CONVERGENCE AND STABILITY PROPERTIES OF MINIMAL POLYNOMIAL AND REDUCED RANK EXTRAPOLATION ALGORITHMS*

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Abstract. The minimal polynomial and reduced rank extrapolation algorithms are two acceleration of convergence methods for sequences of vectors. In a recent survey these methods were tested and compared with the scalar, vector, and topological epsilon algorithms, and were observed to be more efficient than the latter. It was also observed that the two methods have similar convergence properties. The purpose of the present work is to analyze the convergence and stability properties of these methods, and to show that they are bona fide acceleration methods when applied to a class of vector sequences that includes those sequences obtained from systems of linear equations by using matrix iterative methods.

1. Introduction. The minimal polynomial extrapolation algorithm (MPE) of Cabay and Jackson [2] and the reduced rank extrapolation algorithm (RRE) of Eddy [3] and Mešina [4] are two methods that have been devised for accelerating the convergence of sequences of vectors. In a recent survey carried out by Smith, Ford, and Sidi [6], these two methods were tested and compared with the scalar epsilon algorithm of Wynn [7], the vector epsilon algorithm of Wynn [8], and the topological epsilon algorithm of Brezinski [1]. It was observed numerically that the MPE and the RRE have similar convergence properties, and, in general, are more efficient than the three epsilon algorithms.

In the present work we analyze the convergence and stability properties of the MPE and the RRE, and show that they are bona fide convergence acceleration methods when applied to a family of sequences that includes those sequences obtained from systems of linear equations by using matrix iterative methods.

In a recent work of Sidi, Ford, and Smith [5] (this issue, pp. 178–196) a general framework for deriving convergence acceleration methods for vector sequences has been proposed. Within this framework one can derive several methods, some old (including the MPE, RRE, and the topological epsilon algorithm), and some new (including a method that has been designated the modified MPE). The approach of [5] is formulated in general normed linear spaces of finite or infinite dimension. In the present work we use the formulations of the MPE and the RRE as they are given in [5]. We also make the assumption that the normed linear space in which the vector sequence is defined is an inner product space, with the norm being induced by the inner product.

The plan of the present paper is as follows: In § 2 we specify the vector sequences whose convergence we are seeking to accelerate, describe the MPE and the RRE as they were formulated in [5], deriving at the same time determinant representations for them. We shall use § 2 to also introduce much of the notation that we use in the remainder of this work. In § 3 we analyze the convergence properties of both methods and obtain actual rates of acceleration for them. In § 4 we analyze their stability properties. The results of §§ 3 and 4 are helpful in explaining some of the numerical results obtained from the MPE and the RRE. The techniques used in the present work are similar in nature to those developed and used in [5] in the analysis of the modified

MPE and the topological epsilon algorithm. The analysis in the present work, however, is considerably heavier due to the extreme nonlinearity of the MPE and the RRE. Surprisingly, *all* the conclusions that were drawn for the modified MPE hold for MPE and RRE; they are given following Theorem 3.1 and in Theorem 4.1.

2. Notation and description of algorithms. Let B be an inner product space defined over the field of complex numbers. In this work we shall adopt the following convention for the homogeneity property of the inner product. For $y, z \in B$ and α, β complex numbers, the inner product $(\cdot, \cdot)$ is defined such that $(\alpha y, \beta z) = \bar{\alpha}\beta(y, z)$. The norm of a vector $x \in B$ will be defined by $\|x\| = \sqrt{(x, x)}$.

Let us consider a sequence of vectors x_i, $i = 0, 1, \cdots$, in B. We shall assume that

$$(2.1) \qquad\qquad x_m \sim s + \sum_{i=1}^{\infty} v_i \lambda_i^m \quad \text{as } m \to \infty,$$

where s and v_i, $i = 1, 2, \cdots$, are vectors in B, and λ_i, $i = 1, 2, \cdots$, are scalars, such that $\lambda_i \neq 1$, $i = 1, 2, \cdots$, $\lambda_i \neq \lambda_j$ if $i \neq j$, $|\lambda_1| \geq |\lambda_2| \geq \cdots$, and that there can be only a finite number of λ_i whose moduli are equal. Without loss of generality, we assume in (2.1) that $v_i \neq 0$, $\lambda_i \neq 0$ for all $i \geq 1$. The meaning of (2.1) is that for any integer $N > 0$, there exist a positive constant K and a positive integer m_0 that depend only on N, such that for every $m \geq m_0$ the vector

$$\tilde{v}_N(m) = \left(x_m - s - \sum_{i=1}^{N-1} v_i \lambda_i^m \right) \bigg/ \lambda_N^m$$

satisfies

$$(2.1a) \qquad\qquad \|\tilde{v}_N(m)\| \leq K.$$

If $|\lambda_1| < 1$, then $\lim_{m \to \infty} x_m$ exists and is simply s. If $|\lambda_1| \geq 1$, then $\lim_{m \to \infty} x_m$ does not exist, and s is said to be the anti-limit of the sequence x_m, $m = 0, 1, \cdots$. Our problem is to find a good approximation to s, whether it be the limit or the anti-limit of the sequence, from a relatively small number of the vectors x_i, $i = 0, 1, \cdots$.

Example. Let A be a nondefective $M \times M$ (complex) matrix and b, an M-dimensional (complex) vector, and consider the solution of the linear system of equations

$$(2.2) \qquad\qquad x = Ax + b.$$

Let $\lambda_1, \cdots, \lambda_M$ and $v_1, \cdots, v_M$ be the eigenvalues and corresponding eigenvectors of A. Assume also that 1 is not an eigenvalue of A so that (2.2) has a unique solution, which we shall denote by s. For a given x_0, we generate the sequence x_j, $j = 1, 2, \cdots$, by the matrix iterative method

$$(2.3) \qquad\qquad x_{j+1} = Ax_j + b, \qquad j = 0, 1, \cdots.$$

Let $x_0 - s = \sum_{i=1}^{M} \alpha_i v_i$ for some scalars α_i. Then

$$(2.4) \qquad\qquad x_m = s + \sum_{i=1}^{M} \alpha_i v_i \lambda_i^m, \qquad m = 0, 1, \cdots.$$

As is known if $\alpha_1 \neq 0$, which will be the case for the given x_0 in general, then $\lim_{m \to \infty} x_m = s$ provided $|\lambda_1| < 1$; otherwise s is the anti-limit.

Let us denote $u_i = \Delta x_i = x_{i+1} - x_i$, $i = 0, 1, \cdots$, and $w_i = \Delta u_i = \Delta^2 x_i$, $i = 0, 1, \cdots$.

The MPE. Let k be an integer less than or equal to the dimension of the space B. The approximation $s_{n,k}$ to s is given by

$$(2.5) \qquad s_{n,k} = \sum_{j=0}^{k} \gamma_j x_{n+j},$$

where the γ_j are obtained from

$$(2.6) \qquad \gamma_j = \frac{c_j}{\sum_{i=0}^{k} c_i}, \qquad j = 0, 1, \cdots, k,$$

with $c_k = 1$, and $c_0, \cdots, c_{k-1}$ being determined as the solution to the minimization problem

$$(2.7) \qquad \min_{c_0, \cdots, c_{k-1}} \left\| \sum_{i=0}^{k-1} c_i u_{n+j} + u_{n+k} \right\|,$$

provided $\sum_{i=0}^{k} c_i \neq 0$.

For the example above, (2.7) is equivalent to the least squares solution of the overdetermined system of M equations

$$(2.8) \qquad U_{n,k} c = -u_{n+k},$$

where $U_{n,k}$ is the $M \times k$ matrix

$$(2.9) \qquad U_{n,k} = (u_n, u_{n+1}, \cdots, u_{n+k-1}),$$

and c is the column vector $(c_0, c_1, \cdots, c_{k-1})^T$, when $(y, z) = y^* z = \sum_{i=1}^{M} \bar{y}^i z^i$, where $y = (y^1, \cdots, y^M)^T$ and $z = (z^1, \cdots, z^M)^T$, and y^* is the Hermitian conjugate of y. This is the way the MPE was developed originally in [2].

We shall now give a determinant expression for $s_{n,k}$ that will be of use in the remainder of this work. Since $\|y\| = \sqrt{(y, y)}$ for any vector y in B, the c_i that solve the minimization problem in (2.7) satisfy the normal equations

$$(2.10) \qquad \sum_{j=0}^{k-1} (u_{n+i}, u_{n+j}) c_j = -(u_{n+i}, u_{n+k}), \qquad 0 \leq i \leq k-1.$$

Consequently, the γ_j that are defined by (2.6) satisfy the equations

$$(2.11) \qquad \begin{aligned} &\sum_{j=0}^{k} \gamma_j = 1, \\ &\sum_{j=0}^{k} (u_{n+i}, u_{n+j}) \gamma_j = 0, \qquad 0 \leq i \leq k-1, \end{aligned}$$

provided these equations have a solution. Assuming that the determinant of the matrix of equations (2.11) is nonzero, and using Cramer's rule, we can write the solution of (2.11) as

$$(2.12) \qquad \gamma_j = \frac{N_j}{\sum_{i=0}^{k} N_i} = \frac{N_j}{D(1, \cdots, 1)}, \qquad 0 \leq j \leq k,$$

where N_j is the cofactor of σ_j in the determinant

$$(2.13) \qquad D(\sigma_0, \cdots, \sigma_k) = \begin{vmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_k \\ u_{0,0} & u_{0,1} & \cdots & u_{0,k} \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ \vdots & \vdots & & \vdots \\ u_{k-1,0} & u_{k-1,1} & \cdots & u_{k-1,k} \end{vmatrix}$$

with $u_{i,j} \equiv (u_{n+i}, u_{n+j})$, $i, j \geqq 0$. In the first row of the determinant $D(\sigma_0, \cdots, \sigma_k)$ we also allow $\sigma_0, \cdots, \sigma_k$ to be vectors in B, in which case (2.13) is to be interpreted as

$$(2.14) \qquad\qquad D(\sigma_0, \cdots, \sigma_k) = \sum_{i=0}^{k} \sigma_i N_i.$$

Combining (2.5), (2.12), and (2.14), we can now express $s_{n,k}$ as

$$(2.15) \qquad\qquad s_{n,k} = \frac{D(x_n, \cdots, x_{n+k})}{D(1, \cdots, 1)}.$$

The RRE. Let k be an integer less than or equal to the dimension of the space B. The approximation $s_{n,k}$ to s is given by

$$(2.16) \qquad\qquad s_{n,k} = x_n + \sum_{i=0}^{k-1} q_i u_{n+i},$$

where the q_i are determined as the solution of the minimization problem

$$(2.17) \qquad\qquad \min_{q_0, \cdots, q_{k-1}} \left\| u_n + \sum_{i=0}^{k-1} q_i w_{n+i} \right\|.$$

For the example above, (2.17) is equivalent to the least squares solution of the overdetermined system of equations

$$(2.18) \qquad\qquad W_{n,k} q = -u_n,$$

where $W_{n,k}$ is the $M \times k$ matrix

$$(2.19) \qquad\qquad W_{n,k} = (w_n, w_{n+1}, \cdots, w_{n+k-1}),$$

and q is the column vector $q = (q_0, q_1, \cdots, q_{k-1})^T$, when $(y, z) = \sum_{i=1}^{M} \bar{y}_i z_i$, where y and z are again $y = (y^1, \cdots, y^M)^T$ and $z = (z^1, \cdots, z^M)^T$. This is the way the RRE was developed originally in [3].

For this case too $s_{n,k}$ can be expressed as the quotient of two determinants, and we turn to this now. Again by the fact that $\|y\| = \sqrt{(y, y)}$ for any vector y in B, the q_i that solve the minimization problem in (2.17) satisfy the normal equations

$$(2.20) \qquad \sum_{j=0}^{k-1} (w_{n+i}, w_{n+j}) q_j = -(w_{n+i}, u_n), \qquad 0 \leqq i \leqq k-1.$$

Substituting $w_{n+j} = u_{n+j+1} - u_{n+j}$ on the left-hand side of (2.20), and rearranging, we obtain

$$(2.21) \quad (w_{n+i}, u_n)(1 - q_0) + \sum_{j=1}^{k-1} (w_{n+i}, u_{n+j})(q_{j-1} - q_j) + (w_{n+i}, u_{n+k}) q_{k-1} = 0, \ 0 \leqq i \leqq k-1.$$

Let us define

$$(2.22) \qquad \gamma_0 = 1 - q_0, \quad \gamma_k = q_{k-1}, \quad \gamma_j = q_{j-1} - q_j, \qquad 1 \leqq j \leqq k-1.$$

It is easily verified that

$$(2.23) \qquad\qquad \sum_{j=0}^{k} \gamma_j = 1,$$

so that (2.22) and (2.23) establish a one-to-one correspondence between the q_i, $0 \leqq i \leqq k-1$, and the γ_j, $0 \leqq j \leqq k$. Consequently, the linear system of equations (2.20) for the

q_j is equivalent to the linear system

(2.24)
$$\sum_{j=0}^{k} \gamma_j = 1,$$

$$\sum_{j=0}^{k} (w_{n+i}, u_{n+j}) \gamma_j = 0, \qquad 0 \leq i \leq k-1,$$

for the γ_j, by (2.21), (2.22), and (2.23). Similarly, substituting $u_{n+1} = x_{n+i+1} - x_{n+i}$ on the right-hand side of (2.16), rearranging, and invoking (2.22), we see that $s_{n,k}$ for RRE (as for MPE) can be expressed in the form (2.5). Assuming now that the determinant of the matrix of equations (2.24) is nonzero, and using Cramer's rule, we can express the γ_j that solve (2.24) exactly as in (2.12), where N_j is the cofactor of σ_j in the determinant $D(\sigma_0, \cdots, \sigma_k)$ given in (2.13) with $u_{i,j} \equiv (w_{n+i}, u_{n+j})$, $i, j \geq 0$. Again when $\sigma_0, \cdots, \sigma_k$ are vectors in B, $D(\sigma_0, \cdots, \sigma_k)$ is to be interpreted as in (2.14). Consequently $s_{n,k}$, also for RRE, can be expressed as in (2.15).

Before closing this section we shall state a result that will be of use in the remainder of this work.

LEMMA 2.1. *Let $i_0, i_1, \cdots, i_k$ be integers greater than or equal to 1, and assume that the scalars $v_{i_0,\cdots,i_k}$ are odd under an interchange of any two indices $i_0, \cdots, i_k$. Let σ_i, $i \geq 1$, be scalars (or vectors) and let $t_{i,j}$, $i \geq 1$, $1 \leq j \leq k$, be scalars. Define*

(2.25)
$$I_{k,N} = \sum_{i_0=1}^{N} \cdots \sum_{i_k=1}^{N} \sigma_{i_0} \left(\prod_{p=1}^{k} t_{i_p,p} \right) v_{i_0,\cdots,i_k}$$

and

(2.26)
$$J_{k,N} = \sum_{1 \leq i_0 < i_1 < \cdots < i_k \leq N} \begin{vmatrix} \sigma_{i_0} & \sigma_{i_1} & \cdots & \sigma_{i_k} \\ t_{i_0,1} & t_{i_1,1} & \cdots & t_{i_k,1} \\ t_{i_0,2} & t_{i_1,2} & \cdots & t_{i_k,2} \\ \vdots & \vdots & & \vdots \\ t_{i_0,k} & t_{i_1,k} & \cdots & t_{i_k,k} \end{vmatrix} v_{i_0,\cdots,i_k},$$

where the determinant in (2.26) is to be interpreted in the same way as $D(\sigma_0, \cdots, \sigma_k)$ in (2.13). Then

(2.27)
$$I_{k,N} = J_{k,N}.$$

For a proof of Lemma 2.1, see [5, Appendix].

3. Convergence analysis. We have seen in the previous section that $s_{n,k}$, for both the MPE *and* the RRE, is given by

(3.1)
$$s_{n,k} = \frac{D(x_n, \cdots, x_{n+k})}{D(1, \cdots, 1)},$$

where $D(\sigma_0, \cdots, \sigma_k)$ is defined by (2.13) with $u_{i,j} = (u_{n+i}, u_{n+j})$, $i, j \geq 0$, for MPE, and $u_{i,j} = (w_{n+i}, u_{n+j})$, $i, j \geq 0$, for RRE. Subtracting s from both sides of (3.1), and making use of (2.14), we obtain the error formula

(3.2)
$$s_{n,k} - s = \frac{D(x_n - s, \cdots, x_{n+k} - s)}{D(1, \cdots, 1)}.$$

Under the assumption (2.1), we have

$$(3.3) \qquad u_m \sim \sum_{i=1}^{\infty} v_i(\lambda_i - 1)\lambda_i^m \quad \text{as } m \to \infty,$$

and consequently,

$$(3.4) \qquad w_m \sim \sum_{i=1}^{\infty} v_i(\lambda_i - 1)^2\lambda_i^m \quad \text{as } m \to \infty.$$

From (3.3) and (3.4) we see that

$$(3.5) \qquad u_{p,q} = (u_{n+p}, u_{n+q}) \sim \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (v_i, v_j)(\bar{\lambda}_i - 1)(\lambda_j - 1)\bar{\lambda}_i^{n+p}\lambda_j^{n+q} \quad \text{as } n \to \infty,$$

for MPE, and

$$(3.6) \qquad u_{p,q} = (w_{n+p}, u_{n+q}) \sim \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (v_i, v_j)(\bar{\lambda}_i - 1)^2(\lambda_j - 1)\bar{\lambda}_i^{n+p}\lambda_j^{n+q} \quad \text{as } n \to \infty,$$

for RRE. (3.5) and (3.6) can be rewritten in the condensed and unified form

$$(3.7) \qquad u_{p,q} \sim \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} z_{ij}\bar{\lambda}_i^{n+p}\lambda_j^{n+q} \quad \text{as } n \to \infty,$$

where we have defined

$$(3.8) \qquad z_{ij} = (v_i, v_j)(\bar{\lambda}_i - 1)(\lambda_j - 1), \qquad i,j \geq 1,$$

for MPE, and

$$(3.9) \qquad z_{ij} = (v_i, v_j)(\bar{\lambda}_i - 1)^2(\lambda_j - 1), \qquad i,j \geq 1,$$

for RRE.

Note that when the sequence x_j, $j = 0, 1, \cdots$, is generated by a matrix iterative process as described in the example of § 2, then in (3.7) the upper limits on the summations on the right-hand side are replaced by M, and $\sim$ is replaced by $=$.

In Lemma 3.1 and Lemma 3.2 below we derive the asymptotic expansions of $D(x_n - s, \cdots, x_{n+k} - s)$ and $D(1, \cdots, 1)$, respectively, for $n \to \infty$, assuming (2.1). In Theorem 3.1 we give the main result of this paper that shows that both the MPE and RRE are true acceleration methods when applied to sequences of vectors satisfying (2.1), in the sense that $\lim_{n\to\infty} \|s_{n,k} - s\|/\|x_{n+k+1} - s\| = 0$.

In what follows we shall denote asymptotic relations of the form "$\alpha_n \sim \beta_n$ as $n \to \infty$" by "$\alpha_n \sim \beta_n$" for short. Also we shall let

$$\sum_i \equiv \sum_{i=1}^{\infty}, \qquad \sum_{i,j} \equiv \sum_{i=1}^{\infty}\sum_{j=1}^{\infty},$$

and

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k} \equiv \sum_{i_1=1}^{\infty}\sum_{i_2=i_1+1}^{\infty}\cdots\sum_{i_k=i_{k-1}+1}^{\infty},$$

for simplicity.

LEMMA 3.1. *Let the sequence of vectors x_i, $i = 0, 1, 2, \cdots$, be as described in the previous section. For i_p, j_p positive integers, define*

$$(3.10) \qquad R_{j_0, j_1, \cdots, j_k}^{i_1, \cdots, i_k} = \begin{vmatrix} v_{j_0} & v_{j_1} & \cdots & v_{j_k} \\ z_{i_1 j_0} & z_{i_1 j_1} & \cdots & z_{i_1 j_k} \\ z_{i_2 j_0} & z_{i_2 j_1} & \cdots & z_{i_2 j_j} \\ \vdots & \vdots & & \vdots \\ z_{i_k j_0} & z_{i_k j_1} & \cdots & z_{i_k j_k} \end{vmatrix},$$

where the interpretation of this determinant is like that of $D(\sigma_0, \sigma_1, \cdots, \sigma_k)$ in (2.13) and (2.14). Let $V(\xi_0, \xi_1, \cdots, \xi_k)$ be the Vandermonde determinant

$$(3.11) \qquad V(\xi_0, \xi_1, \cdots, \xi_k) = \begin{vmatrix} 1 & \xi_0 & \cdots & \xi_0^k \\ 1 & \xi_1 & \cdots & \xi_1^k \\ \vdots & \vdots & & \vdots \\ 1 & \xi_k & \cdots & \xi_k^k \end{vmatrix} = \prod_{0 \le i < j \le k} (\xi_j - \xi_i).$$

Then $D(x_n - s, \cdots, x_{n+k} - s)$ has the asymptotic expansion given by

$$D(x_n - s, \cdots, x_{n+k} - s)$$

$$(3.12) \qquad \sim \sum_{1 \le j_0 < j_1 < \cdots < j_k} \left(\prod_{p=0}^{k} \lambda_{j_p}^n \right) V(\lambda_{j_0}, \lambda_{j_1}, \cdots, \lambda_{j_k})$$

$$\cdot \sum_{1 \le i_1 < i_2 < \cdots < i_k} \left(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^n \right) V(\bar{\lambda}_{i_1}, \cdots, \bar{\lambda}_{i_k}) R_{j_0, j_1, \cdots, j_k}^{i_1, \cdots, i_k}.$$

Proof. Let us denote $D(x_n - s, \cdots, x_{n+k} - s)$ by P_n for short. By (2.13), (2.1), and (3.7), we have

$$(3.13) \qquad P_n \sim \begin{vmatrix} \sum_{j_0} v_{j_0} \lambda_{j_0}^n & \sum_{j_0} v_{j_0} \lambda_{j_0}^{n+1} & \cdots & \sum_{j_0} v_{j_0} \lambda_{j_0}^{n+k} \\ \sum_{i_1, j_1} z_{i_1 j_1} \bar{\lambda}_{i_1}^n \lambda_{j_1}^n & \sum_{i_1, j_1} z_{i_1 j_1} \bar{\lambda}_{i_1}^n \lambda_{j_1}^{n+1} & \cdots & \sum_{i_1, j_1} z_{i_1 j_1} \bar{\lambda}_{i_1}^n \lambda_{j_1}^{n+k} \\ \sum_{i_2, j_2} z_{i_2 j_2} \bar{\lambda}_{i_2}^{n+1} \lambda_{j_2}^n & \sum_{i_2, j_2} z_{i_2 j_2} \bar{\lambda}_{i_2}^{n+1} \lambda_{j_2}^{n+1} & \cdots & \sum_{i_2, j_2} z_{i_2 j_2} \bar{\lambda}_{i_2}^{n+1} \lambda_{j_2}^{n+k} \\ \vdots & \vdots & & \vdots \\ \sum_{i_k, j_k} z_{i_k j_k} \bar{\lambda}_{i_k}^{n+k-1} \lambda_{j_k}^n & \sum_{i_k, j_k} z_{i_k j_k} \bar{\lambda}_{i_k}^{n+k-1} \lambda_{j_k}^{n+1} & \cdots & \sum_{i_k, j_k} z_{i_k j_k} \bar{\lambda}_{i_k}^{n+k-1} \lambda_{j_k}^{n+k} \end{vmatrix}.$$

By the multilinearity property of determinants we can reexpress (3.13) in the form

$$(3.14) \qquad P_n \sim \sum_{j_0} \sum_{i_1, j_1} \cdots \sum_{i_k, j_k} v_{j_0} \lambda_{j_0}^n \left(\prod_{p=1}^{k} z_{i_p j_p} \bar{\lambda}_{i_p}^{n+p-1} \lambda_{j_p}^n \right) V(\lambda_{j_0}, \lambda_{j_1}, \cdots, \lambda_{j_k}).$$

By changing the order of summation in (3.14), we have

$$(3.15) \quad P_n \sim \sum_{i_1} \cdots \sum_{i_k} \left(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^{n+p-1} \right) \sum_{j_0} \sum_{j_1} \cdots \sum_{j_k} v_{j_0} \left(\prod_{p=1}^{k} z_{i_p j_p} \right) \left(\prod_{p=0}^{k} \lambda_{j_p}^n \right) V(\lambda_{j_0}, \lambda_{j_1}, \cdots, \lambda_{j_k}).$$

Since $\left(\prod_{p=0}^{k} \lambda_{j_p}^n \right) V(\lambda_{j_0}, \lambda_{j_1}, \cdots, \lambda_{j_k})$ is odd under an interchange of any two of the indices j_p, Lemma 2.1 can be applied, and we obtain

$$(3.16) \quad P_n \sim \sum_{i_1} \cdots \sum_{i_k} \left(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^{n+p-1} \right) \sum_{1 \le j_0 < j_1 < \cdots < j_k} R_{j_0, j_1, \cdots, j_k}^{i_1, \cdots, i_k} \left(\prod_{p=0}^{k} \lambda_{j_p}^n \right) V(\lambda_{j_0}, \lambda_{j_1}, \cdots, \lambda_{j_k}).$$

Interchanging the order of the summations in (3.16), we have

$$P_n \sim \sum_{1 \leq j_0 < j_1 < \cdots < j_k} \left(\prod_{p=0}^{k} \lambda_{j_p}^n \right) V(\lambda_{j_0}, \lambda_{j_1}, \cdots, \lambda_{j_k})$$

(3.17)

$$\cdot \sum_{i_1} \cdots \sum_{i_k} \left(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^{p-1} \right) \left(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^n \right) R_{j_0, j_1, \cdots, j_k}^{i_1, \cdots, i_k}.$$

Let now $N_{j_q}^{i_1, \cdots, i_k}$ be the cofactor of v_{j_q} in the determinant expression for $R_{j_0, j_1, \cdots, j_k}^{i_1, \cdots, i_k}$. Then

(3.18)
$$R_{j_0, j_1, \cdots, j_k}^{i_1, \cdots, i_k} = \sum_{q=0}^{k} v_{j_q} N_{j_q}^{i_1, \cdots, i_k}.$$

Observing from (3.10) that the $N_{j_q}^{i_1, \cdots, i_k}$ are odd under an interchange of any two of the indices $i_1, \cdots, i_k$, we can apply Lemma 2.1 again to the multiple sum with respect to the indices $i_1, \cdots, i_k$, with $R_{j_0, j_1, \cdots, j_k}^{i_1, \cdots, i_k}$ replaced by (3.18). The terms that are odd in the indices $i_1, \cdots, i_k$ now are $(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^n) N_{j_q}^{i_1, \cdots, i_k}$. By Lemma 2.1 then

$$P_n \sim \sum_{1 \leq j_0 < j_1 < \cdots < j_k} \left(\prod_{p=0}^{k} \lambda_{j_p}^n \right) V(\lambda_{j_0}, \lambda_{j_1}, \cdots, \lambda_{j_k})$$

(3.19)

$$\cdot \sum_{q=0}^{k} v_{j_q} \sum_{1 \leq i_1 < i_2 < \cdots < i_k} \left(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^n \right) N_{j_q}^{i_1, \cdots, i_k} V(\bar{\lambda}_{i_1}, \cdots, \bar{\lambda}_{i_k}),$$

which, by invoking (3.18), can be reexpressed as (3.12), thus completing the proof. $\square$

LEMMA 3.2. *Let the sequence of vectors* x_i, $i = 0, 1, 2, \cdots$, *be as described in the previous section. For* i_p, j_p *positive integers, define*

(3.20)
$$S_{j_1, \cdots, j_k}^{i_1, \cdots, i_k} = \begin{vmatrix} z_{i_1 j_1} & z_{i_1 j_2} & \cdots & z_{i_1 j_k} \\ z_{i_2 j_1} & z_{i_2 j_2} & \cdots & z_{i_2 j_k} \\ \vdots & \vdots & & \vdots \\ z_{i_k j_1} & z_{i_k j_2} & \cdots & z_{i_k j_k} \end{vmatrix}.$$

Then $D(1, \cdots, 1)$ *has the asymptotic expansion given by*

$$D(1, \cdots, 1) \sim \sum_{1 \leq j_1 < j_2 < \cdots < j_k} \left(\prod_{p=1}^{k} \lambda_{j_p}^n \right) V(1, \lambda_{j_1}, \cdots, \lambda_{j_k})$$

(3.21)

$$\cdot \sum_{1 \leq i_1 < i_2 < \cdots < i_k} \left(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^n \right) V(\bar{\lambda}_{i_1}, \cdots, \bar{\lambda}_{i_k}) S_{j_1, \cdots, j_k}^{i_1, \cdots, i_k}.$$

Proof. Let us denote $D(1, \cdots, 1)$ by Q_n for short. From (2.13) and (3.7), we have

(3.22)

$$Q_n \sim \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum_{i_1, j_1} z_{i_1 j_1} \bar{\lambda}_{i_1}^n \lambda_{j_1}^n & \sum_{i_1, j_1} z_{i_1 j_1} \bar{\lambda}_{i_1}^n \lambda_{j_1}^{n+1} & \cdots & \sum_{i_1, j_1} z_{i_1 j_1} \bar{\lambda}_{i_1}^n \lambda_{j_1}^{n+k} \\ \sum_{i_2, j_2} z_{i_2 j_2} \bar{\lambda}_{i_2}^{n+1} \lambda_{j_2}^n & \sum_{i_2, j_2} z_{i_2 j_2} \bar{\lambda}_{i_2}^{n+1} \lambda_{j_2}^{n+1} & \cdots & \sum_{i_2, j_2} z_{i_2 j_2} \bar{\lambda}_{i_2}^{n+1} \lambda_{j_2}^{n+k} \\ \vdots & \vdots & & \vdots \\ \sum_{i_k, j_k} z_{i_k j_k} \bar{\lambda}_{i_k}^{n+k-1} \lambda_{j_k}^n & \sum_{i_k, j_k} z_{i_k j_k} \bar{\lambda}_{i_k}^{n+k-1} \lambda_{j_k}^{n+1} & \cdots & \sum_{i_k, j_k} z_{i_k j_k} \bar{\lambda}_{i_k}^{n+k-1} \lambda_{j_k}^{n+k} \end{vmatrix}.$$

Again by the multilinearity property of determinants

$$(3.23) \qquad Q_n \sim \sum_{i_1, j_1} \cdots \sum_{i_k, j_k} \left(\prod_{p=1}^{k} z_{i_p j_p} \bar{\lambda}_{i_p}^{n+p-1} \lambda_{j_p}^{n} \right) V(1, \lambda_{j_1}, \cdots, \lambda_{j_k}).$$

By changing the order of summation in (3.23) and observing that $(\prod_{p=1}^{k} \lambda_{j_p}^{n}) V(1, \lambda_{j_1}, \cdots, \lambda_{j_k})$ is odd under an interchange of any two of the indices j_p, we can apply Lemma 2.1 to the summation over $j_1, \cdots, j_k$. The result is

$$(3.24) \qquad \begin{aligned} Q_n \sim & \sum_{i_1} \cdots \sum_{i_k} \left(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^{n+p-1} \right) \\ & \cdot \sum_{1 \leqq j_1 < j_2 < \cdots < j_k} S_{j_1, \cdots, j_k}^{i_1, \cdots, i_k} \left(\prod_{p=1}^{k} \lambda_{j_p}^{n} \right) V(1, \lambda_{j_1}, \cdots, \lambda_{j_k}). \end{aligned}$$

Changing the order of summation again, and observing that $(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^{n}) S_{j_1, \cdots, j_k}^{i_1, \cdots, i_k}$ is odd under an interchange of any two of the indices i_p, we can now apply Lemma 2.1 to the summation over $i_1, \cdots, i_k$. As a result, we obtain (3.21). This completes the proof. $\square$

THEOREM 3.1. *Let the sequence of vectors* x_i, $i = 0, 1, 2, \cdots$, *be as described in the previous section. If, in addition, the* v_i *are linearly independent, and*

$$(3.25) \qquad |\lambda_1| \geqq \cdots \geqq |\lambda_k| > |\lambda_{k+1}| \geqq |\lambda_{k+2}| \geqq \cdots,$$

then, for all sufficiently large n, $s_{n,k}$ *exists, and*

$$(3.26) \qquad s_{n,k} - s = F(n) \lambda_{k+1}^{n} [1 + o(1)] \quad \text{as } n \to \infty,$$

where the vector $F(n)$ *satisfies* $\|F(n)\| \leqq \tilde{K}$, *for some constant* $\tilde{K} > 0$ *independent of* n, *and for all sufficiently large* n. *If, in addition,*

$$(3.27) \qquad |\lambda_{k+1}| > |\lambda_{k+2}|,$$

then

$$(3.28) \qquad F(n) = \frac{R_{1, \cdots, k+1}^{1, \cdots, k}}{S_{1, \cdots, k}^{1, \cdots, k}} \prod_{i=1}^{k} \left(\frac{\lambda_{k+1} - \lambda_i}{\lambda_i - 1} \right) [1 + o(1)] \quad \text{as } n \to \infty.$$

Proof. From (3.20) and (3.8) (or (3.9)),

$$(3.29) \qquad S_{1, \cdots, k}^{1, \cdots, k} = H G(v_1, \cdots, v_k),$$

where

$$H = \left| \prod_{i=1}^{k} (\lambda_i - 1) \right|^2 \text{ for MPE} \left(\text{or } H = \prod_{i=1}^{k} (\bar{\lambda}_i - 1)^2 (\lambda_i - 1) \text{ for RRE} \right),$$

and $G(v_1, \cdots, v_k)$ is the Gram determinant of the vectors $v_1, \cdots, v_k$, given by

$$(3.30) \qquad G(v_1, \cdots, v_k) = \begin{vmatrix} (v_1, v_1) & (v_1, v_2) & \cdots & (v_1, v_k) \\ (v_2, v_1) & (v_2, v_2) & \cdots & (v_2, v_k) \\ \vdots & \vdots & & \vdots \\ (v_k, v_1) & (v_k, v_2) & \cdots & (v_k, v_k) \end{vmatrix}.$$

Since the v_i are linearly independent, $G(v_1, \cdots, v_k)$ is nonzero. Also, since $\lambda_i \neq 1$ for all i, H is nonzero. Consequently, $S_{1, \cdots, k}^{1, \cdots, k}$ is nonzero. Next, since $\lambda_i \neq \lambda_j$ for $i \neq j$, and $\lambda_i \neq 1$ for all i, $V(\bar{\lambda}, \cdots, \bar{\lambda}_k)$ and $V(1, \lambda_1, \cdots, \lambda_k)$ are nonzero on account of (3.11). Finally, we observe that, due to (3.25), the dominant term in the multiple sum on the

right-hand side of (3.21) in Lemma 3.2 would be that for which $i_p = p$, $j_p = p$, $p = 1, \cdots, k$, provided $V(1, \lambda_1, \cdots, \lambda_k)$ and $S_{1, \cdots, k}^{1, \cdots, k}$ are nonzero, which we have already proved to be so. Consequently,

$$D(1, \cdots, 1) = \left| \prod_{p=1}^{k} \lambda_p^n \right|^2 V(1, \lambda_1, \cdots, \lambda_k) V(\bar\lambda_1, \cdots, \bar\lambda_k)$$

(3.31)

$$\cdot S_{1, \cdots, k}^{1, \cdots, k}[1 + o(1)] \quad \text{as } n \to \infty.$$

This also implies that, for all sufficiently large n, $D(1, \cdots, 1) \neq 0$; hence $s_{n,k}$ exists. This completes the proof of the first part of the theorem.

To prove (3.26), we should analyze the behavior of $D(x_n - s, \cdots, x_{n+k} - s)$ for $n \to \infty$. By our assumptions made following (2.1), there is only a finite number of λ_i whose moduli are $|\lambda_{k+1}|$. Let $|\lambda_{k+1}| = \cdots = |\lambda_{k+r}| > |\lambda_{k+r+1}|$. From this and (3.25), it follows that the dominant term on the right-hand side of (3.12)—provided it is nonzero—is the sum of those terms with indices $i_p = p$, $p = 1, \cdots, k$, $j_{p-1} = p$, $p = 1, \cdots, k$, $j_k = k+l$, $l = 1, \cdots, r$, namely

(3.32)

$$D(x_n - s, \cdots, x_{n+k} - s)$$

$$= \left| \prod_{p=1}^{k} \lambda_p^n \right|^2 V(\bar\lambda_1, \cdots, \bar\lambda_k)$$

$$\cdot \sum_{l=1}^{r} \lambda_{k+l}^n V(\lambda_1, \cdots, \lambda_k, \lambda_{k+l}) R_{1, \cdots, k, k+l}^{1, \cdots, k}[1 + o(1)] \quad \text{as } n \to \infty.$$

From (3.10) and (3.20),

(3.33)

$$R_{1, \cdots, k, k+l}^{1, \cdots, k} = \sum_{i=1}^{k} \beta_{li} v_i + (-1)^k S_{1, \cdots, k}^{1, \cdots, k} v_{k+l},$$

so that the summation on the right-hand side of (3.32) becomes

(3.34)

$$\sum_{i=1}^{k} v_i \left[\sum_{l=1}^{r} \lambda_{k+l}^n V(\lambda_1, \cdots, \lambda_k, \lambda_{k+l}) \beta_{li} \right]$$

$$+ (-1)^k S_{1, \cdots, k}^{1, \cdots, k} \sum_{l=1}^{r} \lambda_{k+l}^n V(\lambda_1, \cdots, \lambda_k, \lambda_{k+l}) v_{k+l}.$$

Since $S_{1, \cdots, k}^{1, \cdots, k} \neq 0$, $V(\lambda_1, \cdots, \lambda_k, \lambda_{k+l}) \neq 0$, and the vectors v_i are linearly independent, the second summation in (3.34) is never zero. This proves (3.32). Combining (3.31) and (3.32), (3.26) follows. If (3.27) holds, then $r = 1$. In this case (3.28) follows from (3.31), (3.32), and (3.11). This completes the proof of the theorem. $\square$

The asymptotic error analysis of the MPE and RRE as given in Theorem 3.1, leads us to the following important conclusions:

(1) Under the conditions stated in the theorem, the MPE and RRE are bona fide acceleration methods in the sense that

(3.35)

$$\frac{\|s_{n,k} - s\|}{\|x_{n+k+1} - s\|} = O\left[\left(\frac{\lambda_{k+1}}{\lambda_1} \right)^n \right] \quad \text{as } n \to \infty.$$

This means that if $x_n \to s$ as $n \to \infty$, i.e., $|\lambda_1| < 1$, then $s_{n,k} \to s$ as $n \to \infty$, and more quickly. Also if $\lim_{m \to \infty} x_m$ does not exist, i.e., $|\lambda_1| \geq 1$, then $s_{n,k} \to s$ as $n \to \infty$, provided that $|\lambda_{k+1}| < 1$. The reason that we write x_{n+k+1} in (3.35) is that $s_{n,k}$ in both the MPE and RRE is computed from $x_n, x_{n+1}, \cdots, x_{n+k+1}$.

(2) When the MPE and RRE are being applied to a vector sequence generated by using a matrix iterative method, they will be especially effective when the iteration matrix has a small number of large eigenvalues (k-many when $s_{n,k}$ is being used) that are well separated from the small eigenvalues.

(3) As can be seen from (3.28), a loss of accuracy will take place in $s_{n,k}$ when $\lambda_1, \cdots, \lambda_k$ are close to 1. For sequences of vectors obtained from the iterative solution of linear systems of equations, this means that the matrix of the system is nearly singular.

These conclusions are the same as those for the modified MPE, which has been defined and analyzed in [5].

Finally note that the results of this section (and of § 4) will not change if the MPE and RRE are replaced by any other method giving rise to (3.1), with $D(\sigma_0, \cdots, \sigma_k)$ defined as in (2.13), as long as the $u_{i,j}$ in (2.13) satisfy (3.7) with $z_{ij} = (v_i, v_j)\mu_i \nu_j$, where μ_i and ν_j are fixed nonzero constants.

4. Stability analysis. Let us denote γ_j in (2.5) by $\gamma_j^{(n,k)}$ for both MPE and RRE. We say that $s_{n,k}$ is asymptotically stable if

$$(4.1) \qquad \sup_n \sum_{j=0}^{k} |\gamma_j^{(n,k)}| < \infty.$$

Roughly speaking, this means that if errors are introduced in the vectors x_m, then the error in $s_{n,k}$ stays bounded as $n \to \infty$. Since $\sum_{j=0}^{k} \gamma_j^{(n,k)} = 1$, the most ideal situation is one in which $\gamma_j^{(n,k)} \geq 0$, $0 \leq j \leq k$, for all sufficiently large n, so that

$$\sum_{j=0}^{k} |\gamma_j^{(n,k)}| = \sum_{j=0}^{k} \gamma_j^{(n,k)} = 1.$$

The following theorem gives the stability properties of both the MPE and RRE when they are applied to vector sequences satisfying the conditions of Theorem 3.1. $s_{n,k}$ denotes either that obtained from MPE or RRE.

THEOREM 4.1. *For vector sequences satisfying the conditions stated in Theorem 3.1, $s_{n,k}$ is asymptotically stable. Actually, the following are true:*

(1) *The $\gamma_q^{(n,k)}$ satisfy*

$$(4.2) \qquad \gamma_q^{(n,k)} = \frac{C_q(\lambda_1, \cdots, \lambda_k)}{V(1, \lambda_1, \cdots, \lambda_k)} + o(1) \quad as \; n \to \infty,$$

where

$$(4.3) \qquad C_q(\xi_1, \cdots, \xi_k) = (-1)^q \begin{vmatrix} 1 & \xi_1 & \cdots & \xi_1^{q-1} & \xi_1^{q+1} & \cdots & \xi_1^{k} \\ 1 & \xi_2 & \cdots & \xi_2^{q-1} & \xi_2^{q+1} & \cdots & \xi_2^{k} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \xi_k & \cdots & \xi_k^{q-1} & \xi_k^{q+1} & \cdots & \xi_k^{k} \end{vmatrix};$$

thus

$$(4.4) \qquad \sum_{q=0}^{k} |\gamma_q^{(n,k)}| = \frac{\sum_{q=0}^{k} |C_q(\lambda_1, \cdots, \lambda_k)|}{|V(1, \lambda_1, \cdots, \lambda_k)|} [1 + o(1)] \quad as \; n \to \infty,$$

i.e., (4.1) holds.

(2) *If $\lambda_1, \cdots, \lambda_k$ are real and negative, then*

$$(4.5) \qquad \gamma_q^{(n,k)} > 0, \qquad 0 \leq q \leq k, \quad for \; n \; sufficiently \; large;$$

thus

$$(4.6) \qquad \sum_{q=0}^{k} |\gamma_q^{(n,k)}| = 1 + o(1) \quad \text{as } n \to \infty.$$

(3) *If* $\lambda_1, \cdots, \lambda_k$ *are real and positive, then*

$$(4.7) \qquad \gamma_q^{(n,k)} \gamma_{q+1}^{(n,k)} < 0, \qquad 0 \leq q \leq k-1, \quad \text{for } n \text{ sufficiently large;}$$

thus

$$(4.8) \qquad \sum_{q=0}^{k} |\gamma_q^{(n,k)}| = \left| \prod_{i=1}^{k} \left(\frac{\lambda_i + 1}{\lambda_i - 1} \right) \right| [1 + o(1)] \quad \text{as } n \to \infty.$$

(4) *For any* $\lambda_1, \cdots, \lambda_k$,

$$(4.9) \qquad \lim_{n \to \infty} \sum_{q=0}^{k} \gamma_q^{(n,k)} \lambda^q = \prod_{i=1}^{k} \left(\frac{\lambda_i - \lambda}{\lambda_i - 1} \right).$$

Proof. Since $\gamma_q^{(n,k)}$ is given by (2.12), and we already know the asymptotic behavior of $D(1, \cdots, 1)$ from 3.2, it is sufficient to analyze N_q asymptotically. By deleting the first row and the $(q+1)$st column of the determinant in (3.22), after some manipulation, we have

$$(4.10) \qquad N_q \sim \sum_{i_1, j_1} \cdots \sum_{i_k, j_k} \left(\prod_{p=1}^{k} z_{i_p j_p} \bar{\lambda}_{i_p}^{n+p-1} \lambda_{j_p}^n \right) C_q(\lambda_{j_1}, \cdots, \lambda_{j_k}).$$

Observe that $\left(\prod_{p=1}^{k} \lambda_{j_p}^n \right) C_q(\lambda_{j_1}, \cdots, \lambda_{j_k})$ is odd under an interchange of any two of the indices j_p. Consequently,

$$(4.11) \qquad \begin{aligned} N_q \sim & \sum_{1 \leq j_1 < j_2 < \cdots < j_k} \left(\prod_{p=1}^{k} \lambda_{j_p}^n \right) C_q(\lambda_{j_1}, \cdots, \lambda_{j_k}) \\ & \cdot \sum_{1 \leq i_1 < i_2 < \cdots < i_k} \left(\prod_{p=1}^{k} \bar{\lambda}_{i_p}^n \right) V(\bar{\lambda}_{i_1}, \cdots, \bar{\lambda}_{i_k}) S_{j_1, \cdots, j_k}^{i_1, \cdots, i_k} \end{aligned}$$

follows from (4.10) in exactly the same way (3.21) follows from (3.23). Invoking (3.25),

$$(4.12) \qquad N_q = \left| \prod_{p=1}^{k} \lambda_p^n \right|^2 V(\bar{\lambda}_1, \cdots, \bar{\lambda}_k) S_{1, \cdots, k}^{1, \cdots, k} [C_q(\lambda_1, \cdots, \lambda_k) + o(1)] \quad \text{as } n \to \infty,$$

follows from (4.11) in the same way (3.31) follows from (3.21). Combining (4.12) and (3.31) in (2.12), (4.2) follows. (4.4) follows directly from (4.2). This proves (1).

Note that $V(\lambda, \lambda_1, \cdots, \lambda_k)$ is a polynomial of degree k in λ. From (3.11) and (4.3) we have

$$(4.13) \qquad V(\lambda, \lambda_1, \cdots, \lambda_k) = \sum_{q=0}^{k} C_q(\lambda_1, \cdots, \lambda_k) \lambda^q.$$

Since $V(\lambda, \lambda_1, \cdots, \lambda_k)$ has $\lambda_1, \cdots, \lambda_k$ as its only zeros, we also have

$$(4.14) \qquad V(\lambda, \lambda_1, \cdots, \lambda_k) = C_k(\lambda_1, \cdots, \lambda_k) \prod_{i=1}^{k} (\lambda - \lambda_1).$$

If $\lambda_1, \cdots, \lambda_k$ are real and negative, then (4.13) and (4.14) imply that $C_q(\lambda_1, \cdots, \lambda_k)$, $0 \leq q \leq k$, are all of the same sign. This, along with (4.2), implies (4.5). (4.6) is a direct consequence of (4.5) and (2.11). This proves (2).

If $\lambda_1, \cdots, \lambda_k$ are real and positive, then (4.13) and (4.14) imply that $C_q(\lambda_1, \cdots, \lambda_k) C_{q+1}(\lambda_1, \cdots, \lambda_k) < 0$, $0 \le q \le k-1$. This, along with (4.2), implies (4.7) and

$$
\begin{aligned}
\text{(4.15)} \qquad \sum_{q=0}^{k} |\gamma_q^{(n,k)}| &= \frac{|\sum_{q=0}^{k} (-1)^q C_q(\lambda_1, \cdots, \lambda_k)|}{|V(1, \lambda_1, \cdots, \lambda_k)|} [1 + o(1)] \quad \text{as } n \to \infty \\
&= \left| \frac{V(-1, \lambda_1, \cdots, \lambda_k)}{V(1, \lambda_1, \cdots, \lambda_k)} \right| [1 + o(1)] \quad \text{as } n \to \infty,
\end{aligned}
$$

which, by (3.11), reduces to (4.8). This proves (3).

Multiplying both sides of (4.2) by λ^q, summing over q from 0 to k, and finally making use of (4.13) and (3.11), (4.9) follows, thus proving (4). $\quad \square$

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