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BORELSUMMABILITY AND CONVERGING FACTORS FOR SOME EVERYWHERE DIVERGENT SERIES*

AVRAM SIDI†

Abstract. In this work we deal with the problem of interpretation of certain classes of everywhere divergent power series within the framework of Borel summability, and derive asymptotic expansions for their partial sums and/or their converging factors when the number of terms in the partial sums goes to infinity.

Key words. divergent series, Borel summability, converging factors, asymptotic expansion

AMS(MOS) subject classifications. Primary 34E05, 40A05, 40D15, 40G10, 65B05, 65B10

1. Introduction. Consider the formal power series $F(\zeta) := \sum_{r=1}^{\infty} a_r \zeta^r$ whose terms are of the form

$$a_r = r^p w(r)(r!)^m, \tag{1.1}$$

where $p \geqq 0$ and $m \geqq 1$ are integers, and $w(r)$ is such that

$$w(r) \sim \sum_{i=0}^{\infty} \frac{w_i}{r^{i+\sigma}} \quad \text{as } r \to \infty, \quad \text{for some } \sigma > 0, \tag{1.2}$$

with w_i being some constants independent of r. Obviously $F(\zeta)$ does not converge for any value of ζ. In this work we shall be concerned with the interpretation of the "sum" of $F(\zeta)$, and with the asymptotics of the converging factor of the partial sum $A_n(\zeta) = \sum_{i=1}^{n} a_i \zeta^i$ in the limit $n \to \infty$, for ζ fixed. This problem arises when one tries to apply the t or u transformation of Levin [4] to the sequence $A_j(\zeta)$, $j = 1, 2, \cdots$, to obtain an approximation to the "sum" of $F(\zeta)$, or to the anti-limit of the sequence $\{A_j(\zeta)\}$.

The t and u transformations are nonlinear methods for accelerating the convergence of a slowly converging sequence to its limit, or for effecting convergence of a diverging sequence to its anti-limit. There is ample numerical evidence (see the numerical examples given in [10]) that suggests that in order for the t (or u) transformation to be efficient on a sequence B_i, $i = 1, 2, \cdots$, B_i has to be of the form

$$B_{r-1} = B + R_r f(r), \tag{1.3}$$

where B is the limit or anti-limit of $\{B_i\}$, and $f(r)$ should be such that

$$f(r) \sim \sum_{i=0}^{\infty} \frac{\beta_i}{r^i} \quad \text{as } r \to \infty \tag{1.4}$$

and $R_r = r^{\lambda} b_r$, where $b_1 = B_1$, $b_r = B_r - B_{r-1}$, $r \geqq 2$, and $\lambda = 0$ for the t transformation (or $\lambda = 1$ for the u transformation). From the conjectured behavior of B_i in (1.3) and (1.4), it follows that

$$b_{r+1} = c(r) b_r, \tag{1.5}$$

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† Computer Science Department,Technion–Israel Institute of Technolgy, Haifa, Israel.

where

$$c(r)=\frac{1+r^{\lambda}f(r)}{(r+1)^{\lambda}f(r+1)}\sim\sum_{i=0}^{\infty}\frac{c_i}{r^{i-q}}\quad\text{as } r\to\infty, \tag{1.6}$$

with q being an integer. The solution of (1.5) and (1.6) is known to be, see [3, p. 70], of the form $b_r=x^r v(r)(r!)^q$, with $v(r)$ being such that $v(r)\sim\sum_{i=0}^{\infty}v_i/r^{i+\alpha}$ as $r\to\infty$, for some α, cf. (1.1) and (1.2). With the help of [7, Thm. 6.1], it has been proved in [8, Thm. 2.2] that when $q=0$, and $\lim_{i\to\infty}B_i$ exists, i.e. $|x|>1$, the B_i satisfy (1.3) and (1.4). When $\lim_{i\to\infty}B_i$ does not exist, i.e., $|x|>1$ or $|x|\geqq 1$, it is not known, in general, whether (1.3) and (1.4) still hold, although, under certain circumstances, the techniques of the present work can be used to show that they do. This will be indicated at the end of §2. Using the technique of the proof of [8, Thm. 2.2], (1.3) and (1.4) can be shown to hold for all integers $q\leqq -1$ and for all x, since for this case $\lim_{i\to\infty}B_i$ exists for all x. For $q>0$, in which case $\lim_{i\to\infty}B_i$ does not exist for any x, no result like (1.3) and (1.4) is known in general, and precisely this is the subject of the present work. For $q=1$, (1.3) and (1.4) have been shown to hold for two special cases, see [9].

In the present work we actually show that under certain conditions, $A_i(\zeta)$ is of the form

$$A_{r-1}(\zeta)=A(\zeta)+a_r\zeta^r g(r,\zeta), \tag{1.7}$$

where $A(\zeta)$ is the Borel-type sum of $F(\zeta)$ (to be defined later), and the converging factor $g(r,\zeta)$ has an asymptotic expansion of the form

$$g(r,\zeta)\sim\sum_{i=0}^{\infty}\frac{g_i(\zeta)}{r^i}\quad\text{as } r\to\infty, \tag{1.8}$$

with $g_i(\zeta)$ being polynomials in ζ^{-1}. We also note that all of the numerical examples of everywhere divergent series considered in [11, Tables A3 and A4] are of the form above with $q>0$, and for these examples Levin's transformations produce accurate approximations to their Borel-type sums.

A similar but less general approach to the interpretation of divergent series has been introduced by Dingle in a series of papers, and this approach is summarized in his book [2, Chaps. XXI and XXII]. Dingle is concerned with summing the remainder series $\sum_{i=r}^{\infty}a_i\zeta^i$ for fixed r, whereas our main concern is with the asymptotics of it as $r\to\infty$. Olver's book [6, Chap. 14] contains another approach to the estimation of $\sum_{i=r}^{\infty}a_i\zeta^i$ that was introduced by Stieltjes, and developed further by Airy and J. C. P. Miller; see [6] for further references. Again our problem is different than that considered in Olver's book.

2. Theory.

LEMMA 2.1. *Define*

$$Q_k(r,z)=\left(z\frac{d}{dz}\right)^k\frac{z^r}{1-z}. \tag{2.1}$$

Then

$$Q_k(r,z)=\frac{\sum_{i=0}^{k}g_{k,i}(r)z^{r+i}}{(1-z)^{k+1}}, \tag{2.2}$$

where $g_{k,i}(r)$ are polynomials of degree k in r, satisfying

$$(2.3)\quad \begin{aligned}&g_{k,0}(r)=r^k, \qquad g_{k,k}(r)=(1-r)^k,\\ &g_{k,i}(r)=(r+i)g_{k-1,i}(r)+(k-r-i+1)g_{k-1,i-1}(r), \qquad i=1,\cdots,k-1.\end{aligned}$$

Proof. Equation (2.3) follows easily by induction on k, starting with $k=0$ and $g_{0,0}(r)=1$. □

THEOREM 2.2. *Let a_r be expressible in the form*

$$(2.4)\qquad a_r=r^p w(r)\prod_{i=1}^{m}(\mu_i r+\nu_i)!,$$

where we assume that $p\geqq 0$ is an integer,

$$(2.4a)\qquad w(r)=\int_0^\infty e^{-rt}\varphi(t)\,dt, \qquad r\geqq 1,$$

for some function $\varphi(t)$ such that $\int_0^\infty e^{-t}|\varphi(t)|\,dt<\infty$, and μ_i and ν_i satisfy

$$(2.4b)\qquad \mu_i>0, \quad \mu_i+\nu_i>-1, \quad i=1,\cdots,m.$$

Obviously the power series $F(\zeta):=\sum_{r=1}^\infty a_r\zeta^r$ diverges for all $\zeta\neq 0$. For $0<\theta_0<\pi$ define the bounded sectors $S(\rho,\theta_0)$ in the ζ-plane by

$$(2.5)\qquad S(\rho,\theta_0)=\{\zeta=|\zeta|e^{i\theta}: |\zeta|\leqq\rho,\ \theta_0\leqq\theta\leqq 2\pi-\theta_0\}.$$

Then $F(\zeta)$ is the asymptotic expansion of its Borel-type sum

$$(2.6)\qquad \mathscr{F}(\zeta)=\int_0^\infty\cdots\int_0^\infty \psi(\vec{t})\left(z\frac{d}{dz}\right)^p\left(\frac{z}{1-z}\right)\prod_{i=0}^{m}dt_i,$$

as $\zeta\to 0$, uniformly in $S(\rho,\theta_0)$, for each finite ρ, where $\vec{t}=(t_0,t_1,\cdots,t_m)$, and

$$(2.6a)\qquad \psi(\vec{t})=\exp\left(-\sum_{i=1}^m t_i\right)\left(\prod_{i=1}^m t_i^{\nu_i}\right)\varphi(t_0),$$

$$(2.6b)\qquad z=\zeta e^{-t_0}\prod_{i=1}^m t_i^{\mu_i}.$$

The function $\mathscr{F}(\zeta)$ is analytic in the ζ-plane cut along $[0,\infty)$.

Remarks. (1) If $\varphi(t)\sim\sum_{i=0}^\infty\varphi_i t^{i+\sigma-1}$ as $t\to 0+$, with $\sigma>0$, the application of Watson's lemma, see [6, p. 71], yields $w(r)\sim\sum_{i=0}^\infty\varphi_i(i+\sigma-1)!/r^{i+\sigma}$ as $r\to\infty$, and this is exactly of the form given in (1.2) with $w_i=\varphi_i(i+\sigma-1)!$, $i=0,1,\cdots$. Furthermore, if we take $\mu_i=1$, $\nu_i=0$, $i=1,\cdots,m$, then we are back at (1.1) and (1.2).

(2) There is no loss of generality in assuming $\mu_i+\nu_i>-1$ in (2.4b). For, if $\mu_i+\nu_i>-1$ is not satisfied for all i, we can consider the series $F'(\zeta):=\sum_{r=1}^\infty a'_r\zeta^r$, where $a'_r=a_{r+k}$, with k being chosen such that $\mu_i+(k\mu_i+\nu_i)>-1$, $1\leqq i\leqq m$. Note that $F(\zeta)=A_k(\zeta)+\zeta^kF'(\zeta)$.

(3) The Borel-type sum $\mathscr{F}(\zeta)$ given in (2.6) is obtained by substituting (2.9) (see proof below) in $\sum_{r=1}^\infty a_r\zeta^r$, interchanging the summation with all the integrations, and then summing the geometric-type series $M(z)=\sum_{r=1}^\infty r^pz^r$ to obtain $M(z)=(zd/dz)^p(z/(1-z))$. It can be shown that the Borel sum of $F(\zeta)$, namely $\int_0^\infty e^{-t}(\sum_{r=1}^\infty a_rt^r\zeta^r/r!)\,dt$, see [3, p. 78], is its Borel-type sum when $m=1$, $\mu_1=1$, and $\nu_1=0$.

(4) If in (2.4) $r^p w(r) = Cr^{p'}$, where C is a constant and $p' = p - 1 \geqq 0$, then the Borel-type sum in (2.6) reduces to that obtained from (2.6) by omitting the integration with respect to t_0 after $\varphi(t_0)$ in $\psi(\vec{t})$ has been replaced by C, p has been replaced by p', and the factor e^{-t_0} has been deleted from z. This can be shown by observing that actually $w(r) = C/r$, thus $\varphi(t_0) = C$, and performing the integral with respect to t_0, reducing (2.6) to an m-dimensional integral.

Proof. Using the fact that

$$\frac{z}{1-z} = z + z^2 + \cdots + z^{r-1} + \frac{z^r}{1-z}, \tag{2.7}$$

we have

$$\left(z\frac{d}{dz}\right)^p \frac{z}{1-z} = \sum_{j=1}^{r-1} j^p z^j + Q_p(r,z). \tag{2.8}$$

Letting z be as in (2.6b), and substituting (2.8) in (2.6), and using the fact that

$$j^p \int_0^\infty \cdots \int_0^\infty \psi(\vec{t}) z^j \prod_{i=0}^m dt_i = a_j \zeta^j, \qquad j = 1, 2, \cdots, \tag{2.9}$$

we obtain

$$\mathscr{F}(\zeta) = A_{r-1}(\zeta) + U_r(\zeta), \tag{2.10}$$

where

$$U_r(\zeta) = \int_0^\infty \cdots \int_0^\infty \psi(\vec{t}) Q_p(r,z) \prod_{i=0}^m dt_i. \tag{2.11}$$

From Lemma 2.1

$$Q_p(r,z) = \frac{\sum_{i=0}^p g_{p,i}(r) z^{r+i}}{(1-z)^{p+1}}. \tag{2.12}$$

It is easy to see that

$$|1-z| \geqq \sin\theta_0, \quad \text{all } \zeta \in S(\rho,\theta_0), \quad \text{all } \rho > 0. \tag{2.13}$$

Substituting (2.12) in (2.11), taking the modulus of both sides, and using (2.13), we obtain

$$\begin{aligned} |U_r(\zeta)| \leqq (\sin\theta_0)^{-p-1} \sum_{i=0}^p |g_{p,i}(r)| |\zeta^{r+i}| \\ \times \left(\prod_{j=1}^m [\mu_j(r+i) + \nu_j]! \right) \int_0^\infty e^{-(r+i)t} |\varphi(t)|\, dt, \end{aligned} \tag{2.14}$$

which, for all $\zeta \in S(\rho, \theta_0)$, becomes

$$|U_r(\zeta)| \leqq K_r |\zeta|^r, \tag{2.15}$$

with K_r being *independent* of ζ. This proves the first part of the theorem. The second part of the theorem is obvious. □

We now go on to analyze the "remainder" term $U_r(\zeta)$ in the limit $r \to \infty$.

THEOREM 2.3. *Assume that all the conditions of Theorem 2.2 are satisfied, and, in addition,* $\varphi(t)$ *is continuous in a neighborhood of* 0 *except possibly at* 0, *and satisfies*

$$\varphi(t) \sim \varphi_0 t^{\sigma-1} \quad \text{as } t \to 0^+, \quad \text{for some } \sigma > 0. \tag{2.16}$$

Then, for any integer $k \geq 0$,

$$U_r(\zeta) = -\sum_{j=0}^{k-1} a_{r-1-j}\zeta^{r-1-j} + 0\left(a_{r-1-k}\zeta^{r-1-k-p}\right) \quad \text{as } r \to \infty, \tag{2.17}$$

uniformly in ζ *for* $\zeta \in S(\rho, \theta_0)$, *for each finite* ρ.

Proof. Expressing $Q_p(r,z)$ (see (2.1)) in the form

$$Q_p(r,z) = -\left(z\frac{d}{dz}\right)^p \frac{z^{r-1}}{1-1/z}, \tag{2.18}$$

and making use of (2.7) with z replaced by $1/z$, we have, for any integers $k \geq 1$ and $N \geq k$,

$$Q_p(r,z) = -\sum_{j=0}^{N-1} (r-1-j)^p z^{r-1-j} + Q_p(r-N,z). \tag{2.19}$$

Substituting (2.19) in (2.11), and invoking (2.9), we obtain

$$U_r(\zeta) = -\sum_{j=0}^{N-1} a_{r-1-j}\zeta^{r-1-j} + U_{r-N}(\zeta). \tag{2.20}$$

Now $U_{r-N}(\zeta)$ satisfies (2.14) with r replaced by $r-N$. By (2.16), we conclude that

$$\begin{aligned} &\int_0^\infty e^{-rt}\varphi(t)\,dt \sim \varphi_0(\sigma-1)!/r^\sigma \quad \text{as } r \to \infty, \\ &\int_0^\infty e^{-rt}|\varphi(t)|\,dt \sim |\varphi_0|(\sigma-1)!/r^\sigma \quad \text{as } r \to \infty, \end{aligned} \tag{2.21}$$

see [6, p. 81]. Also, $g_{p,i}(r) = O(r^p)$ as $r \to \infty$, by Lemma 2.1. Consequently, for $\zeta \in S(\rho, \theta_0)$

$$|U_{r-N}(\zeta)| \leq 0\left(r^{p-\sigma}|\zeta|^{r-N} \prod_{j=1}^m \left[\mu_j(r-N+p)+\nu_j\right]!\right) \quad \text{as } r \to \infty, \tag{2.22}$$

uniformly in ζ. Similarly

$$a_{r-N+p} \sim \varphi_0 r^{p-\sigma} \prod_{j=1}^m \left[\mu_j(r-N+p)+\nu_j\right]! \quad \text{as } r \to \infty. \tag{2.23}$$

Comparing (2.22) and (2.23), we obtain

$$|U_{r-N}(\zeta)| \leq 0\left(a_{r-N+p}|\zeta|^{r-N}\right) \quad \text{as } r \to \infty, \tag{2.24}$$

uniformly in ζ for all $\zeta \in S(\rho, \theta_0)$.

Finally, by choosing $N = k+p+1$, and recalling that (see (2.23) above)

$$\lim_{r \to \infty} \frac{a_{r-j}}{a_r} = 0, \qquad j = 1, 2, \cdots, \tag{2.25}$$

(2.17) follows. □

COROLLARY 2.4. *If* $\sum_{i=1}^{m}\mu_i=\mu$, *where* μ *is a positive integer, and if* $\varphi(t)$ *also satisfies*

$$\varphi(t)\sim\sum_{i=0}^{\infty}\varphi_i t^{i+\sigma-1}\quad \text{as } t\to0^+,\quad \text{for some } \sigma>0, \tag{2.26}$$

then $U_r(\zeta)$ *is of the form*

$$U_r(\zeta)\sim a_r\zeta^r\sum_{i=0}^{\infty}\frac{\beta_i(\zeta)}{r^{i+\mu}}\quad \text{as } r\to\infty, \tag{2.27}$$

with $\beta_i(\zeta)$ *being polynomials in* ζ^{-1}. *Furthermore* (2.27) *is uniformly valid in* ζ *for* $\zeta\in T=S(\rho,\theta_0)\backslash\{\zeta\colon |\zeta|<\varepsilon\}$, *for any* $\varepsilon>0$.

Proof. As mentioned in the remark following the statement of Theorem 2.1, (2.4) and (2.26) imply that $w(r)\sim\sum_{i=0}^{\infty}\varphi_i(i+\sigma-1)!/r^{i+\sigma}$ as $r\to\infty$. This, together with the result

$$x^{\beta-\alpha}\frac{(x+\alpha)!}{(x+\beta)!}\sim1+\sum_{i=1}^{\infty}\frac{c_i}{x^i}\quad \text{as } x\to\infty, \tag{2.28}$$

c_i being some constants independent of x (see [1, p. 257, formula 6.1.47]), give

$$\frac{a_{r-1-j}}{a_r}\sim r^{-\mu(1+j)}\left[1+\sum_{i=1}^{\infty}\frac{d_i^{(j)}}{r^i}\right]\quad \text{as } r\to\infty, \tag{2.29}$$

where $d_i^{(j)}$ are constants independent of r. Upon substituting (2.29) in (2.17), we obtain

$$U_r(\zeta)=a_r\zeta^r\left[\sum_{i=0}^{\mu(k+1)-1}\frac{\beta_i(\zeta)}{r^{i+\mu}}+O\left(\frac{1}{r^{\mu(k+1)}}\right)\right]\quad \text{as } r\to\infty, \tag{2.30}$$

with $\beta_i(\zeta)$ being given by

$$\beta_{j\mu+i}(\zeta)=-\sum_{l=0}^{j}d_{l\mu+i}^{(j-l)}\zeta^{-j+l-1},\quad 0\leqq i\leqq\mu-1,\quad j=0,1,\cdots, \tag{2.31}$$

where $d_0^{(j)}=1$, $j=0,1,\cdots$; hence $\beta_0(\zeta)=-\zeta^{-1}$. This completes the proof of the corollary. □

Remark. Under the conditions stated in the corollary above, we have actually shown that the partial sums of the everywhere divergent series $F(\zeta):=\sum_{r=1}^{\infty}a_r\zeta^r$ are of the form given in (1.7) and (1.8), with $A(\zeta)=\mathscr{F}(\zeta)$, the Borel-type sum of $F(\zeta)$, and $g_i(\zeta)=0$, $0\leqq i\leqq\mu-1$.

As an example, consider one of the series given in [11, Table A3], namely $\sum_{r=1}^{\infty}(-1)^{r-1}c_r/x^r$, with $c_1=2$ and $c_r=c_{r-1}(2r-3)^2$, $r\geqq2$. Therefore, $(-1)^{r-1}c_r/x^r=a_r\zeta^r$, with $\zeta=-4/x$ and $a_r=-[(r-3/2)!]^2/(2\pi)$, $r\geqq1$. That is to say, $\mu_1=\mu_2=1$, $\nu_1=\nu_2=-3/2$, and $r^pw(r)=Cr^{p'}$ with $C=-1/(2\pi)$ and $p'=p-1=0$ in Remark (4) following (2.6b). Consequently,

$$\mathscr{F}(\zeta)=-\frac{\zeta}{2\pi}\int_0^{\infty}\int_0^{\infty}e^{-(t_1+t_2)}\frac{(t_1t_2)^{-1/2}}{1-\zeta t_1t_2}\,dt_1\,dt_2.$$

By making the transformation of variables $t_1=\sqrt{x}\,t\cos^2\theta$, $t_2=\sqrt{x}\,t\sin^2\theta$, and performing the integral with respect to θ, we obtain

$$\mathscr{F}(\zeta)=\frac{2}{\sqrt{x}}\int_0^\infty e^{-\sqrt{x}t}\frac{dt}{(1+t^2)^{1/2}}=\frac{\pi}{\sqrt{x}}\left[\mathbf{H}_0(\sqrt{x})-Y_0(\sqrt{x})\right],$$

where $\mathbf{H}_\nu(z)$ is the Struve function of order ν [1, p. 496, formula 12.1.8]. The numerical result given in [11] for $x=4$ is another indication that the u-transformation produces approximations to $\mathscr{F}(\zeta)$.

Finally, we note that when $\mu_i=\nu_i=0$, $1\leqq i\leqq m$, in Theorem 2.2, $\mathscr{F}(\zeta)$ converges for $|\zeta|<1$ and diverges for $|\zeta|>1$. Thus, $\mathscr{F}(\zeta)$ represents an analytic function $u(\zeta)$ within the unit circle. Equation (2.6) now becomes

$$\mathscr{F}(\zeta)=\int_0^\infty \varphi(t)\left[z\frac{d}{dz}\right]^p\frac{z}{1-z}\,dt,\ z=\zeta e^{-t}. \tag{2.32}$$

Since this time $\mathscr{F}(\zeta)$ is analytic in the ζ-plane cut along $[1,\infty)$, it represents the analytic continuation of $u(\zeta)$ outside the unit circle. Furthermore, (2.10) holds with (2.11) replaced by

$$U_r(\zeta)=\int_0^\infty \varphi(t)Q_p(r,z)\,dt. \tag{2.33}$$

Let us now assume that $\varphi(t)$ satisfies (2.26). Then substituting (2.12) in (2.33), and applying Watson's lemma for $r\to\infty$, after some manipulation of the asymptotic expansions that arise, we obtain (2.27) with $\mu=0$ there. Of course, in this case the $\beta_i(\zeta)$ are not necessarily polynomials in ζ^{-1}. In addition, (2.27) with $\mu=0$ holds for all $\zeta\notin[1,\infty)$, for which $F(\zeta)$ converges or diverges. The details are left to the interested reader.

3. Further developments. The results of the previous section have been based mainly on the assumptions of Theorem 2.2, namely (2.4) to (2.4b). It is these assumptions that enable one to express the Borel-type sum $\mathscr{F}(\zeta)$ of $F(\zeta)$ as in (2.6) to (2.6b). One important feature of (2.6) is the function $Q_p(1,z)=(zd/dz)^p(z/(1-z))$, which is very easy to handle. Actually this function has simple expansions about $z=0$ and $z=\infty$, and it is these expansions that lead to the results of Theorem 2.2, Theorem 2.3, and Corollary 2.4. In this section we seek to generalize the conditions of Theorem 2.2 in a way that will enable us to retain the function $Q_p(1,z)$. We note that the developments of this section can readily be applied to generalized hypergeometric functions.

THEOREM 3.1. *Let a_r be expressible in the form*

$$a_r=r^p w(r)\prod_{i=1}^m(\mu_i r+\nu_i)!\prod_{j=1}^n B\left(\kappa_j r+\lambda_j+1,\bar{\kappa}_j r+\bar{\lambda}_j+1\right), \tag{3.1}$$

where p, $w(r)$, μ_i, and ν_i are exactly as in Theorem 2.2,

$$\kappa_j\geqq 0,\quad \bar{\kappa}_j\geqq 0,\quad \kappa_j+\bar{\kappa}_j>0,\quad \kappa_j+\lambda_j>-1,\quad \bar{\kappa}_j+\bar{\lambda}_j>-1,\qquad j=1,\cdots,n, \tag{3.1a}$$

and $B(b,c)$ is the beta function defined by

$$B(b,c)=\int_0^1\tau^{b-1}(1-\tau)^{c-1}d\tau=\frac{(b-1)!(c-1)!}{(b+c-1)!},\qquad \mathrm{Re}\,b>0,\qquad \mathrm{Re}\,c>0. \tag{3.1b}$$

It is clear that the power series $F(\zeta):=\sum_{r=1}^{\infty}a_r\zeta^r$ *diverges for all* $\zeta\neq 0$. *Define* $S(\rho,\theta_0)$ *again as in Theorem* 2.2. *Then* $F(\zeta)$ *is the asymptotic expansion of its Borel-type sum*

$$(3.2)\qquad \mathscr{F}(\zeta)=\underbrace{\int_0^{\infty}\cdots\int_0^{\infty}}_{m+1\ \text{times}}\underbrace{\int_0^{1}\cdots\int_0^{1}}_{n\ \text{times}}\psi(\vec{t},\vec{\tau})Q_p(1,z)\prod_{i=0}^{m}dt_i\prod_{j=1}^{n}d\tau_j,$$

as $\zeta\to 0$, *uniformly in* $S(\rho,\theta_0)$, *for each finite* ρ, *where* $\vec{t}=(t_0,t_1,\cdots,t_m)$, $\vec{\tau}=(\tau_1,\cdots,\tau_n)$, *and*

$$(3.2\text{a})\qquad \psi(\vec{t},\vec{\tau})=\exp\left(-\sum_{i=1}^{m}t_i\right)\left(\prod_{i=1}^{m}t_i^{\nu_i}\right)\varphi(t_0)\left(\prod_{j=1}^{n}\left[\tau_j^{\lambda_j}(1-\tau_j)^{\bar{\lambda}_j}\right]\right),$$

$$(3.2\text{b})\qquad z=\zeta e^{-t_0}\left(\prod_{i=1}^{m}t_i^{\mu_i}\right)\left(\prod_{j=1}^{n}\left[\tau_j^{\kappa_j}(1-\tau_j)^{\bar{\kappa}_j}\right]\right),$$

with the integrals over t_i, $0\leqq i\leqq m$ *being from* 0 *to* ∞ *and those over* τ_j, $1\leqq j\leqq n$, *from* 0 *to* 1. *The function* $\mathscr{F}(\zeta)$ *is analytic in the* ζ*-plane cut along* $[0,\infty)$.

Proof. Similar to that of Theorem 2.2. □

Remark. If in (3.1) $r^pw(r)=Cr^{p'}$, where C is a constant and $p'=p-1\geqq 0$, then the Borel-type sum of $\mathscr{F}(\zeta)$ in (3.2) reduces to that obtained from (3.2) by omitting the integration with respect to t_0 after $\varphi(t_0)$ in $\psi(\vec{t},\vec{\tau})$ has been replaced by C, and p by p', and the factor e^{-t_0} has been deleted from z. (cf. Remark (4) following statement of Theorem 2.2.)

THEOREM 3.2. *Assume that all the conditions of Theorem* 3.1 *are satisfied, and, in addition,* $\varphi(t)$ *is as in Theorem* 2.3. *Then, for any integer* $k\geqq 0$,

$$(3.3)\qquad U_r(\zeta)=\mathscr{F}(\zeta)-A_{r-1}(\zeta)=-\sum_{j=0}^{k-1}a_{r-1-j}\zeta^{r-1-j}+O\left(a_{r-1-k}\zeta^{r-1-k-p}\right)\quad \text{as } r\to\infty,$$

uniformly in ζ *for* $\zeta\in S(\rho,\theta_0)$ *for each finite* ρ.

Proof. Similar to that of Theorem 2.3. □

COROLLARY 3.3. *If* $\sum_{i=1}^{m}\mu_i=\mu$, *where* μ *is an integer, and if* $\varphi(t)$ *is as in Corollary* 2.4, *then* $U_r(\zeta)$ *is of the form given in* (2.27), *with* $\beta_i(\zeta)$ *being polynomials in* ζ^{-1} *again.* (2.27) *is uniformly valid in* ζ *for* $\zeta\in T$.

Proof. Using Stirling's formula, it can be shown that for $r\to\infty$

$$(3.4)\qquad B(\kappa r+\lambda,\bar{\kappa}r+\bar{\lambda})\sim\begin{cases}\left[\dfrac{\kappa^{\kappa}\bar{\kappa}^{\bar{\kappa}}}{(\kappa+\bar{\kappa})^{\kappa+\bar{\kappa}}}\right]^r\displaystyle\sum_{i=0}^{\infty}\frac{e_i}{r^{i+1/2}} & \text{if } \kappa>0,\quad \bar{\kappa}>0,\\ \displaystyle\sum_{i=0}^{\infty}\frac{e_i'}{r^{i+\bar{\lambda}}} & \text{if } \kappa>0,\quad \bar{\kappa}=0,\end{cases}$$

where e_i and e_i' are constants independent of r. With the help of (3.4), the proof of this corollary can now be accomplished as that of Corollary 2.4. □

Note. The results above are applicable to series $F(\zeta)$ for which

$$(3.5)\qquad a_r=r^pw(r)\frac{\prod_{i=1}^{m'}(\varepsilon_i r+\delta_i)!}{\prod_{i=1}^{n'}(\varepsilon_i r+\bar{\delta}_i)!},$$

with p and $w(r)$ as before, $\varepsilon_i > 0$, $\varepsilon_i + \delta_i > -1$, $\bar{\delta}_i > \delta_i$, $1 \leqq i \leqq n'$, $m' > n'$. This is so since a_r can be expressed as in (3.1), due to the fact that

$$\frac{(\varepsilon r + \delta)!}{(\varepsilon r + \bar{\delta})!} = \frac{B(\varepsilon r + \delta + 1, \bar{\delta} - \delta)}{(\bar{\delta} - \delta - 1)!}. \tag{3.6}$$

There is no loss of generality in assuming that $\bar{\delta}_i > \delta_i$, $1 \leqq i \leqq n'$, for if $\bar{\delta}_i \leqq \delta_i$ for some i, say $i = q$, then $(\varepsilon_q r + \bar{\delta}_q)!$ and r^p in (3.5) can be replaced by $[\varepsilon_q r + (k\varepsilon_q + \bar{\delta}_q)]!$ and the polynomial $r^p \prod_{j=1}^{k} (\varepsilon_q r + j\varepsilon_q + \bar{\delta}_q)$ respectively, such that $\hat{\delta}_q = k\varepsilon_q + \bar{\delta}_q > \delta_q$. In general, we can express a_r as $a_r = \sum_{j=1}^{k} h_j a_r^{(j)}$, where

$$a_r^{(j)} = r^{p+j} w(r) \frac{\prod_{i=1}^{m'} (\varepsilon_i r + \delta_i)!}{\prod_{i=1}^{n'} (\varepsilon_i r + \hat{\delta}_i)!}, \qquad 1 \leqq j \leqq k,$$

with $\hat{\delta}_i > \delta_i$, $1 \leqq i \leqq n'$. Now we apply Theorem 3.1, Theorem 3.2 and Corollary 3.3 to each of the series $\sum_{r=1}^{\infty} h_j a_r^{(j)} \zeta^r$ and add the results. The overall result is that Theorem 3.1, Theorem 3.2 and Corollary 3.3 hold for the series $F(\zeta) := \sum_{r=1}^{\infty} a_r \zeta^r$, even though $\bar{\delta}_i > \delta_i$ is not satisfied for all $1 \leqq i \leqq n'$. Thus our results can be applied to the generalized hypergeometric functions ${}_pF_q$, where

$${}_pF_q\left(\begin{matrix} \alpha_1, \cdots, \alpha_p \\ \rho_1, \cdots, \rho_q \end{matrix} \middle| \zeta \right) = \sum_{k=0}^{\infty} \frac{\prod_{h=1}^{p} (\alpha_h)_k \zeta^k}{\prod_{h=1}^{q} (\rho_h)_k k!}$$

with $(c)_k = \Gamma(c+k)/\Gamma(c)$, $k = 0, 1, \cdots$, see [5, p. 155], when $p > q+1$.

Note also that the representation given in (3.2) is somewhat related to the beta transform described in [5, p. 160].

As an example, we consider the asymptotic series $\sum a_r \zeta^r$ with

$$a_r = (\alpha)_{r-1} (1 + \alpha - \beta)_{r-1} / (n-1)! \quad \text{and} \quad \zeta = -1/x.$$

That is to say, a_r is expressible as

$$a_r = -\frac{(r + \alpha - \beta - 1)! B(r + \alpha - 1, 1 - \alpha)}{(-\alpha)!(\alpha - 1)!(\alpha - \beta)!}.$$

By the remark above, $\mathscr{F}(\zeta)$ can be expressed as a double integral that can be reduced to a one-dimensional integral, which, by using some relations among the confluent hypergeometric functions of different parameters, can be shown to be $-x^{\alpha-1} U(\alpha, \beta, x)$. Again [11, Table A3] contains numerical results for different values of α, β, and x, that indicate that the u-transformation produces approximations to $\mathscr{F}(\zeta)$.

4. Concluding remarks. We have shown that under the conditions stated in Corollary 2.4 and Corollary 3.3, the partial sums $A_{r-1}(\zeta) = \sum_{i=1}^{r-1} a_i \zeta^i$ of the everywhere divergent series $F(\zeta) := \sum_{i=1}^{\infty} a_i \zeta^i$ are of the form (1.7) and (1.8), where $A(\zeta)$ is the Borel-type sum of $F(\zeta)$. As mentioned in the introduction to this work, most of the examples of everywhere divergent series considered in [11] satisfy the requirements of Corollary 2.4 and Corollary 3.3; furthermore, after some tedious calculations, involving manipulation of (2.6) and (3.2), one observes for all these examples, that the numbers obtained by applying Levin's t or u transformation to $A_i(\zeta)$, are approximations to the Borel-type sum of $F(\zeta)$. In view of this observation, we conjecture that for the kind of series considered in Corollary 2.4 and Corollary 3.3, Levin's t and u transformations produce approximations that converge to the Borel-type sums of these series.

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