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PREDICTION PROPERTIES OF THE t-TRANSFORMATION*

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Abstract. Some prediction properties of the t-transformation of Levin are analyzed for two kinds of limiting processes. It is shown under certain circumstances that when the t-transformation is applied to a power series, the coefficients of the Maclaurin series of the rational approximation obtained approximate those of the given series with increasing accuracy. Theoretical and numerical examples are appended.

1. Introduction.

Let

$$F(z)=\sum_{r=1}^{\infty} a_r z^{r-1} \tag{1.1}$$

be a formal power series, and

$$A_0(z)=0, \qquad A_j(z)=\sum_{r=1}^{j} a_r z^{r-1}, \qquad j=1,2,\cdots, \tag{1.2}$$

its partial sums. Define the rational approximation $T_{k,n}(z)$ to $F(z)$ by

$$T_{k,n}(z)=\frac{u_{k,n}(z)}{v_{k,n}(z)}=\frac{\sum_{j=0}^{k}\lambda_j^{(k,n)}z^{k-j}A_{n+j-1}(z)}{\sum_{j=0}^{k}\lambda_j^{(k,n)}z^{k-j}}, \qquad \lambda_k^{(k,n)}=1, \tag{1.3}$$

where n and k are positive integers, and $\lambda_j^{(k,n)}$ are constants which may depend on n, k and a_r, but are independent of z. As expressed by (1.3), $T_{k,n}(z)$ is a weighted average of $A_r(z)$, $n-1 \leqq r \leqq n+k-1$. The Padé approximants and the recent d approximations of the authors [7] are of the form (1.3).

Let $T_{k,n}(z)$ have the Maclaurin series expansion

$$T_{k,n}(z)=\sum_{r=1}^{\infty} a_r^{(k,n)} z^{r-1}. \tag{1.4}$$

Let us now assume that the formal power series in (1.1) represents the *function* $F(z)$ asymptotically and that $a_r=F^{(r-1)}(0)/(r-1)!$, $r=1,2,\cdots$. Since $T_{k,n}(z)$ is constructed using information about $F(z)$ at $z=0$, we would expect it to approximate $F(z)$ better in a neighborhood of $z=0$ than elsewhere, and would therefore expect the derivatives of $T_{k,n}(z)$ at $z=0$ to approximate the corresponding derivatives of $F(z)$ at $z=0$. We have the following simple result [7], which will be of use in the remainder of this work:

THEOREM 1.1. *Let $F(z)$, $A_j(z)$ and $T_{k,n}(z)$ be as in* (1.1), (1.2) *and* (1.3) *respectively. Then*

$$v_{k,n}(z)F(z)-u_{k,n}(z)=z^{n+k-1}\sum_{r=0}^{\infty}\varepsilon_r^{(k,n)}z^r, \tag{1.5}$$

where

$$\varepsilon_r^{(k,n)}=\sum_{j=0}^{k}\lambda_j^{(k,n)}a_{n+j+r}, \qquad r=0,1,\cdots; \tag{1.6}$$

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hence

$$F(z)-T_{k,n}(z)=\sum_{r=n+k}^{\infty}(a_r-a_r^{(k,n)})z^{r-1}=O(z^{n+k-1}). \tag{1.7}$$

Let $r'=\min\{r|\varepsilon_r^{(k,n)}\neq 0\}$. *Then*

$$\varepsilon_{r'}^{(k,n)}=a_{n+k+r'}-a_{n+k+r'}^{(k,n)}. \tag{1.8}$$

Although $a_r^{(k,n)}=a_r$ for $r\leqq n+k+r'-1$, this does not necessarily mean that $T_{k,n}(z)$ is a good approximation to $F(z)$ for z not too small, if the $\lambda_j^{(k,n)}$ are assigned arbitrarily. As a matter of fact, one can fix the $\lambda_j^{(k,n)}$ in such a way that $a_r^{(k,n)}$ can take on any preassigned values for some $r\geqq n+k$, having nothing to do with the actual a_r. We therefore argue that if $T_{k,n}(z)$ is to be a good approximation to $F(z)$, then it should be such that: (1) $a_r^{(k,n)}$ is a good approximation to or reproduces a_r for $r\leqq\kappa$; (2) $a_r^{(k,n)}$ predicts a_r closely for $r\geqq\kappa+1$, where κ is the number of terms of the series $F(z)$ that are used in constructing $T_{k,n}(z)$. It has been observed numerically that both the Padé and the d approximations, which are effective means for producing accurate rational approximations, have these properties.

In [7] some theoretical bounds on $|\varepsilon_r^{(k,n)}|$ have been provided for the general d approximations. In the present work we shall restrict ourselves to a special case of the d approximations, namely the t approximations of Levin [3], and prove convergence results for $\varepsilon_r^{(k,n)}$ and $a_r^{(k,n)}$ for two kinds of limiting processes considered in [5] and [6]. We shall do this by deriving actual *rates* of convergence. Our results show that as more terms of the series $F(z)$ are given, the t approximations predict the next unknown terms with increasing accuracy. Finally, we shall illustrate the results by theoretical and numerical examples.

2. Preliminary results for *t* approximations. The t approximations of Levin [3] for the power series $F(z)$ in (1.1) are defined as in (1.3) with

$$\lambda_j^{(k,n)}=(-1)^{k-j}\binom{k}{j}\left(\frac{n+j}{n+k}\right)^{k-1}\frac{a_{n+k}}{a_{n+j}},\qquad j=0,1,\cdots,k, \tag{2.1}$$

provided none of the a_r is zero. Obviously $\lambda_k^{(k,n)}=1$. (In [3] $A_{n+j-1}(z)$ in (1.3) is replaced by $A_{n+j}(z)$, but it is not difficult to show that the two forms are equivalent.)

Substituting (2.1) in (1.6), we have

$$\begin{aligned}\varepsilon_r^{(k,n)}&=\frac{a_{n+k}}{(n+k)^{k-1}}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(n+j)^{k-1}\frac{a_{n+j+r}}{a_{n+j}}\\&=\frac{a_{n+k}}{(n+k)^{k-1}}\Delta^k\left(n^{k-1}\frac{a_{n+r}}{a_n}\right),\end{aligned} \tag{2.2}$$

where Δ is the forward difference operator operating on the index n. Using the facts that $\Delta^k p(n)=0$ whenever $p(n)$ is a polynomial in n of degree at most $k-1$ and that $\Delta p(n)=p(n+1)-p(n)$, we can see immediately that $\varepsilon_0^{(k,n)}=0$; hence (1.5) and (1.7) become respectively

$$v_{k,n}(z)F(z)-u_{k,n}(z)=z^{n+k}\sum_{r=0}^{\infty}\bar{\varepsilon}_r^{(k,n)}z^r, \tag{2.3}$$

where $\bar{\varepsilon}_r^{(k,n)}=\varepsilon_{r+1}^{(k,n)}$, $r=0,1,\cdots$, and

$$F(z)-T_{k,n}(z)=\sum_{r=n+k+1}^{\infty}(a_r-a_r^{(k,n)})z^{r-1}=O(z^{n+k}). \tag{2.4}$$

Defining

$$b_s^{(k,n)} = a_{n+k+s+1} - a_{n+k+s+1}^{(k,n)}, \qquad s = 0, 1, \cdots, \tag{2.5}$$

and using (2.3), (2.4) becomes

$$\sum_{s=0}^{\infty} b_s^{(k,n)} z^s = \frac{\sum_{r=0}^{\infty} \bar{\varepsilon}_r^{(k,n)} z^r}{\sum_{j=0}^{k} \lambda_j^{(k,n)} z^{k-j}}, \tag{2.6}$$

which provides the following recursion relation for $b_s^{(k,n)}$:

$$\bar{\varepsilon}_r^{(k,n)} = \sum_{j=\max(k-r,0)}^{k} \lambda_j^{(k,n)} b_{r+j-k}^{(k,n)}, \qquad r = 0, 1, \cdots. \tag{2.7}$$

In the sequal we shall assume that the a_r satisfy the following condition:

DEFINITION 2.1. The sequence a_r, $r = 1, 2, \cdots$, is said to satisfy *Condition* A if

$$a_{r+1} = c(r) a_r, \qquad r = 1, 2, \cdots, \tag{2.8}$$

where $c(x)$, as a function of the continuous variable x, has a Poincaré-type asymptotic expansion of the form

$$c(x) \sim \sum_{i=0}^{\infty} c_i / x^{i+p} \quad \text{as } x \to \infty, \quad c_0 \neq 0, \tag{2.9}$$

for some integer p.

Remark 2.1. It has been observed numerically that in order for the t approximations to be effective, the sequence a_r, $r = 1, 2, \cdots$, should satisfy Condition A. Condition A is satisfied by sequences for which

$$a_r = r^{\alpha} \zeta^r w(r) / (r!)^p, \tag{2.10}$$

where α and ζ are constants, and $w(x)$, as a function of the continuous variable x, has a Poincaré-type asymptotic expansion of the form

$$w(x) \sim \sum_{i=0}^{\infty} w_i / x^i \quad \text{as } x \to \infty, \quad w_0 \neq 0, \tag{2.11}$$

and vice versa; see [2, p. 70]. For p a negative integer, $F(z)$ is a divergent asymptotic series. For $p = 0$, $F(z)$ has a finite nonzero radius of convergence. The convergence analysis of the T-transformation (of which the t-transformation is a special case) for such series has been taken up in [6]. For p a positive integer, $F(z)$ has an infinite radius of convergence, hence is an entire function of order p.

As a consequence of Condition A we have the following results:

LEMMA 2.1. *The terms a_{r+j} and a_r are related by*

$$a_{r+j} = c^{(j)}(r) a_r, \qquad j = 1, 2, \cdots, \tag{2.12}$$

where

$$c^{(j)}(r) = \prod_{i=0}^{j-1} c(r+i), \qquad j = 1, 2, \cdots; \tag{2.13}$$

hence $c^{(j)}(x)$, as a function of the continuous variable x, has a Poincaré-type asymptotic expansion of the form

$$c^{(j)}(x) \sim \sum_{i=0}^{\infty} c_i^{(j)} / x^{i+jp} \quad \text{as } x \to \infty, \quad c_0^{(j)} \neq 0. \tag{2.14}$$

LEMMA 2.2. *$\varepsilon_r^{(k,n)}$ can be expressed as*

$$\varepsilon_r^{(k,n)} = \frac{a_{n+k}}{(n+k)^{k-1}} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (n+j)^{k-1} c^{(r)}(n+j) \tag{2.15}$$
$$= \frac{a_{n+k}}{(n+k)^{k-1}} \Delta^k (n^{k-1} c^{(r)}(n)).$$

Proof. The proof of (2.12) and (2.13) in Lemma 2.1 follows directly from (2.8), and the proof of (2.14) follows from (2.13) and (2.9). The proof of Lemma 2.2 is achieved by substituting (2.12) in (2.2). □

3. Convergence properties of $\varepsilon_r^{(k,n)}$, $a_r^{(k,n)}$. In this section we shall investigate the convergence properties of $\varepsilon_r^{(k,n)}$ and $a_r^{(k,n)}$ $(r \geqq 1)$ for two kinds of limiting processes that were considered in [5] and [6]:

(1) Process I: k is fixed, $n \to \infty$;

(2) Process II: n is fixed, $k \to \infty$.

Process I.

THEOREM 3.1. *Let k be fixed. Then*

$$\varepsilon_r^{(k,n)} = a_{n+k} O(n^{-q_r}) \quad \text{as } n \to \infty, \tag{3.1}$$

where

$$q_r = \begin{cases} k + pr & \text{if } p < 0, \\ k + \max(k, pr) & \text{if } p \geqq 0. \end{cases} \tag{3.2}$$

Proof. We start with (2.15). Using (2.14) we have

$$n^{k-1} c^{(r)}(n) \sim \sum_{i=0}^{\infty} c_i^{(r)} n^{k-pr-i-1} \quad \text{as } n \to \infty. \tag{3.3}$$

For $p \geqq 0$ and $pr \leqq k-1$, $n^{k-1}c^{(r)}(n)$ can be written as

$$n^{k-1} c^{(r)}(n) = h_1(n) + h_2(n), \tag{3.4}$$

where $h_1(n) = \sum_{i=0}^{k-pr-1} c_i^{(r)} n^{k-pr-i-1}$ is a polynomial in n of degree $k-pr-1 \leqq k-1$ and $h_2(n) = O(n^{-1})$ as $n \to \infty$. Now $\Delta^k h_1(n) = 0$ and $\Delta^k h_2(n) = O(n^{-k-1})$ as $n \to \infty$. Consequently $\Delta^k(n^{k-1}c^{(r)}(n)) = O(n^{-k-1})$ as $n \to \infty$.

For $p < 0$, or $p \geqq 0$ and $pr \geqq k$, we have $\Delta^k(n^{k-1}c^{(r)}(n)) = O(n^{-pr-1})$ as $n \to \infty$.

Combining these with (2.15), we find that (3.1) and (3.2) follow. □

COROLLARY 3.1. *Let a_r be as in Remark* 2.1. *Then* (1) *when $p > 0$, or* (2) *when $p = 0$, $|\zeta| < 1$, or* (3) *$p = 0$, $|\zeta| = 1$ and $2k > \alpha$, then $\varepsilon_r^{(k,n)} \to 0$ as $n \to \infty$. When $p < 0$, $\varepsilon_r^{(k,n)} \to \infty$ as $n \to \infty$.*

Proof. The proof follows by substituting (2.10) in (3.1). □

THEOREM 3.2. *Let k be fixed. Then*

$$b_s^{(k,n)} = a_{n+k} O(n^{-q_{s+1}}) \quad \text{as } n \to \infty, \tag{3.5}$$

where q_r is as defined in (3.2).

Proof. We shall prove (3.5) by induction. (3.5) is true for $s = 0$, since $b_0^{(k,n)} = \bar{\varepsilon}_0^{(k,n)} = \varepsilon_1^{(k,n)} = a_{n+k}O(n^{-q_1})$ as $n \to \infty$. From (2.7),

$$b_r^{(k,n)} = \bar{\varepsilon}_r^{(k,n)} - \sum_{j=\max}^{k-1} \lambda_j^{(k,n)} b_{r+j-k}^{(k,n)} \qquad r = 1, 2, \cdots. \tag{3.6}$$

By (2.1) and Lemma 2.1 we have

$$\lambda_j^{(k,n)} = (-1)^{k-j}\binom{k}{j}\left(\frac{n+j}{n+k}\right)^{k-1} c^{(k-j)}(n+j) \tag{3.7}$$
$$= O(n^{-(k-j)p}) \quad \text{as } n \to \infty.$$

Using the induction hypothesis that (3.5) is true for $s \leqq r-1$, and substituting (3.1), (3.5) and (3.7) in (3.6), the result follows. □

COROLLARY 3.2. *Let a_r be as in Remark* 2.1. *Then* (1) *when $p>0$, or* (2) *when $p=0$ and $|\zeta|<1$, or* (3) *when $p=0$, $|\zeta|=1$ and $2k>\alpha$, then $b_s^{(k,n)} \to 0$ as $n \to \infty$. When $p<0$, $b_0^{(k,n)} \to \infty$ as $n \to \infty$.*

COROLLARY 3.3. *The relative error in $a_{n+k+s+1}^{(k,n)}$ satisfies*

$$\frac{b_s^{(k,n)}}{a_{n+k+s+1}} = O(n^{(s+1)p-q_{s+1}}) = O(n^{-k}) \quad \text{as } n \to \infty. \tag{3.8}$$

Proof. The proof of Corollary 3.2 is identical to that of Corollary 3.1 of Theorem 3.1. The proof of Corollary 3.3 follows from (3.5) and the fact that $a_{n+k}/a_{n+k+s+1} = 1/c^{(s+1)}(n+k) = O(n^{(s+1)p})$ as $n \to \infty$. □

Process II. We now assume that $c^{(j)}(x)$, in addition to being as in (2.14), are of the form

$$c^{(j)}(x) = x^{1-jp}\bar{\phi}^{(j)}(x) = x^{1-jp}\mathscr{L}[\phi^{(j)}(t); x], \tag{3.9}$$

where

$$\mathscr{L}[g(t); x] = \int_0^\infty e^{-xt} g(t)\, dt \tag{3.10}$$

is the Laplace transform. We also assume that the functions $\phi^{(j)}(t)$ are infinitely differentiable for $0 \leqq t < \infty$.

Using Watson's lemma [4, p. 71] we have

$$c^{(j)}(x) \sim \sum_{i=0}^{\infty} \frac{d^i}{dt^i}\phi^{(j)}(t)\bigg|_{t=0} \Big/ x^{i+pj} \quad \text{as } x \to \infty; \tag{3.11}$$

hence we identify $(d^i/dt^i)\phi^{(j)}(t)|_{t=0}$ as $c_i^{(j)}$.

LEMMA 3.1. *$\varepsilon_r^{(k,n)}$ can be expressed as*

$$\varepsilon_r^{(k,n)} = \frac{a_{n+k}}{(n+k)^{k-1}}\left\{\mathscr{L}\left[(e^{-t}-1)^k \frac{d^{k-rp}}{dt^{k-rp}}\phi^{(r)}(t); n\right] + Q_k(n)\right\}, \tag{3.12}$$

where

$$Q_k(n) = \begin{cases} 0 & \text{if } p \geqq 0, \\ \displaystyle\sum_{i=0}^{-rp-1} \frac{d^i}{dt^i}\phi^{(r)}(t)\bigg|_{t=0} \Delta^k(n^{k-rp-1-i}) & \text{if } p<0. \end{cases} \tag{3.13}$$

Proof. Similar to that of [6, Thm. 4.1]. □

THEOREM 3.3. *Let $\phi^{(j)}(t)$ satisfy*

$$\left|\frac{d^i}{dt^i}\phi^{(j)}(t)\right| \leqq M\gamma^i e^{-\delta t}, \qquad 0 \leqq t < \infty, \quad \textit{for all } i \geqq 0, \tag{3.14}$$

where $M>0$, $\gamma>0$ and δ are constants that depend on j only. Then, for n fixed and

$n+\delta>0$,

$$\varepsilon_r^{(k,n)}=a_{n+k}\begin{cases}O(k^{-k-n-\delta+1}\gamma^k) & \text{if } p\geqq 0,\\ O(k^{-2rp-1/2}e^{-k}) & \text{if } p<0,\end{cases}\quad \text{as } k\to\infty. \tag{3.15}$$

Proof. By (3.14) we have

$$I_{i,j}=\left|\mathscr{L}\left[(e^{-t}-1)^k\frac{d^i}{dt^i}\phi^{(j)}(t);n\right]\right|\leqq M\gamma^i\int_0^\infty e^{-(n+\delta)t}(1-e^{-t})^k\,dt. \tag{3.16}$$

Now

$$\begin{aligned}\int_0^\infty e^{-(n+\delta)t}(e^{-t}-1)^k\,dt&=\Delta^k\left(\int_0^\infty e^{-(n+\delta)t}\,dt\right)\\ &=\Delta^k\left(\frac{1}{n+\delta}\right)\\ &=\frac{(-1)^kk!}{(n+\delta)(n+\delta+1)\cdots(n+\delta+k)}.\end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17), we have

$$I_{i,j}\leqq\frac{M\gamma^ik!\,(n+\delta+1)!}{(n+\delta+k)!}. \tag{3.18}$$

Substituting (3.18) in (3.12), and using Stirling's formula, the result follows for $p\geqq 0$. For $p<0$ we have to analyze $Q_k(n)$ for $k\to\infty$. We know that

$$\Delta^kf(n)=\frac{d^k}{dx^k}f(x)\Big|_{x=\tilde{x}}\quad \text{for some } \tilde{x}\in(n,n+k), \tag{3.19}$$

whenever $f(x)$ is k times differentiable on $(n,n+k)$. Using this and (3.14) in (3.13), we have

$$|Q_k(n)|\leqq O((k-rp-1)!\,k^{-rp-1})\quad \text{as } k\to\infty. \tag{3.20}$$

Hence $|Q_k(n)|$ dominates $I_{k-rp,r}$ in (3.12). The result now follows easily. □

COROLLARY 3.4. *Let a_r be as in Remark 2.1. Then for $p\geqq 0$, $\varepsilon_r^{(k,n)}\to 0$ as $k\to\infty$.*

Proof. The proof follows by substituting (2.10) in (3.15). □

THEOREM 3.4. *Let n be fixed and $n+\delta>0$. Then, as $k\to\infty$,*

$$b_s^{(k,n)}=a_{n+k}\begin{cases}O(k^{-k-n-\delta+1}\gamma^k) & \text{if } p>0,\\ O(k^{-k-n-\delta+s+1}\gamma^k) & \text{if } p=0,\\ O(k^{-2(s+1)p-1/2}e^{-k}) & \text{if } p<0.\end{cases} \tag{3.21}$$

Proof. We shall prove (3.21) by induction. (3.21) is true for $s=0$ by $b_0^{(k,n)}=\bar{\varepsilon}_0^{(k,n)}=\varepsilon_1^{(k,n)}$ and Theorem 3.3. Now for r fixed and k sufficiently large, (3.6) becomes

$$b_r^{(k,n)}=\bar{\varepsilon}_r^{(k,n)}-\sum_{j=k-r}^{k}\lambda_j^{(k,n)}b_{r+j-k}^{(k,n)}=\bar{\varepsilon}_r^{(k,n)}-\sum_{s=0}^{r-1}\lambda_{k-r+s}^{(k,n)}b_s^{(k,n)}. \tag{3.22}$$

Whenever j is fixed,

$$\binom{k}{j}=\frac{k(k-1)\cdots(k-j+1)}{j!}=O(k^j)\quad \text{as } k\to\infty. \tag{3.23}$$

Hence, for r fixed and $0 \leqq s \leqq r-1$,

$$\begin{aligned}\lambda_{k-r+s}^{(k,n)} &= (-1)^{r-s}\binom{k}{r-s}\left(\frac{n+k-r+s}{n+k}\right)^{k-1} c^{(r-s)}(n+k-r+s)\\ &= O(k^{(1-p)(r-s)}) \quad \text{as } k \to \infty.\end{aligned} \tag{3.24}$$

Combining (3.15), (3.24) and the induction hypothesis that (3.21) is true for $s \leqq r-1$ in (3.22), the result follows. □

COROLLARY 3.5. *When a_r are as in Remark 2.1, $b_s^{(k,n)} \to 0$ for $p \geqq 0$.*

COROLLARY 3.6. *The relative error in $a_{n+k+s+1}^{(k,n)}$, as $k \to \infty$, satisfies*

$$\frac{b_s^{(k,n)}}{a_{n+k+s+1}} = \begin{Bmatrix} O(k^{-k-n-\delta+(s+1)p+1}\gamma^k) & \text{if } p>0 \\ O(k^{-k-n-\delta+s+1}\gamma^k) & \text{if } p=0 \\ O(k^{-(s+1)p-1/2}e^{-k}) & \text{if } p<0 \end{Bmatrix} = o(1). \tag{3.25}$$

Proof. Similar to that of Corollary 3.3 of Theorem 3.2. □

Remark 3.1. From Corollary 3.3 to Theorem 3.2, it is seen that the relative error in $a_{n+k+s+1}^{(k,n)}$, as $n \to \infty$, behaves like n^{-2k} at best. Corollary 3.6 to Theorem 3.4, on the other hand, tells us that the relative error in $a_{n+k+s+1}^{(k,n)}$, as $k \to \infty$, behaves essentially like e^{-k} at worst. This comparison again brings out the fact that Process II is a much better acceleration method than Process I.

The assumptions (3.9) and (3.14) on $c^{(j)}(x)$ that were used in obtaining our results for Process II may seem artificial at first. However, as we shall show in the next section, they are natural at least for a large class of series of the form (2.10)–(2.11).

In summary, our results for both Processes I and II show under the conditions stated above that, as more terms of the series $F(z)$ are given, the t approximations can predict the next unknown terms with increasing accuracy.

4. Examples. In this section we shall illustrate some of the results of § 3 with examples. In the first two examples we shall show that the conditions in (3.9) and (3.14) are satisfied by some series of interest. In the third example we shall give some numerical results for Process II.

Example 1. We take a_r as in (2.10) with α being an integer and $w(r)$ being a rational function of r. Then the a_r are of the form

$$a_r = \frac{P(r)}{Q(r)}\frac{1}{(r!)^p}, \tag{4.1}$$

with $P(r)$ and $Q(r)$ being polynomials of degrees d_1 and d_2 respectively. Then

$$c^{(j)}(r) = \frac{a_{r+j}}{a_r} = \frac{P(r+j)Q(r)}{Q(r+j)P(r)}\frac{1}{[(r+1)\cdots(r+j)]^p}. \tag{4.2}$$

Therefore, $\bar{\phi}^{(j)}(x)$ is a rational function having the partial fraction decomposition

$$\bar{\phi}^{(j)}(x) = x^{jp-1}c^{(j)}(x) = \sum_{i=1}^{m}\sum_{l=1}^{q_i}\frac{B_{il}}{(x+\delta_i)^l}, \tag{4.3}$$

where $\sum_{i=1}^{m} q_i = d_1 + d_2 + j|p| + 1$. From (4.3) we have

$$\phi^{(j)}(t) = \sum_{i=1}^{m}\sum_{l=1}^{q_i}\frac{B_{il}}{(l-1)!}t^{l-1}e^{-\delta_i t}. \tag{4.4}$$

Now

$$\frac{d^k}{dt^k}(t^l e^{\sigma t}) = \sum_{i=0}^{l} \binom{k}{i} l(l-1)\cdots(l-i+1)t^{l-i}\sigma^{k-i} e^{\sigma t}. \tag{4.5}$$

After some tedious manipulations it can be shown that

$$\left|\frac{d^k}{dt^k}(t^l e^{\sigma t})\right| \leqq B\tilde{\gamma}^k e^{\tilde{\sigma} t} \quad \text{for } 0 \leqq t < \infty, \tag{4.6}$$

for any $\tilde{\sigma} > \sigma$, any $\tilde{\gamma} > \max(|\sigma|, 1)$ and some $B > 0$ independent of k. Combining (4.6) with (4.4), we have

$$\left|\frac{d^k}{dt^k}\phi^{(j)}(t)\right| \leqq M\gamma^k e^{-\delta t} \quad \text{for } 0 \leqq t < \infty, \tag{4.7}$$

for any $\delta > \max\{\delta_1, \cdots, \delta_m\}$, any $\gamma > \max\{|\delta_1|, \cdots, |\delta_m|, 1\}$, and some $M > 0$ independent of k.

When $\bar{\phi}^{(j)}(x)$ has simple poles only, i.e.,

$$\bar{\phi}^{(j)}(x) = \sum_{i=1}^{m} \frac{B_i}{x+\delta_i}, \tag{4.8}$$

we can start with (3.12) and prove that, for $p = 0$,

$$\varepsilon_r^{(k,n)} = (-1)^k k! \frac{a_{n+k}}{(n+k)^{k-1}} \sum_{i=1}^{m} \frac{B_i \delta_i^k}{(n+\delta_i)\cdots(n+\delta_i+k)}, \tag{4.9}$$

in the same way that Theorem 3.3 was proved.

Example 2. $a_r = \sqrt{r}$. For this case

$$\mathscr{L}[\phi^{(j)}(t); n] = \bar{\phi}^{(j)}(n) = \frac{1}{n^{1/2}(n+j)^{1/2}} + \frac{j}{n^{3/2}(n+j)^{1/2}}. \tag{4.10}$$

The inverse transform $\phi^{(j)}(t)$ is then given by

$$\phi^{(j)}(t) = (1+jt)\, e^{-jt/2} I_0\left(\frac{jt}{2}\right) + jt\, e^{-jt/2} I_1\left(\frac{jt}{2}\right) \tag{4.11}$$

[1, p. 1024, formulas 29.3.49 and 29.3.51]. From

$$I_0(z) = \frac{1}{\pi}\int_0^{\pi} e^{\pm z\cos\theta}\, d\theta \tag{4.12}$$

[1, p. 376, formula 9.6.16], for real z we have

$$\left|\frac{d^k}{dz^k} I_0(z)\right| \leqq I_0(z) \quad \text{for all } k. \tag{4.13}$$

Also $I_1(z) = (d/dz)I_0(z)$. Combining these results with those of Example 1, we can show after some manipulation that $\phi^{(j)}(t)$ satisfies the conditions of Theorem 3.3.

Example 3. In this example we give numerical results for $\Delta_r = |(a_r - a_r^{(k,n)})/a_r|$ for the two sequences $a_r = 1/r$, $r = 1, 2, \cdots$, and $a_r = 1/(r-1)!$, $r = 1, 2, \cdots$. Tables 4.1 and 4.2 contain the results for Process II with $n = 1$, $1 \leqq k \leqq 10$. Note that for both examples $\Delta_r = 0$ for $1 \leqq r \leqq n+k$ as implied by (2.4). For $a_r = 1/(r-1)!$, however, we also have $\Delta_{n+k+1} = 0$, which follows from (2.2).

TABLE 4.1.
Relative errors in $a_r^{(k,n)}$, $k+2 \leqq r \leqq k+6$, *for* $n=1$, $1 \leqq k \leqq 10$, *when* $a_r = 1/r$, $r = 1, 2, \cdots$. $a_r^{(k,n)} = a_r$ *for* $1 \leqq r \leqq k+1$. Δ_r *stands for* $|(a_r - a_r^{(k,n)})/a_r|$.

k	Δ_{k+2}	Δ_{k+3}	Δ_{k+4}	Δ_{k+5}	Δ_{k+6}
1	0.25 (0)	0.50 (0)	0.69 (0)	0.81 (0)	0.94 (0)
2	.37 (−1)	.12 (0)	.22 (0)	.33 (0)	.43 (0)
3	.39 (−2)	.18 (−1)	.46 (−1)	.86 (−1)	.14 (0)
4	.32 (−3)	.23 (−2)	.77 (−2)	.18 (−1)	.33 (−1)
5	.21 (−4)	.25 (−3)	.11 (−2)	.31 (−2)	.67 (−2)
6	.12 (−5)	.23 (−4)	.13 (−3)	.45 (−3)	.12 (−2)
7	.60 (−7)	.19 (−5)	.14 (−4)	.60 (−4)	.18 (−3)
8	.28 (−8)	.14 (−6)	.14 (−5)	.72 (−5)	.25 (−4)
9	.10 (−9)	.97 (−8)	.13 (−6)	.78 (−6)	.32 (−5)
10	.36 (−11)	.61 (−9)	.10 (−7)	.79 (−7)	.38 (−6)

TABLE 4.2.
Relative errors in $a_r^{(k,n)}$, $k+3 \leqq r \leqq k+7$, *for* $n=1$, $1 \leqq k \leqq 10$, *when* $a_r = 1/(r-1)!$, $r = 1, 2, \cdots$. $a_r^{(k,n)} = a_r$ *for* $1 \leqq r \leqq k+2$, $k \geqq 2$. Δ_r *stands for* $|(a_r - a_r^{(k,n)})/a_r|$.

k	Δ_{k+3}	Δ_{k+4}	Δ_{k+5}	Δ_{k+6}	Δ_{k+7}
1	0.10 (1)	0.50 (1)	0.23 (2)	0.12 (3)	0.72 (3)
2	.33 (0)	.21 (1)	.88 (1)	.30 (2)	.81 (2)
3	.63 (−1)	.34 (0)	.11 (1)	.29 (1)	.61 (1)
4	.80 (−2)	.21 (−1)	.36 (−1)	.48 (0)	.24 (1)
5	.77 (−3)	.32 (−2)	.40 (−1)	.20 (0)	.67 (0)
6	.59 (−4)	.11 (−2)	.76 (−2)	.23 (−1)	.32 (−1)
7	.38 (−5)	.19 (−3)	.66 (−3)	.12 (−2)	.19 (−1)
8	.21 (−6)	.23 (−4)	.24 (−4)	.95 (−3)	.54 (−2)
9	.10 (−7)	.24 (−5)	.20 (−4)	.19 (−3)	.62 (−3)
10	.42 (−9)	.21 (−6)	.41 (−5)	.21 (−4)	.14 (−4)

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Note. Rational approximations of the form (1.3) are referred to as Padé-type approximations in C. Brezinski's book, *Padé-Type Approximation and General Orthogonal Polynomials* (Birkhäuser Verlag, Basel, 1980). The idea of using the Maclaurin series coefficients of Padé approximants for predicting numerically the next unknown coefficients can also be found in J. Gilewicz's book, *Approximants de Padé* (Springer-Verlag, Berlin, 1978, pp. 424–439). The authors would like to thank one of the referees for drawing their attention to these references.

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