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On the Long Time Behaviour of a Generalized KdV Equation

A. SIDI
Department of Computer Sciences, Technion, Haifa, Israel

C. SULEM
Department of Mathematics, Ben Gurion University, Beersheva, Israel and CNRS, CMA, Ecole Normale Supérieure, Paris, France

and

P. L. SULEM
School of Mathematical Sciences, Tel-Aviv University, Israel and CNRS, Observatoire de Nice, France

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Abstract. We consider the Cauchy problem for the generalized Korteweg–de Vries equation

$$\partial_t u + \partial_x(-\partial_x^2)^\alpha u + \partial_x\left(\frac{u^\lambda}{\lambda}\right) = 0$$

where α is a positive real and λ an integer larger than 1. We obtain the detailed large distance behaviour of the fundamental solution of the linear problem and show that for $\alpha \geqslant \frac{1}{2}$ and $\lambda > \alpha + \frac{3}{2} + (\alpha^2 + 3\alpha + \frac{5}{4})^{1/2}$, solutions of the nonlinear equation with small initial conditions are smooth in the large and asymptotic when $t \to \pm\infty$ to solutions of the linear problem.

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1. Introduction

In this paper we consider the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^\lambda}{\lambda}\right) + \frac{\partial}{\partial x}\left(-\frac{\partial^2}{\partial x^2}\right)^\alpha u = 0 \tag{1.1}$$

which describes the propagation of nonlinear waves in a dispersion medium. In Equation (1.1) λ is a positive integer larger than 1 and α a positive real number. For $\alpha = 1$ and $\lambda = 2$, Equation (1.1) reduces to the celebrated Korteweg–de Vries (KdV) equation which arises in various physical contexts ([1] and references therein). For example, it describes in the continuous limit, the propagation of right going waves in a one-dimensional lattice of masses each connected to the next by nonlinear springs which when compressed or extended, exert a force which is represented by the sum of a term proportional to the deformation and a quadratic term. If, more generally, the nonlinear correction is a positive integer power $\lambda > 0$ of the deformation, we get Equation (1.1) with $\alpha = 1$ and a nonlinearity of the order λ [2]. For $\lambda = 3$, we obtain the 'modified Korteweg–de Vries equation' (MKdV) which has the property that its solution u_3 is related to a solution u_2 of the usual KdV equation by a Riccati-type relation [3]

$$\dot{u}_2 = u_3^2 + i\sqrt{6}\frac{\partial u_3}{\partial x}. \tag{1.2}$$

The KdV and MKdV equations possess the remarkable property of having an infinite number of invariants [4]. This is related to the fact that these equations are integrable by the inverse scattering method [5], which makes the behaviour of the solution very specific.

Another example where Equation (1.1) is obtained occurs in the propagation of waves in a one-dimensional stratified fluid in two limiting cases. In the shallow water limit, the propagation reduces to the KdV equation ($\lambda = 2$, $\alpha = 1$), while in the deep water limit, it reduces to the Benjamin–Ono (BO) equation which corresponds to $\lambda = 2$ and $\alpha = \frac{1}{2}$. This equation has also been shown to possess an infinity of invariants [6, 7], and to be integrable by inverse scattering [8].

For all other values of α and λ, Equation (1.1) is not integrable by the inverse scattering method. For $\lambda = 2$, $\alpha = 2$ (a situation which arises when, under certain critical initial conditions, the fifth-order spatial differentiation turns out to be the dominant dispersive effects), numerical integration shows chaotic dynamics [9]. Indeed, when the special cases of KdV, MKdV, and BO equations are excepted, Equation (1.1) has only three invariant quantities

$$I_1 = \int u \, dx, \tag{1.3a}$$

$$I_2 = \int u^2 \, dx, \tag{1.3b}$$

$$I_3 = \frac{1}{\lambda(\lambda+1)} \int u^{\lambda+1} \, dx + \int |(-\partial_x^2)^\alpha u|^2 \, dx. \tag{1.3c}$$

The third invariant is obtained by multiplying Equation (1.1) by $(u^\lambda/\lambda) + (-\partial_x^2)^\alpha u$ and integrating on the whole space. Equations (1.3b)–(1.3c) and the Sobolev space embedding property

$$|u|_{L^{\lambda+1}} \leq C|u|_{L^2}^{(\lambda-1)/(\lambda(\lambda+1))} |u|_{H^\alpha}^{1-(\lambda-1)/(\lambda(\lambda+1))}$$

leads to the following *a-priori* estimate [10]

$$|u|_{H^\alpha}^2 \leq |I_3| + C I_2^{\lambda+1-((\lambda-1)/2\alpha)} |u|_{H^\alpha}^{(\lambda-1)/2\alpha} \tag{1.5}$$

under the condition $\lambda \leq (1+2\alpha)/(1-2\alpha)$ if $\alpha < \frac{1}{2}$ (no condition otherwise).

2. Existence and Regularity Properties

Several papers have been devoted to the existence and regularity of solutions of the KdV equation. The existence and uniqueness of a global solution has been proved in $L^\infty(0, \infty, H^2)$ [11]. Existence without uniqueness in $L^\infty(0, \infty, H^1)$ was proved in [12]. Cohen–Murray [13] used an analysis based on the inverse scattering method to establish a relationship between the regularity of the solution for $t > 0$ and the decay at infinity of the initial conditions. He proved the existence of solutions when the initial conditions are piecewise of Class C^4 and decay, together with the first four derivatives at an algebraic rate. The faster the decay of the initial condition, the smoother the solution will be for $t > 0$. In particular, if the initial condition and its four first derivatives decay faster than $|x|^{-N}$ for all N, the solution will be infinitely differentiable for $t > 0$.

Existence of the solution to Equation (1.1) with $\alpha = 1$ and arbitrary $\lambda > 0$ was considered by Kato [14]. Existence of the weak solution in $L^\infty(0, \infty, H^1)$ results from Equation (1.5). Existence of classical solutions in $L^\infty(0, \infty, H^2)$ was shown as follows: differentiation of Equation (1.1) with $\alpha = 1$ twice in space, multiplication by ∂_x^2 and integration on the whole space lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |\partial_x^2 u|^2\,\mathrm{d}x - \tfrac{1}{2}\int a'''(u)(\partial_x u)^5\,\mathrm{d}x - 10\int a(u)(\partial_{xxx}u)(\partial_{xx}u)\,\mathrm{d}x = 0, \tag{2.1}$$

where $a(u) = u^{\lambda-1}$ and the primes denote derivatives with respect to u. Equation (1.1) is then used to eliminate the third derivative which appears in Equation (2.1). After some computations, the following equality is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int |\partial_x^2 u|^2\,\mathrm{d}x - \tfrac{5}{4}\int a(u)(\partial_x u)^2\,\mathrm{d}x\right\}$$
$$= \tfrac{1}{12}\int a'''(u)(\partial_x u)^5\,\mathrm{d}x + \tfrac{5}{3}\int a(u)a'(u)(\partial_x u)^3\,\mathrm{d}x. \tag{2.2}$$

Using the inequalities

$$\int |\partial_x u|^5\,\mathrm{d}x \leqslant |\partial_x u|_{L^2}^{7/2}|\partial_x^2 u|_{L^2}^{3/2},$$

$$\int |\partial_x u|^3\,\mathrm{d}x \leqslant |\partial_x u|_{L^2}^{5/2}|\partial_x^2 u|_{L^2}^{1/2}$$

and the uniform boundedness of $|u(t)|_{H^1}$ which results from Equation (1.5), Equation (2.2) ensures that $|u(t)|_{H^2}$ does not grow faster than exponentially in time.

Equation (1.1) with more general α and λ was considered in [10]. Estimate (1.5) ensures the existence of a weak solution in $L^\infty(R^+, H^\alpha(R))$ with no additional requirement when $\lambda < 4\alpha + 1$ and under the condition that the L^2-norm of the initial data $|u_0|_{L^2}$ be small enough when $\lambda = 4\alpha + 1$ [10]. For smooth initial conditions, the above solutions are classical when $\alpha > \frac{3}{2}$. For $\lambda > 4\alpha + 1$, the same estimate also insures the existence of solutions (weak solutions for $\alpha < \frac{3}{2}$ or classical solutions for $\alpha > \frac{3}{2}$) when the H^α-norm of the initial condition is sufficiently small.

In the next section, using a different approach we show that a more precise result can be obtained in the case of strong nonlinearity

$$(\lambda > \alpha + \tfrac{3}{2} + (\alpha^2 + 3\alpha + \tfrac{5}{4})^{1/2}, \qquad \text{where } \alpha \geq \tfrac{1}{2}).$$

In this case, if the initial conditions are 'sufficiently small', there exist global classical solutions of Equation (1.1) which are asymptotics when $t \to \infty$ to solutions of the linear equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{\partial^2}{\partial x^2}\right)^\alpha u = 0. \tag{2.3}$$

This extends the result of Strauss [15, 16] given in the case of $\alpha = 1$. The method is based on the fact that when $\alpha \geq \frac{1}{2}$, the Green function of Equation (2.3) decays in time. This dispersive effect insures the decay in time of the $L^p(p > 2)$-norms of the solutions of the linear problem. For the nonlinear problem, the idea is to rewrite Equation (1.1) in an integral form and to compensate the nonlinearities by the dispersion. This method has been used for other dispersive equations like the nonlinear wave equation, the nonlinear Klein–Gordon equation, the nonlinear Schrödinger equation, and an hydrodynamic equation for classical spins [17–20]. More recently, a similar approach has been used for incompressible magnetohydrodynamics when dispersion is replaced by differential transport [21].

3. The Linear Problem

Let us consider the linear equation

$$\partial_t u + \partial_x(-\partial_x^2)^\alpha u = 0, \quad u(x, 0) = u_0(x). \tag{3.1}$$

The solution reads

$$u(x, t) = \frac{2}{t^{1/(2\alpha+1)}} \int_{-\infty}^{+\infty} g\left(\frac{x - x'}{t^{1/(2\alpha+1)}}\right) u_0(x')\,\mathrm{d}x' \tag{3.2}$$

with

$$g(x) = \int_0^\infty \cos(\xi^{2\alpha+1} + x\xi)\,\mathrm{d}\xi \tag{3.3}$$

provided $\alpha > 0$. This condition guarantees the existence of the integral in (3.3). When $\alpha = 1$, $g(x) = 3^{-1/3}\pi\mathrm{Ai}(3^{-1/3}x)$, where $\mathrm{Ai}(z)$ is the well-known Airy function. Thus, $g(x) = O(x^{-1/4})$ as $x \to -\infty$ and $g(x)$ decreases exponentially as $x \to +\infty$. The asymptotic behaviour of $g(x)$ for $x \to -\infty$ and $x \to +\infty$ for any $\alpha > 0$ is given in Theorem 3.1 below.

THEOREM 3.1. *Let* $\beta \equiv 2\alpha + 1$ *and* $\alpha > 0$. *Then*

(1) *as* $x \to -\infty$:

$$g(x) \sim \frac{\sqrt{\pi}}{\frac{\beta(\beta-1)}{2}\left(\frac{|x|}{\beta}\right)^{(\beta-2)/2(\beta-1)}} \cos\left[(1-\beta)\left(\frac{|x|}{\beta}\right)^{\beta/(\beta-1)} + \frac{\pi}{4}\right] +$$

$$+ o(|x|^{-(\beta-2)/2(\beta-1)}) \tag{3.4}$$

(2) *as* $x \to +\infty$

(a) *if* $\beta = 2n$ $(n = 1, 2, \ldots)$

$$g(x) \sim (-)^{n+1} \sum_{k=0}^{\infty} (-)^k \frac{[2n(2k+1)]!}{(2k+1)!\, x^{2n(2k+1)+1}}, \quad \beta = 2n, \quad n = 1, 2, \ldots, \tag{3.5}$$

(b) *if* $\beta = 2n + 1$ $(n = 1, 2, \ldots)$

$$g(x) \sim \pi \frac{(2n+1)^{-1/4}}{2n^{1/2}} x^{-((2n-1)/4n)} \times$$

$$\times \sum_{j=1}^{n} \exp\left\{-\frac{2n}{2n+1}(2n+1)^{-(1/2n)} x^{1+(1/2n)} \times\right.$$

$$\left.\times \exp\left[\frac{i(2j-1)}{2n}\pi\right] + \frac{(2j-1)}{4n}\pi - \frac{\pi}{4}\right\} \tag{3.6}$$

(c) *if* β *is not an integer*

$$g(x) \sim -\frac{\beta!\cos(\beta\pi/2)}{x^{\beta+1}} \tag{3.7}$$

As a result of (3.4)–(3.7), $g(x)$ *belongs to* $L^\infty(R)$ *for* $\alpha \geq \frac{1}{2}$.

Proof. We start by writing (3.3) in the form

$$g(x) = \operatorname{Re} \int_0^\infty \exp[i(x\xi + \xi^\beta)\, \mathrm{d}\xi]. \tag{3.8}$$

1. *Behaviour of* $g(x)$ *for* $x \to -\infty$. Defining

$$s = \left(\frac{|x|}{\beta}\right)^{\beta/(\beta-1)}$$

and making the change of the variable of integration $\xi = (1+\tau)s^{1/\beta}$, Formula (3.8) becomes

$$g(x) = s^{1/\beta} \operatorname{Re} G(s), \tag{3.9}$$

where

$$G(s) = \int_{-1}^{\infty} \exp[isp(\tau)]\, \mathrm{d}\tau, \tag{3.10}$$

with

$$p(\tau) = (1+\tau)^{\beta} - \beta(1+\tau). \tag{3.11}$$

We now write $G(s) = G_1(s) + G_2(s)$, where

$$\begin{aligned} G_1(s) &= \int_0^{\infty} \exp[isp(\tau)]\,\mathrm{d}\tau, \\ G_2(s) &= \int_{-1}^{0} \exp[isp(\tau)]\,\mathrm{d}\tau. \end{aligned} \tag{3.12}$$

We first treat $G_1(s)$. By (3.11) it can be verified that all the conditions of Theorem 13.1 in Olver [22], p. 101, for applicability of the stationary phase method are satisfied. (The parameters a, $p(a)$, P, μ, Q and λ in the above-mentioned theorem take on the values 0, $1-\beta$, $\beta(\beta-1)/2$, 2, 1, and 1, respectively.) Consequently

$$G_1(s) = \tfrac{1}{2}\Gamma(\tfrac{1}{2})\,\mathrm{e}^{i\pi/4}\frac{\exp is(1-\beta)}{[(\frac{1}{2}\beta(\beta-1))s]^{1/2}} + o(s^{-1/2}) \quad \text{as } s \to +\infty. \tag{3.13}$$

For $G_2(s)$ we make the transformation of the variable of integration $\tau = -\tau'$ in (3.12). Also in this case, Theorem 13.1 in [22] can be applied. The result is $G_2(s) = G_1(s) + o(s^{-1/2})$ as $s \to +\infty$. Thus, $G(s) = 2G_1(s) + o(s^{-1/2})$ as $s \to +\infty$. Invoking now the definition of s, (3.4) follows

2. *Behaviour of* $g(x)$ *when* $x \to +\infty$. Defining $s = x^{\beta/(\beta-1)}$ and making a change of the variable of integration $\xi = \tau s^{1/\beta}$, (3.8) becomes

$$g(x) = s^{1/\beta}\,\mathrm{Re}\,G(s), \tag{3.14}$$

where

$$G(s) = \int_0^{\infty} \exp[isp(\tau)] \tag{3.15}$$

with

$$p(\tau) = \tau + \tau^{\beta}. \tag{3.16}$$

We observe that $p(\tau)$ is a monotonically increasing function of τ on $[0, \infty)$ and maps the interval $[0, \infty)$ in a one-to-one manner onto itself. Thus, the inverse function of $p(\tau)$ exists and we call it $h(\sigma)$, i.e., $\tau = h(\sigma)$. Since $\beta > 1$, $p'(\tau)$ exists for all τ in $[0, \infty)$ and $p'(\tau) \neq 0$ there. This guarantees the existence of $h'(\sigma)$ for all σ in $[0, \infty)$. Thus, making a change of the variable of integration $\sigma = p(\tau)$, (3.15) becomes

$$G(s) = \int_0^{\infty} \exp[is\sigma]h'(\sigma)\,\mathrm{d}\sigma. \tag{3.17}$$

In the sequel we make use of the following result:

Let the function $v(\xi)$ be $m-1$ times continuously differentiable over $[0, \infty)$

such that $\lim_{\xi\to\infty} v^{(k)}(\xi) = 0$, $0 \leq k \leq m-1$. Then, for r real,

$$\int_0^\infty \exp[ir\xi]v(\xi)\,d\xi = \sum_{k=0}^{m-1} \frac{v^{(k)}(0)}{(-ir)^{k+1}} + \int_0^\infty \frac{\exp[ir\xi]}{(-ir)^m} v^{(m)}(\xi)\,d\xi, \tag{3.18}$$

provided the integral on the right-hand side exists. Also, if $\int_0^\infty |v^{(m)}(\xi)|\,d\xi < \infty$, then the integral on the right-hand side of (3.18) is $O(r^{-m})$ as $r\to\infty$. If, in addition, $m=\infty$, then

$$\int_0^\infty \exp[ir\xi]v(\xi)\,d\xi \sim \sum_{k=0}^{\infty} \frac{v^{(k)}(0)}{(-ir)^{k+1}} \quad \text{as } r\to\infty. \tag{3.19}$$

Now the function $h(\sigma)$ is infinitely differentiable for all $\sigma\in(0,\infty)$ as $p(\tau)$ is for all $\tau\in(0,\infty)$. Furthermore, $\lim_{\sigma\to\infty} h^{(k)}(\sigma)=0$, $k=1,2,\ldots$. To see this observe that

$$h^{(k)}(\sigma) = \left(\frac{1}{p'(\tau)}\frac{d}{d\tau}\right)^{k-1}\frac{1}{p'(\tau)}, \quad k=1,2,\ldots, \tag{3.20}$$

and that $p'(\tau) = O(\tau^{\beta-1})$ as $\tau\to\infty$, since $\beta>1$.

Let us assume that β is an integer greater than 1. Then $h^{(k)}(0)$, $k=1,2,\ldots$, all exist, and by (3.19)

$$G(s) \sim \sum_{k=0}^{\infty} \frac{h^{(k+1)}(0)}{(-is)^{k+1}} \quad \text{as } s\to\infty. \tag{3.21}$$

Since only Re $G(s)$ contributes to $g(x)$, we have ($s = x^{\beta/\beta-1}$)

$$g(x) \sim s^{1/\beta} \sum_{k=1}^{\infty} \frac{h^{(2k)}(0)}{(-is)^{2k}} \quad \text{as } x\to+\infty. \tag{3.22}$$

When $\beta = 2n$, $n=1,2,\ldots$, using the Lagrange–Bürmann formula, we have

$$h(\sigma) = \sum_{j=0}^{\infty} (-)^j \frac{(2nj)!}{j!}\frac{\sigma^{j(2n-1)+1}}{[j(2n-1)+1]!} \tag{3.23}$$

Combining (3.22) with (3.23), (3.6) follows.

When $\beta = 2n+1$, $n=1,2,\ldots$, $h(\sigma)$ is an odd function of α, then $h^{(2k)}(0)=0$, $k=1,2,\ldots$, therefore, the summation on the right-hand side of (3.22) disappears, indicating that $g(x) = o(x^{-\mu})$ for any $\mu>0$. The exact asymptotic behaviour of $g(x)$ for $x\to+\infty$ can be obtained by the method of steepest descent. For this case $g(x)$ can be reexpressed as

$$g(x) = \tfrac{1}{2}s^{1/\beta}\tilde{G}(s), \tag{3.24}$$

where

$$\tilde{G}(s) = \int_{-\infty}^{+\infty} \exp[isp(\tau)]\,d\tau, \tag{3.25}$$

where s and τ are still as defined prior to (3.14), and $p(\tau)$ is as in (3.16). First we note that in the upper half of the τ-plane $\exp[isp(\tau)]$ tends to zero exponentially as $\tau \to \infty$ along the rays

$$\arg \tau = \theta_k = (4k+1)\pi/(4n+2), \quad k = 0, 1, \dots, n.$$

The saddle points of $p(\tau)$ are the roots of the equation

$$p'(\tau) = 1 + (2n+1)\tau^{2n} = 0,$$

and the ones in the upper half of the τ-plane are

$$\tau_j^* = (2n+1)^{-1/(2n)} \exp[i\theta_j^*] \quad \text{with } \theta_j^* = (2j-1)\pi/(2n).$$

Notice that

$$0 < \theta_0 < \theta_1^* < \theta_1 < \theta_2^* < \cdots < \theta_n^* < \theta_n < \pi. \tag{3.26}$$

We can now deform the contour of integration such that $\tilde{G}(s)$ can be expressed as

$$\tilde{G}(s) = \sum_{k=1}^{n} \int_{\Gamma_k} \exp[isp(\tau)]\, d\tau, \tag{3.27}$$

where Γ_k is the steepest descent contour that passes through τ_k^* and approaches the rays $\arg \tau = \theta_{k-1}$ and $\arg \tau = \theta_k$ asymptotically as $\tau \to \infty$; its direction being such that the two rays are to its right. For the case $n = 5$, see Figure 1. Using the facts that

$$p(\tau_k^*) = \frac{2n}{2n+1}\tau_k^* \quad \text{and} \quad p''(\tau_k^*) = -\frac{2n}{\tau_k^*}, \quad k = 1, \dots, n,$$

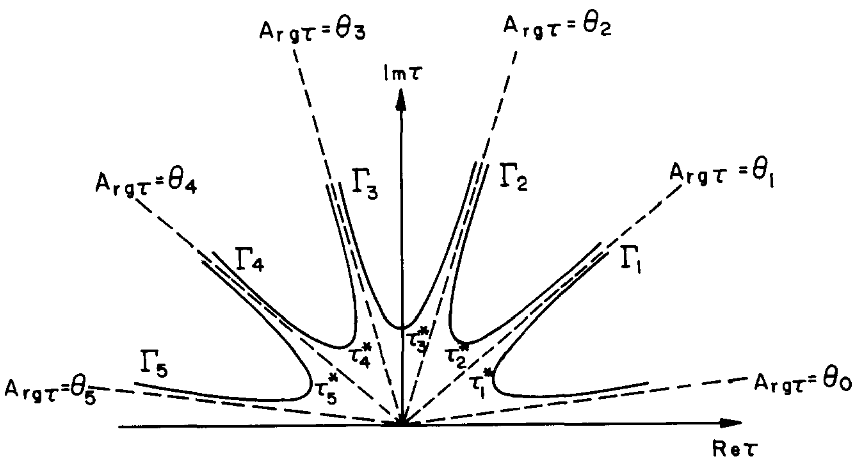

Fig. 1.

this results in

$$G(s) \sim \sum_{k=1}^{n} \left(\frac{\pi|\tau_k^*|}{ns}\right)^{1/2} \exp\left[i\left(\frac{2n}{2n+1}\tau_k^* s + \frac{\theta_k^*}{2} - \frac{\pi}{4}\right)\right] \quad \text{as } s \to \infty \tag{3.28}$$

from which we can easily obtain the asymptotic behaviour of $g(x)$ as $x \to +\infty$.

We finally assume that β is not an integer and that $\beta = m+1-\delta$, $m = 1, 2, \ldots$, and $0 < \delta < 1$. Therefore, $p(\tau)$ is m times continuously differentiable on $[0, \infty)$ implying that $h'(\sigma)$ is $m-1$ times continuously differentiable on $[0, \infty)$. Also,

$$p'(\tau) = 1 + \beta\tau^{\beta-1} \quad \text{and} \quad p''(\tau) = \beta(\beta-1)\tau^{\beta-2}$$

imply that $h'(0) = 1$, and, for $m \geqslant 2$, $h^{(k)}(0) = 0$, $2 \leqslant k \leqslant m$. Thus invoking (3.18), we have

$$G(s) = \frac{1}{-is} + \int_0^\infty \frac{\exp[is\sigma]}{(-is)^m} h^{(m+1)}(\sigma)\, d\sigma. \tag{3.29}$$

By (3.20) it can be shown that for $k \geqslant 2$,

$$h^{(k)}(\sigma) \sim -\beta(\beta-1)\cdots(\beta-k+1)\sigma^{\beta-k} \quad \text{as } \sigma \to 0,$$

which implies that the conditions of Theorem 13.1 in [22] are satisfied by the integrand on the right-hand side of (3.24). Consequently

$$G(s) \sim \frac{1}{-is} - \frac{\exp[i\pi(\beta-m)/2]}{(is)^m} \beta(\beta-1)\cdots(\beta-m)\frac{\Gamma(\beta-m)}{s^{\beta-m}} \quad \text{as } s \to \infty. \tag{3.30}$$

Substituting (3.30) in (3.14), (3.7) follows. Note that when β is an even integer, the right-hand side of (3.7) is identical to the leading term of the asymptotic series in (3.6), as expected.

COROLLARY 3.1. *For initial data u_0 in $L^p(R) \cap H^1(R)$, the solution of Equation* (3.1) *with $\alpha \geqslant \frac{1}{2}$ satisfies*

$$|u(t)|_{L^q} \leqslant C(1+t)^{1/(2\alpha+1)(1-(2/q))}(|u_0|_{L^p} + |u_0|_{H^1}) \tag{3.31}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \textit{and} \quad 1 \leqslant p \leqslant 2.$$

Proof. Using Theorem 3.1, we have for $t > 0$

$$|u(t)|_{L^\infty} \leqslant Ct^{-(1/2\alpha+1)}|u_0|_{L^1}.$$

Furthermore, one easily checks that

$$|u(t)|_{L^2} = |u_0|_{L^2}.$$

Interpolation between L^2 and L^∞ ([23], p. 179) then leads to Equation (3.31). □

4. *A-Priori* Estimates for the Nonlinear Problem

It is well known that for initial data $u_0 \in H^s(R)$, $s > \frac{3}{2}$, there exists a time T such that Equation (1.1) has a unique solution in $L^\infty([0, T], H^s(R))$ [10].

PROPOSITION 4.1. *For initial condition* $u_0 \in H^3(R)$, *the above solution satisfies*

$$|u(t)|_{H^2} \leq |u_0|_{H^3} \exp\left\{C \int_0^t |u(\tau)|^{\lambda-1}_{W^{2,2\lambda}} \, d\tau\right\}, \quad \lambda > 0 \tag{4.1}$$

Proof. We start with

$$\frac{1}{2}\frac{d}{dt}\int |\partial_x^3 u|^2 \, dx + \int \partial_x^3 u \, \partial_x^3(u^{\lambda-1} \, \partial_x u) \, dx = 0. \tag{4.2}$$

When the second term in the left-hand side of Equation (4.2) is developed, four terms are obtained which may be estimated as follows:

$$\text{(i)} \quad \left|\int u^{\lambda-2} \, \partial_x u (\partial_x^3 u)^2 \, dx\right| \leq |u|^{\lambda-2} |\partial_x u|_{L^\infty} |u|^2_{H^3}$$

$$\leq C|u|^{\lambda-1}_{W^{2,q}} |u|^2_{H^3} \quad (q > 1), \tag{4.3}$$

$$\text{(ii)} \quad \left|\int u^{\lambda-2} (\partial_x^2 u) \, \partial_x^3 u \, dx\right| \leq |u|^{\lambda-2}_{L^{2\lambda}} |\partial_x^2 u|^2_{L^{2\lambda}} |\partial_x^3 u|_{L^2}, \tag{4.4}$$

by Hölder inequalities. Furthermore,

$$|\partial_x^2 u|_{L^{2\lambda}} \leq |\partial_x^2 u|_{H^1} \quad (\lambda > 1) \tag{4.5}$$

which leads to

$$\int u^{\lambda-2} (\partial_x^2 u)^2 (\partial_x^3 u) \, dx \leq C|u|^{\lambda-1}_{W^{2,2\lambda}} |u|^2_{H^3}. \tag{4.6}$$

$$\text{(iii)} \quad \left|\int u^{\lambda-3} (\partial_x u)^2 \, \partial_x^2 u \, \partial_x^3 u \, dx\right|$$

$$\leq C|\partial_x^3 u|_{L^2} |\partial_x^2 u|_{L^{2\lambda}} |\partial_x u|^2_{L^{2\lambda}} |u|^{\lambda-3}_{L^{2\lambda}}. \tag{4.7}$$

$$\text{(iv)} \quad \left|\int \partial_x^3 u \, u^{\lambda-4} (\partial_x u)^4 \, dx\right| \leq C|\partial_x^3 u|_{L^2} |u|^{\lambda-4}_{L^{2\lambda}} |\partial_x u|^4_{L^{2\lambda}}. \tag{4.8}$$

This leads to the estimate

$$\frac{d}{dt}|u|_{H^3} \leq C|u|^{\lambda-1}_{W^{2,2\lambda}} |u|_{H^3}. \tag{4.9}$$

PROPOSITION 2. *For* $\alpha \geq \frac{1}{2}$ *and* $\lambda > \alpha + \frac{3}{2} + (\alpha^2 + 3\alpha + \frac{5}{4})^{1/2}$, *there exists a constant* $\delta > 0$ *such that for initial conditions*

$$u_0 \in W^{2,p}(R) \cap H^3(R) \quad \text{with } p = \frac{2\lambda}{2\lambda - 1},$$

satisfying the conditions $|u_0|_{W^{2,p}} \cap |u_0|_{H^3} < \delta$, *the solution u of* (1.1) *is such that*

$$M(t) = \sup_{0 \leq s \leq t} (1+s)^{(1-1/\lambda)/(2\alpha+1)} |u(s)|_{W^{2,2\lambda}} \tag{4.10}$$

is bounded for all $t \in [0, T]$, *independently of* T.

Proof. Equation (1.1) is rewritten

$$u(x, t) = G(t)u_0(x) + \int_0^t G(t-s)\partial_x \left(\frac{u^\lambda}{\lambda}\right)(x, s)\, ds \tag{4.11}$$

where the operator $G(t)$ denotes the convolution in space by the function

$$\frac{1}{t^{1/2\alpha+1}}\, g\left(\frac{\cdot}{t^{1/(2\alpha+1)}}\right).$$

Then, using (3.31), we have

$$|u(t)|_{W^{2,2\lambda}} \leq C(1+t)^{-(1-1/\lambda)/(2\alpha+1)}(|u_0|_{H^3} + |u_0|_{W^{2,2\lambda/(2\lambda-1)}}) + \\ + \int_0^t (t-s)^{-(1-1/\lambda)/(2\alpha+1)} |u^{\lambda-1}\, \partial_x u|_{W^{2,2\lambda/(2\lambda-1)}}\, ds. \tag{4.12}$$

By the Hölder inequalities we have

$$|u^{\lambda-1}\, \partial_x u|_{W^{2,2\lambda/(2\lambda-1)}} \leq C |u|^{\lambda}_{W^{2,2\lambda}} |u|_{H^3}. \tag{4.13}$$

We now use (4.9) to estimate $|u|_{H^3}$ in terms of $|u|_{W^{2,2\lambda}}$ and get

$$|u(s)|_{H^3} \leq |u_0|_{H^3} \exp \int_0^s f(\tau) M(\tau)^{\lambda-1}\, d\tau \tag{4.14}$$

with

$$f(s) = \int_0^s (1+\tau)^{-(\lambda-1)^2/\lambda(2\alpha+1)}\, d\tau. \tag{4.15}$$

Substituting (4.13) and (4.14) in (4.12), we obtain

$$M(t) \leq C(|u_0|_{H^3} + |u_0|_{W^{2,2\lambda/(2\lambda-1)}}) + \\ + C|u_0|_{H^3} \exp[f(t)M(t)^{\lambda-1}] M(t)^{\lambda-1} h(t) \tag{4.16}$$

where

$$h(t) = \int_0^t \frac{(1+t)^{(1-1/\lambda)/(2\alpha+1)}}{(t-s)^{(1-1/\lambda)/(2\alpha+1)}} (1+s)^{-(\lambda-1)^2/\lambda(2\alpha+1)}\, ds. \tag{4.17}$$

Under the condition

$$\frac{(\lambda-1)^2}{\lambda(2\alpha+1)} > 1,$$

the functions f and h are uniformly bounded and (4.16) is rewritten

$$M(t) \leq C\,\delta\{1 + C_1 M(t)^{\lambda-1} \exp[C_2 M(t)^{\lambda-1}]\} \tag{4.18}$$

with

$$\delta = |u_0|_{H^3} + |u_0|_{W^{2,2\lambda/(2\lambda-1)}}.$$

Consider now the function

$$\varphi(m) = c(1 + c_1 m^{\lambda-1} \exp[c_2 m^{\lambda-1}]) - m. \tag{4.19}$$

If δ is sufficiently small, φ has a positive zero m_1. If, moreover, $\delta < m_1$, then $M(0) < m_1$ and consequently, $M(t) \leq m_1$ for all t. This completes the proof of Proposition 4.2. □

THEOREM 4.1. *If* $\alpha \geq \frac{1}{2}$ *and* $\lambda > \alpha + \frac{3}{2} + (\alpha^2 + 3\alpha + \frac{5}{4})^{1/2}$, *there exists a constant* $\delta > 0$ *such that if the initial condition* u_0 *belongs to*

$$u_0 \in W^{2,p}(R) \cap H^3(R),$$

with

$$p = 2\lambda/(2\lambda - 1) \quad \textit{and} \quad |u_0|_{W^{2,p}} + |u_0|_{H^3} < \delta,$$

there exists a unique solution u of (1.1) *in* $L^\infty(R^+, H^3(R))$ *which satisfies*

$$|u(t)|_{W^{2,2\lambda}} \leq c(1+t)^{-(1-1/\lambda)/(2\alpha+1)}. \tag{4.20}$$

Moreover, the problem is asymptotically free in the sense that there exist solutions $u_\pm$ *to the linear problem* (3.1) *such that*

$$|u(t) - u_\pm(t)|_{H^2} \to 0 \quad \textit{when } t \to \pm\infty.$$

Proof. The existence generally of a solution of Equation (1.1) results from the following estimates, valid for all $t \in [0, T]$

$$|u(t)|_{H^3} \leq C|u_0|_{H^3} \exp[CM(T)] \leq C|u_0|_{H^3} \exp[CM_0] \leq K. \tag{4.21}$$

Asymptotic freedom is proved as in [17–19]. One defines

$$u_\pm(t) = u(t) + \int_t^{\pm\infty} G(t-s)\partial_x\left(\frac{u^\lambda}{\lambda}\right)(s)\,\mathrm{d}s. \tag{4.22}$$

The integral in the left-hand side of (4.22) exists since

$$\left|\int_t^{\pm\infty} G(t-s)\,\partial_x\left(\frac{u^2}{\lambda}\right)(s)\,\mathrm{d}s\right|_{H^2} \leq C\int_t^{\pm\infty} |u(s)|_{W^{2,q}}^{\lambda-1} |u(s)|_{H^3}\,\mathrm{d}s$$
$$\leq CKM_0^{\lambda-1}\int_t^{\pm\infty} (1+s)^{-((\lambda-1)^2)/(\lambda(2\alpha+1))}\,\mathrm{d}s.$$

$u_\pm(t)$ are thus well defined and $|u(t) - u_\pm(t)|_{H^2} \to 0$ when $t \to \pm\infty$. One easily checks that $u_\pm$ satisfy the linear equation. □

REMARK. Theorem 4.1 is easily extended to a more general class of equations where the nonlinear term is a polynomial of degree λ in u and $\partial_x u$ that does not ensure conservation of the L^2-norm of the solution [18, 19].

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