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# Rational Approximations from the *d*-Transformation

AVRAM SIDI

*Computer Science Department,*
*Technion–Israel Institute of Technology, Haifa, Israel*

AND

DAVID LEVIN

*School of Mathematical Sciences,*
*Tel Aviv University, Ramat Aviv, Israel*

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Recently the authors have presented the $d$-transformation which has proved to be very efficient in accelerating the convergence of a large class of infinite series. In this work the $d$-transformation is modified in a way that suits power series. An economical method for computing the rational approximations arising from the modified transformation is developed. Some properties of these approximants, similar to those of the Padé approximants, are derived. In the course of development a class of power series to which these rational approximations can be applied efficiently is characterized. A numerical example, showing the strong convergence properties of the approximations is appended and a comparison with the corresponding Padé approximants is given.

## 1. Introduction

RECENTLY the authors have developed some non-linear methods for accelerating the convergence of infinite integrals and series, namely the $D$-transformation for integrals and the $d$-transformation for series, see Levin & Sidi (1981). The convergence analysis of these methods has been taken up in a series of papers by Sidi (1979*a*, 1979*b*, 1980). The numerical examples given in Levin & Sidi (1981), and those given in Levin (1973) and in Smith & Ford (1979) for a special case, namely the $T$-transformation, indicate that these methods are very powerful and, in many cases, more efficient than Shanks' (1955) $e$-transformations. For a comparison of Levin's $T$-transformation and Shanks' transformations (or their equivalent $\varepsilon$-algorithm of Wynn, 1956) and also the $\theta$-algorithm of Brezinski (1971), see Smith & Ford (1979).

As is shown in Shanks (1955), the application of the $e$-transformation to a power series gives the Padé table of that series. Similarly, in Levin & Sidi (1981), it is mentioned that the application of the $d$-transformation to a power series gives rise to rational approximations to the power series too. The observation that the $d$-transformation is, in many cases, more efficient than the $e$-transformation leads us to expect that the rational approximations obtained from the $d$-transformation are better than the Padé approximants.



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In this work we modify the definition of the $d$-transformation as given in Levin & Sidi (1981), in a way that suits power series; the new definition, unlike the original one, enables us to give a very economical method of computing the rational approximations. In the course of development, a class of power series, for which these approximations are appropriate, is characterized. We derive some properties of these approximations, similar to those for Padé approximants. Finally, we give a numerical example showing the strong convergence properties of them and comparison with the Padé approximants.

## 2. Modification of the *d*-Transformation and Application to Power Series

We shall start by reviewing the main results of Levin & Sidi (1981) which bear relevance to the present work. The notation used is that of Sidi (1979) and is slightly different from that used in Levin & Sidi (1981).

*Definition* 2.1 A function $\alpha(x)$ is said to belong to the set $A^{(\gamma)}$, if, as $x \to \infty$, it has a Poincaré-type asymptotic expansion of the form

$$\alpha(x) \sim x^{\gamma} \sum_{i=0}^{\infty} \alpha_i/x^i. \tag{2.1}$$

*Remark* 2.1 From this definition it follows that $A^{(\gamma)} \supset A^{(\gamma-1)} \supset \ldots$.

*Remark* 2.2 If $f \in A^{(\gamma)}$ and $g \in A^{(\delta)}$, then $fg \in A^{(\gamma+\delta)}$, and if, in addition, $g \notin A^{(\delta-1)}$, then $f/g \in A^{(\gamma-\delta)}$. If $\gamma-\delta$ is an integer, then $f+g \in A^{(\sigma)}$, where $\sigma = \max\{\gamma, \delta\}$.

THEOREM 2.1 *Let the elements of the sequence* $\{f_r\}_{r=1}^{\infty}$ *satisfy a homogeneous linear difference equation of order m of the form*

$$f_r = \sum_{k=1}^{m} p_k(r)\Delta^k f_r, \tag{2.2}$$

*where* $\Delta$ *is the forward difference operator operating on the index r, and* $p_k(x)$, *as functions of the continuous variable x, are in* $A^{(i_k)}$ *but not in* $A^{(i_k-1)}$, *such that* $i_k$ *are integers satisfying* $i_k \leqslant k$, $1 \leqslant k \leqslant m$. *Let also*

$$\lim_{r\to\infty} [\Delta^{i-1}p_k(r)][\Delta^{k-i}f_r] = 0, \quad i \leqslant k \leqslant m, \quad 1 \leqslant i \leqslant m. \tag{2.3}$$

*If for every integer* $s = -1, 1, 2, 3, \ldots$,

$$\sum_{k=1}^{m} s(s-1), \ldots, (s-k+1)\bar{p}_k \neq 1, \tag{2.4}$$

*where*

$$\bar{p}_k = \lim_{x\to\infty} x^{-k}p_k(x), \quad 1 \leqslant k \leqslant m, \tag{2.5}$$

*then*

$$\sum_{r=1}^{\infty} f_r = \sum_{r=1}^{R-1} f_r + \sum_{k=0}^{m-1} R^{\rho_k}(\Delta^k f_R)\beta_k(R), \tag{2.6}$$

*where*

$$\beta_k \in A^{(0)}, \quad 0 \leqslant k \leqslant m-1,$$

*and*

$$\rho_k \leqslant \max(i_{k+1}, i_{k+2}-1, \ldots, i_m-m+k+1), \quad 0 \leqslant k \leqslant m-1. \tag{2.7}$$

The proof of this theorem can be found in Levin & Sidi (1981). A detailed proof for the case $m = 1$ has been given in Sidi (1979$a$).

*Remark* 2.3 It follows from the proof of Theorem 2.1 that equality holds in (2.7) if

$$\max_{1 \leqslant s \leqslant m-k} \{i_{k+s} - s + 1\}$$

is achieved only by one of the integers $(i_{k+s} - s + 1)$. Consequently, $\rho_{m-1} = i_m$. In particular, for $m = 1$ we have $\rho_0 = i_1$ (see Sidi, 1979$a$). Furthermore, without loss of generality, the inequality in (2.7) can be replaced by an equality, by remark 2.1.

The definition of the $d$-transformation given in Levin & Sidi (1981) is based on the result in (2.6). For power series, however, it is more convenient to express (2.6) in a different manner and use this new form of (2.6) to modify the $d$-transformation.

*Corollary*. Equation (2.6) can be re-expressed in the form

$$\sum_{r=1}^{\infty} f_r = \sum_{r=1}^{R-1} f_r + \sum_{k=0}^{m-1} R^{w_k} f_{R+k} \theta_k(R), \tag{2.8}$$

where $\theta_k \in A^{(0)}$, $0 \leqslant k \leqslant m-1$, and $w_k$ are integers satisfying

$$w_k \leqslant \max(i_{k+1}, i_{k+2}, \ldots, i_m), \quad 0 \leqslant k \leqslant m-1. \tag{2.9}$$

*Proof.* Using the fact that

$$\Delta^k f_R = \sum_{j=0}^{k} \alpha_{k,j} f_{R+j}, \tag{2.10}$$

where

$$\alpha_{k,j} = (-1)^{k-j} \binom{k}{j}, \quad 0 \leqslant j \leqslant k,$$

we can express the second sum on the right-hand side of (2.6) in the form

$$Q = \sum_{k=0}^{m-1} R^{\rho_k} (\Delta^k f_R) \beta_k(R) = \sum_{k=0}^{m-1} R^{\rho_k} \sum_{j=0}^{k} \alpha_{k,j} f_{R+j} \beta_k(R). \tag{2.11}$$

Upon interchanging the summations on $k$ and $j$ in this last equality we obtain

$$Q = \sum_{j=0}^{m-1} f_{R+j} \sum_{k=j}^{m-1} \alpha_{k,j} R^{\rho_k} \beta_k(R). \tag{2.12}$$

Defining now

$$w_j = \max(\rho_j, \rho_{j+1}, \ldots, \rho_{m-1}), \quad 0 \leqslant j \leqslant m-1, \tag{2.13}$$

we can write

$$\sum_{k=j}^{m-1} \alpha_{k,j} x^{\rho_k} \beta_k(x) = x^{w_j} \theta_j(x), \quad 0 \leqslant j \leqslant m-1, \tag{2.14}$$

where $\theta_j \in A^{(0)}$, $0 \leqslant j \leqslant m-1$. Substituting (2.14) in (2.12) and (2.12) in (2.6), (2.8) follows. (2.9) follows by using (2.7) in (2.13). ■

We now define a modification of the $d$-transformation which is based on the above corollary and hence is different from and yet completely analogous to the $d$-transformation given in Levin & Sidi (1981).

*Definition* 2.2 Let the sequence $\{f_r\}_{r=1}^{\infty}$, be as in Theorem 2.1. Define $n = (n_0, n_1, \ldots, n_{m-1})$ with $n_i$ non-negative integers. Then $d_n^{(m,j)}$, the approximation to

$$\sum_{r=1}^{\infty} f_r,$$

together with the parameters $\bar{\theta}_{k,i}$, $0 \leqslant i \leqslant n_k$, $0 \leqslant k \leqslant m-1$, are defined as the solution of the set of linear equations

$$d_n^{(m,j)} = \sum_{r=1}^{R-1} f_r + \sum_{k=0}^{m-1} R^{w_k} f_{R+k} \sum_{i=0}^{n_k} \bar{\theta}_{k,i}/R^i, \quad 1 \leqslant j \leqslant R \leqslant j+N, \tag{2.15}$$

where

$$\sum_{r=1}^{0} f_r = 0,$$

and

$$N = \sum_{k=0}^{m-1} (n_k+1), \tag{2.16}$$

provided the determinant of the system in (2.15) is non-zero.

For the case of power series, i.e.

$$f_r = a_r z^{r-1}, \quad a_r \neq 0, \quad r = 1, 2, \ldots,$$

equations (2.15) take the form

$$d_n^{(m,j)}(z) = A_{R-1} + \sum_{k=0}^{m-1} R^{w_k} a_{R+k} z^{R+k-1} \sum_{i=0}^{n_k} \bar{\theta}_{k,i}/R^i, \quad 1 \leqslant j \leqslant R \leqslant j+N, \tag{2.17}$$

where the $A_s$ are defined as

$$A_0 = 0, \quad A_s = \sum_{r=1}^{s} a_r z^{r-1}, \quad s = 1, 2, \ldots. \tag{2.18}$$

If the equations in (2.17) are solved directly to obtain $d_n^{(m,j)}(z)$, then this would mean a great loss of computational efficiency since for each $z$ these equations have to be solved again. This is exactly the deficiency that the original definition of the $d$-transformation suffers from and, in general, it cannot be overcome. However, the equations in (2.17), for the modified $d$-transformation, can be re-expressed in a form that enables one to overcome the deficiency above, and this is done below.

Multiplying both sides of (2.17) by $z^{j+N-R}$ we can re-express it in the form

$$z^{j+N-R} d_n^{(m,j)}(z) = z^{j+N-R} A_{R-1} + \sum_{k=0}^{m-1} R^{w_k} a_{R+k} \sum_{i=0}^{n_k} \frac{\theta'_{k,i}}{R^i}, \quad 1 \leqslant j \leqslant R \leqslant j+N, \tag{2.19}$$

where

$$\theta'_{k,i} = \bar{\theta}_{k,i} z^{j+N+k-1}.$$

This time the unknowns are $d_n^{(m,j)}(z)$ and the $\theta'_{k,i}$ and, except for the column corresponding to $d_n^{(m,j)}(z)$, all the other columns in the matrix of equations (2.19) are independent of $z$. Now using Cramer's rule, $d_n^{(m,j)}(z)$ can be expressed as the quotient of two determinants in the form

$$d_n^{(m,j)}(z) = \frac{\det P}{\det Q}. \tag{2.20}$$

Here $P$ is the matrix whose $(s+1)$th column is the $(N+1)$-dimensional vector

$$\begin{aligned}&[z^{N-s}A_{j+s-1}, (j+s)^{w_0}a_{j+s}v_{n_0}(j+s), (j+s)^{w_1}a_{j+s+1}v_{n_1}(j+s), \ldots,\\ &\quad (j+s)^{w_{m-1}}a_{j+s+m-1}v_{n_{m-1}}(j+s)]^T, \quad s = 0, 1, 2, \ldots, N,\end{aligned} \tag{2.21}$$

where $T$ denotes transpose, and $v_l(q)$ is the $(l+1)$-dimensional row vector

$$v_l(q) = (1, q^{-1}, q^{-2}, \ldots, q^{-l}). \tag{2.22}$$

The matrix $Q$, on the other hand, is obtained from $P$ by replacing the first row of $P$, i.e. the row vector

$$(z^N A_{j-1}, z^{N-1}A_j, z^{N-2}A_{j+1}, \ldots, zA_{j+N-2}, A_{j+N-1}), \tag{2.23}$$

by

$$(z^N, z^{N-1}, \ldots, z, 1) \tag{2.24}$$

and leaving the other rows of $P$ unchanged.

THEOREM 2.2 *$d_n^{(m,j)}(z)$ is a rational function in $z$ whose numerator has degree $\leqslant j+N-2$ and whose denominator has degree $\leqslant N$. Actually, $d_n^{(m,j)}(z)$ is of the form*

$$d_n^{(m,j)}(z) = \frac{\sum_{i=0}^{N} \delta_i z^{N-i} A_{j+i-1}}{\sum_{i=0}^{N} \delta_i z^{N-i}}, \tag{2.25}$$

*where $\delta_i$ is the cofactor of $z^{N-i}$ in the first row of the matrix $Q$.*

*Proof.* Expand $\det P$ and $\det Q$ with respect to their first rows given in (2.23) and (2.24), respectively. By using the fact that the cofactors of the first rows of $P$ and $Q$ are identical, (2.25) now follows. Clearly the denominator has degree $\leqslant N$. Since the degree of $A_k$ is $k-1$, it is easily seen that the numerator of $d_n^{(m,j)}(z)$ is of degree $\leqslant j+N-2$. This completes the proof. ■

As can be seen from (2.25), once the $\delta_i$, which are independent of $z$, have been computed, the approximation $d_n^{(m,j)}$ is known for all $z$ with very little additional effort. From (2.25) we also see that the $\delta_i$ can be multiplied by a non-zero constant without changing $d_n^{(m,j)}$. Now the $\delta_i$ satisfy the system of linear equations $U\delta = 0$, where $U = (u_0, u_1, u_N)$ is the $N \times (N+1)$ matrix whose columns $u_s$ are given by

$$\begin{aligned}u_s = {} & [(j+s)^{w_0}a_{j+s}v_{n_0}(j+s), (j+s)^{w_1}a_{j+s+1}v_{n_1}(j+s), \ldots,\\ & (j+s)^{w_{m-1}}a_{j+s+m-1}v_{n_{m-1}}(j+s)]^T, \quad s = 0, 1, \ldots, N,\end{aligned} \tag{2.26}$$

and $\delta = (\delta_0, \delta_1, \ldots, \delta_N)^T$. This is so since the $\delta_i$ are the cofactors of the elements of the first row of the matrix $P$ (or $Q$). Combining these facts we can now state the following useful result:

THEOREM 2.3 *Let* rank $(U) = N$. *Then the vector* $\delta$ *can be determined uniquely (up to a multiplicative constant) by solving the system of* $N+1$ *equations*

$$V\delta = e, \tag{2.27}$$

*where* $e = (1, 0, 0, \ldots, 0)^T$ *and* $V = (v/U)$, *i.e.* $v$ *is the first row of* $V$ *and* $U$ *is the rest of* $V$, *with* $v$ *being any* $(N+1)$ *dimensional vector such that* $V$ *is non-singular*

## 3. Some Properties of $d_n^{(m,j)}(z)$

3.1 *A Characterization Result for a Class of Infinite Series to which the d-Transformation can be Applied*

Let us recall that the derivation of $d_n^{(m,j)}$ in the previous section has been based on Theorem 2.1 and/or its corollary. The most important condition in Theorem 2.1 is (2.2), which has been formulated in the form of a linear homogeneous difference equation in the $f_r$, with the coefficients $p_k(x)$ of this equation being in $A^{(i_k)}$, $k_k \leqslant k$, $i_k$ integers. For the case of power series, i.e. $f_r = a_r z^{r-1}$, it is desirable to have a set of conditions on the $a_r$ (rather than $f_r$) that will guarantee that all the conditions in Theorem 2.1 and hence (2.6) hold. It turns out that such a set of sufficient conditions can be formulated and it characterizes a large class of power series

$$\sum_{r=1}^{\infty} a_r z^{r-1}$$

to which Theorem 2.1 applies and consequently for which the approximations $d_n^{(m,j)}(z)$ are appropriate. This is done in lemma 3.1 below.

LEMMA 3.1 *Let* $f_r = a_r z^{r-1}$, $r = 1, 2, \ldots$, *and let the* $a_r$ *satisfy the linear homogeneous* $(m+1)$-*term recursion relation*

$$a_{r+m} = \sum_{k=0}^{m-1} c_k(r) a_{r+k}, \tag{3.1}$$

*where* $c_k(x)$, *as a function of the continuous variable* $x$, *belongs to* $A^{(\mu_k)}$ *for some integer* $\mu_k$, $k = 0, 1, \ldots, m-1$. *Define* $c_m(r) \equiv -1$, *i.e.* $\mu_m = 0$ *and*

$$i_k = \max_{k \leqslant r \leqslant m} \{\mu_r\} - \max_{0 \leqslant r \leqslant m} \{\mu_r\} \leqslant 0, \quad k = 1, 2, \ldots, m. \tag{3.2}$$

*Then, for general* $z$, *the* $f_r$ *satisfy the linear homogeneous difference equation of order* $m$

$$f_r = \sum_{k=1}^{m} p_k(r, z) \Delta^k f_r, \tag{3.3}$$

*where* $p_k(x, z)$ *are rational functions in* $z$ *given by*

$$p_k(x, z) = -\frac{\sum_{j=k}^{m} \binom{j}{k} c_j(x) z^{m-j}}{\sum_{j=0}^{m} c_j(x) z^{m-j}}, \quad k = 1, \ldots, m, \tag{3.4}$$

*and $p_k(x, z)$, as functions of the continuous variable $x$, belong to $A^{(i_k)}$.*

*Proof.* Let us first express (3.1) in the form

$$\sum_{k=0}^{m} c_k(r) a_{r+k} = 0.$$

Using the equality $a_r = z^{-r+1} f_r$ and the relation

$$b_{r+k} = (1+\Delta)^k b_r = \sum_{i=0}^{k} \binom{k}{i} \Delta^i b_r, \tag{3.5}$$

we can write this recursion relation as

$$\sum_{k=0}^{m} c_k(r) z^{-r-k+1} \sum_{i=0}^{k} \binom{k}{i} \Delta^i f_r = 0. \tag{3.6}$$

Multiplying equation (3.6) by $z^{r+m-1}$ and rearranging we obtain

$$\sum_{i=0}^{m} \left[ \sum_{k=i}^{m} \binom{k}{i} c_k(r) z^{m-k} \right] \Delta^i f_r = 0. \tag{3.7}$$

Finally, (3.3) and (3.4) follow from (3.7). From remark 2.2, the numerator of $p_k(x, z)$ in (3.4), for general $z$, is in $A^{(\gamma_k)}$ but not in $A^{(\gamma_k - 1)}$, where

$$\gamma_k = \max_{k \leqslant r \leqslant m} \{\mu_r\}$$

and its denominator, for general $z$, is in $A^{(\gamma)}$ but not in $A^{(\gamma-1)}$, where

$$\gamma = \max_{0 \leqslant r \leqslant m} \{\mu_r\}.$$

By remark 2.2 again we conclude that $p_k \in A^{(\gamma_k - \gamma)}$. But $\gamma_k - \gamma = i_k$ from (3.2) above, hence $p_k \in A^{(i_k)}$. This completes the proof of the lemma. ■

*Remark* 3.1 It seems that the result of lemma 3.1 holds for every $z$ which is not a point of singularity of the function represented by

$$\sum_{r=1}^{\infty} a_r z^{r-1},$$

although there is no proof of this yet.

Using lemma 3.1 we can now prove the following useful result:

THEOREM 3.1 *Let the sequence $\{a_r\}_{r=1}^{\infty}$ be as in lemma 3.1 with the same notation, and assume further that the series*

$$\sum_{r=1}^{\infty} a_r z^{r-1}$$

*has a non-zero radius of convergence $\zeta$. Then, whenever $|z| < \zeta$, the sequence $\{f_r\}_{r=1}^{\infty}$, where $f_r = a_r z^{r-1}$, satisfies all the conditions of Theorem 2.1, hence (2.8) holds with $w_k \leqslant 0$, $0 \leqslant k \leqslant m-1$.*

*Proof.* From lemma 3.1 it follows that (2.2) is satisfied with $i_k \leqslant 0$, $0 \leqslant k \leqslant m-1$. Therefore, $\bar{p}_k = 0$, $0 \leqslant k \leqslant m-1$, consequently (2.4) is satisfied. Since $p_k(x, z)$, as a function of $x$, is in $A^{(i_k)}$, $\Delta^s p_k(r, z) = O(r^{i_k - s})$ as $r \to \infty$, hence $\Delta^s p_k(x, z)$ in is $A^{(i_k - s)}$. We also have that $\Delta^s f_r \to 0$ as $r \to \infty$ due to the fact that

$$\Delta^s f_r = \sum_{j=0}^{s} \alpha_{s,j} f_{r+j},$$

and that $f_r \to 0$ as $r \to \infty$ whenever $|z| < \zeta$. Now $q(r)f_r \to 0$ as $r \to \infty$, whenever $|z| < \zeta$, if $q \in A^{(\gamma)}$ for some $\gamma$. Combining these results we can see that (2.3) is satisfied too. Hence, we have shown that all the sufficient conditions in Theorem 2.1 are satisfied. The rest follows trivially. ■

*Remark* 3.1 From remark 2.3 it is clear that the $\rho_k$ in (2.6) are all zero. Many examples that we have examined indicate that $w_k = 0$, $0 \leqslant k \leqslant m-1$, although there is no proof of this so far. For the case $m = 1$, however, $w_0 = 0$ trivially since $w_0 \equiv \rho_0$ and $\rho_0 = 0$.

*Remark* 3.2 Theorem 3.1 characterizes a large class of infinite power series to which the $d$-transformation can be successfully applied. This class seems to contain a large number of power series which, inside their circles of convergence, represent functions that have a finite number of poles and branch points in the complex $z$-plane. See the example in Section 4.

### 3.2 *Some Padé-like Properties of* $d_n^{(m,j)}(z)$

We now turn to the derivation of some properties of the approximations $d_n^{(m,j)}(z)$ which are similar to those of the Padé approximants. We recall that the $(p/q)$ Padé approximant $R_{p,q}(z)$ to

$$\sum_{r=1}^{\infty} a_r z^{r-1}$$

is the rational function

$$R_{p,q}(z) = \frac{U(z)}{V(z)} = \frac{\sum_{i=0}^{p} \alpha_i z^i}{\sum_{j=0}^{q} \beta_j z^j}, \quad \beta_0 = 1$$

which satisfies

$$V(z) \sum_{r=1}^{\infty} a_r z^{r-1} - U(z) = O(z^K)$$

as $z \to 0$, where $K = p+q+1$. As such $R_{p,q}(z)$ can be determined from $a_1, a_2, \ldots, a_K$. We now show that $d_n^{(m,j)}(z)$ have a similar property. We start with the following general result.

THEOREM 3.2 *Let* $T(z) = u(z)/v(z)$, *where*

$$u(z) = \sum_{i=0}^{s} \lambda_i z^{s-i} A_{j+i-1}(z)$$

*and*

$$v(z) = \sum_{i=0}^{s} \lambda_i z^{s-i},$$

*where $A_r(z)$ are as in* (2.18) *and $\lambda_i$ are constants. Then*

$$v(z)\sum_{i=1}^{\infty} a_i z^{i-1} - u(z) = O(z^{j+s-1}), \tag{3.8}$$

*and if $\lambda_s \neq 0$, then* (3.8) *is equivalent to*

$$\sum_{r=1}^{\infty} a_r z^{r-1} - T(z) = O(z^{j+s-1}). \tag{3.8a}$$

*Proof.* Using the expressions for $u(z)$ and $v(z)$ we have

$$\begin{aligned} v(z)\sum_{i=1}^{\infty} a_i z^{i-1} - u(z) &= z^{j+s-1}\sum_{i=0}^{s}\lambda_i \sum_{r=i+j}^{\infty} a_r z^{r-(i+j)} \\ &= z^{j+s-1}\sum_{r=0}^{\infty}\varepsilon_r z^r \end{aligned} \tag{3.9}$$

where

$$\varepsilon_r = \sum_{i=0}^{s}\lambda_i a_{i+j+r}, \quad r = 0, 1, \ldots, \tag{3.10}$$

and this proves (3.8). If we now divide both sides of (3.8) [or (3.9)] by $v(z)$ and make use of the assumption that $\lambda_s \neq 0$ (or equivalently $v(0) \neq 0$), then (3.8a) follows.

Recalling (2.25) we obtain from Theorem 3.2 the following:

*Corollary*. Let $u(z)$ and $v(z)$ be the numerator and denominator, respectively, of the rational approximation $d_n^{(m,j)}$. Then Theorem 3.2 applies to $d_n^{(m,j)}(z)$ with $s = N$ and $\lambda_i = \delta_i$, $i = 0, 1, \ldots, N$.

The result stated in this corollary can be sharpened under certain circumstances, which have been observed to be valid in many cases of interest, and in Theorem 3.3 we actually show that, like the Padé approximants, the $d_n^{(m,j)}(z)$ too have Maclaurin series expansions that agree with

$$\sum_{r=1}^{\infty} a_r z^{r-1}$$

through *all* the terms that go into their construction, provided $\delta_N \neq 0$.

Let us consider again the rational functions $T(z)$ which have been defined in Theorem 3.2. Let

$$\sum_{r=1}^{\infty} \bar{a}_r z^{r-1}$$

be the Maclaurin series of $T(z)$. From Theorem 3.2 we have $\bar{a}_r = a_r$, $r = 1, \ldots, j+s-1$. Let $r'$ be such that $j+s+r' = \min\{r \mid \bar{a}_r \neq a_r\}$. Obviously $r' \geqslant 0$. Now

$$\sum_{r=1}^{\infty} a_r z^{r-1} - T(z) = \sum_{r=j+s+r'}^{\infty} (a_r - \bar{a}_r) z^{r-1},$$

hence, one measure of the closeness of $T(z)$ to $\sum_{r=1}^{\infty} a_r z^{r-1}$ is the closeness to zero of

$(a_r - \bar{a}_r)$, $r \geqslant j+s+r'$. If we normalize the denominator $v(z)$ of $T(z)$ such that $\lambda_s = v(0) = 1$, then we can easily see that $\varepsilon_r = a_{j+s+r'} - \bar{a}_{j+s+r'}$ with $\varepsilon_i$ as defined in (3.10). This suggests that another measure of the closeness of $T(z)$ to

$$\sum_{r=1}^{\infty} a_r z^{r-1}$$

is the closeness to zero of the $\varepsilon_i$ provided we take $\lambda_s = 1$. In Theorem 3.3 we provide bounds on the $\varepsilon_i$ for some cases of interest, which, to a certain extent, suggest that $\varepsilon_i \to 0$ as $n_k \to \infty$, $0 \leqslant k \leqslant m-1$.

THEOREM 3.3 Let $w_k \geqslant 0$, $0 \leqslant k \leqslant m-1$ in (2.17) and let $u(z)$ and $v(z)$ be the numerator and denominator of $d_n^{(m,j)}(z)$ as given in (2.25). For $n_k \geqslant w_k$ we have

$$v(z) \sum_{r=1}^{\infty} a_r z^{r-1} - u(z) = z^{j+N+m-1} \sum_{i=0}^{\infty} \bar{\varepsilon}_i z^i, \tag{3.11}$$

where $\bar{\varepsilon}_i = \varepsilon_{m+i}$, $i = 0, 1, \ldots$, and

$$\varepsilon_i = \sum_{r=0}^{N} \delta_r a_{i+j+r}, \quad r = 0, 1, \ldots .$$

Furthermore, when the sequence $\{a_r\}_{r=1}^{\infty}$ is as in lemma 3.1 with the notation therein, such that $\mu_k \leqslant 0$, $0 \leqslant k \leqslant m-1$, and the functions $c_k(x)$ in (3.1) are infinitely differentiable for $j \leqslant x \leqslant \infty$, then

$$|\bar{\varepsilon}_s| \leqslant \left( \sum_{i=0}^{N} \sum_{k=0}^{m-1} |\delta_i a_{j+i+k}| \right) o(\tilde{n}^{-t}) \quad \text{as } \tilde{n} \to \infty \text{ for all } t > 0, \tag{3.12}$$

where $\tilde{n} = \min(n_0, n_1, \ldots, n_{m-1})$.

*Remark* 3.3 If $w_k \geqslant 0$, $k = 0, 1, \ldots, m-1$ and $\delta_N \neq 0$, then (3.11) implies that the Maclaurin series of $d_n^{(m,j)}(z)$ agrees with

$$\sum_{r=1}^{\infty} a_r z^{r-1}$$

through the terms $a_r z^{r-1}$, $r = 1, 2, \ldots, j+N+m-1$. On the other hand, from equations (2.19) that define $d_n^{(m,j)}(z)$ we can see that the coefficients $a_r$ that determine $d_n^{(m,j)}(z)$ are $a_r$, $r = 1, 2, \ldots, j+N+m-1$. In this respect our rational approximations $d_n^{(m,j)}(z)$ resemble the Padé approximants. If $w_k = 0$, $n_k = 0$, $0 \leqslant k \leqslant m-1$, then $d_n^{(m,j)}(z)$ is simply the $(j+m-2/m)$ Padé approximant, $R_{j+m-2,m}(z)$.

*Proof.* In order to prove (3.11) all we have to do is show that $\varepsilon_r = 0$, $0 \leqslant r \leqslant m-1$, in the corollary to Theorem 3.2. Now

$$\varepsilon_r = \sum_{i=0}^{N} \delta_i a_{j+i+r} = \det V_r,$$

where $V_r$ is the matrix obtained from $P$ by replacing its first row by the row vector $(a_{j+r}, a_{j+r+1}, \ldots, a_{j+r+N})$, the rest of $P$ staying the same. For $n_k \geqslant w_k \geqslant 0$, $0 \leqslant k \leqslant m-1$, and $0 \leqslant r \leqslant m-1$, the first row of $V_r$ is identical to one of the remaining rows as can be seen by analysing the structure of $P$ with the help of (2.21) and (2.22). Hence $\varepsilon_r = 0$, $0 \leqslant r \leqslant m-1$.

In order to prove (3.12) we proceed as follows. Since the elements $a_r$, $r = 1, 2, \ldots$, satisfy (3.1) with $\mu_k \leqslant 0$, $0 \leqslant k \leqslant m-1$, it can be shown by induction that

$$a_{r+m+s} = \sum_{k=0}^{m-1} c_k^{(s)}(r) a_{r+k}, \quad r \geqslant 0, \tag{3.13}$$

where

$$\begin{aligned} c_k^{(0)}(x) &= c_k(x), & 0 \leqslant k \leqslant m-1 \\ c_k^{(s+1)}(x) &= c_{k-1}^{(s)}(x+1) + c_{m-1}^{(s)}(x+1) c_k(x), & 0 \leqslant k \leqslant m-1, \end{aligned} \tag{3.14}$$

with $c_{-1}^{(s)}(x) \equiv 0$ for all $s$. Since the functions $c_k(x)$ belong to $A^{(0)}$ and are infinitely differentiable for $j \leqslant x \leqslant \infty$, so are the $c_k^{(s)}(x)$.

Let us map the interval $j \leqslant x \leqslant \infty$ to $0 \leqslant \xi \leqslant 1$ by the transformation $\xi = j/x$ and let

$$\pi_{k,p}^{(s)}(\xi) = \sum_{i=0}^{p} \pi_{k,p,i}^{(s)} \xi^i$$

be the best polynomial approximation of degree $p$ to $c_k^{(s)}(j/\xi)$ on $0 \leqslant \xi \leqslant 1$. Since $c_k^{(s)}$ are infinitely differentiable for $j \leqslant x \leqslant \infty$, $c_k^{(s)}(j/\xi)$ are infinitely differentiable for $0 \leqslant \xi \leqslant 1$. Therefore,

$$\max_{j \leqslant x \leqslant \infty} |c_k^{(s)}(x) - \pi_{k,p}^{(s)}(j/x)| = o(p^{-t}), \tag{3.15}$$

as $p \to \infty$ for any $t > 0$, by a standard result in approximation theory. Now

$$\bar{\varepsilon}_s = \det V_{s+m} = \sum_{i=0}^{N} \delta_i a_{j+m+s+i}.$$

Substituting (3.13) in this expression, with $r$ replaced by $j+i$, we obtain

$$\bar{\varepsilon}_s = \sum_{i=0}^{N} \delta_i \sum_{k=0}^{m-1} c_k^{(s)}(j+i) a_{j+i+k}. \tag{3.16}$$

Let $W_q^k$ be the matrix obtained from $P$ by replacing its first row by the row vector

$$[a_{j+k} j^{-q}, a_{j+1+k}(j+1)^{-q}, \ldots, a_{j+N+k}(j+N)^{-q}],$$

then

$$\det W_q^k = \sum_{i=0}^{N} \delta_i a_{j+i+k}(j+i)^{-q} = 0$$

for $0 \leqslant q \leqslant n_k$, since $W_q^k$, for $0 \leqslant q \leqslant n_k$, has two identical rows, one of them being the first row. Therefore

$$\sum_{i=0}^{N} \delta_i \sum_{k=0}^{m-1} \pi_{k,n_k}^{(s)}\left(\frac{j}{j+i}\right) a_{j+i+k} = 0. \tag{3.17}$$

Finally, subtracting (3.17) from (3.16) we obtain

$$\bar{\varepsilon}_s = \sum_{i=0}^{N} \delta_i \sum_{k=0}^{m-1} a_{j+i+k} \left[ c_k^{(s)}(j+i) - \pi_{k,n_k}^{(s)}\left(\frac{j}{j+i}\right)\right]. \tag{3.18}$$

Taking absolute values on both sides of (3.18) and using (3.15), (3.12) now follows. ■

The Padé-like properties of $d_n^{(m,j)}(z)$ are similar to those of the rational approximations presented by Brezinski (1979). In both these classes of approximations one is making use of some extra assumption about the given formal power series; in Brezinski's approximations the poles, or some of them, are predetermined, while in the $d_n^{(m,j)}$ approximations we assume that the coefficients of the power series satisfy a certain type of recursion relation. An example by Brezinski (1979) shows that the Padé approximants are not always optimal and better rational approximations can be obtained by a proper choice of the poles. As reported to us by the referee, numerical tests show that for the example from Brezinski's paper, $z^{-1}\log(1+z)$, the $d_n^{(m,j)}$ approximations perform even better than Brezinski's approximations. We are grateful to the referee for carrying out this example.

## 4. Practical Implementation of $d_n^{(m,j)}(z)$ and Numerical Examples

Let

$$\sum_{r=1}^{\infty} a_r z^{r-1}$$

be a series with a non-zero radius of convergence. In Theorem 2.1 we gave a set of sufficient conditions on the sequence $\{f_r = a_r z^{r-1}\}_{r=1}^{\infty}$ that enables us to apply the $d$-transformation to this series and obtain good approximations to

$$\sum_{r=1}^{\infty} a_r z^{r-1}.$$

The most important condition in this set seems to be (2.2). We note that in order to apply the $d$-transformation to

$$\sum_{r=1}^{\infty} a_r z^{r-1}$$

we do *not* have to know the difference equation in (2.2) explicitly; mere knowledge of its existence and of its order $m$ is enough. Even $m$ does not have to be known exactly, an upper bound for $m$, if available, can be used instead of $m$. If no knowledge of $m$ is available, one can try, instead of $m$, the values $m' = 1, 2, \ldots,$ until meaningful (quickly converging) approximations $d_n^{(m',j)}(z)$ are obtained for some value of $m'$ which is then taken to be the true value of $m$. Besides $m$ what goes into the $d$-transformation is the set of integers $\{w_0, \ldots, w_{m-1}\}$. Now in order to know what these $w_k$ are, one has to have some knowledge of the difference equation (2.2) as can be seen from (2.9). Actually, one has to know the $i_k$. Even then (2.9) provides us with upper bounds to the $w_k$ in general. Now it turns out that one does *not* have to know the $w_k$ exactly in the computation of $d_n^{(m,j)}(z)$; one can replace $w_k$ by $w'_k$ that satisfies $w_k \leqslant w'_k \leqslant m-1$, $k = 0, 1, \ldots, m-1$. However, for the *power series* considered in this work it seems that $w'_k = 0$, $k = 0, 1, \ldots, m-1$ is an appropriate choice. Furthermore, in many cases of interest $w_k = 0$ for all $k$.

We now give some useful results that are helpful in determining the value of $m$.

*Definition* 4.1 A sequence $\{a_r\}_{r=1}^{\infty}$ whose elements satisfy an $(m+1)$-term recursion relation of the form given in (3.1) with $c_k(x)$ in $A^{(\mu_k)}$ for some *integers* $\mu_k$ is said to belong to $G^{(m)}$.

LEMMA 4.1 *Let $\{a_r\}_{r=1}^{\infty}$ and $\{b_r\}_{r=1}^{\infty}$ belong to $G^{(m)}$ and $G^{(m')}$, respectively. Then $\{a_r+b_r\}_{r=1}^{\infty}$ and $\{a_r b_r\}_{r=1}^{\infty}$ belong to $G^{(m_1)}$ and $G^{(m_2)}$, respectively, for some $m_1 \leqslant m+m'$ and $m_2 \leqslant mm'$.*

LEMMA 4.2 *Let $\{a_r\}_{r=1}^{\infty}$ belong to $G^{(m)}$. Then $\{a_r^2\}_{r=1}^{\infty}$ belongs to $G^{(\bar{m})}$ where $\bar{m} \leqslant m(m+1)/2$.*

These lemmas are simple consequences of lemma 3 and its first two corollaries in Levin & Sidi (1981).

Numerous examples done by the authors suggest that if the sequences $\{a_r\}_{r=1}^{\infty}$ and $\{b_r\}_{r=1}^{\infty}$ considered in the two lemmas above satisfy all the conditions given in lemma 3.1 (i.e. in addition to being in $G^{(m)}$ and $G^{(m')}$ they satisfy the remaining conditions of lemma 3.1), then so do $\{a_r+b_r\}_{r=1}^{\infty}$, $\{a_r b_r\}_{r=1}^{\infty}$, and $\{a_r^2\}_{r=1}^{\infty}$ with $m$ replaced by $m_1$, $m_2$, and $\bar{m}$, respectively. Usually, $m_1 = m+m'$, $m_2 = mm'$, and $\bar{m} = m(m+1)/2$.

LEMMA 4.3 *Let $\{a_r\}_{r=1}^{\infty}$ satisfy all the conditions given in lemma 3.1 for some $m$ and let $g(x)$ be in $A^{(\gamma)}$ for some $\gamma$. Then $\{g(r)a_r\}_{r=1}^{\infty}$ satisfies all the conditions given in lemma 3.1 with the same value of $m$.*

The proof of this lemma is easy and will be omitted.

*Example*. Consider the function

$$F(z) = z^{-1}\log(1+z)+(z^2+\alpha)^{-\beta}, \tag{4.1}$$

which has three branch points at $z=-1$ and $z = \pm i\sqrt{\alpha}$. $F(z)$ has a Maclaurin series expansion of the form

$$F(z) = \sum_{r=1}^{\infty} a_r z^{r-1},$$

where $a_r = b_r + \bar{b}_r$ with

$$b_r = \frac{(-1)^{r-1}}{r}, \quad r = 1, 2, \ldots,$$

$$\bar{b}_1 = \alpha^{-\beta},\ \bar{b}_{2r-1} = \frac{(-1)^{r-1}}{\alpha^{\beta+r-1}}\frac{\beta(\beta+1)\ldots(\beta+r-2)}{(r-1)!}, \quad r = 2, 3, \ldots, \tag{4.2}$$

$$\bar{b}_{2r} = 0, \quad r = 1, 2, \ldots,$$

and this series converges for

$$|z| < \zeta = \min(1, \sqrt{|\alpha|}).$$

It can easily be shown that

$$b_{r+1} = -\frac{r}{r+1}b_r \quad \text{and} \quad \bar{b}_{r+2} = -\frac{1}{\alpha}\frac{2\beta+r-1}{r+1}\bar{b}_r.$$

Hence by definition 4.1 the sequences $\{b_r\}_{r=1}^{\infty}$ and $\{\bar{b}_r\}_{r=1}^{\infty}$ are in $G^{(1)}$ and $G^{(2)}$, respectively. By lemma 4.1 the sequence $\{a_r = b_r + \bar{b}_r\}_{r=1}^{\infty}$ is in $G^{(3)}$. Actually it can be shown that the sequence $\{a_r\}_{r=1}^{\infty}$ satisfies

$$a_{r+3} = c_0(r)a_r + c_1(r)a_{r+1} + c_2(r)a_{r+2}$$

with $c_0(x)$, $c_1(x)$, $c_2(x)$ being all in $A^{(0)}$. Since

$$\sum_{r=1}^{\infty} a_r z^{r-1}$$

has a non-zero radius of convergence, by Theorem 3.1, (2.8) holds with $m = 3$ and $w_k \leqslant 0$, $k = 0, 1, 2$. Therefore, the approximations $d_n^{(3,j)}$ are appropriate for this case. In our numerical experiments we took $w_0 = w_1 = w_2 = 0$ in definition 2.2.

The numerical results given in Levin & Sidi (1981) and the general convergence theory given in Sidi (1979b) suggest that the sequence of approximations $d_n^{(m,j)}(z)$ with $j = 1$ and $n = (\nu, \ldots, \nu)$, $\nu = 0, 1, 2, \ldots$, have the best convergence properties. We denote these approximations by $d_\nu^{(m,1)}$. As for the Padé approximants it is well known that the sequence of diagonal approximants $R_{k,k}(z)$ has the best convergence properties. Therefore, any comparison between the two methods has to be made between those $d_\nu^{(m,1)}(z)$ and $R_{k,k}(z)$ which are obtained from approximately the same number of terms of the series

$$\sum_{r=1}^{\infty} a_r z^{r-1},$$

i.e. for $m(\nu+2) \approx 2k+1$.

In Table 1 we give the approximations $d_5^{(3,1)}(z)$ and $R_{10,10}(z)$ to the function $F(z)$ given in (4.1) with $\alpha = 1$ and $\beta = \frac{1}{2}$ at several points in the complex $z$-plane. Note that both approximants are obtained from the first 21 terms of the Maclaurin series of $F(x)$.

One of the important applications of rational approximations is to approximate the

TABLE 1

| $z$ | $R_{10,10}(z)$ | $d_5^{(3,1)}(z)$ | $F(z)$ |
|---|---|---|---|
| 0·5 | 1·705357407213 | 1·705357407216 | 1·705357407216 |
| 1·0 | 1·40025390 | 1·400253961753 | 1·400253961747 |
| 2·0 | 0·99648 | 0·99651956 | 0·996519739834 |
| 5·0 | 0·5529 | 0·554489 | 0·554468028983 |
| −0·5 | 2·2807215514 | 2·280721552119 | 2·280721552120 |
| −0·9 | 3·298 | 3·3015 | 3·301722 |
| −0·95 | 3·848 | 3·8737 | 3·87840 |
| 2+2$i$ | 0·7326 −0·4064$i$ | 0·73166 −0·40665$i$ | 0·731645 −0·406695$i$ |
| −0·2+$i$ | 1·995 +0·656$i$ | 1·98366 +0·64811$i$ | 1·98345 +0·64816$i$ |
| 0·9$i$ | 3·1079 −0·331$i$ | 3·10837 −0·329614$i$ | 3·10839 −0·329626$i$ |

TABLE 2

| Poles of $R_{10,10}(z)$ | Poles of $d_5^{(3,1)}(z)$ | Singular points of $F(z)$ |
|---|---|---|
| $-1{\cdot}046$ | $-1{\cdot}024$ | $-1$ |
| $-0{\cdot}005 \pm 1{\cdot}016i$ | $-0{\cdot}0002 \pm 1{\cdot}0066i$ | $\pm i$ |

location of the singular points of the functions being approximated. It turns out that some of the poles of the rational approximations $d_v^{(m,j)}(z)$ and $R_{k,k}(z)$, as $v \to \infty$ and $k \to \infty$, tend to the singular points of the function being approximated. Table 2 gives poles of the approximations $d_5^{(3,1)}(z)$ and $R_{10,10}(z)$ for $F(z)$.

Similar testing was done for several other values of $\alpha$ and $\beta$ and the conclusions are in general as above. Only for integer $\beta$ the poles of the Padé approximants approximate the locations of the poles of $F(z)$ better than those of the $d_n^{(m,j)}(z)$ approximations.

We also computed the first few $\bar{\varepsilon}_r$ [see (3.11)] both for $R_{10,10}(z) = U(z)/V(z)$ and $d_5^{(3,1)}(z) = u(z)/v(z)$ with $V(0) = 1$ and $v(0) = 1$. The results are as follows:

$$V(z)F(z) - U(z) = (-0{\cdot}652 \times 10^{-3})z^{21} + (0{\cdot}204 \times 10^{-2})z^{22} + (-0{\cdot}223 \times 10^{-2})z^{23} + \ldots,$$

and

$$v(z)F(z) - u(z) = (-0{\cdot}104 \times 10^{-6})z^{21} + (-0{\cdot}406 \times 10^{-6})z^{22} + (0{\cdot}536 \times 10^{-6})z^{23} + \ldots.$$

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