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Convergence of Exponential Interpolation for Completely Bounded Functions

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The authors study convergence of certain exponential sums that interpolate to functions which are the Laplace–Stieltjes transform of a signed or complex distribution. They prove uniform convergence in unbounded sectors of the right half plane and establish rates of convergence in certain cases. For the special case where the function is completely monotone, the results generalize and improve theorems of D. W. Kammler [*J. Math. Anal. Appl.* **57** (1977), 560–570].

1. INTRODUCTION

In [5], Kammler investigated interpolation by exponential sums to functions $f(t)$ completely monotone on $[0, \infty]$, that is, functions $f(t)$ having a representation

$$f(t) = \int_0^{\infty} e^{-tx}\, d\alpha(x), \qquad t \in [0, \infty],$$

where $\alpha(x)$ is real valued, monotone increasing and $\lim_{x\to\infty} \alpha(x) = \alpha(\infty) < \infty$. We can assume $\alpha(0) = 0$. In Theorem 3 of [5], Kammler proved uniform convergence of the interpolating sums in $[0, \infty]$ for quite general choices of interpolation points.

The question arises: To what extent can the positivity of the distribution $d\alpha(x)$ be relaxed? In this note, the authors show that $d\alpha(x)$ may be replaced by a signed or complex distribution $d\beta(x)$, provided the exponents of the exponential sums are chosen in a certain way, and provided one interpolates at a smaller number of points.

A second question arises: Do the exponential sums converge in the right half plane, in view of the analyticity of Laplace transforms in a half plane?

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In fact, it follows from the uniform boundedness of the exponential sums in Theorem 3 in [5] that they converge uniformly in compact subsets of the right half plane. Here we prove uniform convergence in unbounded sectors, under more general conditions. Further, for certain choices of the interpolation points, we establish rates of convergence.

2. Notation

Definition 2.1. A function $\beta(x)$: $[0, \infty] \to \mathbb{C}$ is said to be of bounded variation in $[0, \infty]$ if

(i) $\beta(x)$ is of bounded variation in each compact subinterval of $[0, \infty)$,

(ii) $\beta(x)$ is right continuous in $[0, \infty)$,

(iii) $\beta(0) = 0$,

(iv) $\beta(\infty) = \lim_{x \to \infty} \beta(x)$ is a finite complex number.

We use $\beta^{r+}(x)$, $\beta^{r-}(x)$, $\beta^{i+}(x)$, $\beta^{i-}(x)$ to denote, respectively, the real and imaginary upper and lower variations of $\beta(x)$, so that they are monotone increasing and right continuous in $[0, \infty)$ and

$$\beta(x) = \{\beta^{r+}(x) - \beta^{r-}(x)\} + i\{\beta^{i+}(x) - \beta^{i-}(x)\}, \qquad x \in [0, \infty].$$

We assume the four functions are normalized to have the value zero at $x = 0$. The total variation function of $\beta(x)$ is $|\beta|(x) = \beta^{r+}(x) + \beta^{r-}(x) + \beta^{i+}(x) + \beta^{i-}(x)$, $x \in [0, \infty]$.

Throughout $\alpha(x)$ will denote a function of bounded variation in $[0, \infty]$ that is also real valued and monotone increasing there.

Definition 2.2. A function $\beta(x)$ of bounded variation in $[0, \infty]$ is said to be absolutely continuous with respect to $\alpha(x)$ if there is a (complex valued) function $d\beta/d\alpha(x)$, defined for almost all x in $[0, \infty)$ such that

$$\beta(x) = \int_0^x \frac{d\beta}{d\alpha}(u)\, d\alpha(u), \qquad x \in [0, \infty],$$

except possibly at discontinuities of $\alpha(x)$, and such that

$$\int_0^\infty \left| \frac{d\beta}{d\alpha}(u) \right|^2 d\alpha(u) < \infty. \tag{2.1}$$

DEFINITION 2.3. We shall denote the Laplace–Stieltjes transform by $\mathscr{L}$. More precisely,

$$\mathscr{L}[d\alpha](t) = \int_0^\infty e^{-tx}\, d\alpha(x), \qquad \mathrm{Re}(t) \geqslant 0,$$

is a function completely monotone in $[0, \infty[$, while if $\beta(x)$ is of bounded variation in $[0, \infty[$, then we shall say that

$$\mathscr{L}[d\beta](t) = \int_0^\infty e^{-tx}\, d\beta(x), \qquad \mathrm{Re}(t) \geqslant 0,$$

is a function completely bounded in $[0, \infty[$.

DEFINITION 2.4. For $n = 1, 2, 3, \ldots$ let positive integers $m(n)$ and interpolation points $0 \leqslant t_{n1} < t_{n2} < \cdots < t_{nm(n)} < \infty$ be given. We shall say $\{t_{nj}\}_{n,j}$ is a uniqueness set if there exist numbers $\tau_k > r_k > 0$, $k = 1, 2, 3 \ldots$ such that

(i) the intervals $(\tau_k - r_k, \tau_k + r_k)$, $k = 1, 2, 3, \ldots$, are all disjoint;

(ii) if $l_k = \liminf_{n \to \infty} |(\tau_k - r_k, \tau_k + r_k) \cap \{t_{n1}, t_{n2}, \cdots t_{nm(n)}\}|$, $k = 1, 2, 3, \ldots$, then

$$\sum_{k=1}^{\infty} l_k(\tau_k - r_k)/\{1 + (\tau_k + r_k)^2\} = \infty. \tag{2.2}$$

Here $|\;|$ denotes the cardinality of a set, so that l_k is an asymptotic lower bound for the number of interpolation points in $(\tau_k - r_k, \tau_k + r_k)$. When some $l_k = \infty$, the series in (2.2) is interpreted as ∞.

In Theorem 3 of [5], Kammler used a restricted form of the above uniqueness condition: he assumed all $r_k = 0$ and all $l_k = 1$, in effect. The motivation for the term "uniqueness set" will become clearer in Corollary 3.2. For the moment we note:

LEMMA 2.5. *Let $f(z)$ be bounded and analytic in $\{z: \mathrm{Re}(z) > 0\}$ with zeroes z_k, $k = 1, 2, 3, \ldots$, repeated according to multiplicity. If*

$$\sum_{k=1}^{\infty} \mathrm{Re}(z_k)/(1 + |z_k|^2) = \infty$$

then $f(z) \equiv 0$ in $\{z: \mathrm{Re}(z) > 0\}$.

Proof. See Hille [3, Theorem 19.2.6]. ∎

It is easy to see that the following choices of interpolation points all yield uniqueness sets:

$$t_{nj} = j, \qquad j = 1, 2, \ldots n; n = 1, 2, \ldots;$$

$$t_{nj} = 1 + (j-1)/n, \qquad j = 1, 2, \ldots n; n = 1, 2, \ldots;$$

$$t_{nj} = 1 + (j-1)/(n \log n), \qquad j = 1, 2, \ldots n; n = 1, 2, \ldots,$$

$$t_{nj} = 1/j, j = 1, 2, \ldots n; n = 1, 2, \ldots.$$

The last two sets of interpolation points do not satisfy the conditions in Theorem 3 of [5].

Given $0 < \theta_0 < \pi$, the sector with vertex at 0, angle $2\theta_0$, that is symmetric with respect to the positive real axis is

$$\mathscr{S}(\theta_0) = \{z = re^{i\theta} : |\theta| < \theta_0\},$$

while given $\varepsilon > 0$, we let

$$\mathscr{S}(\theta_0, \varepsilon) = \{z = re^{i\theta} : r > \varepsilon > 0 \text{ and } |\theta| < \theta_0\}.$$

Finally, given a function $g(x)$ defined on $[0, 1]$, we set

$$\|g\|_{\infty} = \sup\{|g(x)| : x \in [0, 1]\}$$

and

$$\|g\|_p = \left(\int_0^1 |g(x)|^p \, dx\right)^{1/p}, \qquad 0 < p < \infty,$$

whenever this is defined and finite.

3. Uniform Convergence

We first establish a convergence result for Laplace–Stieltjes transforms which is possibly of independent interest.

THEOREM 3.1. *Suppose $\beta(x)$, $\beta_n(x)$, $n = 1, 2, \ldots$ are complex valued functions of bounded variation in $[0, \infty]$. Denote their total variation functions by $|\beta|(x)$, $|\beta|_n(x)$, $n = 1, 2, \ldots$ Assume that*

$$\lim_{n \to \infty} \beta_n(x) = \beta(x); \qquad \lim_{n \to \infty} |\beta|_n(x) = |\beta|(x) \tag{3.1A}$$

for almost all $x \in [0, \infty)$, *and*

$$\lim_{n \to \infty} |\beta|_n(\infty) = |\beta|(\infty). \tag{3.1B}$$

Let $f(t) = \mathscr{L}|d\beta|(t)$ *and* $f_n(t) = \mathscr{L}|d\beta_n|(t)$, $\operatorname{Re}(t) \geqslant 0$ *and* $n = 1, 2, \ldots$ *Then*

$$\lim_{n \to \infty} f_n(t) = f(t), \qquad \operatorname{Re}(t) \geqslant 0, \tag{3.2}$$

and the convergence is uniform in $\mathscr{S}(\theta_0)$ *for any* $0 < \theta_0 < \pi/2$. *If only* (3.1A) *holds for almost all* $x \in [0, \infty)$, *and* $\sup_n |\beta|_n(\infty) < \infty$, *then*

$$\lim_{n \to \infty} f_n(t) = f(t), \qquad \operatorname{Re}(t) > 0,$$

and the convergence is uniform in $\mathscr{S}(\theta_0, \varepsilon)$ *for any* $0 < \theta_0 < \pi/2$ *and* $\varepsilon > 0$.

Proof. The convergence (3.2) is an immediate consequence of Helly's Convergence Theorem (Freud [2, p. 56]), of (3.1B), and of the uniform boundedness of e^{-tx} for $\operatorname{Re}(t) \geqslant 0$ and $x \geqslant 0$. Assuming (3.1A, B) we shall prove uniform convergence in $\mathscr{S}(\theta_0)$ for fixed $0 < \theta_0 < \pi/2$. Let $\eta > 0$. Firstly, we can choose $0 < S < T < \infty$ such that (3.1A) holds for $x = S, T$ and

$$\int_0^S d(|\beta|(x) + |\beta|_n(x)) < \eta/2, \qquad n > N, \tag{3.3A}$$

$$\int_T^\infty d(|\beta|(x) + |\beta|_n(x)) < \eta/2, \qquad n > N. \tag{3.3B}$$

Here N is a positive integer and we have used (3.1A, B). Then we see by (3.3A, B),

$$\left| \int_0^S e^{-tx}\, d\beta(x) - \int_0^S e^{-tx}\, d\beta_n(x) \right| + \left| \int_T^\infty e^{-tx}\, d\beta(x) - \int_T^\infty e^{-tx}\, d\beta_n(x) \right| < \eta, \qquad \operatorname{Re}(t) \geqslant 0. \tag{3.4}$$

Next for $t \in \mathscr{S}(\theta_0)$ and $x \in [S, T]$, we see

$$\begin{aligned} |te^{-tx}| &\leqslant \sec(\theta_0) \operatorname{Re}(t)\, e^{-\operatorname{Re}(t)x} \\ &\leqslant \sec(\theta_0) \max\{ue^{-ux} : u > 0\} \\ &= \sec(\theta_0)(ex)^{-1}. \end{aligned}$$

Then integrating by parts, we have, for all $t \in \mathscr{S}(\theta_0)$,

$$\begin{aligned}
&\left|\int_S^T e^{-tx}\,d\beta(x)-\int_S^T e^{-tx}\,d\beta_n(x)\right| \\
&\quad= \left|\{e^{-tx}(\beta(x)-\beta_n(x))\}_{x=S}^{x=T}+\int_S^T te^{-tx}(\beta(x)-\beta_n(x))\,dx\right| \\
&\quad\leqslant |\beta(T)-\beta_n(T)|+|\beta(S)-\beta_n(S)| \\
&\qquad+\sec(\theta_0)\int_S^T (ex)^{-1}\,|\beta(x)-\beta_n(x)|\,dx.
\end{aligned} \tag{3.5}$$

The right member of (3.5) is independent of t and converges to 0 as $n \to \infty$ by Lebesgue's Dominated Convergence Theorem and our assumptions on S and T. Then by (3.4) and (3.5), we obtain for all large enough n, and for all $t \in \mathscr{S}(\theta_0)$,

$$\begin{aligned}
|f(t)-f_n(t)| &= \left|\left[\int_0^S+\int_S^T+\int_T^\infty\right] e^{-tx}\,d(\beta(x)-\beta_n(x))\right| \\
&< 2\eta.
\end{aligned}$$

This completes the first part of the theorem.

Remark. When (3.1A) holds for almost all $x \in [0, \infty)$ but (3.1B) does not, then there is no longer necessarily uniform convergence in $\mathscr{S}(\theta_0)$ nor even convergence for $Re(t)=0$, as shown by the following example: Let $d\beta_n(x)$ be a Dirac delta of unit mass at $x=n$, $n=1, 2, \ldots$ and $\beta(x)=0$, $x \in [0, \infty]$. Then $f_n(t)=\mathscr{L}[d\beta_n](t)=e^{-nt}$ and $f_n(1/n)=e^{-1}$ and $f_n(0)=1$, $n=1, 2, \ldots$ while $f(t)\equiv 0$ so $f_n(t)$ does not converge uniformly in $\mathscr{S}(\theta_0)$ and $f_n(0)$ does not converge to $f(0)$.

Finally, suppose (3.1A) holds with $\sup_n |\beta|_n(\infty) < \infty$, and fix $0<\theta_0<\pi/2$ and $\varepsilon>0$. We write

$$|f(t)-f_n(t)| = \left|\left[\int_0^S+\int_S^\infty\right] e^{-tx}\,d(\beta(x)-\beta_n(x))\right|$$

and use (3.3A) to estimate $\int_0^S \cdots$ as before. For the second integral we use an integration by parts to deduce

$$\left|\int_S^\infty e^{-tx}\,d(\beta(x)-\beta_n(x))\right| \leqslant |\beta(S)-\beta_n(S)|+\int_S^\infty |te^{-tx}|\,|\beta(x)-\beta_n(x)|\,dx.$$

Further, we use the estimate

$$\begin{aligned} |te^{-tx}| &\leqslant \sec(\theta_0)\,\mathrm{Re}(t)\,e^{-\mathrm{Re}(t)\,x/2}e^{-\mathrm{Re}(t)\,x/2} \\ &\leqslant 2\sec(\theta_0)(ex)^{-1}, \qquad t \in \mathscr{S}(\theta_0, \varepsilon), \end{aligned}$$

and the result follows as before. ∎

The following corollary generalizes and strengthens Theorem 3 in Kammler [5].

COROLLARY 3.2. *Let* $f(t) = \mathscr{L}[d\alpha](t)$, $\mathrm{Re}(t) \geqslant 0$. *For* $n = 1, 2, \ldots$ *let* $2n$ *interpolation points* $0 \leqslant t_{n1} < t_{n2} < \cdots < t_{n,2n}$ *be given and assume* $\{t_{nj}\}_{n,j}$ *is a uniqueness set. Let* $f_n(t)$ *be the real exponential sum involving* n *exponents such that* $f_n(t_{nj}) = f(t_{nj})$, $j = 1, 2, \ldots 2n$; $n = 1, 2, \ldots$. *Then*

$$\lim_{n \to \infty} f_n(t) = f(t), \qquad \mathrm{Re}(t) \geqslant 0,$$

and the convergence is uniform in the unbounded sector $\mathscr{S}(\theta_0)$ *for any* $0 < \theta_0 < \pi/2$.

Proof. In Lemmas 4 and 5 in [4], Kammler shows that $f_n(t) = \mathscr{L}[d\alpha_n](t)$, $n = 1, 2, 3, \ldots$, where $\alpha_n(x)$ is of bounded variation in $[0, \infty]$ and is, further, real valued and monotone increasing with

$$\alpha_n(\infty) \leqslant \alpha(\infty), \qquad n = 1, 2, \ldots \tag{3.6}$$

(His notation is different.) By Helly's Selection Theorem (Freud [2, p. 56]), we can find a sequence of integers $\mathscr{I}$ and a monotone increasing function $\gamma(x)$, also of bounded variation in $[0, \infty]$, such that

$$\lim_{\substack{n \to \infty \\ n \in \mathscr{I}}} \alpha_n(x) = \gamma(x) \qquad \text{for almost all } x \in [0, \infty).$$

Then by Theorem 3.1,

$$\lim_{\substack{n \to \infty \\ n \in \mathscr{I}}} f_n(t) = \mathscr{L}[d\gamma](t) = g(t) \qquad \text{uniformly in } \mathscr{S}(\theta_0, \varepsilon), \tag{3.7}$$

any $\varepsilon > 0$, $0 < \theta_0 < \pi/2$. Now let $\{\tau_k\}$ and $\{r_k\}$ be as in Definition 2.4. Fix a positive integer k. By hypothesis $f(t) - f_n(t)$ has at least l_k zeroes in $(\tau_k - r_k, \tau_k + r_k)$ for large n. Then by Hurwitz' Theorem, $f(t) - g(t)$ has at least l_k zeroes, counting multiplicity, in $\{z : |z - \tau_k| \leqslant r_k\}$, in view of (3.7).

These zeroes have real parts $\geqslant \tau_k - r_k$, and moduli $\leqslant \tau_k + r_k$. Thus if $z_1, z_2, \ldots$ are the zeroes of $f(t) - g(t)$,

$$\sum_{k=1}^{\infty} \frac{\operatorname{Re}(z_k)}{1+|z_k|^2} \geqslant \sum_{k=1}^{\infty} \frac{l_k(\tau_k - r_k)}{1+(\tau_k + r_k)^2} = \infty$$

by (2.2). By Lemma 2.5,

$$\mathscr{L}|d(\alpha - \gamma)|(t) = f(t) - g(t) = 0, \qquad \operatorname{Re}(t) \geqslant 0,$$

and by Theorem 19.4.2 in Hille [3], $\alpha(x) \equiv \gamma(x)$. Since the same argument applies to all subsequences of $\{1, 2, 3 \ldots\}$ we have shown

$$\lim_{n \to \infty} \alpha_n(x) = \alpha(x), \qquad \text{almost all } x \in [0, \infty).$$

Finally, for almost all $x > 0$, (3.6) yields

$$\alpha(\infty) \geqslant \limsup_{n \to \infty} \alpha_n(\infty) \geqslant \liminf_{n \to \infty} \alpha_n(\infty) \geqslant \lim_{n \to \infty} \alpha_n(x) = \alpha(x),$$

and letting $x \to \infty$, we obtain

$$\lim_{n \to \infty} \alpha_n(\infty) = \alpha(\infty).$$

Thus (3.1A, B) hold for $\beta = \alpha$, $\beta_n = \alpha_n$, $n = 1, 2, \ldots$, and the result follows from Theorem 3.1. ∎

4. Completely Bounded Functions

First we establish the existence of exponential sums involving n exponents, which interpolate to completely bounded functions $\mathscr{L}|d\beta|$ at n points, and for which the sum of the moduli of the "weights" is bounded independent of n. It is interesting to note that both the exponents and interpolation points do not depend on the distribution $d\beta(x)$.

Theorem 4.1. *Let $\beta(x)$ be of bounded variation in $[0, \infty)$ and absolutely continuous with respect to $\alpha(x)$. Let*

$$f(t) = \mathscr{L}|d\beta|(t), \qquad \operatorname{Re}(t) \geqslant 0.$$

Let n be a positive integer and let nonnegative t_{n1} and positive h_n be given. Let

$$t_{nj} = t_{n1} + (j-1)\,h_n, \qquad j = 1, 2 \ldots n, \tag{4.1}$$

and

$$\gamma_n(\sigma) = \int_{-(\log \sigma)/h_n}^{\infty} e^{-2t_{n1}x}\, d\alpha(x), \qquad \sigma \in [0, 1]. \tag{4.2}$$

Let $0 < \sigma_{n1} < \sigma_{n2} < \cdots < \sigma_{nn} < 1$ be the abscissas in the Gauss–Jacobi quadrature of order n for $d\gamma_n(\sigma)$ and let

$$\mu_{nj} = -(\log \sigma_{nj})/h_n, \qquad j = 1, 2 \ldots n. \tag{4.3}$$

Then there is an exponential sum

$$f_n(t) = \sum_{j=1}^{n} w_{nj}\, e^{-t\mu_{nj}} \tag{4.4}$$

satisfying

$$f_n(t_{nj}) = f(t_{nj}), \qquad j = 1, 2 \ldots n, \tag{4.5}$$

and

$$\sum_{j=1}^{n} |w_{nj}| \leqslant \Gamma = \left\{ \alpha(\infty) \int_0^{\infty} \left| \frac{d\beta}{d\alpha}(x) \right|^2 d\alpha(x) \right\}^{1/2}. \tag{4.6}$$

Proof. Firstly

$$\begin{aligned} f(t_{nj}) &= \int_0^{\infty} e^{-(t_{n1}+(j-1)h_n)x}\, d\beta(x) \\ &= \int_0^{\infty} \{e^{-h_n x}\}^{j-1} \left\{ \frac{d\beta}{d\alpha}(x)\, e^{t_{n1}x} \right\} e^{-2t_{n1}x}\, d\alpha(x) \\ &= \int_0^{1} \sigma^{j-1} H(\sigma)\, d\gamma_n(\sigma), \qquad j = 1, 2 \ldots n, \end{aligned} \tag{4.7}$$

where $\sigma = e^{-h_n x}$ and

$$\begin{aligned} H(\sigma) &= \frac{d\beta}{d\alpha}(x)\, e^{t_{n1}x} \\ &= \frac{d\beta}{d\alpha}(-(\log \sigma)/h_n)\, \sigma^{-t_{n1}/h_n}, \qquad \sigma \in [0, 1]. \end{aligned}$$

and where we have used (4.2). Note that

$$\int_0^1 |H(\sigma)|^2 \, d\gamma_n(\sigma) = \int_0^\infty \left| \frac{d\beta}{d\alpha}(x) \right|^2 d\alpha(x) < \infty. \tag{4.8}$$

Now as $\gamma_n(\sigma)$ is monotone increasing, there are positive weights λ_{nj} and abscissas σ_{nj} such that

$$\sum_{j=1}^{n} \lambda_{nj} P(\sigma_{nj}) = \int_0^1 P(\sigma)\, d\gamma_n(\sigma) \tag{4.9}$$

whenever $P(\sigma)$ is a polynomial of degree at most $2n-1$. Further, $\gamma_n(\sigma)$ generates a sequence of orthonormal polynomials and $H(\sigma)$ has an (formal) orthonormal series expansion in these polynomials in view of (4.8). Let $s(\sigma)$ denote the partial sum of the first n terms of this expansion, so that $s(\sigma)$ is a polynomial of degree at most $n-1$. Let

$$w_{nj} = \lambda_{nj} s(\sigma_{nj})\, e^{u_{nj} t_{n1}}, \qquad j = 1, 2 \ldots n.$$

Then if $f_n(t)$ is given by (4.4),

$$f_n(t_{nj}) = \sum_{k=1}^{n} w_{nk}\, e^{-(t_{n1}+(j-1)h_n)u_{nk}}$$

$$= \sum_{k=1}^{n} \lambda_{nk}\, s(\sigma_{nk})\, \sigma_{nk}^{j-1}$$

(by (4.3))

$$= \int_0^1 s(\sigma)\, \sigma^{j-1}\, d\gamma_n(\sigma)$$

(by (4.9) and as $s(\sigma)\,\sigma^{j-1}$ has degree $\leqslant 2n-2$)

$$= \int_0^1 H(\sigma)\, \sigma^{j-1}\, d\gamma_n(\sigma)$$

(as $H(\sigma) - s(\sigma)$ is orthogonal to $1, \sigma, \sigma^2 \cdots \sigma^{n-1}$)

$$= f(t_{nj}), \qquad j = 1, 2 \ldots n,$$

by (4.7), so establishing (4.5). Next we prove (4.6). By the Cauchy–Schwarz inequality

$$\sum_{j=1}^{n} |w_{nj}| \leqslant \left\{ \sum_{j=1}^{n} \lambda_{nj}\, e^{2u_{nj}t_{n1}} \right\}^{1/2} \left\{ \sum_{j=1}^{n} \lambda_{nj} |s(\sigma_{nj})|^2 \right\}^{1/2}$$

$$= \left\{ \sum_{j=1}^{n} \lambda_{nj}\, \sigma_{nj}^{-2t_{n1}/h_n} \right\}^{1/2} \left\{ \int_0^1 |s(\sigma)|^2\, d\gamma_n(\sigma) \right\}^{1/2} \tag{4.10}$$

by (4.3), (4.9) and as $|s(\sigma)|^2 = s(\sigma)\bar{s}(\sigma)$ is a polynomial of degree at most $2n-2$. Next, the function $g(\sigma) = \sigma^{-2t_{n1}/h_n}$ has all its even order derivatives non-negative in $(0, 1]$. By an inequality of Shohat (Freud [2, Lemma 1.5, p. 92]),

$$\sum_{j=1}^{n} \lambda_{nj}\sigma_{nj}^{-2t_{n1}/h_n} \leqslant \int_0^1 \sigma^{-2t_{n1}/h_n}\, d\gamma_n(\sigma) = \alpha(\infty) \tag{4.11}$$

by (4.2). Further, by Bessel's inequality (Hille [3, pp. 328–329]),

$$\int_0^1 |s(\sigma)|^2\, d\gamma_n(\sigma) \leqslant \int_0^1 |H(\sigma)|^2\, d\gamma_n(\sigma). \tag{4.12}$$

Finally, (4.8), (4.10), (4.11), and (4.12) yield (4.6). ∎

Note that having fixed the exponents μ_{nk}, the existence and uniqueness of $f_n(t)$ satisfying (4.4) and (4.5) follow from the fact that $\{e^{-t\mu_{n1}}, ..., e^{-t\mu_{nn}}\}$ is a Chebyshev system. However, the essential feature of the above theorem is that the bound in (4.6) is independent of n and the interpolation points. The above result is connected to the theory of "product integration" rules (Sloan and Smith [8]) but we shall not expand on this.

THEOREM 4.2. *For each positive integer n, let non-negative t_{n1} and positive h_n be given and define t_{nj}, $j = 1, 2... n$, by* (4.1). *Assume $\{t_{nj}\}_{n,j}$ is a uniqueness set. Further, let $f(t), f_n(t)$ be as in Theorem* 4.1 *so that* (4.5) *holds*, $n = 1, 2....$ *Then*

$$\lim_{n\to\infty} f_n(t) = f(t), \qquad \mathrm{Re}(t) > 0,$$

and the convergence is uniform in the unbounded sector $\mathscr{S}(\theta_0, \varepsilon)$ for any $0 < \theta_0 < \pi/2$ and $\varepsilon > 0$.

Proof. Let $\beta_n(x) = \sum_{\mu_{nj}\leqslant x} w_{nj}$, $x \in [0, \infty]$, $n = 1, 2...$, so that $f_n(t) = \mathscr{L}[d\beta_n](t)$, $n = 1, 2....$ Let $\beta_n^{r+}(x)$, $\beta_n^{r-}(x)$, $\beta_n^{i+}(x)$, $\beta_n^{i-}(x)$ denote, respectively, the real and imaginary upper and lower variations of $\beta_n(x)$, so that, for example,

$$\beta_n^{r+}(x) = \sum_{\substack{\mu_{nj}\leqslant x \\ \mathrm{Re}(w_{nj})>0}} \mathrm{Re}(w_{nj}), \qquad x \in [0, \infty], n = 1, 2...,$$

as in Definition 2.1. Then if $|\beta|_n(x)$ is the total variation function of $\beta_n(x)$, we see

$$|\beta|_n(\infty) = \sum_{j=1}^{n} (|\mathrm{Re}(w_{nj})| + |\mathrm{Im}(\omega_{nj})|) \leqslant \sqrt{2} \sum_{j=1}^{n} |w_{nj}| \leqslant \sqrt{2}\, \Gamma.$$

$n = 1, 2, 3\ldots$ by (4.6). By Helly's Selection Theorem, we can extract a subsequence of integers $\mathcal{N}$ and monotone increasing functions $\gamma^{r+}(x)$, $\gamma^{r-}(x)$, $\gamma^{i+}(x)$, $\gamma^{i-}(x)$ which are, respectively, the limits of $\beta_n^{r+}(x)$, $\beta_n^{r-}(x)$, $\beta_n^{i+}(x)$, $\beta_n^{i-}(x)$ for almost all $x \in [0, \infty)$ as $n \to \infty$, $n \in \mathcal{N}$. This defines a function $\gamma(x)$, and its total variation function $|\gamma|$ as in Definition 2.1. Let $g(t) = \mathcal{L}|d\gamma|(t)$, $\text{Re}(t) \geqslant 0$. As

$$\lim_{\substack{n \to \infty \\ n \in \mathcal{N}}} \beta_n(x) = \gamma(x), \qquad \lim_{\substack{n \to \infty \\ n \in \mathcal{N}}} |\beta|_n(x) = |\gamma|(x)$$

for almost all $x \in [0, \infty)$, Theorem 3.1 shows

$$\lim_{\substack{n \to \infty \\ n \in \mathcal{N}}} f_n(t) = g(t), \qquad \text{Re}(t) > 0,$$

the convergence being uniform in $\mathcal{S}(\theta_0, \varepsilon)$ for any $0 < \theta_0 < \pi/2$ and $\varepsilon > 0$. As in Corollary 3.2, we deduce $g(t) \equiv f(t)$ and $\gamma(x) \equiv \beta(x)$, and the proof is completed as in Corollary 3.2. ∎

Theorems 4.1 and 4.2 show that if $f(t)$ is completely bounded in $[0, \infty]$, then it is the limit in $(0, \infty)$ of a sequence of exponential sums

$$f_n(t) = \sum_{j=1}^{n} w_{nj} e^{-u_{nj} t}; \quad \sum_{j=1}^{n} |w_{nj}| \leqslant \Gamma, \qquad n = 1, 2, \ldots \tag{4.13}$$

Conversely, if $f(t)$: $[0, \infty) \to \mathbb{C}$ is the limit of a sequence of sums $f_n(t)$ satisfying (4.13), then it is easily seen using Helly's Theorem that $f(t)$ is completely bounded in $[0, \infty]$.

Nira Dyn of Tel Aviv University has informed the authors (oral communication) that provided $d\beta(x)$ is real, it is possible to obtain convergent exponential interpolation for other choices of exponents and interpolation points.

5. Rates of Convergence

When the interpolation points are equidistant and satisfy certain asymptotic assumptions, one can establish rates of convergence of $f_n(t)$ to $f(t)$ using standard theorems on the degree of approximation by polynomials. It is also possible to establish convergence rates using complex analysis methods from the theory of rational approximation. However, for simplicity, we omit the latter. Our analysis is based upon:

Lemma 5.1. *Let $\beta(x)$, $\gamma(x)$ be of bounded variation in $[0, \infty]$ and let*

$f(t)=\mathscr{L}[d\beta](t)$, $g(t)=\mathscr{L}[d\gamma](t)$, $\mathrm{Re}(t)\geqslant 0$. Let $a, h>0$ and n be a positive integer. Let $t_j=a+(j-1)h, j=1, 2 \ldots n$. Assume $f(t_j)=g(t_j)$, $j=1, 2 \ldots n$. Then

(i) $|f(t)-g(t)| \leqslant (|\beta|(\infty)+|\gamma|(\infty)) \|\sigma^{(t-a)/h}-P(\sigma)\|_\infty, \quad \mathrm{Re}(t)\geqslant a,$

whenever $P(\sigma)$ is a polynomial of degree at most $n-1$:

(ii) $|f(t)-g(t)| \leqslant (|\beta|(\infty)+|\gamma|(\infty)) |t-a| h^{-1} \|\sigma^{(t-a)/h-1}-Q(\sigma)\|_1, \quad \mathrm{Re}(t)>a,$

whenever $Q(\sigma)$ is a polynomial of degree at most $n-2$.

Proof.

$$f(t)-g(t)=\int_0^\infty (e^{-hx})^{(t-a)/h} e^{-ax} \, d(\beta-\gamma)(x)$$

$$=\int_0^1 \sigma^{(t-a)/h} \, d\Delta(\sigma),$$

where $\sigma=e^{-hx}$ and

$$\Delta(\sigma)=\int_{-(\log\sigma)/h}^\infty e^{-ax} d(\beta-\gamma)(x), \qquad \sigma\in[0, 1].$$

In particular,

$$0=f(t_j)-g(t_j)=\int_0^1 \sigma^{j-1} \, d\Delta(\sigma), \qquad j=1, 2 \ldots n.$$

Then if $P(\sigma)$ is a polynomial of degree at most $n-1$,

$$\begin{aligned} |f(t)-g(t)| &= \left|\int_0^1 (\sigma^{(t-a)/h}-P(\sigma)) \, d\Delta(\sigma)\right| \\ &\leqslant \|\sigma^{(t-a)/h}-P(\sigma)\|_\infty \, |\Delta|(1). \end{aligned} \tag{5.1}$$

But $|\Delta|(1) \leqslant \int_0^\infty e^{-ax} \, d(|\beta|+|\gamma|)(x) \leqslant |\beta|(\infty)+|\gamma|(\infty)$ and (i) follows. To obtain (ii), we integrate (5.1) by parts:

$$\begin{aligned} |f(t)-g(t)| \leqslant{}& |1-P(1)| \, |\Delta(1)| \\ &+ \int_0^1 |(t-a) h^{-1} \sigma^{(t-a)/h-1}-P'(\sigma)| \, |\Delta(\sigma)| \, d\sigma. \end{aligned}$$

Given any polynomial $Q(\sigma)$ of degree at most $n-2$, let $P(\sigma)=(t-a) h^{-1} \int_0^\sigma Q(u) \, du + C$, where C is chosen so that $P(1)=1$. Then we obtain $|f(t)-g(t)| \leqslant |t-a|\, h^{-1}\, |\Delta|(1) \int_0^1 |\sigma^{(t-a)/h-1} - Q(\sigma)|\, d\sigma$ and (ii) follows. ∎

First we obtain rates of convergence for interpolation at equally spaced points with fixed stepsize.

THEOREM 5.2. *Let $\beta(x)$ be of bounded variation in $[0, \infty)$ and absolutely continuous with respect to $\alpha(x)$. Let $f(t) = \mathscr{L}\,|d\beta|(t)$, $\mathrm{Re}(t) \geqslant 0$. Fix positive h and nonnegative a. Let $t_{nj} = a + (j-1)h$, $j = 1, 2, \ldots, n$; $n = 1, 2, \ldots$ Let $f_n(t)$ be given by (4.4) so that (4.5) holds, $n = 1, 2, \ldots$ Then if $q = (\mathrm{Re}(t) - a)/h > 0$,*

$$|f(t) - f_n(t)| = O(n^{-q}) \qquad as \quad n \to \infty.$$

Proof. Let $z = (t-a)/h$ so that $q = \mathrm{Re}(z)$. Let l be the largest integer $\leqslant q$ and let $\eta = z - l$. By Lemma 5.1(i) and Theorem 4.1, we have

$$|f(t) - f_n(t)| \leqslant (|\beta|(\infty) + \sqrt{2}\,\Gamma) \min_{\deg(P) < n} \|\sigma^z - P(\sigma)\|_{L_\infty}. \tag{5.2}$$

Now $d^l/d\sigma^l\,\{\sigma^z\} = z(z-1)\cdots(z-l+1)\,\sigma^\eta$ is bounded in $[0, 1]$ and further belongs to $\mathrm{Lip}(q-l)$ in $[0, 1]$. For, if $0 < \sigma < \sigma' < 1$ and $\sigma' \geqslant 1.5\sigma$, then

$$\begin{aligned} |\sigma^\eta - \sigma'^\eta| / |\sigma - \sigma'|^{q-l} &\leqslant \{1 + (\sigma/\sigma')^{q-l}\}/\{1 - (\sigma/\sigma')\}^{q-l} \\ &\leqslant 2/\{1 - (2/3)\}^{q-l}, \end{aligned}$$

while if $0 < \sigma < \sigma' < 1$ and $\sigma' < 1.5\sigma$, then $\sigma'/\sigma = 1 + \delta$, where $0 < \delta < 1/2$ and

$$\begin{aligned} |\sigma^\eta - \sigma'^\eta| / |\sigma - \sigma'|^{q-l} &= |1 - (1+\delta)^\eta| / \delta^{q-l} \\ &\leqslant \delta K \delta^{-(q-l)} \leqslant K, \end{aligned}$$

where $K = \sum_{j=1}^\infty |\binom{\eta}{j}|\, 2^{1-j}$ and as $q - l < 1$. Thus $d^l/d\sigma^l\,\{\sigma^z\} \in \mathrm{Lip}(q-l)$. By standard results (Rivlin [7, pp. 22–23]),

$$\min_{\deg(P) < n} \|\sigma^z - P(\sigma)\|_{L_\infty} = O(n^{-q}) \qquad \text{as} \quad n \to \infty \tag{5.3}$$

and the result now follows from (5.2) and (5.3). ∎

In the same way, one obtains:

THEOREM 5.3. *Let $f(t) = \mathscr{L}\,|d\alpha|(t)$, $\mathrm{Re}(t) \geqslant 0$. Fix positive h and nonnegative a and let*

$$t_{nj} = a + (j-1)h, \qquad j = 1, 2, \ldots, 2n;\ n = 1, 2, \ldots$$

Let $f_n(t)$ be the exponential sum involving n exponents that interpolates to $f(t)$ at $t = t_{nj}$, $j = 1, 2 \ldots 2n$, $n = 1, 2 \ldots$. Then if $q = (\mathrm{Re}(t) - a)/h > 0$, $|f(t) - f_n(t)| = 0(n^{-q})$ as $n \to \infty$.

LEMMA 5.4. *Let $z \in \mathbb{C}$ satisfy $\mathrm{Re}(z) > 0$. Let d be real and a nonnegative integer k be given. Then if*

$$e_n = \min_{\deg(P) \leqslant n-k} \|\sigma^{nz+d} - P(\sigma)\|_2,$$

$$\limsup_{n \to \infty} e_n^{1/n} \leqslant \rho(z) = \exp\left\{\int_0^1 \log\left|\frac{z-x}{z+x}\right| dx\right\} \tag{5.4}$$

with equality if $\mathrm{Im}(z) \neq 0$ *or* $\mathrm{Re}(z) > 1$.

Proof. It is easy to modify the arguments in Cheney [1, pp. 194–196] from the real to the complex case to show that for complex v, such that $\mathrm{Re}(v) > -1/2$,

$$\min_{\deg(P) \leqslant n} \|\sigma^v - P(\sigma)\|_2 = (2\mathrm{Re}(v) + 1)^{-1/2} \prod_{j=0}^{n} \left|\frac{v-j}{v+j+1}\right|.$$

Hence

$$e_n = (2n\,\mathrm{Re}(z) + 2d + 1)^{-1/2} \prod_{j=0}^{n-k} \left|\frac{nz+d-j}{nz+d+j+1}\right|.$$

Thus

$$\begin{aligned} n^{-1} \log e_n = &-(2n)^{-1} \log(2n\,\mathrm{Re}(z) + 2d + 1) \\ &+ n^{-1}(\log|nz+d| - \log|nz+d+n-k+1|) \\ &+ n^{-1} \sum_{j=1}^{n-k} \log\left|\frac{z-(j-d)/n}{z+(j+d)/n}\right|. \end{aligned} \tag{5.5}$$

If $\mathrm{Im}(z) \neq 0$ or $\mathrm{Re}(z) > 1$, the sum in (5.5) has the nature of a Riemann sum of a function continuous in [0, 1] and in such a case, we see

$$\lim_{n \to \infty} n^{-1} \log e_n = \int_0^1 \log\left|\frac{z-x}{z+x}\right| dx.$$

If $\mathrm{Im}(z) = 0$ and $0 < z \leqslant 1$, all terms in the sum in (5.5) are nonpositive and we deduce that for any $\eta > 0$,

$$\begin{aligned} n^{-1} \log e_n &\leqslant n^{-1} \sum_{\substack{j=1 \\ |z-(j-d)/n| \geqslant \eta}}^{n-k} \log \left| \frac{z-(j-d)/n}{z-(j+d)/n} \right| \\ &\to \int_{[0,1] \setminus [z-\eta, z+\eta]} \log \left| \frac{z-x}{z+x} \right| dx \end{aligned}$$

as $n \to \infty$. Using Lebesgue's Dominated Convergence Theorem, we can let $\eta \to 0$, to deduce

$$\limsup_{n \to \infty} n^{-1} \log e_n \leqslant \int_0^1 \log \left| \frac{z-x}{z+x} \right| dx, \qquad 0 < z \leqslant 1,$$

and the result follows. ∎

Different results concerning e_n, when z is real, may be found in [6]. The following theorem gives rates of convergence for equidistant interpolation in a fixed interval.

THEOREM 5.5. *Let $\beta(x)$ be of bounded variation in $[0, \infty)$ and absolutely continuous with respect to $\alpha(x)$. Let $f(t) = \mathscr{L}\,|d\beta|(t)$, $\mathrm{Re}(t) \geqslant 0$. Fix positive b and nonnegative a. Let*

$$t_{nj} = a + (j-1)\, b/n, \qquad j = 1, 2, \ldots n;\ n = 1, 2, \ldots$$

Let $f_n(t)$ be given by (4.4) so that (4.5) holds, $n = 1, 2, \ldots$. Then given t such that $\mathrm{Re}(t) > a$, we have

$$\limsup_{n \to \infty} |f(t) - f_n(t)|^{1/n} \leqslant \rho(z) < 1.$$

where $z = (t-a)/b$ and $\rho(z)$ is given by (5.4).

Proof. By Lemma 5.1(ii) and Theorem 4.1,

$$\begin{aligned} &|f(t) - f_n(t)| \\ &\quad \leqslant (|\beta|(\infty) + \sqrt{2}\, \Gamma)(t-a)\, n b^{-1} \min_{\deg(Q) \leqslant n-2} \|\sigma^{[n(t-a)/b]-1} - Q(\sigma)\|_1. \end{aligned}$$

If we use monotonicity of $\|\cdot\|_p$ in p, and Lemma 5.4, the result follows. ∎

For completely monotone functions, we have similarly:

THEOREM 5.6. *Let* $f(t) = \mathscr{L}|d\alpha|(t)$, $\mathrm{Re}(t) \geqslant 0$. *Fix positive* b *and nonnegative* a. *Let*

$$t_{nj} = a + (j-1)\, b/(2n), \qquad j = 1, 2 \ldots 2n;\ n = 1, 2, \ldots.$$

Let $f_n(t)$ *be the exponential sum involving* n *exponents that interpolates to* $f_n(t)$ *at* $t = t_{nj}$, $j = 1, 2 \ldots 2n$; $n = 1, 2 \ldots$. *Then given* t *such that* $\mathrm{Re}(t) > a$, *we have*

$$\limsup_{n \to \infty} |f(t) - f_n(t)|^{1/n} \leqslant \rho^2(z),$$

where $z = (t-a)/b$ *and* $\rho(z)$ *is given by* (5.4).

The above convergence rates improve those of Kammler [5, p. 565] for individual t.

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