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ASYMPTOTIC EXPANSIONS OF MELLIN TRANSFORMS AND ANALOGUES OF WATSON'S LEMMA*

AVRAM SIDI†

Abstract. In this paper the asymptotic behavior of the Mellin transform $\hat{f}(x)=\int_0^\infty t^{x-1}f(t)\,dt$ of $f(t)$, for $x\to+\infty$, is analyzed. In particular, it is shown for certain classes of functions $u_k(t)$, $k=0,1,\cdots$, that form asymptotic sequences for $t\to+\infty$, that if $f(t)\sim\sum_{k=0}^\infty A_k u_k(t)$ as $t\to+\infty$, then $\hat{f}(x)\sim\sum_{k=0}^\infty A_k\hat{u}_k(x)$ as $x\to+\infty$. In this sense the results of this paper are analogues of Watson's lemma for Laplace integrals. Several illustrative examples involving summation of everywhere divergent moment series and special functions are appended.

1. Introduction. Let $f(t)$ be a function that is locally integrable for $0<t<+\infty$ such that, for some real constant σ, $t^{\sigma-1}f(t)$ is absolutely integrable in any finite interval of the form $[0,a]$, and

$$f(t)=O(t^{-\mu})\quad \text{as } t\to+\infty,\quad \text{any } \mu>0. \tag{1.1}$$

Then the Mellin transform $\hat{f}(x)$ of $f(t)$, defined by

$$\hat{f}(x)=\int_0^\infty t^{x-1}f(t)\,dt, \tag{1.2}$$

exists for all sufficiently large x.

The purpose of this work is to give an asymptotic analysis of $\hat{f}(x)$ for $x\to+\infty$. Surprisingly, this problem does not seem to have received much attention. Doetsch [2, Vol. 2, Chap. 5] has considered the problem of analytic continuation of the Mellin transform beyond the strip in which its integral representation converges, and has obtained results on the singularity structure of it. Riekstinš [8] has considered the asymptotic expansion of the inverse Mellin transform. Wagner [11] has obtained some Tauberian theorems for Mellin transforms. Handlesman and Lew [3], [4], [5] have used techniques involving the Mellin transform for obtaining asymptotic expansions for other integral transforms.

The results of this work can be summarized in an informal way as follows. Consider the sequence of functions $\{u_0(t),u_1(t),\cdots\}$ and the sequence of the corresponding Mellin transforms $\{\hat{u}_0(x),\hat{u}_1(x),\cdots\}$. Assume that

$$\lim_{t\to+\infty}\frac{u_q(t)}{u_k(t)}=0,\qquad q>k,\quad k=0,1,\cdots, \tag{1.3}$$

and

$$\lim_{x\to+\infty}\frac{\hat{u}_q(x)}{\hat{u}_k(x)}=0,\qquad q>k,\quad k=0,1,\cdots, \tag{1.4}$$

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† Computer Science Department, Technion–Israel Institute of Technology, Haifa, Israel. The research for this paper was done while the author was a National Research Council NASA Research Associate at Lewis Research Center, Cleveland, Ohio 44135.

i.e., that $\{u_0(t), u_1(t), \cdots\}$ and $\{\hat{u}_0(x), \hat{u}_1(x), \cdots\}$ are asymptotic sequences as $t \to +\infty$ and $x \to +\infty$ respectively. If the function $f(t)$ has an asymptotic expansion of the form

$$f(t) \sim \sum_{k=0}^{\infty} A_k u_k(t) \quad \text{as } t \to +\infty, \tag{1.5}$$

then, for some choices of the $u_k(t)$, $\hat{f}(x)$ has the asymptotic expansion

$$\hat{f}(x) \sim \sum_{k=0}^{\infty} A_k \hat{u}_k(x) \quad \text{as } x \to +\infty, \tag{1.6}$$

i.e., the asymptotic expansion of $\hat{f}(x)$ is that obtained by taking the Mellin transform of the right-hand side of (1.5) term by term.

In the sense of (1.5) and (1.6) the results of the present work are analogues of Watson's lemma for Laplace transforms. Recall that essentially Watson's lemma concerns sequences $\{w_0(t), w_1(t), \cdots\}$, where $w_k(t) = t^{\gamma_k}$, $k = 0, 1, \cdots$, and $-1 < \operatorname{Re}\gamma_0 < \operatorname{Re}\gamma_1 < \cdots$, and states that if the function $g(t)$ has an asymptotic expansion of the form $g(t) \sim \sum_{k=0}^{\infty} B_k w_k(t)$ as $t \to 0+$, then the Laplace transform $\bar{g}(s) = \int_0^\infty e^{-st} g(t)\,dt$ of $g(t)$ has the asymptotic expansion $\bar{g}(s) \sim \sum_{k=0}^{\infty} B_k \bar{w}_k(s)$ as $s \to +\infty$. For Watson's lemma and its generalizations see Olver [7].

The use of the asymptotic expansion of $f(t)$ as $t \to +\infty$ to obtain that of $\hat{f}(x)$ as $x \to +\infty$ can be heuristically justified as follows. Consider the integral

$$I(b;x) = \int_0^b t^{x-1} f(t)\,dt, \qquad 0 < b < +\infty. \tag{1.7}$$

Making the change of variable of integration $\xi = \log(b/t)$, (1.7) becomes

$$I(b;x) = b^x \int_0^\infty e^{-x\xi} f(be^{-\xi})\,d\xi. \tag{1.8}$$

Now the asymptotic expansion of $I(b;x)$ for $x \to +\infty$ can be obtained by expanding $f(be^{-\xi})$ asymptotically for $\xi \to 0+$ and applying Watson's lemma or its generalizations. But expanding $f(be^{-\xi})$ for $\xi \to 0+$ is equivalent to expanding $f(t)$ for $t \to b-$. This and the fact that $\hat{f}(x) = I(+\infty; x)$ suggest that one should consider expanding $f(t)$ for $t \to +\infty$ in order to analyze the asymptotic behavior of $\hat{f}(x)$ for $x \to +\infty$.

Finally note that by making the change of variable of integration $t = e^{-\eta}$, (1.2) becomes

$$\hat{f}(x) = \int_{-\infty}^{+\infty} e^{-x\eta} f(e^{-\eta})\,d\eta, \tag{1.9}$$

which is a two-sided Laplace transform. Hence, our results carry over to such transforms naturally. This point has been noted by various authors.

The main results of this work are given in the next section. These results are illustrated in §3 with examples that involve the summation of everywhere divergent moment series and some special functions.

2. Main results. Let the function $f(t)$ be as in the first paragraph of §1. In Theorems 2.1 and 2.2 of the present section we show that for some choices of the functions $u_k(t)$, $k = 0, 1, \cdots$, the sequences $\{u_0(t), u_1(t), \cdots\}$ and $\{\hat{u}_0(x), \hat{u}_1(x), \cdots\}$ are asymptotic sequences as $t \to +\infty$ and $x \to +\infty$ respectively, and that (1.5) implies (1.6). In the proofs of our results we make use of the following simple observations.

LEMMA 2.1. *For any fixed $T>0$*

$$\int_0^T t^{x-1} f(t)\,dt = O(T^x) \quad \text{as } x \to +\infty. \tag{2.1}$$

Proof. (2.1) is a consequence of the assumption that, for some real constant σ, $t^{\sigma-1}f(t)$ is absolutely integrable in any finite interval of the form $[0,a]$. □

LEMMA 2.2. *Assume $\{u_0(t), u_1(t), \cdots\}$ is an asymptotic sequence as $t\to+\infty$ and $f(t)$ has the asymptotic expansion in* (1.5). *Set*

$$r_n(t) = f(t) - \sum_{k=0}^{n-1} A_k u_k(t), \qquad n=1,2,\cdots. \tag{2.2}$$

Then for each positive integer n, there exist positive constants K and T that depend only on n, such that

$$|r_n(t)| \le K|u_n(t)|, \qquad t\ge T. \tag{2.3}$$

Proof. (2.3) follows from the fact that $\lim_{t\to+\infty}[r_n(t)/u_n(t)] = A_n$, which in turn is a consequence of $r_n(t) = A_n u_n(t) + O(u_{n+1}(t))$ as $t\to+\infty$ and (1.3). □

THEOREM 2.1. *Let*

$$u_k(t) = t^{-\lambda_k}\exp\left(-\alpha_k t^{\beta_k}\right), \qquad k=0,1,\cdots, \tag{2.4}$$

where

$$\lambda_k \text{ real}, \quad \mathrm{Re}\,\alpha_k>0, \quad k=0,1,\cdots, \quad 0<\beta_0\le\beta_1\le\beta_2\le\cdots, \tag{2.5}$$

(2.6) *when $k<q$, $\beta_k=\beta_q$ implies either one of the four combinations* (a *and* c), (a *and* d), (b *and* c), *and* (b *and* d), *with*
a) $\mathrm{Re}\,\alpha_k < \mathrm{Re}\,\alpha_q$, b) $\mathrm{Re}\,\alpha_k = \mathrm{Re}\,\alpha_q$ *and* $\lambda_k<\lambda_q$,
c) $|\alpha_k|<|\alpha_q|$, d) $|\alpha_k|=|\alpha_q|$ *and* $\lambda_k<\lambda_q$,

and no restrictions are imposed on λ_k and α_k when $\beta_k<\beta_q$. Then $\{u_0(t), u_1(t),\cdots\}$ and $\{\hat u_0(x), \hat u_1(x), \cdots\}$ are asymptotic sequences as $t\to+\infty$ and $x\to+\infty$ respectively. If, for any nonnegative integer n, there exists an integer $N>n$, such that

(2.7)
either a) $\beta_n<\beta_N$
or b) $\beta_n=\beta_N$ *and* $|\alpha_n|<\mathrm{Re}\,\alpha_N$,
or c) $\beta_n=\beta_N$, $|\alpha_n|=\mathrm{Re}\,\alpha_N$ *and* $\lambda_n\leqq\lambda_N$

then (1.5) *implies* (1.6).

Proof. The first part of the theorem is a direct consequence of (2.4)–(2.6),

$$\hat u_k(x) = \beta_k^{-1}\alpha_k^{-(x-\lambda_k)/\beta_k}\Gamma\left(\frac{x-\lambda_k}{\beta_k}\right), \tag{2.8}$$

and Stirling's formula for the gamma function.

For the second part of the theorem it is sufficient to show that, for each positive integer n,

$$\hat r_n(x) = O(\hat u_n(x)) \quad \text{as } x\to+\infty, \tag{2.9}$$

where $r_n(t)$ has been defined in (2.2). For a given positive integer n, let N be as in the statement of the theorem. Then by Lemma 2.2 there exist positive constants K and T

that depend only on N, hence only on n, for which $|r_N(t)|\le K|u_N(t)|$ when $t\ge T$. Now for sufficiently large x

$$\hat{r}_N(x)=\int_0^T t^{x-1}f(t)\,dt-\sum_{k=0}^{N-1}A_k\int_0^T t^{x-1}u_k(t)\,dt+\int_T^\infty t^{x-1}r_N(t)\,dt. \tag{2.10}$$

Each one of the integrals $\int_0^T t^{x-1}f(t)\,dt$ and $\int_0^T t^{x-1}u_k(t)\,dt$, $k=0,1,\cdots,N-1$, is $O(T^x)$ as $x\to+\infty$, by Lemma 2.1. Furthermore,

$$\left|\int_T^\infty t^{x-1}r_N(t)\,dt\right|\le\int_T^\infty t^{x-1}|r_N(t)|dt\le K\int_T^\infty t^{x-1}|u_N(t)|dt$$
$$<K\int_0^\infty t^{x-1}|u_N(t)|dt=K\beta_N^{-1}(\operatorname{Re}\alpha_N)^{-(x-\lambda_N)/\beta_N}\Gamma\left(\frac{x-\lambda_N}{\beta_N}\right). \tag{2.11}$$

Invoking now (2.7), (2.11) can be replaced by

$$\int_T^\infty t^{x-1}r_N(t)\,dt=O(\hat{u}_n(x))\quad\text{as } x\to+\infty. \tag{2.12}$$

Thus (2.10) becomes

$$\hat{r}_N(x)=O(T^x)+O(\hat{u}_n(x))=O(\hat{u}_n(x))\quad\text{as } x\to+\infty, \tag{2.13}$$

by (2.8) and Stirling's formula. But

$$\hat{r}_n(x)=\hat{r}_N(x)+\sum_{k=n}^{N-1}A_k\hat{u}_k(x), \tag{2.14}$$

and $\hat{u}_k(x)=O(\hat{u}_n(x))$ as $x\to+\infty$ for each $k\ge n$ since $\{\hat{u}_0(x),\hat{u}_1(x),\cdots\}$ is an asymptotic sequence as $x\to+\infty$. Combining this and (2.13) in (2.14), (2.9) follows. This completes the proof of the theorem. □

The special case of (1.5) with $u_k(t)$ as given by (2.4) in Theorem 2.1, such that $\alpha_k=\alpha_{k+1}$ and $\beta_k=\beta_{k+1}$ for all $k=0,1,\cdots$, is of importance, and we turn to this case in Theorems 2.2 and 2.3 below. We first note that for this special case $u_k(t)$ is of the form

$$u_k(t)=t^{-\lambda_k}\exp(-\alpha t^\beta),\qquad k=0,1,\cdots, \tag{2.15}$$

and, consequently

$$\hat{u}_k(x)=\beta^{-1}\alpha^{-(x-\lambda_k)/\beta}\Gamma\left(\frac{x-\lambda_k}{\beta}\right),\qquad k=0,1,\cdots. \tag{2.16}$$

THEOREM 2.2. *Let $u_k(t)$ and $\hat{u}_k(x)$ be as in* (2.15) *and* (2.16), *where α,β and λ_k are all real, and*

$$\alpha>0,\beta>0,\qquad \lambda_0<\lambda_1<\lambda_2<\cdots. \tag{2.17}$$

Then $\{u_0(t),u_1(t),\cdots\}$ and $\{\hat{u}_0(x),\hat{u}_1(x),\cdots\}$ are asymptotic sequences as $t\to+\infty$ and $x\to+\infty$ respectively, and (1.5) *implies* (1.6).

Proof. We observe that, with the present $u_k(t)$, all the conditions of Theorem 2.1 are satisfied with $\alpha_k=\alpha$, $\beta_k=\beta$, $k=0,1,\cdots$, and with $N=n+1$ for each nonnegative integer n. Therefore, Theorem 2.1 holds. This proves the theorem. □

If α in Theorem 2.2 is not real, then the proof of this theorem is no longer valid since (2.7) is not satisfied for any $N>n$. However, different arguments establish essentially the same result when α is complex with positive real part if $f(t)$ satisfies further conditions in the complex t-plane. This is given in Theorem 2.3 below.

THEOREM 2.3. *Let* $u_k(t)$ *and* $\hat{u}_k(x)$ *be as in* (2.15) *and* (2.16), *where* α *is now complex, and*

$$\text{(2.18)} \qquad \operatorname{Re}\alpha>0, \quad \beta>0, \quad \lambda_0<\lambda_1<\lambda_2<\cdots.$$

Then $\{u_0(t),u_1(t),\cdots\}$ *and* $\{\hat{u}_0(x),\hat{u}_1(x),\cdots\}$ *are asymptotic sequences as* $t\to\infty$ (*along any path in the complex t-plane possibly cut along the negative real axis*) *and* $x\to+\infty$ *respectively. Denote* $\theta=\arg t$, $\omega=\arg\alpha$, $\theta_1=\min(0,-\omega/\beta)$, *and* $\theta_2=\max(0,-\omega/\beta)$. *Assume that for some* $T_0\geq 0$ *and* $\delta>0$ *the function* $f(t)$ *is analytic in the set* $D=\{t:|t|\geq T_0,\theta\in S\}$, *where* $S=(\theta_1-\delta,\theta_2+\delta)$, *and that*

$$\text{(2.19)} \qquad f(t)\sim\sum_{k=0}^{\infty}A_k u_k(t) \quad \text{as } t\to\infty,\ \theta\in S.$$

If, in addition, for sufficiently large x

$$\text{(2.20)} \qquad \lim_{R\to\infty}\int_{L(R)} t^{x-1}f(t)\,dt=0,$$

where $L(\rho)=\{t:t=\rho e^{i\theta},\theta$ *goes from* 0 *to* $-\omega/\beta\}$, *then* (1.6) *holds.*

Proof. As in Theorem 2.1, the first part of the present theorem is a direct consequence of (2.15), (2.16), (2.18), and Stirling's formula.

To prove that (1.6) holds we proceed as follows. Since $f(t)$ is analytic in D and satisfies (2.20), we can write

$$\text{(2.21)} \qquad \hat{f}(x)=\left(\int_0^{T_0}+\int_{L(T_0)}+\int_C\right)t^{x-1}f(t)\,dt,$$

where $C=\{t:t=\rho e^{-i\omega/\beta}$, ρ goes from T_0 to $+\infty\}$. By Lemma 2.1

$$\text{(2.22)} \qquad \int_0^{T_0}t^{x-1}f(t)\,dt=O(T_0^x) \quad \text{as } x\to+\infty.$$

Similarly, by analyticity properties of $f(t)$,

$$\text{(2.23)} \qquad \left|\int_{L(T_0)}t^{x-1}f(t)\,dt\right|\leq\frac{|\omega|}{\beta}\left(\max_{\theta_1\leq\theta\leq\theta_2}|f(T_0e^{i\theta})|\right)T_0^x.$$

Now the integral along C can be reexpressed as

$$\text{(2.24)} \qquad \int_C t^{x-1}f(t)\,dt=e^{-i\omega x/\beta}\int_{T_0}^{\infty}\rho^{x-1}f(\rho e^{-i\omega/\beta})\,d\rho,$$

and the function $F(\rho)=f(\rho e^{-i\omega/\beta})$ satisfies

$$\text{(2.25)} \qquad F(\rho)\sim\exp(-|\alpha|\rho^{\beta})\sum_{k=0}^{\infty}A_k\exp\left(\frac{i\omega\lambda_k}{\beta}\right)\rho^{-\lambda_k} \quad \text{as } \rho\to\infty$$

by (2.19). Thus the function $F_1(\rho)=H(\rho-T_0)F(\rho)$, where $H(y)$ is the Heaviside unit function, satisfies all the requirements of Theorem 2.2. Consequently

(2.26)

$$\int_{T_0}^{\infty}\rho^{x-1}F(\rho)\,d\rho=\hat{F}_1(\rho)=\sum_{k=0}^{n-1}A_k\exp\left(\frac{i\omega\lambda_k}{\beta}\right)\hat{v}_k(x)+O(\hat{v}_n(x))\quad \text{as } x\to+\infty,$$

where

$$\hat{v}_k(x)=\beta^{-1}|\alpha|^{-(x-\lambda_k)/\beta}\Gamma\left(\frac{x-\lambda_k}{\beta}\right),\qquad k=0,1,\cdots. \tag{2.27}$$

Combining (2.22)–(2.27) in (2.21), (1.6) follows. □

Remark. In Theorem 2.3, if we assume that (2.19) holds *uniformly* for $\theta\in S$, then (2.20) is automatically satisfied. To see this observe that, under this condition, there exist positive constants K and $T>T_0$ independent of t, such that

$$|f(t)|\le K|u_0(t)|,\qquad t\ge T,\quad \theta\in S. \tag{2.28}$$

Thus, for $R\ge T$,

$$\begin{aligned}\left|\int_{L(R)}t^{x-1}f(t)\,dt\right|&\le K\left|\int_{L(R)}|t|^{x-1}|u_0(t)||dt|\right|\\&=KR^{x-\lambda_0}\int_{\theta_1}^{\theta_2}\exp\left[-|\alpha|R^{\beta}\cos(\omega+\beta\theta)\right]d\theta.\end{aligned} \tag{2.29}$$

But for $\theta\in[\theta_1,\theta_2]$ we have $|\omega+\beta\theta|\le|\omega|<\pi/2$, thus $\cos(\omega+\beta\theta)\ge\cos\omega>0$. Consequently

$$\left|\int_{L(R)}t^{x-1}f(t)\,dt\right|\le K\frac{|\omega|}{\beta}R^{x-\lambda_0}\exp(-|\alpha|R^{\beta}\cos\omega), \tag{2.30}$$

and (2.20) follows by letting $R\to\infty$.

The corollary below gives a reformulation or rearrangement of the asymptotic expansion in (1.6) when $u_k(t)$ are as in Theorem 2.2 or Theorem 2.3 and $(\lambda_{k+1}-\lambda_k)/\beta$ is a fixed rational number for all $k=0,1,\cdots$. The form of the asymptotic expansion that is given by this corollary is more familiar and revealing than (1.6) itself, and we make extensive use of it in Examples 2–5.

COROLLARY. *Let p and q be two positive relatively prime integers, and let*

$$\lambda_{k+1}-\lambda_k=\frac{p}{q}\beta,\qquad k=0,1,\cdots, \tag{2.31}$$

in Theorem 2.2 *or Theorem* 2.3. *Then there exist constants* B_j, $j=0,1,\cdots$, *such that, for any positive integer n*,

$$\hat{f}(x)=\alpha^{-x/\beta}\Gamma\left(\frac{x-\lambda_0}{\beta}\right)\left[\sum_{j=0}^{n-1}\frac{B_j}{x^{j/q}}+O\left(\frac{1}{x^{n/q}}\right)\right]\quad \text{as } x\to+\infty, \tag{2.32}$$

with $B_0=A_0\beta^{-1}\alpha^{\lambda_0/\beta}$.

Proof. Starting with (2.9), we have, for any positive integer n,

$$\hat{f}(x)=\hat{u}_0(x)\left[\sum_{k=0}^{n-1} A_k \frac{\hat{u}_k(x)}{\hat{u}_0(x)}+O\left(\frac{\hat{u}_n(x)}{\hat{u}_0(x)}\right)\right] \quad \text{as } x\to+\infty. \tag{2.33}$$

Making use of (2.31) and the formula (see [1, formula 6.1.47, p. 257])

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)}\sim 1+\sum_{j=1}^{\infty}\frac{c_j}{z^j} \quad \text{as } z\to\infty, \tag{2.34}$$

where c_j are constants independent of z, we have from (2.16)

$$\begin{aligned}\frac{\hat{u}_k(x)}{\hat{u}_0(x)}&=\alpha^{(\lambda_k-\lambda_0)/\beta}\frac{\Gamma((x-\lambda_k)/\beta)}{\Gamma((x-\lambda_0)/\beta)}\\ &\sim x^{-kp/q}\sum_{j=0}^{\infty}\frac{d_{kj}}{x^j} \quad \text{as } x\to+\infty,\end{aligned} \tag{2.35}$$

where d_{kj} are constants independent of x. From (2.35) it also follows that

$$\frac{\hat{u}_n(x)}{\hat{u}_0(x)}=O(x^{-np/q}) \quad \text{as } x\to+\infty. \tag{2.36}$$

Combining (2.35) and (2.36) in (2.33), (2.32) follows. □

Finally we can introduce integral powers of $\log t$ in the functions $u_k(t)$ and still retain Theorems 2.1–2.3. The only additional results that one needs for proving this are

$$\int_0^{\infty} t^{x-1}(\log t)^m \exp(-\alpha t^{\beta})\,dt=\left(-\frac{\partial}{\partial x}\right)^m\left[\beta^{-1}\alpha^{-x/\beta}\Gamma\left(\frac{x}{\beta}\right)\right] \tag{2.37}$$

and the asymptotic behavior of the psi function and its derivatives (see [1, formulas 6.3.18, 6.4.11, pp. 259–260]). We shall not pursue this further, as the results and the techniques for proving them are now obvious.

Note also that all the results of this section hold true if the integral $\int_0^\infty t^{x-1}f(t)\,dt$ is replaced by $\int_a^\infty t^{x-1}f(t)\,dt$ for any $a>0$, as the proofs depend solely on the asymptotic behavior of $f(t)$ for $t\to+\infty$ or $t\to\infty$ in a sector in the complex t-plane. We have already used this in the proof of Theorem 2.3, and shall use it in some of the examples in the next section.

3. Examples. We shall illustrate the results of the previous section by several examples. The first example is a straightforward application of Theorem 2.1. The second example arises in applying the T-transformation of Levin [6] to the partial sums of the everywhere divergent moment series

$$H(z)\sim\sum_{i=1}^{\infty}\frac{\mu_i}{z^i} \quad \text{as } z\to\infty, \tag{3.1}$$

where

$$H(z)=\int_0^{\infty}\frac{w(t)}{z-t}\,dt \tag{3.2}$$

and

$$\mu_i=\int_0^\infty w(t)t^{i-1}\,dt, \qquad i=1,2,\cdots. \tag{3.3}$$

Ultimately, one is interested in the asymptotic behavior of the partial sum $\sum_{i=1}^{r-1}\mu_i/z^i$ as $r\to\infty$. It is easy to show that

$$\sum_{i=1}^{r-1}\frac{\mu_i}{z^i}=H(z)-\frac{1}{z^r}\int_0^\infty\frac{w(t)t^{r-1}}{1-t/z}\,dt. \tag{3.4}$$

The integral on the right is simply a Mellin transform, and the problem is to find its asymptotic expansion as $r\to\infty$. This integral is actually related to the converging factor for the series in (3.1). In [10] the cases $w(t)=t^\gamma e^{-t}$, $\gamma>-1$, and $w(t)=t^\gamma E_m(t)$, $\gamma>-1$, $\gamma+m>0$, where $E_m(t)$ is the exponential integral, were considered. The results of [10] were used in [9] in the derivation of new numerical quadrature formulas for infinite range integrals with $w(x)$ above as the weight functions. For further details see [9], [10]. Finally, the rest of the examples deal with some special functions when their orders tend to infinity.

Example 1.

$$I(x)=\int_0^\infty\frac{t^{x-1}e^{-ct}}{1-ze^{-t}}\,dt, \qquad \operatorname{Re}c>0. \tag{3.5}$$

Here $f(t)=e^{-ct}/(1-ze^{-t})$ satisfies all the requirements of Theorem 2.1 with $A_k=z^k$, $\lambda_k=0$, $\alpha_k=c+k$, $\beta_k=1$, $k=0,1,\cdots$. Hence

$$I(x)\sim\Gamma(x)\sum_{k=0}^\infty\frac{z^k}{(c+k)^x} \quad \text{as } x\to+\infty. \tag{3.6}$$

It is worth noting that for $|z|<1$ this series converges and $\sim$ can be replaced by $=$. For $|z|>1$, however, the series diverges, but by Theorem 2.1, it represents $I(x)$ asymptotically as $x\to\infty$. A special case of this example is $I(x)=\Gamma(x)\zeta(x)$, where $\zeta(x)$ is the Riemann Zeta function, and is obtained by setting $c=1$, $z=1$.

Example 2.

$$I(x)=\int_0^\infty\frac{t^{x-1}w(t)}{z-t}\,dt, \qquad z\notin[0,\infty). \tag{3.7}$$

We assume that

$$w(t)\sim e^{-t}\sum_{k=0}^\infty\frac{c_k}{t^{k+\sigma}} \quad \text{as } t\to+\infty. \tag{3.8}$$

Therefore,

$$f(t)=\frac{w(t)}{z-t}\sim e^{-t}\sum_{k=0}^\infty\frac{c_k}{t^{k+\sigma+1}}\sum_{j=0}^\infty\frac{z^j}{t^j}=e^{-t}\sum_{k=0}^\infty\frac{A_k(z)}{t^{k+\sigma+1}} \quad \text{as } t\to+\infty, \tag{3.9}$$

where $A_k(z)$ are polynomials in z. Thus $f(t)$ satisfies all the conditions of Theorem 2.2 and the corollary, with $\alpha=1$, $\beta=1$, $\lambda_k=k+\sigma+1$, $k=0,1,\cdots$. Hence

$$I(x) \sim \Gamma(x-\sigma-1)\sum_{j=0}^{\infty}\frac{B_j(z)}{x^j} \quad \text{as } x\to+\infty, \tag{3.10}$$

where $B_j(z)$ are polynomials in z, and are independent of x. The $B_j(z)$ can be determined in terms of the c_k and z, but we shall not go into this. Furthermore, this expansion is valid for all $z\notin[0,\infty)$.

For the cases (a) $w(t)=t^{\gamma}e^{-t}$, and (b) $w(t)=t^{\gamma}E_m(t)$, where $E_m(t)=\int_1^{\infty}e^{-ty}/y^m\,dy$ is the exponential integral, (3.8) holds. For (a) (3.8) holds with $\sigma=-\gamma$ and $c_0=1$, $c_k=0$, $k\geq 1$. For (b) (3.8) holds with $\sigma=-\gamma+1$ and $c_k=(-1)^k(m)_k$, $k=0,1,\cdots$, where $(m)_k$ is the Pochhamer symbol, see [1, formula 5.1.51, p. 231]. Using entirely different techniques, in [10], (3.10) was shown to be valid for all $z\notin[0,\infty)$ for case (a) and for z with $\operatorname{Re} z<0$ for case (b). It is now obvious that (3.10) is valid for all $z\notin[0,\infty)$ as long as $w(t)$ satisfies (3.8).

Example 3. Asymptotic expansion of $K_\nu(z)$ as $\nu\to+\infty$.

Here $\operatorname{Re} z>0$ and $K_\nu(z)$ is the modified Bessel function of the second kind of order ν and has the integral representation, see [1, formula 9.6.23, p. 376],

$$K_\nu(z)=\frac{\pi^{1/2}(z/2)^{\nu}}{\Gamma(\nu+1/2)}\int_1^{\infty}e^{-zt}(t^2-1)^{\nu-1/2}dt. \tag{3.11}$$

Making the change of variable or integration $t^2-1=\xi^2$, we have

$$\begin{aligned} q_\nu(z)&=\int_1^{\infty}e^{-zt}(t^2-1)^{\nu-1/2}dt \\ &=\int_0^{\infty}\exp\left[-z(1+\xi^2)^{1/2}\right]\frac{\xi^{2\nu}}{(1+\xi^2)^{1/2}}d\xi. \end{aligned} \tag{3.12}$$

Now

$$\frac{\exp\left[-z(1+\xi^2)^{1/2}\right]}{(1+\xi^2)^{1/2}}=\frac{e^{-z\xi}}{\xi}\left\{\frac{\exp\left(z\left[\xi-(1+\xi^2)^{1/2}\right]\right)}{(1+1/\xi^2)^{1/2}}\right\}. \tag{3.13}$$

It is not difficult to see that the term inside the curly brackets has a convergent expansion of the form

$$q(\xi;z)=\frac{\exp\left(z\left[\xi-(1+\xi^2)^{1/2}\right]\right)}{(1+1/\xi^2)^{1/2}}=\sum_{k=0}^{\infty}\frac{A_k(z)}{\xi^k}, \qquad \xi>1 \tag{3.14}$$

with $A_k(z)$ being polynomials in z and $A_0(z)=1$. Therefore, the integrand of (3.12) satisfies all the conditions of Theorem 2.3 and the corollary, with $x=2\nu$, $\alpha=z$, $\beta=1$, $\lambda_k=k$, $k=0,1,\cdots$. Consequently

$$q_\nu(z)\sim z^{-2\nu}\Gamma(2\nu)\sum_{j=0}^{\infty}\frac{B_j(z)}{\nu^j} \quad \text{as } \nu\to+\infty, \tag{3.15}$$

where the $B_j(z)$ are polynomials in z and $B_0(z)=1$. By (3.15) and the duplication formula for the gamma function, see [1, formula 6.1.18, p. 256], we obtain

$$K_\nu(z) \sim \frac{1}{2}\left(\frac{2}{z}\right)^\nu \Gamma(\nu)\left[1+\sum_{j=1}^{\infty}\frac{B_j(z)}{\nu^j}\right] \quad \text{as } \nu \to +\infty, \tag{3.16}$$

which, for integer ν, has also been derived in [12] by analyzing the power series of $K_\nu(z)$ for small z.

Example 4. Asymptotic expansion of $Y_\nu(z)$ as $\nu \to +\infty$.

Here $\operatorname{Re} z>0$ and $Y_\nu(z)$ is the Bessel function of the second kind of order ν and has the integral representation, see [1, formula 9.1.22, p. 360],

$$Y_\nu(z)=\frac{1}{\pi}\int_0^\pi \sin(z\sin\theta-\nu\theta)\,d\theta-\frac{1}{\pi}\int_0^\infty \{e^{\nu t}+e^{-\nu t}\cos(\nu\pi)\}e^{-z\sinh t}\,dt. \tag{3.17}$$

Integrating by parts once, we see that the first integral is $O(\nu^{-1})$ as $\nu \to +\infty$. In the second integral we have two contributions:

$$\begin{aligned} I_1&=-\frac{1}{\pi}\int_0^\infty e^{\nu t}e^{-z\sinh t}\,dt,\\ I_2&=-\frac{1}{\pi}\cos(\nu\pi)\int_0^\infty e^{-\nu t}e^{-z\sinh t}\,dt. \end{aligned} \tag{3.18}$$

Using Watson's lemma, we can show that $I_2 \sim \cos(\nu\pi)\sum_{j=1}^{\infty} b_j(z)/\nu^j$ as $\nu \to +\infty$, with $b_j(z)$ being independent of ν. In I_1 we make the change of variable of integration $e^t=\xi$, and obtain

$$I_1=-\frac{1}{\pi}\int_1^\infty \xi^{\nu-1}e^{-z\xi/2}e^{-z/(2\xi)}\,d\xi. \tag{3.19}$$

We can now apply Theorem 2.3 and the corollary, and obtain

$$I_1 \sim -\frac{1}{\pi}\left(\frac{z}{2}\right)^{-\nu}\Gamma(\nu)\left\{1+\sum_{j=1}^{\infty}\frac{c_j(z)}{\nu^j}\right\} \quad \text{as } \nu \to +\infty, \tag{3.20}$$

where the $c_j(z)$ are independent of ν. Consequently

$$Y_\nu(z) \sim -\frac{1}{\pi}\left(\frac{2}{z}\right)^\nu \Gamma(\nu)[1+O(\nu^{-1})] \quad \text{as } \nu \to +\infty. \tag{3.21}$$

Example 5. Asymptotic expansion of $H_\nu(z)-Y_\nu(z)$ as $\nu \to +\infty$.

Here $\operatorname{Re} z>0$ and $H_\nu(z)$ is the Struve function of order ν. We have, see [1, formula 12.1.18, p. 496],

$$H_\nu(z)-Y_\nu(z)=\frac{2(z/2)^\nu}{\pi^{1/2}\Gamma(\nu+1/2)}\int_0^\infty e^{-zt}(1+t^2)^{\nu-1/2}\,dt. \tag{3.22}$$

Making the change of variable of integration $1+t^2=\xi^2$, we have

$$q_\nu(z)=\int_0^\infty e^{-zt}(1+t^2)^{\nu-1/2}\,dt=\int_1^\infty \exp\left[-z(\xi^2-1)^{1/2}\right]\frac{\xi^{2\nu}}{(\xi^2-1)^{1/2}}\,d\xi. \tag{3.23}$$

As in Example 3,

$$\frac{\exp\left[-z(\xi^2-1)^{1/2}\right]}{(\xi^2-1)^{1/2}}=\frac{e^{-z\xi}}{\xi}\sum_{k=0}^{\infty}\frac{A_k(z)}{\xi^k},\qquad \xi>1, \tag{3.24}$$

where the $A_k(z)$ are polynomials in z and $A_0(z)=1$. Following Example 3, we obtain

$$H_\nu(z)-Y_\nu(z)\sim\frac{1}{\pi}\left(\frac{2}{z}\right)^\nu\Gamma(\nu)\left[1+O(\nu^{-1})\right]\quad \text{as } \nu\to+\infty. \tag{3.25}$$

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