

CONVERGING FACTORS FOR SOME ASYMPTOTIC MOMENT SERIES THAT ARISE IN NUMERICAL QUADRATURE

AVRAM SIDI¹

(Received 25 August 1980; revised 22 December 1981)

Abstract

In this work the asymptotic behavior of the partial sums of the divergent asymptotic moment series $\sum_{i=1}^{\infty} \mu_i/z^i$, where μ_i are the moments of the weight functions $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, and $w(x) = x^\alpha E_m(x)$, $\alpha > -1$, $m + \alpha > 0$, on the interval $[0, \infty)$, is analyzed. Expressions for the converging factors are derived. These converging factors form the basis of some very accurate numerical quadrature formulas derived by the author for the infinite range integrals $\int_0^\infty w(x)f(x) dx$ with $w(x)$ as given above.

1. Introduction

Recently a new approach to numerical quadrature has been presented by the author, see [5]. In this approach numerical quadrature formulae $I_k[u]$ for the integral $I[u]$, where

$$\begin{aligned} I[u] &= \int_a^b w(x)u(x) dx, \\ I_k[u] &= \sum_{i=1}^k A_{ki}u(x_{ki}), \end{aligned} \tag{1.1}$$

are derived by forming a sequence of rational approximations $H_k(z)$, $k = 1, 2, \dots$, to the function $H(z)$, where

$$\begin{aligned} H(z) &= \int_a^b \frac{w(x)}{z-x} dx, \\ H_k(z) &= \sum_{i=1}^k \frac{A_{ki}}{z-x_{ki}}. \end{aligned} \tag{1.2}$$

¹ Computer Science Department, Technion-Israel Institute of Technology, Haifa, Israel.
© Copyright Australian Mathematical Society 1982

The function $H(z)$ is analytic in the complex z -plane cut along the interval $[a, b]$.

We shall not deal with the motivation of this approach in this work, as that has been done in detail in [5]. We shall only state that from the motivation in the work above one could conclude heuristically that $I_k[u]$ would be a good approximation to $I[u]$ if $H_k(z)$ is a good approximation to $H(z)$ in the complex z -plane cut along $[a, b]$, and that $I_k[u] \rightarrow I[u]$ quickly as $k \rightarrow \infty$ if $H_k(z) \rightarrow H(z)$ quickly as $k \rightarrow \infty$ in the complex z -plane cut along $[a, b]$.

The rational approximations $H_k(z)$ in [5] are obtained by applying some modified version of the T -transformation of [1] to the moment series of $H(z)$, namely to the series

$$H(z) \sim \sum_{i=1}^{\infty} \frac{\mu_i}{z^i}, \quad \text{as } z \rightarrow \infty, z \notin [a, b], \tag{1.3}$$

where μ_i are the moments of $w(x)$,

$$\mu_i = \int_a^b w(x)x^{i-1}dx, \quad i = 1, 2, \dots \tag{1.4}$$

It can easily be seen that if $[a, b]$ is finite, then the series (1.3) converges for $|z| > \max(|a|, |b|)$. If, however, $[a, b]$ is infinite, like $[0, \infty)$, then (1.3) is a divergent asymptotic series.

As explained in [5] (see also [3] and [4], in which convergence properties of the T -transformation are analyzed), the T -transformation, when applied to the sequence of the partial sums of the infinite series in (1.3), produces very good approximations to $H(z)$, provided that

$$H(z) = \sum_{i=1}^{n-1} \frac{\mu_i}{z^i} + R_n f(n), \tag{1.5}$$

where R_n are numbers related to the moments, and $f(y)$; as a function of the continuous variable y , has an asymptotic expansion of the form

$$f(y) \sim \sum_{i=0}^{\infty} \frac{\beta_i}{y^i} \quad \text{as } y \rightarrow \infty, \tag{1.6}$$

and is infinitely differentiable up to $y = \infty$. We notice that the term $R_n f(n)$ in (1.5) serves as a ‘‘converging factor’’ for the series in (1.3).

For $a = 0, b = 1$, and $w(x) = (1 - x)^\alpha x^\beta (-\log x)^\nu, \alpha + \nu > -1, \beta > -1$, it has been shown in [5] that (1.5) holds with $R_n = 1/(n^{\alpha+\nu+1}z^n)$ and

$$f(y) = y^{\alpha+\nu+1} \int_0^\infty e^{-y't} t^{\alpha+\nu} g(t) dt,$$

where

$$g(t) = [(1 - e^{-t})/t]^\alpha e^{-\beta t} / (1 - e^{-t}/z) = \sum_{i=0}^{\infty} g_i t^i;$$

hence (1.6) is also satisfied with $f(y) \sim \sum_{i=0}^{\infty} g_i(\alpha + \nu + i)!/y^i$ as $y \rightarrow \infty$, for all $z \notin [0, 1]$. (Actually R_n , $f(n)$, $g(t)$, and g_i should be written as $R_n(z)$, $f(n, z)$, $g(t, z)$, and $g_i(z)$. However, z is fixed and we do not perform any operation with respect to z . Hence we omit z without fear of confusion.) This expansion can be obtained by applying Watson's lemma, see [2, page 71], to the integral representation of $f(y)$. Since (1.5) and (1.6) are satisfied, the numerical quadrature formulae for this $w(x)$ turn out to be very accurate. For more details and numerical results see [5].

We note that $f(y)/y$ is the Laplace transform of a function $\varphi(t)$ which is analytic at $t = 0$ and has a Maclaurin series with a finite radius of convergence (that of the Maclaurin series of $g(t)$). This follows from a more general result which is given in the appendix to the present work.

In a recent work, [6], very accurate numerical quadrature formulae for infinite range integrals with $a = 0$, $b = \infty$, and weight functions $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, and $w(x) = x^\alpha E_m(x)$, $\alpha > -1$, $m + \alpha > 0$, where $E_m(x)$ is the exponential integral, have been developed using the approach of [5]. These formulae are based on the results of Section 2 and Section 3 of the present work, which show that also in these cases the functions $H(z)$ satisfy (1.5) and (1.6). Furthermore, for $\text{Re } z < 0$, it is shown that $f(y)/y$ for these cases are Laplace transforms of entire functions $\varphi(t)$, which, we believe, should be of some importance in the convergence analysis of $H_k(z)$ to $H(z)$.

2. The case $w(x) = x^\alpha e^{-x}$, $\alpha > -1$

Let $w(x) = x^\alpha e^{-x}$, $\alpha > -1$. Then from (1.4) we have

$$\mu_i = \int_0^\infty e^{-x} x^{\alpha+i-1} dx = (\alpha + i - 1)!, \quad i = 1, 2, \dots \quad (2.1)$$

By substituting the relation

$$\frac{1}{z-x} = \sum_{i=1}^{n-1} \frac{x^{i-1}}{z^i} + \frac{1}{z^n} \frac{x^{n-1}}{1-x/z} \quad (2.2)$$

and (2.1) in (1.2), we have

$$H(z) = \int_0^\infty \frac{x^\alpha e^{-x}}{z-x} dx = \sum_{i=1}^{n-1} \frac{\mu_i}{z^i} + \frac{J(\alpha + n - 1, z)}{z^n}, \quad (2.3)$$

where

$$J(p, z) = \int_0^\infty \frac{e^{-x} x^p}{1-x/z} dx, \quad (2.4)$$

which is analytic for all z not on the positive real axis.

All we have to analyze then is $J(p, z)$. Now for $\text{Re } z < 0$, we have $\text{Re}[e^{i \arg(-z)}(1 - x/z)] > 0$ whenever $0 \leq x < \infty$. It then follows that

$$\frac{e^{-x}}{1 - x/z} = e^{-z} \int_{\substack{-z \\ (L)}}^{\infty} e^{-\tau(1-x/z)} d\tau, \tag{2.5}$$

where the integral is taken along the straight line path L in the τ -plane, which starts at $\tau = -z$ and extends to $\tau = \infty$, and along which $\arg \tau = \arg(-z)$, see Figure 1. Substituting (2.5) in (2.4), and changing the order of integration, we have

$$J(p, z) = e^{-z} \int_{\substack{-z \\ (L)}}^{\infty} d\tau e^{-\tau} \int_0^{\infty} dx e^{-(\tau/z)x} x^p, \tag{2.6}$$

which, upon using the fact that

$$\int_0^{\infty} e^{-ix} x^q dx = \frac{q!}{i^{q+1}}, \quad q > -1, \quad \text{Re } i > 0, \tag{2.7}$$

becomes

$$J(p, z) = e^{-z} p! (-z)^{p+1} \int_{\substack{-z \\ (L)}}^{\infty} d\tau \frac{e^{-\tau}}{\tau^{p+1}}, \tag{2.8}$$

where τ^s , for s real, is taken to be positive real for τ positive real, with its branch cut along the negative real axis, *i.e.*, $\tau^s = |\tau|^s e^{is \arg(\tau)}$, $|\arg(\tau)| < \pi$, similarly for $(-z)^s$. The change of order of integration in (2.6) is allowed since the integrand is absolutely integrable both at $x = \infty$ and $\tau = \infty e^{i \arg(-z)}$.

Now (2.8) can also be expressed as

$$J(p, z) = e^{-z} p! (-z)^{p+1} \int_{\substack{-z \\ (\Gamma)}}^{\infty} d\tau \frac{e^{-\tau}}{\tau^{p+1}}, \tag{2.9}$$

where the integral this time is taken along the contour Γ that starts at $\tau = -z$ and approaches the real τ -axis asymptotically, see Figure 1. Since the integral in (2.9)

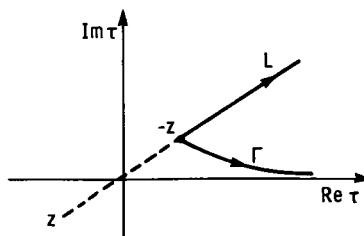


Figure 1. The contours Γ and L in the τ -plane for the case $\text{Re } z < 0$.

converges both when $\text{Re } z < 0$ and $\text{Re } z \geq 0$, by analytic continuation, $J(p, z)$ is given by (2.9) for all z , such that $z \notin [0, \infty)$.

LEMMA 1. Let C be a contour in the complex plane that extends from ξ to infinity and define

$$M[F; \xi] = \int_{\xi}^{\infty} \frac{F(\tau)}{\tau^{q+1}} d\tau. \tag{2.10}$$

(C)

Assume that along C the function $F(\tau)$ is infinitely differentiable and that $(S^i F)(\tau) = O(\tau^q)$ as $\tau \rightarrow \infty$ along C , where the operators S^i are defined as follows: $(SF)(\tau) = \tau F'(\tau)$, $S^0 F = F$, $S^i F = S(S^{i-1} F)$, $i = 1, 2, \dots$. Then for any positive integer N ,

$$M[F; \xi] = \frac{1}{\xi^q} \sum_{i=0}^{N-1} \frac{(S^i F)(\xi)}{q^{i+1}} + \frac{1}{q^N} M[S^N F; \xi]. \tag{2.11}$$

PROOF. Integrating (2.10) by parts, we obtain

$$M[F; \xi] = \frac{1}{\xi^q} \frac{F(\xi)}{q} + \frac{1}{q} M[SF; \xi]. \tag{2.12}$$

Equation (2.11) now follows by repeated application of (2.12) to $S^i F$, $i = 1, 2, \dots, N - 1$.

COROLLARY 2. Let the contour C be such that

$$\int_{\xi}^{\infty} \frac{|d\tau|}{|\tau|^{q+1}} = O(|\xi|^{-q}) \quad \text{as } q \rightarrow \infty, \tag{2.13}$$

(C)

and let $(S^i F)(\tau) = o(1)$ as $\tau \rightarrow \infty$ along C , $i = 0, 1, 2, \dots$. Then $M[F; \xi]$, as $q \rightarrow \infty$, has an asymptotic expansion given by

$$M[F; \xi] \sim \frac{1}{\xi^q} \sum_{i=0}^{\infty} \frac{(S^i F)(\xi)}{q^{i+1}}. \tag{2.14}$$

PROOF. It is enough to show that in (2.11)

$$\xi^q M[S^N F; \xi] = O(q^{-1}) \quad \text{as } q \rightarrow \infty. \tag{2.15}$$

Now from (2.12)

$$\xi^q M[S^N F; \xi] = \frac{(S^N F)(\xi)}{q} + \frac{1}{q} \xi^q M[S^{N+1} F; \xi]. \tag{2.16}$$

Since $(S^i F)(\tau) = o(1)$ as $\tau \rightarrow \infty$ along C for any i , there exist finite positive constants λ_i such that $\lambda_i = \max_{\tau \in C} |(S^i F)(\tau)|$, $i = 0, 1, \dots$. Substituting in

(2.16) the integral representation of $M[S^{N+1}F; \zeta]$ and taking the modulus of both sides, we obtain

$$|\zeta^q M[S^N F; \zeta]| \leq \frac{\lambda_N}{q} + \frac{\lambda_{N+1}}{q} |\zeta|^q \int_{(C)}^{\infty} \frac{|d\tau|}{|\tau|^{q+1}}. \tag{2.17}$$

Using now (2.13) in (2.17), the result follows.

REMARK 3. *Parametric representations for two types of contours C , for which (2.13) is valid, are given below:*

(Type 1) $\tau = \zeta r$, r real, $1 \leq r < \infty$,

(Type 2) $C = C_1 \cup C_2$, where C_1 is defined parametrically by $\tau = \zeta e^{i\theta}$, $\min(0, \theta_0) \leq \theta \leq \max(0, \theta_0)$ for some fixed θ_0 such that $0 < |\theta_0| < \pi$, and C_2 is defined parametrically by $\tau = \zeta r e^{i\theta_0}$, r real, $1 \leq r < \infty$. For contours of Type 1

$$\int_{(C)}^{\infty} \frac{|d\tau|}{|\tau|^{q+1}} = \frac{1}{|\zeta|^q},$$

whereas for those of Type 2

$$\int_{(C)}^{\infty} \frac{|d\tau|}{|\tau|^{q+1}} = \frac{|\theta_0|}{|\zeta|^q} + \frac{1}{q|\zeta|^q}.$$

Note that the contour L in Figure 1 is a contour of Type 1 with $\zeta = -z$.

Going back to (2.9), we can see that the above lemma and its corollary can be applied quite easily to the integral on the right hand side by letting $F(\tau) = e^{-\tau}$, $q = p$, and $\zeta = -z$. Since the integrand is analytic everywhere except on the branch cut along the negative real axis, the contour Γ can be deformed to a contour of Type 2 as in Figure 2.

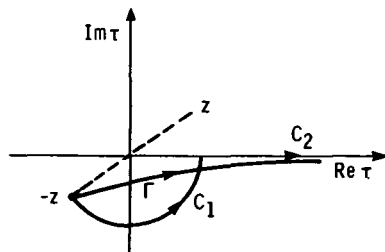


Figure 2. Deformation of Γ to a contour of Type 2.

Applying now Lemma 1 and Corollary 2 to (2.9), we obtain

$$J(p, z) \sim -ze^{-z}p! \sum_{i=0}^{\infty} \frac{\gamma_i(z)}{p^{i+1}} \quad \text{as } p \rightarrow \infty, \tag{2.18}$$

where

$$\gamma_i(z) = (S^i F)(\tau)|_{\tau=-z} = \left(z \frac{d}{dz}\right)^i e^z, \quad i = 0, 1, 2, \dots \tag{2.19}$$

Using the Maclaurin series expansion of e^z , $\gamma_i(z)$ can be expressed by the infinite series

$$\gamma_i(z) = \sum_{k=0}^{\infty} \frac{k^i}{k!} z^k, \quad i = 0, 1, 2, \dots \tag{2.20}$$

As explained in the Introduction, what goes into the T -transformation is a converging factor in terms of an infinite series in inverse powers of y . Such an expansion can also be found quite easily. Letting $p = \alpha + y - 1$, we can express (2.9) in the form

$$J(\alpha + y - 1, z) = e^{-z}(\alpha + y - 1)! (-z)^{\alpha+y-1} \int_{-z}^{\infty} d\tau \frac{(e^{-\tau}/\tau^{\alpha-1})}{\tau^{y+1}}. \tag{2.21}$$

Applying now Lemma 1 and Corollary 2 with $F(\tau) = e^{-\tau}/\tau^{\alpha-1}$, $q = y$ and $\zeta = -z$, with the contour Γ deformed as above, we obtain

$$J(\alpha + y - 1, z) \sim e^{-z}(\alpha + y - 1)! (-z)^{\alpha-1} \sum_{i=0}^{\infty} \frac{\epsilon_i(z)}{y^{i+1}} \quad \text{as } y \rightarrow \infty, \tag{2.22}$$

where

$$\epsilon_i(z) = \left(\tau \frac{d}{d\tau}\right)^i \frac{e^{-\tau}}{\tau^{\alpha-1}} \Big|_{\tau=-z}, \quad i = 0, 1, 2, \dots \tag{2.23}$$

We can therefore conclude that (1.5) and (1.6) are satisfied with

$$R_n = \frac{(\alpha + n - 1)!}{nz^n} = \frac{\mu_n/z^n}{n} \tag{2.24}$$

and

$$f(y) \sim (-z)^{\alpha-1} e^{-z} \sum_{i=0}^{\infty} \frac{\epsilon_i(z)}{y^i} \quad \text{as } y \rightarrow \infty. \tag{2.25}$$

REMARK 4. When Levin's t -transformation is applied to the sequence of the partial sums of the series (1.3), one takes $R_n = \mu_n/z^n$. From (2.18) we see that R_n should actually be $(\mu_n/z^n)n^{-1}$. However, the results of using either R_n in the T -transformation are about the same, see [5].

We now wish to show that the function $f(y)$ can be expressed in terms of a Laplace transform of an *entire* function whenever $\text{Re } z < 0$. Making the change of variable $\tau = -ze^\xi$ in the integral in (2.8), we obtain

$$J(p, z) = -ze^{-z} p! \int_0^\infty e^{-p\xi} \exp(ze^\xi) d\xi, \tag{2.26}$$

or

$$J(\alpha + y - 1, z) = -ze^{-z}(\alpha + y - 1)! \int_0^\infty e^{-y\xi} h(\xi, z) d\xi, \tag{2.27}$$

where $h(\xi, z) = e^{(1-\alpha)\xi} \exp(ze^\xi)$ is an entire function of ξ . Hence by (1.5) and (2.24)

$$f(y) = -ze^{-z} y \int_0^\infty e^{-y\xi} h(\xi, z) d\xi,$$

which is a Laplace transform. Using Watson's lemma in (2.26) (or (2.27)), the existence of the asymptotic expansions in (2.18) (or (2.22)) and the expressions given in (2.20) and (2.23) can be re-established.

3. The case $w(x) = x^\alpha E_m(x)$, $\alpha > 1$, $m + \alpha > 0$

Let $w(x) = x^\alpha E_m(x)$, where $E_m(x) = \int_1^\infty (e^{-xt}/t^m) dt$. Then for $\alpha > -1$ and $m + \alpha > 0$

$$\mu_i = \int_0^\infty dx x^{\alpha+i-1} \int_1^\infty dt \frac{e^{-xt}}{t^m}, \tag{3.1}$$

which, upon changing the order of integration and using (2.7), becomes

$$\mu_i = \frac{(\alpha + i - 1)!}{m + \alpha + i - 1}, \quad i = 1, 2, \dots \tag{3.2}$$

Substituting again (2.2) in (1.2), and using (3.2), we obtain

$$H(z) = \int_0^\infty \frac{x^\alpha E_m(x)}{z - x} dx = \sum_{i=1}^{n-1} \frac{\mu_i}{z^i} + \frac{K(\alpha + n - 1, z)}{z^n}, \tag{3.3}$$

where

$$K(p, z) = \int_0^\infty \frac{x^p E_m(x)}{1 - x/z} dx = \int_0^\infty dx \frac{x^p}{1 - x/z} \int_1^\infty dt \frac{e^{-xt}}{t^m}, \tag{3.4}$$

which, upon changing the order of integration and then making the change of variable $xt = u$ in the integral with respect to x (t fixed), is seen to be

$$K(p, z) = \int_1^\infty \frac{dt}{t^{m+p+1}} J(p, tz), \tag{3.5}$$

where $J(p, z)$ is as defined in the previous section.

We now consider the case $\text{Re } z < 0$. Since in (3.5) $1 \leq t < \infty$, for this case $\text{Re}(tz) < 0$ too. Hence we can make use of the integral representation for $J(p, z)$ given in (2.26), with z replaced by tz . We then obtain

$$K(p, z) = -zp! \int_1^\infty \frac{dt}{t^{m+p}} e^{-tz} \int_0^\infty d\xi e^{-p\xi} \exp(tze^\xi). \tag{3.6}$$

Making the change of variable $t = e^\sigma$ in (3.6), we have

$$K(p, z) = -zp! \int_0^\infty d\sigma e^{-(m+p-1)\sigma} \exp(-ze^\sigma) \int_0^\infty d\xi e^{-p\xi} \exp(ze^{\xi+\sigma}). \tag{3.7}$$

Making the further change of variables $(\xi, \sigma) \rightarrow (\omega, \sigma')$, where $\omega = \xi + \sigma$, and $\sigma' = \sigma$, we can express (3.7) as

$$K(p, z) = -zp! \int_0^\infty d\omega e^{-p\omega} \exp(ze^\omega) B(\omega, z), \tag{3.8}$$

where

$$B(\omega, z) = \int_0^\omega d\sigma e^{(1-m)\sigma} \exp(-ze^\sigma). \tag{3.9}$$

Note that $B(\omega, z)$ is an entire function of ω . Letting $p = \alpha + y - 1$, and writing the integrand in equation (3.8) in the form $e^{-y\omega} A(\omega, z)$, it is very easy to see that $A(\omega, z)$ is an entire function of ω , hence

$$K(\alpha + y - 1, z) \sim \frac{(\alpha + y - 1)!}{y} \sum_{i=0}^\infty \frac{-z\varepsilon_i(z)}{y^i} \text{ as } y \rightarrow \infty, \tag{3.10}$$

where

$$\varepsilon_i(z) = \partial^i A(\omega, z) / \partial \omega^i |_{\omega=0}, \quad i = 0, 1, \dots \tag{3.11}$$

A similar result for the case $\text{Re } z \geq 0$ could probably be obtained, however this seems to be rather complicated and shall not be pursued further in this work.

We have shown that, at least for $\text{Re } z < 0$, $H(z)$ satisfies (1.5) with (1.6), where $R_n = [(\alpha + n - 1)!/z^n]n^{-1}$, such that the expansion $f(y)/y$ is the Laplace transform of an entire function. Note that this R_n is independent of m and is the same as that obtained for $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, whereas in the t -transformation of Levin $R_n = \mu_n/z^n$, hence this R_n depends on m through $\mu_n = (\alpha + n - 1)!/(m + \alpha + n - 1)$. These observations have some important implications in the development of the new numerical quadrature formulae as explained in [6].

Appendix

The Laplace transform $\bar{u}(p)$ of a function $u(t)$ is defined by

$$\bar{u}(p) = \mathcal{L}[u(t); p] = \int_0^\infty e^{-pt} u(t) dt. \tag{A.1}$$

THEOREM. *Let $g(t)$ be an analytic function at $t = 0$ and let its Maclaurin series*

$$g(t) = \sum_{i=0}^{\infty} g_i t^i \tag{A.2}$$

have radius of convergence ρ . Define

$$G(p) = \mathcal{L}[t^\sigma g(t); p], \quad \sigma > -1. \tag{A.3}$$

Then there exists a function $h(t)$, which is analytic at $t = 0$ and has a Maclaurin series with radius of convergence ρ , such that

$$\bar{h}(p) = p^\sigma G(p). \tag{A.4}$$

Actually for $|t| < \rho$

$$h(t) = \sum_{i=0}^{\infty} \frac{(\sigma + i)!}{i!} g_i t^i. \tag{A.5}$$

PROOF. We shall deal with the case $-1 < \sigma < 0$ first. Since $-\sigma - 1 > -1$, $p^\sigma = \mathcal{L}[t^{-\sigma-1}/(-\sigma-1)!; p]$. Hence $p^\sigma G(p)$ is the Laplace transform of the convolution

$$h(t) = \int_0^t \frac{(t-\tau)^{-\sigma-1}}{(-\sigma-1)!} \tau^\sigma g(\tau) d\tau. \tag{A.6}$$

Let $t \leq \bar{\rho} < \rho$. Then substituting (A.2) in (A.6), and changing the order of summation and integration, we obtain

$$h(t) = \sum_{i=0}^{\infty} g_i \int_0^t \frac{(t-\tau)^{-\sigma-1}}{(-\sigma-1)!} \tau^{\sigma+i} d\tau. \tag{A.7}$$

This is allowed since the Maclaurin series of $g(t)$ converges absolutely and uniformly for $|t| \leq \bar{\rho} < \rho$. Using the fact that the integral in the summation is a convolution, (A.5) is easily obtained. Using the ratio test, the infinite power series in (A.5) can be shown to have the same radius of convergence as that of (A.2), namely ρ . This completes the proof for the case $-1 < \sigma < 0$.

If we let $\sigma = 0$, then (A.4) and (A.5) reduce to $h(t) = g(t)$, which is trivially true. If σ is a positive integer, (A.4) and (A.5) are again true, for in this case $h(t) = (d^\sigma/dt^\sigma)[t^\sigma g(t)]$. The case $\sigma > 0$, σ not an integer, can be dealt with as follows: Define the function $\tilde{g}(t)$ by $t^\sigma g(t) = t^{\bar{\sigma}} \tilde{g}(t)$, where $\bar{\sigma} = \sigma - [\sigma] - 1$. Then $-1 < \bar{\sigma} < 0$ and $\tilde{g}(t) = t^{[\sigma]+1} g(t)$. Then as we have shown above, (A.4) and (A.5) hold with σ and $g(t)$ replaced by $\bar{\sigma}$ and $\tilde{g}(t)$, respectively. Using the definitions of $\bar{\sigma}$ and $\tilde{g}(t)$, (A.4) and (A.5) can now be easily obtained. This completes the proof of the theorem.

References

- [1] D. Levin, "Development of non-linear transformations for improving convergence of sequences", *Internat. J. Comput. Math.* B3 (1973), 371–388.
- [2] F. W. J. Olver, *Asymptotics and special functions* (Academic, New York, 1974).
- [3] A. Sidi, "Convergence properties of some non-linear sequence transformations", *Math. Comp.* 33 (1979), 315–326.
- [4] A. Sidi, "Analysis of convergence of the T -transformation for power series", *Math. Comp.* 35 (1980), 833–850.
- [5] A. Sidi, "Numerical quadrature and non-linear sequence transformations; unified rules for the efficient computation of integrals with end-point singularities", *Math. Comp.* 35 (1980), 851–874.
- [6] A. Sidi, "Numerical quadrature rules for some infinite range integrals.", *Math. Comp.* 38 (1982), 127–142.