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Interpolation by a Sum of Exponential Functions When Some Exponents Are Preassigned

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In a recent paper (A. Sidi, *J. Approx. Theory* **34** (1982), 194–210) the author has given the solutions to the problems of interpolation at equidistant points and confluent interpolation by a sum of exponential functions when none of the exponents is known. In the present work we generalize these results by specifying some of the exponents. Necessary and sufficient conditions for existence and uniqueness of solutions are given, and the solutions are provided in closed form. The connection of these problems with Padé approximants is exploited to prove a limit result.

1. INTRODUCTION

In a recent paper [4] the author has given complete constructive solutions to the following problems of interpolation by sums of exponential functions:

(1) Find $u(x; h) \in U_n^h$ satisfying

$$u(x_0 + ih; h) = c_i, \qquad i = 0, 1, \ldots, 2n-1, \tag{1.1}$$

where $h \neq 0$, and c_i are given constants, and

$$\begin{aligned} U_n^h = \Big\{ w(x) = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} B_{j,k} x^{k-1} \zeta_j^{x/h} \mid \zeta_j \text{ distinct}, \\ -\pi < \arg \zeta_j \leqslant \pi,\ \zeta_j^{y} \text{ takes on its principal value}, \\ \sum_{j=1}^{r} \lambda_j \leqslant n \Big\}. \end{aligned} \tag{1.2}$$

(2) Find $v(x) \in U_n$ satisfying

$$v^{(i)}(x_0) = \gamma_i, \qquad i = 0, 1, \ldots, 2n-1, \tag{1.3}$$

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where γ_i are given constants, and

$$U_n = \left\{ w(x) = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} B_{j,k} x^{k-1} e^{\sigma_j x} \,|\, \sigma_j \text{ distinct}, \sum_{j=1}^{r} \lambda_j \leqslant n \right\}. \tag{1.4}$$

These problems were solved by observing their connection with appropriate Padé approximants. Necessary and sufficient conditions for the existence and uniqueness of their solutions and explicit formulas for these solutions were all given in terms of the corresponding Padé approximants.

Note that, for the two problems described in (1) and (2), the ζ_j (see (1.2)) or the σ_j (see (1.4)), their respective multiplicities λ_j, and the constants $B_{j,k}$ are not known, and they are determined from the interpolation conditions (1.1) or (1.3). In the present work we generalize the problems in (1) and (2) by assuming that some of the ζ_j or σ_j are known and that we have "some knowledge" of their multiplicities λ_j, the corresponding $B_{j,k}$ being unknown. Surprisingly, these problems can also be solved by using Padé approximants and the techniques employed in [4], provided they are formulated appropriately, and this is done below:

(1a) Find $u(x; h) \in U_n^h$, for which s of the ζ_j's, say, $\zeta_1, ..., \zeta_s$, are preassigned, and their respective multiplicities are *bounded from below* by the integers $\mu_1, ..., \mu_s$, such that $\sum_{j=1}^{s} \mu_j \leqslant n$; i.e., $u(x; h)$ is of the form

$$u(x; h) = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} B_{j,k} x^{k-1} \zeta_j^{x/h}, \qquad \lambda_j \geqslant \mu_j,\ j = 1, ..., s,\ \sum_{j=1}^{r} \lambda_j \leqslant n, \tag{1.5}$$

and satisfying the interpolation conditions

$$u(x_0 + ih; h) = c_i, \qquad i = 0, 1, ..., 2n - 1 - \sum_{j=1}^{s} \mu_j. \tag{1.6}$$

We shall call this problem the discrete interpolation problem.

(2a) Find $v(x) \in U_n$, for which s of the σ_j's, say, $\sigma_1, ..., \sigma_s$, are preassigned, and their respective multiplicities are *bounded from below* by the integers $\mu_1, ..., \mu_s$, such that $\sum_{j=1}^{s} \mu_j \leqslant n$; i.e., $v(x)$ is of the form

$$v(x) = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} B_{j,k} x^{k-1} e^{\sigma_j x}, \qquad \lambda_j \geqslant \mu_j,\ j = 1, ..., s,\ \sum_{j=1}^{r} \lambda_j \leqslant n, \tag{1.7}$$

and satisfying the interpolation conditions

$$v^{(i)}(x_0) = \gamma_i, \qquad i = 0, 1, ..., 2n - 1 - \sum_{j=1}^{s} \mu_j. \tag{1.8}$$

We shall call this problem the confluent interpolation problem.

Note that in both the discrete and confluent interpolation problems, the coefficients $B_{j,k}$, $1 \leqslant k \leqslant \mu_j$, $1 \leqslant j \leqslant s$, in (1.5) and in (1.7), are unknowns, and some or all of them may eventually turn out to be zero.

Examples 1 and 2 below will help to clarify and motivate the formulation of the discrete interpolation problem. Example 3 demonstrates that this problem does not always have a solution. Similar examples can be constructed for the confluent interpolation problem.

In all three examples below we take $x_0 = 0$ and $h = 1$.

EXAMPLE 1. Find $u(x; 1) \in U_2^1$, when $u(x; 1) = A 1^x + B\zeta^x$, and $c_0 = 2$, $c_1 = 5$, $c_2 = 13$. This problem has the unique solution $u(x; 1) = \frac{1}{5} + \frac{9}{5}(\frac{8}{3})^x$. If we consider the same problem with $c_0 = 1$, $c_1 = 2$, $c_2 = 4$, then $u(x; 1) = 2^x$ is the unique solution, and does not contain in it the part 1^x; i.e., the coefficient A of 1^x is zero.

EXAMPLE 2. Find $u(x; 1) \in U_3^1$, when $u(x; 1) = (A + Bx)\, 2^x + C\zeta^x$, and $c_0 = 0$, $c_1 = 2$, $c_2 = 16$, $c_3 = 72$. This problem does not have a solution. However, there does exist a unique solution of the form $u(x; 1) = (A + Bx + Cx^2)\, 2^x$, and it is given by $u(x; 1) = x^2 2^x$.

EXAMPLE 3. Find $u(x; 1) \in U_2^1$, when $u(x; 1)$ has in it the component 2^x, and $c_0 = 1$, $c_1 = 1$, $c_2 = 2$. This problem has no solution, of either of the forms $u(x; 1) = A2^x + B\zeta^x$ or $u(x; 1) = (A + Bx)\, 2^x$, which are the only possible forms that contain in them the component 2^x.

We note that, for the discrete interpolation problem, the case in which $\mu_j = 1$, $1 \leqslant j \leqslant s$, can be found in Hamming [2, pp. 623–624], and is solved there by extending Prony's method for the problem described in (1). This extension, however, does not work for the general discrete interpolation problem described in (1a).

Note also that the special case, in which $\sum_{j=1}^{s} \mu_j = n$, is simply interpolation with *all* th exponents specified, and it is easy to verify that for this case solutions to both the discrete and the confluent interpolation problems exist, and are unique.

The remainder of this work is planned as follows: In Section 2 we define and give the relevant properties of Padé approximants with preassigned poles, since they are used to solve our problems. In Section 3 we give the solutions to the two interpolation problems under consideration, and at the same time provide contour integral representations for them. Finally, in Section 4 we consider the case $c_i = f(x_0 + ih)$ and $\gamma_i = f^{(i)}(x_0)$, where the function $f(x)$ is sufficiently smooth in a neighbourhood of x_0. For this case we show, roughly speaking, that whenever the solution to the confluent interpolation problem exists, the solution to the discrete interpolation problem also exists (for h sufficiently close to zero) and that the latter tends to the former as $h \to 0$. The techniques used are similar to those of [4], although the presence of the known exponents complicates some of the proofs considerably.

2. Padé Approximants with Preassigned Poles

A brief outline of Padé approximants and some of their properties, which are of relevance to our problems, have been given in Section 2 of [4]. We now extend that outline to Padé approximants some of whose poles are preassigned. We again start with the formal power series

$$g(z) = \sum_{i=0}^{\infty} c_i z^i, \tag{2.1}$$

and define its (m/n) Padé approximant, with the preassigned non-zero poles $z_1, ..., z_s$, with multiplicities $\mu_1, ..., \mu_s$, respectively, and total multiplicity $\mu = \sum_{j=1}^{s} \mu_j$, if it exists, to be the rational function

$$\tilde{g}_{m,n}(z) = \frac{P(z)}{q(z)\, Q(z)}, \qquad Q(0) = 1, \tag{2.2}$$

where

$$q(z) = \prod_{j=1}^{s} (z - z_j)^{\mu_j}, \tag{2.3}$$

and $P(z)$ and $Q(z)$ are polynomials in z of degree at most m and $n - \mu$, respectively, such that the Maclaurin series of $\tilde{g}_{m,n}(z)$ agrees with that in (2.1), up to and including the term $c_{m+n-\mu} z^{m+n-\mu}$, i.e.,

$$g(z) - \tilde{g}_{m,n}(z) = O(z^{m+n-\mu+1}) \qquad \text{as} \quad z \to 0. \tag{2.4}$$

Lemma 2.1. *Let*

$$G(z) = q(z)\, g(z) = \sum_{i=0}^{\infty} \tilde{c}_i z^i. \tag{2.5}$$

Then the rational function $H(z) = P(z)/Q(z)$, with $P(z)$ and $Q(z)$ as in (2.2), is the $(m/n-\mu)$ Padé approximant $G_{m,n-\mu}(z)$ to $G(z)$.

Proof. Multiplying both sides of (2.4) by $q(z)$, we obtain

$$G(z) - H(z) = O(z^{m+n-\mu+1}) \qquad \text{as} \quad z \to 0. \tag{2.6}$$

The result now follows from the fact that $\deg P(z) \leqslant m$ and $\deg Q(z) \leqslant n-\mu$, and the definition of Padé approximants. ∎

Note that since we know how to compute the ordinary Padé approximants easily, by Lemma 2.1 we can compute the rational function $\tilde{g}_{m,n}(z)$ with no difficulty.

THEOREM 2.1. *If $\tilde{g}_{m,n}(z)$ in (2.2) exists, it is unique.*

Proof. This follows easily from the uniqueness of the ordinary Padé approximant $G_{m,n-\mu}(z)$ in Lemma 2.1, and Lemma 2.1 itself. ∎

As in [4], we say that the rational function $v(z)$ has property R if the degree of its numerator polynomial is strictly less than that of its denominator polynomial.

The $\tilde{g}_{m,n}(z)$ that are relevant to our work are those for which $m=n-1$, and we turn to those now.

If $\tilde{g}_{n-1,n}(z)$ has property R, then, after canceling the common factors in its numerator and denominator, it will have the form $\tilde{g}_{n-1,n}(z)=\bar{P}(z)/\bar{Q}(z)$, where $\deg \bar{P}(z) \leqslant n'-1$ and $\deg \bar{Q}(z)=n'$, with $n' \leqslant n$. This is possible if and only if the ordinary Padé approximant $G_{n-1,n-\mu}(z)$, after canceling the common factors of its numerator and denominator, has the form $G_{n-1,n-\mu}(z)=\hat{P}(z)/\hat{Q}(z)$, with $\deg \hat{P}(z) \leqslant n''-1$ and $\deg \hat{Q}(z)=n''-\mu$, such that $n'' \leqslant n$. As mentioned in [4], whether or not this is true can be established by analyzing the C-table of $G(z)$, hence we consider this problem totally solved.

THEOREM 2.2. *Let $\tilde{g}_{n-1,n}(z)$ have property R, and let its partial fraction decomposition be*

$$\tilde{g}_{n-1,n}(z)=\sum_{j=1}^{r}\sum_{k=1}^{\lambda_j}\frac{A_{j,k}}{(z-z_j)^k}. \tag{2.7}$$

Then $A_{j,k}$ and z_j satisfy the equations

$$c_i=\sum_{j=1}^{r}\sum_{k=1}^{\lambda_j}(-1)^k\binom{k+i-1}{k-1}\frac{A_{j,k}}{z_j^{k+i}}, \qquad i=0, 1,..., 2n-\mu-1. \tag{2.8}$$

Conversely, given the constants $c_i, 0 \leqslant i \leqslant 2n-\mu-1$, let $z_j \neq 0$ and $A_{j,k}$, $1 \leqslant k \leqslant \lambda_j$, $1 \leqslant j \leqslant r$, for some λ_j and r such that $\sum_{j=1}^{r}\lambda_j \leqslant n$, satisfy Eqs. (2.8). Then the rational function $\sum_{j=1}^{r}\sum_{k=1}^{\lambda_j}A_{j,k}/(z-z_j)^k$ is $\tilde{g}_{n-1,n}(z)$ with z_j, μ_j, $1 \leqslant j \leqslant s$, preassigned, and $\sum_{j=1}^{s}\mu_j=\mu$.

Proof. Similar to those of Theorem 2.2 and Theorem 2.3 in [4]. ∎

3. Solution of the Interpolation Problems

(1) *The Discrete Interpolation Problem*

THEOREM 3.1. *Using the notation of* (1a) *in Section* 1, *let* $q(z)=\prod_{j=1}^{s}(z-z_j)^{\mu_j}$, *where* $z_j=\zeta_j^{-1}$, $1\leqslant j\leqslant s$, $\mu=\sum_{j=1}^{s}\mu_j$, $F(z)=\sum_{i=0}^{2n-\mu-1}c_i z^i$, *and* $G(z)=q(z)\,F(z)$. *Then the discrete interpolation problem described in* (1a) *of Section* 1 *has a unique solution* $u(x;h)$ *if and only if the* $(n-1/n-\mu)$ *Padé approximant* $G_{n-1,n-\mu}(z)$ *to* $G(z)$ *exists, and* $z^{-\mu}G_{n-1,n-\mu}(z)$ *has property R. Then the* $(n-1/n)$ *Padé approximant* $\tilde{F}_{n-1,n}(z)$ *to* $F(z)$, *with preassigned poles* z_j *of multiplicity* μ_j, $1\leqslant j\leqslant s$, *exists and has property R*; *let its partial fraction decomposition be*

$$\tilde{F}_{n-1,n}(z)=\sum_{j=1}^{r}\sum_{k=1}^{\lambda_j}\frac{A_{j,k}}{(z-z_j)^k}. \tag{3.1}$$

Then $u(x;h)$ *is given by*

$$u(x;h)=\sum_{j=1}^{r}\sum_{k=1}^{\lambda_j}E_{j,k}\binom{k+((x-x_0)/h)-1}{k-1}\zeta_j^{(x-x_0)/h}, \tag{3.2}$$

where

$$\zeta_j=z_j^{-1},\qquad E_{j,k}=(-1)^k A_{j,k}z_j^{-k},\qquad 1\leqslant k\leqslant\lambda_j,\ 1\leqslant j\leqslant r, \tag{3.3}$$

and $\binom{a}{k}$ *is the binomial coefficient. Furthermore, if none of the* z_j *lies on* $(-\infty,0]$, $u(x;h)$ *has the contour integral representation*

$$u(x;h)=-\frac{1}{2\pi i}\int_C\tilde{F}_{n-1,n}(z)\,z^{-\omega}\,dz, \tag{3.4}$$

where C is a simple closed Jordan curve whose interior contains all the poles of $\tilde{F}_{n-1,n}(z)$ *and never touches the line* $(-\infty,0]$, $\omega=(x-x_0)/h+1$, *and* $z^{-\omega}$ *takes on its principal value and has a branch cut along the line* $(-\infty,0]$.

Proof. The proof can be achieved by using Theorem 2.2, and is similar to those of Theorem 3.1 and Theorem 5.1 in [4]. ∎

(2) *The Confluent Interpolation Problem*

THEOREM 3.2. *Using the notation of* (2a) *in Section* 1, *let* $t(\sigma)=\prod_{j=1}^{s}(\sigma-\sigma_j)^{\mu_j}$, $\mu=\sum_{j=1}^{s}\mu_j$, $V(\tau)=\sum_{i=0}^{2n-\mu-1}\gamma_i\tau^i$, *and* $W(\tau)=\tau^{\mu}t(\tau^{-1})\,V(\tau)$. *Then the confluent interpolation problem described in* (2a) *of Section* 1 *has a unique solution* $v(x)$ *if and only if the* $(n-1/n-\mu)$ *Padé approximant* $W_{n-1,n-\mu}(\tau)$ *to* $W(\tau)$ *exists. Then the* $(n-1/n)$ *Padé approximant*

$\tilde{V}_{n-1,n}(\tau) = W_{n-1,n-\mu}(\tau)/[\tau^{\mu}t(\tau^{-1})]$ *to* $V(\tau)$ *with preassigned finite poles* σ_j^{-1} *of multiplicity* μ_j, $1 \leq j \leq s$, *exists, and* $\hat{V}_{n-1,n}(\sigma) = \sigma^{-1}\tilde{V}_{n-1,n}(\sigma^{-1})$ *has property R. Furthermore, letting* $\hat{V}(\sigma) = \sigma^{-1}V(\sigma^{-1})$, *we have*

$$\hat{V}(\sigma) - \hat{V}_{n-1,n}(\sigma) = O(\sigma^{-2n+\mu-1}) \quad \text{as} \quad \sigma \to \infty. \tag{3.5}$$

Let the partial fraction decomposition of $\hat{V}_{n-1,n}(\sigma)$ *be*

$$\hat{V}_{n-1,n}(\sigma) = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} \frac{B_{j,k}}{(\sigma - \sigma_j)^k}. \tag{3.6}$$

Then $v(x)$ *is given by*

$$v(x) = \sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} \frac{B_{j,k}}{(k-1)!} (x - x_0)^{k-1} e^{\sigma_j(x - x_0)}. \tag{3.7}$$

Furthermore, $v(x)$ *has the contour integral representation*

$$v(x) = \frac{1}{2\pi i} \int_D \hat{V}_{n-1,n}(\sigma)\, e^{\sigma(x - x_0)}\, d\sigma, \tag{3.8}$$

where D is a simple closed Jordan curve whose interior contains all the poles of $\hat{V}_{n-1,n}(\sigma)$.

Proof. Similar to those of Theorem 4.1 and Theorem 5.2 in [4]. ∎

When all the exponents in both interpolation problems are specified, it is known that a unique solution exists. This can be recovered from Theorem 3.1 and Theorem 3.2 above, simply by observing that for this case $\mu = n$, and that the Padé approximants $G_{n-1,0}(z)$ and $W_{n-1,0}(\tau)$ exist, and are given by $G_{n-1,0}(z) = \sum_{i=0}^{n-1} \tilde{c}_i z^i$ and $W_{n-1,0}(\tau) = \sum_{i=0}^{n-1} \tilde{\gamma}_i \tau^i$, and that $\tilde{F}_{n-1,n}(z)$ automatically has property R.

4. A Limit Result

Throughout this section we shall be using the notation of the previous sections.

Let $f(x)$ be a function, which is $2n - \mu - 1$ times continuously differentiable in a neighbourhood of x_0, and let $c_i = f(x_0 + ih)$ and $\gamma_i = f^{(i)}(x_0)$, $i = 0, 1, \ldots, 2n - \mu - 1$. Furthermore assume that the preassigned exponents in the discrete and confluent interpolation problems are the same, i.e., $\zeta_j = e^{h\sigma_j}$, $1 \leq j \leq s$, and so are their respective multiplicities. We would now like to show that if $v(x)$ exists, then, for h sufficiently small, $u(x; h)$ exists and $\lim_{h \to 0} u(x; h) = v(x)$. For this we need several preliminary results.

Lemma 4.1. *Let* $q(z) = \prod_{j=1}^{s} (z - z_j)^{\mu_j} = \sum_{j=0}^{\mu} \alpha_j z^j$, *where* $z_j = \zeta_j^{-1}$,

$1 \leqslant j \leqslant s$, *and let* $t(\sigma) = \prod_{j=1}^{s} (\sigma - \sigma_j)^{\mu_j} = \sum_{j=0}^{\mu} a_j \sigma^j$. *Let* $z = e^{-h\sigma}$. *For* σ *and* σ_j, $1 \leqslant j \leqslant s$, *fixed, we have*

$$\lim_{h \to 0} \frac{q(z)}{h^\mu} = (-1)^\mu t(\sigma) \tag{4.1}$$

and

$$\lim_{h \to 0} \frac{\sum_{j=0}^{\mu-p} \binom{\mu-j}{p} \alpha_j}{h^{\mu-p}} = (-1)^\mu a_p, \qquad 0 \leqslant p \leqslant \mu. \tag{4.2}$$

(*Note that the* α_j *are functions of* h.)

Proof. The proof of (4.1) is easy and we omit it. To prove (4.2) we proceed as follows: Define

$$\eta = (1 - z)/hz. \tag{4.3}$$

Obviously η is a function of h when z is; and for $z = e^{-h\sigma}$ we have $\eta = \sigma \sum_{k=0}^{\infty} (h\sigma)^k/(k+1)!$, and

$$\lim_{h \to 0} z = 1 \qquad \text{and} \qquad \lim_{h \to 0} \eta = \sigma. \tag{4.4}$$

Also

$$z = 1/(1 + h\eta). \tag{4.5}$$

Using (4.5) we can express $q(z)/h^\mu$ as

$$\begin{aligned} \frac{q(z)}{h^\mu} &= \frac{z^\mu}{h^\mu} \sum_{j=0}^{\mu} \alpha_j (1 + h\eta)^{\mu - j} \\ &= z^\mu \sum_{p=0}^{\mu} \left\{ \frac{1}{h^{\mu-p}} \sum_{j=0}^{\mu-p} \binom{\mu-j}{p} \alpha_j \right\} \eta^p. \end{aligned} \tag{4.6}$$

Furthermore, it is easy to see that

$$(-1)^\mu p!\, a_p = \frac{d^p}{d\sigma^p} \lim_{h \to 0} \left. \frac{q(z)}{(hz)^\mu} \right|_{\sigma=0} = \lim_{h \to 0} \frac{d^p}{d\sigma^p} \left. \frac{q(z)}{(hz)^\mu} \right|_{\sigma=0}, \qquad p = 0, 1, \ldots, \mu. \tag{4.7}$$

Substituting (4.6) in (4.7), and using the fact that

$$\begin{aligned} \left. \frac{d^p}{d\sigma^p} \eta^j \right|_{\sigma=0} &= 0 && \text{if} \quad j > p \\ &= p! && \text{if} \quad j = p \\ &= o(1) \text{ as } h \to 0 && \text{if} \quad j < p, \end{aligned} \tag{4.8}$$

(4.2) can be proved by induction on p, starting with $p = 0$. ∎

LEMMA 4.2. *Let $G(z)$ in Theorem 3.1 and $W(\tau)$ in Theorem 3.2 have the expansions*

$$G(z)=\sum_{i=0}^{2n-1} \tilde{c}_i z^i \qquad \textit{and} \qquad W(\tau)=\sum_{i=0}^{2n-1} \tilde{\gamma}_i \tau^i. \tag{4.9}$$

Therefore,

$$\tilde{c}_i=\sum_{j=0}^{\min(i,\mu)} c_{i-j}\alpha_j, \qquad i=0, 1,..., 2n-\mu-1 \tag{4.10}$$

and

$$\tilde{\gamma}=\sum_{k=\max(0,\mu-i)}^{\mu} \gamma_{i+k-\mu} a_k, \qquad i=0, 1,..., 2n-\mu-1. \tag{4.11}$$

Then

$$\lim_{h\to 0}\frac{\Delta^i\tilde{c}_\mu}{h^{\mu+i}}=(-1)^\mu\,\tilde{\gamma}_{u+i}, \qquad i=0, 1,..., 2n-2\mu-1, \tag{4.12}$$

where Δ is the forward difference operator operating on μ.

Proof. From Taylor's theorem with remainder, we have

$$c_k=f(x_0+kh)=\sum_{m=0}^{2n-\mu-1}\frac{\gamma'_m}{m!}(kh)^m, \tag{4.13}$$

where $\gamma'_m=\gamma_m$, $0\leqslant m\leqslant 2n-\mu-2$, and $\gamma'_{2n-\mu-1}=\gamma_{2n-\mu-1}+o(1)$ as $h\to 0$. Combining (4.10) and (4.13), we have

$$\Delta^i\tilde{c}_\mu=\sum_{p=0}^{\mu}\alpha_p\sum_{m=0}^{2n-\mu-1}\frac{\gamma'_m}{m!}[\Delta^i(\mu-p)^m]\,h^m. \tag{4.14}$$

Now

$$h^m(\mu-p)^m=\frac{d^m}{d\sigma^m}e^{(\mu-p)h\sigma}\bigg|_{\sigma=0}, \qquad m=0, 1,.... \tag{4.15}$$

Substituting (4.15) in (4.14), and rearranging, we obtain

$$\begin{aligned}\Delta^i\tilde{c}_\mu&=\sum_{m=0}^{2n-\mu-1}\frac{\gamma'_m}{m!}\frac{d^m}{d\sigma^m}\sum_{p=0}^{\mu}\alpha_p[\Delta^i e^{(\mu-p)h\sigma}]\bigg|_{\sigma=0}\\&=\sum_{m=0}^{2n-\mu-1}\frac{\gamma'_m}{m!}\frac{d^m}{d\sigma^m}\sum_{p=0}^{\mu}\alpha_p e^{(\mu-p)h\sigma}(e^{h\sigma}-1)^i\bigg|_{\sigma=0}\\&=\sum_{m=0}^{2n-\mu-1}\frac{\gamma'_m}{m!}\frac{d^m}{d\sigma^m}\{e^{\mu h\sigma}(e^{h\sigma}-1)^i\,q(e^{-h\sigma})\}\bigg|_{\sigma=0}.\end{aligned} \tag{4.16}$$

Finally, dividing both sides of (4.16) by $h^{\mu+i}$ and using the fact that $\lim_{h\to 0}\lim_{\sigma\to 0}(d^m/d\sigma^m)=\lim_{\sigma\to 0}(d^m/d\sigma^m)\lim_{h\to 0}$, and Lemma 4.1, we obtain

$$\lim_{h\to 0}\frac{\Delta^i\tilde{c}_\mu}{h^{\mu+i}}=(-1)^\mu\sum_{m=0}^{2n-\mu-1}\frac{\gamma_m}{m!}\frac{d^m}{d\sigma^m}\{\sigma^i t(\sigma)\}\bigg|_{\sigma=0}. \tag{4.17}$$

The result now follows by invoking (4.11). ∎

Denote $\bar{n}=n-\mu$, and for the sequences A_m, $m=0, 1,\ldots,$ and d_i, $i=0, 1,\ldots,$ define

$$D(A_m; d_i)=\begin{vmatrix} A_0 & A_1 & \cdots & A_{\bar{n}} \\ d_0 & d_1 & \cdots & d_{\bar{n}} \\ d_1 & d_2 & \cdots & d_{\bar{n}+1} \\ \vdots & \vdots & & \vdots \\ d_{\bar{n}-1} & d_{\bar{n}} & \cdots & d_{2\bar{n}-1}\end{vmatrix}. \tag{4.18}$$

LEMMA 4.3. *Let* $\rho=(\prod_{i=0}^{n-\mu}h^{\mu+i})(\prod_{i=1}^{n-\mu}h^{i-1})$. *Then*

$$D(A_m; d_{\mu+i})=\rho D\left(\frac{\Delta^m A_0}{h^{\mu+m}};\frac{\Delta^i d_\mu}{h^{\mu+i}}\right), \tag{4.19}$$

where

$$\Delta^m A_0=\sum_{i=0}^{m}(-1)^{m-j}\binom{m}{j}A_j \quad \text{and} \quad \Delta^i d_\mu=\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}d_{\mu+j}. \tag{4.20}$$

Proof. The proof of (4.19) can be achieved by performing elementary row and column transformations on $D(A_m; d_{\mu+i})$. For details see the proof of Theorem 5.4 in [4]. ∎

The following result is an extension of Theorem 5.4 in [4].

THEOREM 4.1. *Let the polynomials* $\psi(\sigma)$ *and* $Q(z; h)$ *be given by*

$$\psi(\sigma)=D(\sigma^m;\tilde{\gamma}_{\mu+i}) \qquad \text{and} \qquad Q(z;h)=D(z^{n-\mu-m};\tilde{c}_{\mu+i}). \tag{4.21}$$

Let $\psi(\sigma)$ *have exactly* $n-\mu$ *zeros, which we denote by* $\bar{\sigma}_i$, $i=1,\ldots,n-\mu$, *counting multiplicities. Then for* h *sufficiently close to zero,* $Q(z;h)$ *has exactly* $n-\mu$ *zeros, which we denote by* $\bar{z}_i(h)$, $i=1,\ldots,n-\mu$, *counting multiplicities, with the property that* $\bar{z}_i(h)$ *are continuous in a neighbourhood of* $h=0$, *and differentiable at* $h=0$, *such that*

$$\bar{z}_i(0)=1, \qquad \frac{d\bar{z}_i(h)}{dh}\bigg|_{h=0}=-\bar{\sigma}_i, \qquad i=1,\ldots,n-\mu, \tag{4.22}$$

with proper ordering.

Proof. By Lemma 4.3 and (4.3), we have

$$Q(z;h) = \frac{\rho z^{n-\mu}}{h^\mu} D\left(\eta^m; \frac{\Delta^i \tilde{c}_\mu}{h^{\mu+i}}\right). \tag{4.23}$$

By recalling (4.12) in Lemma 4.2, the proof can be completed exactly as that of Theorem 5.4 in [4]. ∎

LEMMA 4.4. *Let* $z = e^{-\sigma h}$ *and*

$$S_k = \sum_{i=0}^{k} \tilde{c}_i z^i, \qquad T_k = \sum_{i=0}^{k} \tilde{\gamma}_i/\sigma^i, \qquad k = 0, 1, \ldots. \tag{4.24}$$

Then

$$\lim_{h\to 0} \frac{S_{\mu-1}}{h^{\mu-1}} = (-1)^\mu \sigma^{\mu-1} T_{\mu-1}. \tag{4.25}$$

Proof. Making use of (4.5), we have

$$\begin{aligned} S_{\mu-1} &= z^{\mu-1} \sum_{\nu=0}^{\mu-1} \tilde{c}_\nu (1+h\eta)^{\mu-1-\nu} \\ &= z^{\mu-1} \sum_{k=0}^{\mu-1} (h\eta)^k \sum_{\nu=0}^{\mu-1-k} \binom{\mu-1-\nu}{k} \tilde{c}_\nu. \end{aligned} \tag{4.26}$$

By (4.4), all we have to show now is that

$$\lim_{h\to 0} \frac{\sum_{\nu=0}^{\mu-1-k} \binom{\mu-1-\nu}{k} \tilde{c}_\nu}{h^{\mu-1-k}} = (-1)^\mu \tilde{\gamma}_{\mu-1-k}, \qquad k = 0, 1, \ldots, \mu-1. \tag{4.27}$$

Recalling (4.10), we have

$$R_k = \sum_{\nu=0}^{\mu-1-k} \binom{\mu-1-\nu}{k} \tilde{c}_\nu = \sum_{p=0}^{\mu-1-k} \alpha_p \sum_{\nu=p}^{\mu-1-k} \binom{\mu-1-\nu}{k} c_{\nu-p}. \tag{4.28}$$

Making use of the fact that $c_m = \sum_{i=0}^{m} \binom{m}{i} \Delta^i c_0$, $m = 0, 1, \ldots$, we can express (4.28) as

$$R_k = \sum_{p=0}^{\mu-1-k} \alpha_p \sum_{q=0}^{\mu-1-k-p} Z_q \Delta^q c_0, \tag{4.29}$$

where

$$Z_q = \sum_{\nu=p+q}^{\mu-1-k} \binom{\mu-1-\nu}{k} \binom{\nu-p}{q}. \tag{4.30}$$

But, see Knuth [3, p. 58, Eq. (25)], $Z_q = \binom{\mu-p}{k+1+q}$. Substituting this in (4.29), changing the order of summation, and dividing by $h^{\mu-1-k}$, we obtain

$$\frac{R_k}{h^{\mu-1-k}} = \sum_{q=0}^{\mu-1-k} \frac{\Delta^q c_0}{h^q} \left\{ \frac{1}{h^{\mu-1-k-q}} \sum_{p=0}^{\mu-1-k-q} \binom{\mu-p}{k+1+q} \alpha_p \right\}. \tag{4.31}$$

The result now follows by using Lemma 4.1, and the fact that $\lim_{h\to 0} \Delta^q c_0/h^q = \gamma_q$, $q = 0, 1,..., 2n-\mu-1$, and (4.11). ∎

LEMMA 4.5. *Let $z = e^{-\sigma h}$ and let S_k and T_k be as in* (4.24). *Then*

$$\lim_{h\to 0} \frac{\Delta^k(z^{-\mu} S_{\mu-1})}{h^{\mu+k-1}} = (-1)^\mu \sigma^{\mu+k-1} T_{\mu+k-1}, \qquad k = 0, 1,..., n-\mu, \tag{4.32}$$

where Δ is the forward difference operator operating on μ.

Proof. From (4.24) we have, after some manipulation,

$$\begin{aligned}\Delta^k(z^{-\mu} S_{\mu-1}) &= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} z^{-\mu-i} \left(S_{\mu-1} + \sum_{v=\mu}^{\mu+i-1} \tilde{c}_v z^v \right) \\ &= S_{\mu-1} z^{-\mu}(z^{-1}-1)^k + \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} z^{-i} \sum_{j=0}^{i-1} \tilde{c}_{\mu+j} z^j\end{aligned} \tag{4.33}$$

with $\sum_{j=0}^{i-1} \tilde{c}_{\mu+j} z^j$ defined to be zero for $i = 0$.

Now from Lemma 4.4, and (4.3) and (4.4), we have

$$\lim_{h\to 0} \frac{S_{\mu-1} z^{-\mu}(z^{-1}-1)^k}{h^{\mu+k-1}} = (-1)^\mu \sigma^{\mu+k-1} T_{\mu-1}, \qquad k = 0, 1,.... \tag{4.34}$$

From (4.12) in Lemma 4.2, and from Theorem 5.5 and Appendix in [4], it follows that

$$\lim_{h\to 0} \frac{\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} z^{-i} \sum_{j=0}^{i-1} \tilde{c}_{\mu+j} z^j}{h^{\mu+k-1}} = (-1)^\mu \sigma^{k-1} \sum_{i=0}^{k-1} \frac{\tilde{\gamma}_{\mu+i}}{\sigma^i}. \tag{4.35}$$

Dividing both sides of (4.33) by $h^{\mu+k-1}$, and making use of (4.34) and (4.35), (4.32) follows.

We now give the main result of this section.

THEOREM 4.2. *Let the polynomial $D(\sigma^m; \tilde{\gamma}_{\mu+i})$ be of degree precisely $n-\mu$. Then $v(x)$, the solution to the confluent interpolation problem, exists.*

Also, for h sufficiently close to zero, $u(x; h)$, the solution to the discrete interpolation problem, exists, and

$$\lim_{h \to 0} u(x; h) = v(x). \tag{4.36}$$

Proof. The fact that $D(\sigma^m; \tilde{\gamma}_{\mu+i})$ has degree precisely $n-\mu$ implies that the $(n-1/n-\mu)$ Padé approximant $W_{n-1,n-\mu}(\tau)$ to $W(\tau)$ (see Theorem 3.2) exists, which, by Theorem 3.2, implies that $v(x)$ exists. Using the determinant representations of Padé approximants, see Baker [1, Chap. 1], $\hat{V}_{n-1,n}(\sigma)$ can be expressed as

$$\hat{V}_{n-1,n}(\sigma) = \frac{\sigma^{\mu-1}}{t(\sigma)} \frac{D(\sigma^m T_{\mu-1+m}; \tilde{\gamma}_{\mu+i})}{D(\sigma^m; \tilde{\gamma}_{\mu+i})}. \tag{4.37}$$

From Theorem 4.1, the assumption that $D(\sigma^m; \tilde{\gamma}_{\mu+i})$ is of degree precisely $n-\mu$ implies that the polynomial $D(z^{n-\mu-m}; \tilde{c}_{\mu+i})$ has precisely $n-\mu$ zeros that tend to 1 as $h \to 0$, hence this polynomial is non-zero at $z=0$, whenever h is sufficiently close to zero. This implies that the $(n-1/n-\mu)$ Padé approximant $G_{n-1,n-\mu}(z)$ to $G(z)$ (see Theorem 3.1) exists and that $\tilde{F}_{n-1,n}(z)$ has property R. Consequently, by Theorem 3.1, $u(x; h)$ exists. Again using the determinant representations of Padé approximants, $\tilde{F}_{n-1,n}(z)$ can be expressed as

$$\tilde{F}_{n-1,n}(z) = \frac{1}{q(z)} \frac{D(z^{n-\mu-m} S_{\mu-1+m}; \tilde{c}_{\mu+i})}{D(z^{n-\mu-m}; \tilde{c}_{\mu+i})}. \tag{4.38}$$

Since, for h sufficiently close to zero, the poles of $\tilde{F}_{n-1,n}(z)$ are close to 1, the contour integral representation of $u(x; h)$, given in (3.4), holds, where the contour C can be taken to be a circle $\bar{C}$ of radius less than 1 with center at 1. Making the change of integration variable $z = e^{-\sigma h}$ in (3.4), we have

$$u(x; h) = \frac{1}{2\pi i} \int_{\bar{C}} h\tilde{F}_{n-1,n}(e^{-\sigma h})\, e^{\sigma(x-x_0)}\, d\sigma. \tag{4.39}$$

Now by (4.38), Lemma 4.3, and (4.23) in Theorem 4.1, we have

$$h\tilde{F}_{n-1,n}(e^{-\sigma h}) = \frac{1}{q(z)/h^{\mu}} \frac{D\left(\dfrac{\Delta^m [z^{n-\mu} S_{\mu-1}]}{h^{\mu-1+m}}; \dfrac{\Delta^i \tilde{c}_{\mu}}{h^{\mu+i}}\right)}{z^{n-\mu} D\left(\eta^m; \dfrac{\Delta^i \tilde{c}_{\mu}}{h^{\mu+i}}\right)}, \tag{4.40}$$

where η is as in (4.3). From (4.40), Lemma 4.1, and Theorem 4.1 it is clear that the poles of $h\tilde{F}_{n-1,n}(e^{-\sigma h})$ in the complex σ-plane are approaching those of $\hat{V}_{n-1,n}(\sigma)$, hence we can fix the contour $\bar{C}$ to be a circle enclosing all the poles of $\hat{V}_{n-1,n}(\sigma)$, for h sufficiently close to zero, and furthermore

$$\lim_{h \to 0} h\tilde{F}_{n-1,n}(e^{-\sigma h}) = \hat{V}_{n-1,n}(\sigma), \tag{4.41}$$

this limit being attained uniformly on $\bar{C}$. Equation (4.36) now follows by letting $h \to 0$ in (4.39), and comparing the result with (3.8). ∎

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