \frac{A_{j,k}}{(z - z_j)^k}. \tag{2.6}$$

THEOREM 2.2. *Let $g(z)$ and its $(n - 1/n)$ Padé approximant $g_{n-1,n}(z)$ be given by* (2.1) and (2.6), *respectively. Then the parameters z_j and $A_{j,k}$ in* (2.6) *satisfy the set of non-linear equations*

$$c_i = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} (-1)^k \binom{k+i-1}{k-1} \frac{A_{j,k}}{z_j^{k+i}}, \qquad i = 0, 1,..., 2n - 1, \tag{2.7}$$

where

$$\binom{t}{0} = 1, \qquad \binom{t}{r} = \frac{t(t-1)\cdots(t-r+1)}{r!}, \qquad r = 1, 2,.... \tag{2.8}$$

Proof. Expanding $g_{n-1,n}(z)$ as, given in (2.6), in its Maclaurin series, we obtain

$$g_{n-1,n}(z) = \sum_{i=0}^{\infty} \left\{ \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} (-1)^{i+k} \binom{-k}{i} \frac{A_{j,k}}{z_j^{i+k}} \right\} z^i. \tag{2.9}$$

Upon using the fact that

$$\binom{-k}{i} = (-1)^i \binom{k+i-1}{i} = (-1)^i \binom{k+i-1}{k-1} \tag{2.10}$$

in (2.9), and recalling (2.3), we obtain (2.7). ∎

We now state the converse of Theorem 2.2.

THEOREM 2.3. *Let $c_0, c_1, \ldots, c_{2n-1}$ be given numbers. Let s and μ_j, $j = 1, \ldots, s$, be positive integers such that $\sum_{j=1}^{s} \mu_j \leqslant n$. Suppose that $z_j \neq 0$, $A_{j,k}$, $1 \leqslant k \leqslant \mu_j$, $1 \leqslant j \leqslant s$, are the solution to the set of non-linear equations given in* (2.7).

Then the rational function

$$v(z) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} \frac{A_{j,k}}{(z - z_j)^k} \tag{2.11}$$

is simply the $(n - 1/n)$ Padé approximant to the power series $\sum_{i=0}^{2n-1} c_i z^i$. Consequently, this Padé approximant has property R.

Proof. Identical to that of Theorem 2.2. ∎

COROLLARY. *Given $c_0, c_1, \ldots, c_{2n-1}$, there is at most one choice of the integers $\mu_1, \ldots, \mu_s$ with $\sum_{j=1}^{s} \mu_j \leqslant n$, and the parameters $z_j \neq 0$, $A_{j,k}$, $1 \leqslant k \leqslant \mu_j$, $1 \leqslant j \leqslant s$, which satisfy* (2.7).

Proof. Suppose that there is more than one choice. This implies the existence of more than one $(n - 1/n)$ Padé approximant to $\sum_{i=0}^{2n-1} c_i z^i$, according to Theorem 2.2. The result now follows from Theorem 2.1 and the uniqueness of the partial fraction decomposition of rational functions. ∎

We note that whether the Padé approximant $g_{n-1,n}(z)$ exists and has property R can be decided by analyzing the C-table of (2.1), and this is connected with the normality property and the block structure of the Padé table. Necessary conditions for $g_{n-1,n}(z)$ to exist and to have property R can be formulated in terms of the C-determinants, but we shall not do this here. We shall only state that a sufficient condition for $g_{n-1,n}(z)$ to have property R is that the Padé table of $g(z)$ should be normal. For the definition of normality, the C-table, and the block structure of the Padé table see Baker [1, Chap. 2].

3. Solution of the Interpolation Problem

DEFINITION 3.1. The sets of functions U_n and U_n^h for $h \neq 0$ are defined by

$$U_n = \left\{ u(x) = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} B_{j,k} x^{k-1} e^{\sigma_j x} \mid \sigma_j \text{ distinct}, \sum_{j=1}^{r} \lambda_j \leqslant n \right\},$$

$$U_n^h = \left\{ u(x) = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} B_{j,k} x^{k-1} \zeta_j^{x/h} \mid \zeta_j \text{ distinct}, -\pi < \arg \zeta_j \leqslant \pi, \right.$$

$$\left. \zeta_j^y \text{ takes on its principal value}, \sum_{j=1}^{r} \lambda_j \leqslant n \right\}. \tag{3.1}$$

Clearly $U_n^h \subset U_n$.

THEOREM 3.1. *There exists a unique function $u(x)$ in U_n^h, which solves the interpolation problem*

$$c_i = u(x_0 + ih), \qquad i = 0, 1,..., 2n-1, \text{ for some } h \neq 0, \tag{3.2}$$

if and only if the $(n-1/n)$ Padé approximant $F_{n-1,n}(z)$ to $F(z) = \sum_{i=0}^{2n-1} c_i z^i$ exists and has property R. If $F_{n-1,n}(z)$ has the partial fraction decomposition

$$F_{n-1,n}(z) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} \frac{A_{j,k}}{(z - z_j)^k}, \tag{3.3}$$

then

$$u(x) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} E_{j,k} \binom{k + \frac{x - x_0}{h} - 1}{k-1} \zeta_j^{(x-x_0)/h}, \tag{3.4}$$

where

$$\zeta_j = z_j^{-1}, \qquad E_{j,k} = (-1)^k A_{j,k} z_j^{-k}, \qquad 1 \leqslant k \leqslant \mu_j, \ 1 \leqslant j \leqslant s. \tag{3.5}$$

Proof. Suppose that $F_{n-1,n}(z)$ exists, and has property R. Then $F_{n-1,n}(z)$ has a partial fraction decomposition; assume it is the one given in (3.3). We would like to show that $u(x)$ in (3.4)–(3.5) solves the interpolation problem. Now $\binom{at+b}{m}$ is a polynomial of degree m in t, as is seen from Eq. (2.8). Consequently $u(x)$ is in U_n^h. Let us substitute $x = x_0 + ih$, $i = 0, 1,..., 2n-1$, in (3.4). Using (3.5), we obtain

$$u(x_0 + ih) = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} (-1)^k \binom{k+i-1}{k-1} \frac{A_{j,k}}{z_j^{k+i}}, \qquad i = 0, 1,..., 2n-1. \tag{3.6}$$

But the z_j and $A_{j,k}$, being the parameters of the partial fraction decomposition of $F_{n-1,n}(z)$, by Theorem 2.2, satisfy the equations

$$c_i = \sum_{j=1}^{s} \sum_{k=1}^{\mu_j} (-1)^k \binom{k+i-1}{k-1} \frac{A_{j,k}}{z_j^{k+i}}, \qquad i = 0, 1, \ldots, 2n-1. \tag{3.7}$$

Thus $u(x)$ satisfies (3.2).

Suppose now that there exists $u(x)$ in U_n^h that satisfies equations (3.2); assume that it is given by (3.4). Define the parameters z_j and $A_{j,k}$, $1 \leqslant k \leqslant \mu_j$, $1 \leqslant j \leqslant s$, through Eqs. (3.5). Equations (3.2) imply that the z_j and $A_{j,k}$ satisfy Eqs. (3.7). Invoking now Theorem 2.3, we conclude that $F_{n-1,n}(z)$ exists and has property R.

As for the uiqueness of $u(x)$, only one set of ζ_j's and $E_{j,k}$'s can exist, since from the corollary to Theorem 2.2, only one set of z_j's and $A_{j,k}$'s can satisfy Eqs. (3.7). This completes the proof. ∎

Note. There does not exist a unique solution to the interpolation problem above from U_n. For if $u(x)$ in (3.4) is a solution from U_n, then $\bar{u}(x)$, which is obtained from $u(x)$ by adding to $\arg \zeta_j$ in (3.4) arbitrary multiplies of $2\pi i$, is also a solution. Also when $\sum_{j=1}^{s} \mu_j \leqslant n-2$, we can add to $u(x)$ in (3.4) $u_1(x) = C \sin(m\pi(x-x_0)/h)$, where C is an arbitrary constant and m is an arbitrary integer, and $u(x) + u_1(x)$ solves the interpolation problem, and is in U_n.

The method of construction of $u(x)$ that satisfies the interpolation conditions in (3.2) is now clear. First we obtain the $(n-1/n)$ Padé approximant of $F(z) = \sum_{i=0}^{2n-1} c_i z^i$ in its reduced form (and make sure that it has property R), then find its poles and form its partial fraction decomposition, and finally form the sum in (3.4) with the help of the relations between the ζ_j and z_j and $E_{j,k}$ and $A_{j,k}$.

COROLLARY. *If $c_0, c_1, \ldots, c_{2n-1}$ are real numbers, then $u(x)$ in Theorem 3.1 is a real function, provided none of the z_j is in $(-\infty, 0]$.*

Proof. Since $c_0, c_1, \ldots, c_{2n-1}$ are real, the Padé approximant $F_{n-1,n}(z)$ is a real analytic function. Hence if z_j is a complex pole of multiplicity μ_j, so is its complex conjugate. Let z_p be the complex conjugate of z_j. Then $A_{p,k}$ is the complex conjugate of $A_{j,k}$, $1 \leqslant k \leqslant \mu_j = \mu_p$. If, on the other hand, z_j is a real pole of multiplicity μ_j, then the $A_{j,k}$, $1 \leqslant k \leqslant \mu_j$, are all real. The rest of the proof now is obvious. ∎

4. THE CONFLUENT PROBLEM

So far we have considered the problem of interpolating a function $f(x)$ by functions from U_n^h at equidistant points. We now turn to the confluent problem in which the points of interpolation coincide, i.e., $x_0 = x_1 = \cdots = x_{2n-1}$. Then the problem is to find a function $v(x)$ in U_n such that

$$\gamma_i = f^{(i)}(x_0) = v^{(i)}(x_0), \qquad i = 0, 1, \ldots, 2n-1. \tag{4.1}$$

It turns out that the solution to this problem too is closely connected with Padé approximants as the following theorem shows.

THEOREM 4.1. *There exists a unique function $v(x)$ in U_n, which solves the problem described by Eqs.* (4.1), *if and only if the* $(n-1/n)$ *Padé approximant $V_{n-1,n}(\tau)$ to the power series $V(\tau) = \sum_{i=0}^{2n-1} \gamma_i \tau^i$ exists. When $V_{n-1,n}(\tau)$ exists, $\bar{V}_{n-1,n}(\sigma) = \sigma^{-1} V_{n-1,n}(\sigma^{-1})$ has property R; let its partial fraction decomposition be*

$$\bar{V}_{n-1,n}(\sigma) = \sum_{j=1}^{r} \sum_{k=1}^{\nu_j} \frac{B_{j,k}}{(\sigma - \sigma_j)^k}. \tag{4.2}$$

Then $v(x)$, the solution to the confluent interpolation problem, is given by

$$v(x) = \sum_{j=1}^{r} \sum_{k=1}^{\nu_j} \frac{B_{j,k}}{(k-1)!} (x - x_0)^{k-1} e^{\sigma_j (x - x_0)}. \tag{4.3}$$

Proof. Suppose that $V_{n-1,n}(\tau)$ exists. Consequently, $V_{n-1,n}(0)$ is finite. Then it can easily be verified that $\bar{V}_{n-1,n}(\sigma)$ has property R, and its denominator polynomial has degree at most n. Let its partial fraction decomposition be the one given in (4.2). Define $\bar{V}(\sigma) = \sigma^{-1} V(\sigma^{-1})$. With the help of (2.3) it can be shown that

$$\bar{V}(\sigma) - \bar{V}_{n-1,n}(\sigma) = O(\sigma^{-2n-1}) \qquad \text{as} \quad \sigma \to \infty. \tag{4.4}$$

As a consequence of (4.4), the parameters σ_j and $B_{j,k}$ in (4.2) satisfy the equations

$$\gamma_i = \sum_{j=1}^{r} \sum_{k=1}^{\nu_j} B_{j,k} \binom{i}{k-1} \sigma_j^{i-k+1}, \quad i = 0, 1, \ldots, 2n-1. \tag{4.5}$$

Now we want show that $v(x)$ as given in (4.3) is the required solution. First of all, it is clear that $v(x)$ is in U_n. Next, after some algebra and with the help of Eqs. (4.5), it can be shown that $v(x)$ satisfies Eqs. (4.1).

It is now possible to prove results similar to Theorem 2.3 and its corollary. With these results the rest of the theorem can be proved in a manner analogous to Theorem 3.1. We omit the details. ∎

COROLLARY. *If $\gamma_0, \gamma_1, \ldots, \gamma_{2n-1}$ are real numbers, then $v(x)$ in Theorem* 4.1, *if it exists, is a real function.*

Proof. Identical to that of the corollary to Theorem 3.1.

5. Integral Representations and a Limit Theorem

We now give some contour integral representations for the solutions to the two interpolation problems that were dealt with in the previous sections.

THEOREM 5.1. *Let the function $u(x)$ be as in Theorem 3.1. Then $u(x)$ has the integral representation*

$$u(x) = -\frac{1}{2\pi i}\int_C F_{n-1,n}(z)\, z^{-\omega}\, dz, \tag{5.1}$$

where C is a simple closed Jordan curve whose interior contains all the poles of $F_{n-1,n}(z)$ and such that it never touches the line $(-\infty, 0]$, $\omega = (x - x_0)/h + 1$, and $z^{-\omega}$ takes on its principal value and has a branch cut along the line $(-\infty, 0]$.

Proof. Substituting the partial fraction decomposition of $F_{n-1,n}(z)$ in (5.1), we have

$$u(x) = -\frac{1}{2\pi i}\int_C \sum_{j=1}^{s}\sum_{k=1}^{\mu_j} \frac{A_{j,k}}{(z - z_j)^k}\, z^{-\omega}\, dz. \tag{5.2}$$

Now

$$\begin{aligned} z^{-\omega} &= [(z - z_j) + z_j]^{-\omega} \\ &= z_j^{-\omega}[1 + (z - z_j)/z_j]^{-\omega} \\ &= z_j^{-\omega} \sum_{i=0}^{\infty} \binom{-\omega}{i} \left(\frac{z - z_j}{z_j}\right)^i, \end{aligned} \tag{5.3}$$

where the last equality holds when z is sufficiently close to z_j. Substituting (5.3) in (5.2), and using the residue theorem, we obtain

$$u(x) = -\sum_{j=1}^{s}\sum_{k=1}^{\mu_j} A_{j,k} z_j^{-\omega} \binom{-\omega}{k-1} z_j^{-(k-1)}, \tag{5.4}$$

which, upon using (2.10) and the fact that $\omega = (x - x_0)/h + 1$, reduces to (3.4) – (3.5). ∎

THEOREM 5.2. *The function $v(x)$ in Theorem 4.1 has the integral representation*

$$v(x) = \frac{1}{2\pi i} \int_D \bar{V}_{n-1,n}(\sigma)\, e^{\sigma(x-x_0)}\, d\sigma, \tag{5.5}$$

where D is a simple closed Jordan curve whose interior contains all the poles of $\bar{V}_{n-1,n}(\sigma)$.

Proof. Substituting the partial fraction decomposition of $\bar{V}_{n-1,n}(\sigma)$ in (5.5) and computing residues, (5.5) can be shown to be identical to (4.3). ∎

The following determinant representations for $F_{n-1,n}(z)$ and $\bar{V}_{n-1,n}(\sigma)$ will be useful in the remainder of this section.

THEOREM 5.3. *If $F_{n-1,n}(z)$ exists, then it is given by*

$$F_{n-1,n}(z) = \frac{\begin{vmatrix} z^n S_{-1}(z) & z^{n-1} S_0(z) & \cdots & z S_{n-2}(z) & z^0 S_{n-1}(z) \\ c_0 & c_1 & \cdots & c_{n-1} & c_n \\ c_1 & c_2 & \cdots & c_n & c_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} & c_{2n-1} \end{vmatrix}}{\begin{vmatrix} z^n & z^{n-1} & \cdots & z & 1 \\ c_0 & c_1 & \cdots & c_{n-1} & c_n \\ c_1 & c_2 & \cdots & c_n & c_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} & c_{2n-1} \end{vmatrix}}, \tag{5.6}$$

where

$$S_{-1}(z) = 0, \qquad S_k = \sum_{i=0}^{k} c_i z^i, \qquad k = 0, 1, \ldots, \tag{5.7}$$

provided the cofactor of 1 in the first row of the denominator determinant is non-zero. This will be the case, for example, when the denominator polynomial of $F_{n-1,n}(z)$ has degree exactly n, and, apart from a constant, has no common factor with the numerator polynomial; in this case $F_{n-1,n}(z)$ has property R too. ∎

For a proof of (5.6) see Baker [1].

COROLLARY. *If $V_{n-1,n}(\tau)$ exists, then $\bar{V}_{n-1,n}(\sigma)$ is given by*

$$\bar{V}_{n-1,n}(\sigma) = \frac{\begin{vmatrix} 0 & \sigma^0 T_0(\sigma) & \cdots & \sigma^{n-2} T_{n-2}(\sigma) & \sigma^{n-1} T_{n-1}(\sigma) \\ \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} & \gamma_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n & \gamma_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \gamma_{n-1} & \gamma_n & \cdots & \gamma_{2n-2} & \gamma_{2n-1} \end{vmatrix}}{\begin{vmatrix} 1 & \sigma & \cdots & \sigma^{n-1} & \sigma^n \\ \gamma_0 & \gamma_1 & \cdots & \gamma_{n-1} & \gamma_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n & \gamma_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \gamma_{n-1} & \gamma_n & \cdots & \gamma_{2n-2} & \gamma_{2n-1} \end{vmatrix}}, \tag{5.8}$$

where

$$T_k(\sigma) = \sum_{i=0}^{k} \gamma_i/\sigma^i, \qquad k = 0, 1, \ldots, \tag{5.9}$$

provided the cofactor of σ^n in the first row of the denominator determinant is non-zero. This will be the case, for example, when the denominator polynomial of $\bar{V}_{n-1,n}(\sigma)$ has degree exactly n, and, apart from a constant, has no common factors with the numerator polynomial; in this case $\bar{V}_{n-1,n}(\sigma)$ has property R too. ∎

THEOREM 5.4. *Let $\psi(\sigma)$ be the polynomial given by*

$$\psi(\sigma) = \begin{vmatrix} 1 & \sigma & \cdots & \sigma^n \\ \gamma_0 & \gamma_1 & \cdots & \gamma_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{n+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n-1} & \gamma_n & \cdots & \gamma_{2n-1} \end{vmatrix}, \tag{5.10}$$

and suppose that $\psi(\sigma)$ has exactly n zeros, which we denote by $\bar{\sigma}_i$, $i = 1, \ldots, n$, counting multiplicities. Let $Q(z; h)$ be the polynomial in z given by

$$Q(z; h) = \begin{vmatrix} z^n & z^{n-1} & \cdots & 1 \\ c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \end{vmatrix}. \tag{5.11}$$

Then for h sufficiently close to zero, $Q(z; h)$ has exactly n zeros, which we denote by $\bar{z}_i(h)$, $i = 1, \ldots, n$, counting multiplicities, with the property that $\bar{z}_i(h)$

are continuous in a neighborhood of $h = 0$, *and differentiable at* $h = 0$, *such that*

$$\bar{z}_i(0) = 1, \qquad i = 1,..., n, \tag{5.12}$$

and

$$\left.\frac{d\bar{z}_i(h)}{dh}\right|_{h=0} = -\bar{\sigma}_i, \qquad i = 1,..., n, \tag{5.13}$$

with proper ordering.

Proof. Let us define the forward difference operators Δ^k by $\Delta^0 a_r = a_r$, $\Delta^k a_r = \Delta^{k-1} a_{r+1} - \Delta^{k-1} a_r$, $k = 1, 2,....$ Then it is known that

$$\Delta^k a_r = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} a_{r+i}. \tag{5.14}$$

If $a_r = z^{n-r}$, $r = 0, 1,..., n$, then

$$\Delta^k a_0 = z^{n-k}(1-z)^k, \qquad k = 0, 1,.... \tag{5.15}$$

By simple row transformations, with the help of (5.14), it is easy to obtain from (5.11)

$$Q(z; h) = \begin{vmatrix} z^n & z^{n-1} & \cdots & z & 1 \\ c_0 & c_1 & \cdots & c_{n-1} & c_n \\ \Delta c_0 & \Delta c_1 & \cdots & \Delta c_{n-1} & \Delta c_n \\ \vdots & \vdots & & \vdots & \vdots \\ \Delta^{n-1} c_0 & \Delta^{n-1} c_1 & \cdots & \Delta^{n-1} c_{n-1} & \Delta^{n-1} c_n \end{vmatrix}. \tag{5.16}$$

By simple column transformations, and using (5.15), (5.16) can be expressed as

$$Q(z, h) = \begin{vmatrix} z^n & z^{n-1}(1-z) & \cdots & z(1-z)^{n-1} & (1-z)^n \\ c_0 & \Delta c_0 & \cdots & \Delta^{n-1} c_0 & \Delta^n c_0 \\ \Delta c_0 & \Delta^2 c_0 & \cdots & \Delta^n c_0 & \Delta^{n+1} c_0 \\ \vdots & \vdots & & \vdots & \vdots \\ \Delta^{n-1} c_0 & \Delta^n c_0 & \cdots & \Delta^{2n-2} c_0 & \Delta^{2n-1} c_0 \end{vmatrix}. \tag{5.17}$$

Dividing the jth column by $h^{j-1}, j = 1, 2,..., n+1$, then dividing the first row by z^n, and the ith row by h^{i-2}, $i = 2,..., n+1$, and defining

$$\delta_p = \Delta^p c_0 / h^p, \qquad p = 0, 1,..., 2n-1, \tag{5.18}$$

and η by

$$\eta = (1-z)/(hz), \tag{5.19}$$

we obtain from (5.17)

$$Q(z;h)=h^{n^2}z^n\phi(\eta;h), \tag{5.20}$$

where $\phi(\eta;h)$ is the polynomial defined by

$$\phi(\eta;h)=\begin{vmatrix} 1 & \eta & \cdots & \eta^n \\ \delta_0 & \delta_1 & \cdots & \delta_n \\ \delta_1 & \delta_2 & \cdots & \delta_{n+1} \\ \vdots & \vdots & & \vdots \\ \delta_{n-1} & \delta_n & \cdots & \delta_{2n-1} \end{vmatrix}. \tag{5.21}$$

Recalling that $c_i=f(x_0+ih)$, $\gamma_i=f^{(i)}(x_0)$, $i=0,1,\ldots,2n-1$, and that $f(x)$ has $2n-1$ continuous derivatives in a neighborhood of x_0, we have, from a well known result on divided differences,

$$\delta_p=\gamma_p+\varepsilon_p, \qquad \text{with} \quad \varepsilon_p=o(1) \text{ as } h\to 0,\ p=0,1,\ldots,2n-1. \tag{5.22}$$

Therefore, by continuity, the coefficients of the polynomial $\phi(\eta;h)=\sum_{i=0}^n a_i(h)\,\eta^i$ are related to those of $\psi(\sigma)=\sum_{i=0}^n b_i\sigma^i$, by $a_i(h)=b_i+o(1)$ as $h\to 0$. Since $\psi(\sigma)$ has exactly n zeros, $b_n\neq 0$; consequently, by continuity, for h sufficiently close to zero, $a_n(h)\neq 0$, hence $\phi(\eta;h)$ has exactly n zeros, $\bar{\eta}_i(h)$, $i=1,\ldots,n$, which can be ordered so that

$$\bar{\eta}_i(h)=\bar{\sigma}_i+o(1) \qquad \text{as} \quad h\to 0,\ i=1,\ldots,n, \tag{5.23}$$

so that $\bar{\eta}_i(h)$ are continuous in a neighborhood of $h=0$. Let us now define

$$\bar{\zeta}_i(h)=\frac{1}{1+h\bar{\eta}_i(h)}, \qquad i=1,\ldots,n. \tag{5.24}$$

The $\bar{\zeta}_i(h)$ are also continuous in a neighborhood of $h=0$, and satisfy

$$\bar{\zeta}_i(0)=1, \qquad i=1,\ldots,n. \tag{5.25}$$

Solving (5.24) for $\bar{\eta}_i(h)$, we obtain

$$\bar{\eta}_i(h)=\frac{1-\bar{\zeta}_i(h)}{h\bar{\zeta}_i(h)}, \qquad i=1,\ldots,n. \tag{5.26}$$

Since $\lim_{h\to 0}\bar{\eta}_i(h)=\bar{\sigma}_i$ and $\lim_{h\to 0}\bar{\zeta}_i(h)=1\neq 0$, we see from (5.26) that $d\bar{\zeta}_i(h)/dh|_{h=0}$ exists and is given by

$$\left.\frac{d\bar{\zeta}_i(h)}{dh}\right|_{h=0}=-\bar{\sigma}_i, \qquad i=1,\ldots,n. \tag{5.27}$$

Writing now $\phi(\eta; h) = a_n(h) \prod_{i=1}^{n} (\eta - \bar{\eta}_i(h))$, and substituting (5.19) and (5.26) in this expression, we obtain from (5.20)

$$Q(z; h) = a_n(h)\, h^{n^2-n} \left[\prod_{i=1}^{n} \bar{\zeta}_i(h)\right]^{-1} \prod_{i=1}^{n} (\bar{\zeta}_i(h) - z). \tag{5.28}$$

Hence whenever h is sufficiently close to zero, $Q(z; h)$ has exactly n zeros $\bar{z}_i(h)$, and these are simply $\bar{\zeta}_i(h)$, $i = 1,\ldots, n$.

This completes the proof of the theorem. ∎

Collecting the results of Theorem 3.1, Theorem 4.1, Theorem 5.3 and its corollary, and Theorem 5.4, we can state the following result:

THEOREM 5.5. *Let the denominator polynomial of $\bar{V}_{n-1,n}(\sigma)$ be of degree exactly n, and assume that, apart from a constant, it has no common factors with the numerator polynomial. Then*

(1) *$\bar{V}_{n-1,n}(\sigma)$ has property R and is given by (5.8)–(5.9); i.e., its denominator is $\psi(\sigma)$ in Theorem 5.4. Consequently $v(x)$ exists.*

(2) *For h sufficiently close to zero, $F_{n-1,n}(z)$ exists, has property R, and has no poles in $(-\infty, 0]$, and is given by (5.6)–(5.7); i.e., its denominator is $Q(z; h)$ in Theorem 5.4. Consequently, $u(x)$ exists and we now denote it by $u(x; h)$.*

(3) *The following limit result is true:*

$$\lim_{h \to 0} u(x; h) = v(x). \tag{5.29}$$

Proof. The proofs of (1) and (2) are trivial and we shall omit them. For the proof of (3) we proceed as follows: Let us apply the column and row transformations that led from (5.11) to (5.20)–(5.21), to the numerator determinant of $F_{n-1,n}(z)$ as given in (5.6)–(5.7). Then

$$F_{n-1,n}(z) = \frac{z^{-1}}{h} \frac{\rho(\eta; h)}{\phi(\eta; h)}, \tag{5.30}$$

where

$$\rho(\eta; h) = \frac{z^{-n+1}}{h} \begin{vmatrix} 0 & (\Delta \bar{S}_{-1})/h^0 & \cdots & (\Delta^n \bar{S}_{-1})/h^{n-1} \\ \delta_0 & \delta_1 & \cdots & \delta_n \\ \delta_1 & \delta_2 & \cdots & \delta_{n+1} \\ \vdots & \vdots & & \vdots \\ \delta_{n-1} & \delta_n & \cdots & \delta_{2n-1} \end{vmatrix}, \tag{5.31}$$

where

$$\bar{S}_k = z^{n-k-1} S_k(z), \qquad k = -1, 0, \ldots, n-1, \tag{5.32}$$

and (5.19) has been used. In the Appendix to this work it is proved that

$$\frac{z^{-n+1}(\Delta^k \bar{S}_{-1})}{h^{k-1}} = \eta^{k-1} \sum_{i=0}^{k-1} \delta_i/\eta^i. \tag{5.33}$$

Since, for h sufficiently close to zero, all the poles of $F_{n-1,n}(z)$ are close to 1, Theorem 5.1 applies and $u(x; h)$ is given by (5.1), where C can be taken to be a circle with its center at $z = 1$ and radius $r < 1$. Let us now make the change of variable $z = e^{-\sigma h}$ in (5.1). Then, with the help of (5.30), (5.1) becomes

$$u(x; h) = \frac{1}{2\pi i} \int_{\bar{C}} e^{\sigma h} \frac{\rho(\eta; h)}{\phi(\eta; h)} e^{\sigma(x - x_0)} \, d\sigma, \tag{5.34}$$

where η is given in terms of σ by

$$\eta = \frac{1 - z}{hz} = \frac{e^{\sigma h} - 1}{h}. \tag{5.35}$$

The contour $\bar{C}$ can now be taken to be a fixed simple closed Jordan curve which contains in its interior all the poles of $\bar{V}_{n-1,n}(\sigma)$, since the poles of $\rho(\eta; h)/\phi(\eta; h)$ in the σ-plane are approaching those of $\bar{V}_{n-1,n}(\sigma)$ as $h \to 0$. Furthermore, $\bar{C}$ is positively oriented. Now for σ fixed

$$\lim_{h \to 0} \frac{\rho(\eta; h)}{\phi(\eta; h)} = \bar{V}_{n-1,n}(\sigma), \tag{5.36}$$

since $\delta_i \to \gamma_i$, $i = 0, 1, \ldots, 2n-1$, and $\eta \to \sigma$, and hence $z^{-n+1}(\Delta^k \bar{S}_{-1}/h^{k-1}) \to \sigma^{k-1} T_{k-1}(\sigma)$, $k = 1, \ldots, n$, as $h \to 0$, and these limits are attained uniformly in σ on $\bar{C}$.

Consequently, from (5.34)

$$\lim_{h \to 0} u(x; h) = \frac{1}{2\pi i} \int_{\bar{C}} \lim_{h \to 0} \left[e^{\sigma h} \frac{\rho(\eta; h)}{\phi(\eta; h)} \right] e^{\sigma(x - x_0)} \, d\sigma, \tag{5.37}$$

the right hand side being nothing but $v(x)$. This completes the proof of the Theorem. ∎

APPENDIX: Proof of (5.33)

It is sufficient to show that

$$z^{-n+1}\Delta^k\bar{S}_{-1}=\sum_{i=0}^{k-1}(\Delta^i c_0)(z^{-1}-1)^{k-i-1}$$
$$=\sum_{i=0}^{k-1}(\Delta^{k-i-1}c_0)(z^{-1}-1)^i. \tag{A.1}$$

Now, making use of (5.14), we have

$$z^{-n+1}\Delta^k\bar{S}_{-1}=\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\sum_{j=0}^{i-1}c_j z^{-(i-j)+1}. \tag{A.2}$$

Making the substitution $i-j=p+1$, and changing the order of summation, (A.2) becomes

$$z^{-n+1}\Delta^k\bar{S}_{-1}=\sum_{p=0}^{k-1}z^{-p}\left[\sum_{i=p+1}^{k}(-1)^{k-i}\binom{k}{i}c_{i-p-1}\right]. \tag{A.3}$$

Substituting the identity

$$z^{-p}=[1+(z^{-1}-1)]^p=\sum_{r=0}^{p}\binom{p}{r}(z^{-1}-1)^r \tag{A.4}$$

in (A.3), and changing the order of summation, we obtain

$$z^{-n+1}\Delta^k\bar{S}_{-1}=\sum_{r=0}^{k-1}(z^{-1}-1)^r\left[\sum_{p=r}^{k-1}\binom{p}{r}\sum_{i=p+1}^{k}(-1)^{k-i}\binom{k}{i}c_{i-p-1}\right]. \tag{A.5}$$

All we have to show now is that the double sum inside the square brackets, which we now denote by D, is just $\Delta^{k-r-1}c_0$. Making the substitution $q=i-p-1$, and changing the order of summation, this double sum becomes

$$D=\sum_{q=0}^{k-1-r}c_q\left[\sum_{p=r}^{k-1-q}\binom{p}{r}\binom{k}{q+p+1}(-1)^{k-p-q-1}\right]. \tag{A.6}$$

The proof of (5.33) will be complete if we show that

$$A_{r,q}=\sum_{p=r}^{k-1-q}\binom{p}{r}\binom{k}{q+p+1}(-1)^{p-r}=\binom{k-1-r}{q},$$
$$0\leqslant q\leqslant k-1-r, \qquad 0\leqslant r\leqslant k-1. \tag{A.7}$$

Let us define for any a

$$\binom{a}{m} = 0, \qquad m \text{ a negative integer.} \tag{A.8}$$

Then it is well known that

$$\binom{a+1}{m} = \binom{a}{m} + \binom{a}{m-1} \qquad \text{for all integers } m. \tag{A.9}$$

LEMMA A.1. *The $A_{r,q}$ satisfy the 3-term recursion relation*

$$A_{r,q} = A_{r+1,q} + A_{r+1,q-1}, \qquad 0 \leqslant q \leqslant k-1-r, \quad 0 \leqslant r \leqslant k-1. \tag{A.10}$$

Proof. From (A.7), $A_{r+1,q-1}$ is given by

$$A_{r+1,q-1} = \sum_{p=r+1}^{k-q} \binom{p}{r+1}\binom{k}{p+q}(-1)^{p-r-1}. \tag{A.11}$$

Making the substitution $p = m+1$, (A.11) becomes

$$A_{r+1,q-1} = \sum_{m=r}^{k-q-1} \binom{m+1}{r+1}\binom{k}{m+q+1}(-1)^{m-r}. \tag{A.12}$$

From (A.9) we have

$$\binom{m+1}{r+1} = \binom{m}{r+1} + \binom{m}{r}. \tag{A.13}$$

Substituting (A.13) in (A.12), we obtain

$$\begin{aligned} A_{r+1,q-1} = &\sum_{m=r}^{k-q-1} \binom{m}{r+1}\binom{k}{m+q+1}(-1)^{m-r} \\ &+ \sum_{m=r}^{k-q-1} \binom{m}{r}\binom{k}{m+q+1}(-1)^{m-r}. \end{aligned} \tag{A.14}$$

Now the second sum is just $A_{r,q}$. In the first sum, the first term, i.e., that with $m = r$, is zero since $\binom{r}{r+1} = 0$, hence this sum actually starts with the term $m = r+1$. Consequently, the first sum is nothing but $-A_{r+1,q}$. From these observations (A.10) follows. ∎

LEMMA A.2. *Equation* (A.7) *holds for* $q = k-1-r$, $0 \leqslant r \leqslant k-1$.

Proof. By inspection of (A.7). ∎

LEMMA A.3. *Equation* (A.7) *holds for* $r=0$, $0 \leqslant q \leqslant k-1$.

Proof. We have to prove that

$$\sum_{p=0}^{k-1-q} \binom{k}{q+p+1} (-1)^p = \binom{k-1}{q}, \qquad 0 \leqslant q \leqslant k-1. \tag{A.15}$$

That (A.15) is true follows from the relation

$$\sum_{m=0}^{M} \binom{b}{m} (-1)^m = (-1)^M \binom{b-1}{M}, \qquad \text{for any } b, \tag{A.16}$$

see Knuth [5, p. 57, Eq. (18)]. ∎

From Lemma A.1 it is clear that $A_{r,q}$ satisfy the 3-term recursion relation satisfied by $\binom{k-r-1}{q}$ too. From Lemma A.2 and Lemma A.3 we see that $A_{r,q}$ have the same values as $\binom{k-r-1}{q}$ on two sides of the triangle bounded by the straight lines $r=0$, $q=0$, $r+q=k-1$, in the $q-r$ plane. These boundary values together with the 3-term recursion relation are enough for determining $A_{r,q}$ uniquely at all grid points of the triangle above. This completes the proof of (A.7).

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