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Extrapolation methods for divergent oscillatory infinite integrals that are defined in the sense of summability

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Abstract: In a recent work by the author an extrapolation method, the W-transformation, was developed, by which a large class of oscillatory infinite integrals can be computed very efficiently. The results of this work are extended to a class of divergent oscillatory infinite integrals in the present paper. It is shown in particular that these divergent integrals exist in the sense of Abel summability and that the W-transformation can be applied to them without any modifications. Convergence results are stated and numerical examples given.

1. Introduction

In some problems in physics and engineering one encounters oscillatory infinite integrals that do not exist in the ordinary sense due to their integrands' not being integrable at infinity. These integrals may, however, exist in the summability sense and represent physical quantities, and one would like to know their numerical values. One such integral was given in [3], and it is

$$\int_0^\infty xc^{-1}y^{-1}\exp(-y\sin\alpha)[2(c^2+x^2)J_0(x\cos\alpha)-2x\sec\alpha J_1(x\cos\alpha)]\,\mathrm{d}x\ ,$$

where

$$y=(x^2+\tfrac{1}{4}R^2)^{1/2}\ ,\quad c=Ry+\tfrac{1}{2}R^2$$

and R is a positive constant, with $\alpha=0$. This integral arises in fluid mechanics in the study of particle interaction in a slow viscous flow. The angle α relates the particle position to the flow direction and R is the Reynolds number. See the references in [3].

When $\alpha=0$ the integrand in the integral above belongs to the set B_d, a general family of oscillatory functions, which we define precisely in Section 2. The set B_d is closely related (and complementary) to another set of functions that we shall term B_c that was considered in a recent work [7]. The difference between the two sets is that functions in B_c are integrable at infinity, whereas those in B_d are not. In fact if $f(x)$ is in B_d, then $f(x)/x^p$, for some positive integer p, is in B_c. (For $\alpha \neq 0$, the integrand above is in B_c). In [7] we developed an extrapolation procedure, the W-transformation, for numerically evaluating the infinite integrals $\int_a^\infty f(t)\,dt$, $a \geq 0$, $f \in B_c$. In the present work we consider the problem of numerically evaluating the divergent integrals $\int_a^\infty f(t)\,dt$, $a \geq 0$, $f \in B_d$, that are defined in the sense of Abel summability. We show that the W-transformation of [7] can be used in evaluating these integrals efficiently.

To give an idea about the kind of integrands we shall consider, we end this section with a description of the subset $\tilde{B}_d$ of the set B_d. We start this with the following definition of the set $A^{(\gamma)}$.

Definition 1.1. We say that a function $\alpha(x)$, defined for $x > a \geq 0$, belongs to $A^{(\gamma)}$, if it is infinitely differentiable for all $x > a$, and if, as $x \to \infty$, it has a Poincaré-type asymptotic expansion of the form

$$\alpha(x) \sim x^\gamma \sum_{i=0}^{\infty} \alpha_i / x^i\,, \tag{1.1}$$

and all its derivatives, as $x \to \infty$, have Poincaré-type asymptotic expansions, which are obtained by differentiating the right hand side of (1.1) term by term.

As consequences of Definition 1.1, we have

(1) $A^{(\gamma)} \supset A^{(\gamma-1)} \supset A^{(\gamma-2)} \supset \cdots$.

(2) If $\alpha \in A^{(\gamma)}$ and $\beta \in A^{(\delta)}$, then $\alpha\beta \in A^{(\gamma+\delta)}$, and if, in addition $\beta \notin A^{(\delta-1)}$, then $\alpha/\beta \in A^{(\gamma-\delta)}$.

(3) If $\alpha \in A^{(0)}$, then α is infinitely differentiable for all $x > a$ up to and including $x = \infty$, although not necessarily analytic at $x = \infty$.

Definition 1.2. The subset $\tilde{B}_d$ of B_d is the collection of all functions $f(x)$ that are defined for $x > a \geq 0$ and are expressible in the form

$$f(x) = \exp(\mathrm{i}\vartheta(x))h(x)\,, \tag{1.2}$$

where

(1) $\vartheta(x)$ is a real function in $A^{(m)}$, m being a positive integer,

(2) $h(x)$ is a (complex) function in $A^{(\gamma)}$ for some $\gamma \geq m-1$.

We require $\gamma \geq m-1$ so that $f(x)$ is not integrable at $x = \infty$. (For $\gamma < m-1$, $f(x)$ is integrable at infinity and belongs to B_c.)

Example. $f(x) = \exp[\mathrm{i}(x^3 + \sqrt{x^4 + 2x^3 + x})](x + 2/\sqrt{x^2+1})^{7/3}$. Here $m = 3$ and $\gamma = \frac{7}{3}$.

Let us also define

$$F(x) = \int_a^x f(t)\,dt \tag{1.3}$$

and

$$I[f]=\int_a^\infty f(t)\,\mathrm{d}t\,. \tag{1.4}$$

In the next section we show that $I[f]$ exists in the Abel summability sense, and we derive an asymptotic expansion for its 'tail' $\int_x^\infty f(t)\,\mathrm{d}t$ as $x\to\infty$. This asymptotic expansion is valid also for all functions $f(x)$ in the set B_d, which we define precisely in the next section. Based on this asymptotic expansion, we devise an extrapolation method for computing $I[f]$, and this method turns out to be the W-transformation of [7]. We recall that the W-transformation produces approximations to $I[f]$ by using a small number of the finite integrals $F(x_l)$, $l=0,1,\ldots$, for some carefully selected values of x_l. In Section 3 we give some numerical examples to illustrate the use of the extrapolation procedure. A convergence result is also given.

2. Theory

Let $f(x)$ be in $\tilde{B}_d$. Then $f(x)$ is expressible as

$$f(x)=\exp(\mathrm{i}\vartheta(x))h(x)\,, \tag{2.1}$$

where $\vartheta(x)$ is real and belongs to $A^{(m)}$ for some positive integer m, and $h\in A^{(\gamma)}$ for some $\gamma\geqslant m-1$. Define

$$f_\epsilon(x)=\mathrm{e}^{-\epsilon x}f(x)\,,\quad \epsilon>0\,, \tag{2.2}$$

and let

$$F_\epsilon(x)=\int_a^x f_\epsilon(t)\,\mathrm{d}t\,, \tag{2.3}$$

and consider

$$I[f_\epsilon]=\int_a^\infty f_\epsilon(t)\,\mathrm{d}t\,, \tag{2.4}$$

which exists in the ordinary sense.

It can be shown that $f_\epsilon(x)$ satisfies the homogeneous linear first order differential equation

$$f_\epsilon(x)=s(x)f_\epsilon'(x)\,, \tag{2.5}$$

with

$$1/s(x)=\mathrm{i}\vartheta'(x)-\epsilon+h'(x)/h(x)\,. \tag{2.6}$$

Since $h'/h\in A^{(-1)}$, $\vartheta'\in A^{(m-1)}$, $m\geqslant 1$, and $\vartheta(x)$ and ϵ are real, we see that $1/s\in A^{(m-1)}$ and $1/s\not\in A^{(m-2)}$, for *all* ϵ including $\epsilon=0$. Thus $s\in A^{(-m+1)}$ for *all* $\epsilon\geqslant 0$.

Substituting (2.5) in the integral

$$I[f_\epsilon]-F_\epsilon(x)=\int_x^\infty f_\epsilon(t)\,\mathrm{d}t\,, \tag{2.7}$$

and integrating by parts once, we obtain

$$\int_x^\infty f_\epsilon(t)\,\mathrm{d}t = -s(x)f_\epsilon(x) - \int_x^\infty s'(t)f_\epsilon(t)\,\mathrm{d}t\,. \tag{2.8}$$

Next defining

$$s_1(x) = s(x)\,, \qquad s_k(x) = s(x)s'_{k-1}(x)\,, \quad k = 2, 3, \ldots, \tag{2.9}$$

substituting (2.5) in the integral on the right hand side of (2.8), and integrating by parts $N-1$ times, we obtain

$$\int_x^\infty f_\epsilon(t)\,\mathrm{d}t = \left[\sum_{k=1}^{N} (-1)^k s_k(x)\right] f_\epsilon(x) + (-1)^N \int_x^\infty s'_N(t)f_\epsilon(t)\,\mathrm{d}t\,. \tag{2.10}$$

Now $s_1 = s \in A^{(-m+1)}$ implies that $s_2 = ss' \in A^{(-2m+1)}$, and, in general, $s_k \in A^{(-km+1)}$ for all $\epsilon \geqslant 0$. The integrand $s'_N(t)f_\epsilon(t)$ of the integral on the right hand side of (2.10) is of the form $\exp(-\epsilon t + \mathrm{i}\vartheta(t))\hat{h}(t)$, where $\hat{h}(t) = h(t)s'_N(t)$ and $\hat{h} \in A^{(\gamma - Nm)}$ for all $\epsilon \geqslant 0$. Therefore, the integral $\int_x^\infty s'_N(t)f_\epsilon(t)\,\mathrm{d}t$ exists in the ordinary sense when $\epsilon = 0$ provided $\gamma - Nm < m - 1$, or $N > -1 + (\gamma + 1)/m \equiv \nu$. In addition, this integral is absolutely convergent when $\epsilon = 0$ provided $\gamma - Nm < -1$ or $N > \nu+1$. Also for $0 \leqslant \epsilon \leqslant \epsilon_0$, from some fixed ϵ_0, and x sufficiently large, we can show that there exists a function $M(t)$ that is independent of ϵ, and is in $A^{(\gamma - Nm)}$, such that $|\hat{h}(t)| \leqslant M(t)$ for all $t \geqslant x$. Let $N > \nu + 1$. Then all the conditions of Theorem 25.14 in [2, p. 352] are satisfied, and consequently

$$\lim_{\epsilon \to 0+} \int_x^\infty s'_N(t)f_\epsilon(t)\,\mathrm{d}t = \int_x^\infty [\lim_{\epsilon \to 0+} s'_N(t)]f(t)\,\mathrm{d}t\,. \tag{2.11}$$

Combining (2.7), (2.10), and (2.11), we finally have

$$\lim_{\epsilon \to 0+} I[f_\epsilon] = F(x) + \left[\sum_{k=1}^{N} (-1)^k \lim_{\epsilon \to 0+} s_k(x)\right] f(x) + (-1)^N \int_x^\infty [\lim_{\epsilon \to 0+} s'_N(t)]f(t)\,\mathrm{d}t\,, \tag{2.12}$$

thus proving that $I[f]$ is defined in the Abel summability sense.

Let us denote $\lim_{\epsilon \to 0+} s_k(x) = q_k(x)$. As mentioned in the previous paragraph $q_k \in A^{(-km+1)}$. We now reexpress (2.12) in the form

$$\lim_{\epsilon \to 0+} I[f_\epsilon] = F(x) + \left[\sum_{k=1}^{N} (-1)^k q_k(x)\right] f(x) + (-1)^N \int_x^\infty g(t)\,\mathrm{d}t\,, \tag{2.13}$$

where $g \in B_c$ and is of the form

$$g(x) = \exp(\mathrm{i}\vartheta(x))r(x)\,, \quad r \in A^{(\gamma - Nm)}\,. \tag{2.14}$$

From Theorem 2.2 in [7]

$$\begin{aligned}\int_x^\infty g(t)\,\mathrm{d}t &= g(x)\lambda(x)\,, \qquad \lambda \in A^{(-m+1)}\,,\\ &= f(x)\mu_N(x)\,, \qquad \mu_N \in A^{(-Nm-m+1)}\,.\end{aligned} \tag{2.15}$$

Thus, (2.13) becomes

$$\lim_{\epsilon\to 0+} I[f_\epsilon] = F(x) + \left[\sum_{k=1}^{N} (-1)^k q_k(x) + (-1)^N \mu_N(x)\right] f(x) . \tag{2.16}$$

It is easy to see that the expression inside the square brackets is in $A^{(-m+1)}$. With this observation we now state the main result of this section.

Theorem 2.1. *Let $f \in \tilde{B}_d$ be expressible as in the first paragraph of this section. Then $\int_a^\infty f(t)\,\mathrm{d}t$, $a \geqslant 0$, is defined in the sense of Abel summability, and for $x > a$, there exists a function $\beta(x)$ in $A^{(0)}$ such that*

$$\lim_{\epsilon\to 0+} I[f_\epsilon] = F(x) + x^{-m+1}\beta(x)f(x) . \tag{2.17}$$

Remark 1. Surprisingly, the result in (2.17) is identical to the one that was obtained for $f \in B_c$ with $\gamma < m-1$, see Theorem 2.2 in [7]. Thus for *all* γ we have

$$I[f] = F(x) + x^{-m+1}\beta(x)f(x) , \quad \beta \in A^{(0)} , \tag{2.18}$$

where $I[f]$ is to be interpreted as the value of $\int_a^\infty f(t)\,\mathrm{d}t$ in the sense of Abel summability whenever $f \in \tilde{B}_d$.

From this point on we follow closely the treatment given in [7] $\vartheta \in A^{(m)}$ implies

$$\vartheta(x) \sim x^m \sum_{i=0}^{\infty} \vartheta_i / x^i \quad \text{as } x \to \infty . \tag{2.19}$$

We can then express $\vartheta(x)$ in the form $\vartheta(x) = \bar{\vartheta}(x) + \Delta(x)$, where

$$\bar{\vartheta}(x) = \sum_{i=0}^{m-1} \vartheta_i x^{m-i} , \quad \Delta(x) \sim \sum_{i=0}^{\infty} \vartheta_{m+i}/x^i \quad \text{as } x \to \infty . \tag{2.20}$$

Notice that $\bar{\vartheta}(x)$ is a polynomial of degree m, and $\mathrm{e}^{\mathrm{i}\Delta(x)}$ is in $A^{(0)}$ since $\Delta \in A^{(0)}$. Thus (2.18) can be re-expressed in the form

$$I[f] = F(x) + x^{\gamma-m+1}\,\mathrm{e}^{\mathrm{i}\bar{\vartheta}(x)}\beta^*(x) , \quad \beta^* \in A^{(0)} , \tag{2.21}$$

where $\beta^*(x) = x^{-\gamma}h(x)\beta(x)\,\mathrm{e}^{\mathrm{i}\Delta(x)}$.

We now define the set B_d.

Definition 2.1. B_d is the set of all functions $f(x)$ that can be expressed in the form

$$f(x) = \sum_{j=1}^{r} f_j(x) , \tag{2.22}$$

where each $f_j(x)$ is of the form

$$f_j(x) = u_j(\vartheta_j(x))h_j(x) , \tag{2.23}$$

such that

(1) $u_j(z)$ is either $\mathrm{e}^{\mathrm{i}z}$ or $\mathrm{e}^{-\mathrm{i}z}$ or any linear combination of these (like $\cos z$ or $\sin z$);

(2) $\vartheta_j \in A^{(m)}$ for all j, and $\bar{\vartheta}_j(x) \equiv \bar{\vartheta}_{j'}(x) \equiv \bar{\vartheta}(x)$ for $j \neq j'$;
(3) $h_j \in A^{(\gamma_j)}$ such that $\gamma_j - \gamma_{j'} =$ integer for $j \neq j'$ (hence $h_j \in A^{(\gamma)}$ for all j, where $\gamma = \max\{\gamma_1, \ldots, \gamma_r\}$), and $\gamma \geq m-1$. (If $\gamma < m-1$, then $f \in B_c$.)

Comparing with B_c (see Lemma 2.1 in [7]), we see that $B_c \cup B_d$ is simply B_c with no restrictions on the γ_j.

The result in (2.21) can now be extended to functions in B_d.

Theorem 2.2. *Let $f \in B_d$ with the notation of Definition* 2.1. *Then $I[f]$ is defined in the sense of Abel summability, and*

$$I[f] = F(x) + x^{\gamma-m+1}[\cos(\bar{\vartheta}(x))b_1(x) + \sin(\bar{\vartheta}(x))b_2(x)]\,, \tag{2.24}$$

where $b_1, b_2 \in A^{(0)}$.

Remark 2. Theorem 2.2 is an extension of Lemma 2.1 in [7] in the sense that Theorem 2.1 is an extension of Theorem 2.2 in [7]. Thus, (2.24) is valid for *all* γ_j provided $I[f]$ is interpreted as $\int_a^\infty f(t)\,dt$ in the sense of Abel summability. The W-transformation of [7] is based solely on (2.24), thus it can be used for all functions $f(x)$, whether in B_c or in B_d.

For the sake of completeness, we shall briefly recall the main points of the W-transformation.

Let x_0 be the smallest zero of $\sin(\bar{\vartheta}(x))$ greater than a so that x_0 is a root of the equation $\bar{\vartheta}(x) = q\pi$ for some integer q. Then determine $x_0 < x_1 < x_2 < \cdots$, where x_l is root of $\bar{\vartheta}(x) = (q+l)\pi$. (In a similar manner, we can reverse the roles of sin and cos, starting with x_0 as the root of $\cos(\bar{\vartheta}(x)) = 0$ or $\bar{\vartheta}(x) = (q+\frac{1}{2})\pi$.) Set

$$\psi(x_l) = (-1)^l x_l^{\gamma-m+1}\,, \quad l = 0, 1, \ldots, \tag{2.25}$$

and solve the system of linear equations

$$W_n^{(j)} = F(x_l) + \psi(x_l)\sum_{i=0}^{n} \bar{\beta}_i / x_l^i\,, \quad j \leq l \leq j+n+1\,, \tag{2.26}$$

for $W_n^{(j)}$, the approximation to $I[f]$. The $W_n^{(j)}$ can be computed very efficiently in a recursive manner by the W-algorithm of [8], which is summaried below: Let

$$M_{-1}^{(s)} = F(x_s)/\psi(x_s)\,, \quad N_{-1}^{(s)} = 1/\psi(x_s)\,, \qquad s = 0, 1, \ldots. \tag{2.27}$$

Compute for $s = 0, 1, \ldots$, and $k = 0, 1, \ldots$,

$$\begin{aligned} M_k^{(s)} &= (M_{k-1}^{(s)} - M_{k-1}^{(s+1)})/(x_s^{-1} - x_{s+k+1}^{-1})\,,\\ N_k^{(s)} &= (N_{k-1}^{(s)} - N_{k-1}^{(s+1)})/(x_s^{-1} - x_{s+k+1}^{-1})\,,\\ W_k^{(s)} &= M_k^{(s)}/N_k^{(s)}\,. \end{aligned} \tag{2.28}$$

We finally state convergence results on the $W_n^{(j)}$ for two types of limiting processes that have been designated Process I and Process II in [6] and [7]. In Process I n is fixed and $j \to \infty$, while in Process II j is fixed and $n \to \infty$.

Theorem 2.3. *For Process* I

$$I[f] - W_n^{(j)} = O(x_j^{\gamma-m-n}) = O(j^{(\gamma-m-n)/m}) \quad \text{as } j\to\infty, \tag{2.29}$$

while for Process II

$$I[f] - W_n^{(j)} = O(n^{-\mu}) \quad \text{as } n\to\infty, \quad \text{any } \mu>0. \tag{2.30}$$

The proof of this theorem is very similar to those of Theorems 4 and 5 and their corollary in [6], the only additional factors being that for $f\in B_d$

$$\max_{j\le l\le j+n+1} |\psi(x_l)| = x_{j+n+1}^{\gamma-m+1} = \begin{cases} O(x_j^{\gamma-m+1}) = O(j^{(\gamma-m+1)/m}) & \text{as } j\to\infty, \\ O(x_n^{\gamma-m+1}) = O(n^{(\gamma-m+1)/m}) & \text{as } n\to\infty. \end{cases}$$

Formula (2.29) implies that Process I converges provided $n>\gamma-m$, while (2.30) implies that Process II always converges and much more quickly than Process I.

3. Numerical examples

In this section we apply the W-transformation to several integrals whose integrands are in B_d. The transformation is implemented using the W-algorithm. Only the approximations $W_n^{(0)}$ are tabulated. The computations for these examples were done in double precision arithmetic on the IMB-370 computers at Technion, Haifa and NASA Lewis Research Center, Cleveland, Ohio.

Example 3.1.

$$I = \int_0^\infty \exp[\mathrm{i}\vartheta(x)]\vartheta(x)\vartheta'(x)\,\mathrm{d}x = \exp[\mathrm{i}\vartheta(0)][-1+\mathrm{i}\vartheta(0)].$$

For $\vartheta\in A^{(m)}$, $m>0$ an integer, the integrand $f(x)=\exp[\mathrm{i}\vartheta(x)]\vartheta(x)\vartheta'(x)$ is in $\tilde{B}_d$. Numerical results were obtained for the choice $\vartheta(x)=x^2-2+2\sqrt{x^2+x+1}$ so that $\vartheta(0)=0$ and $I=-1$. For this choice of $\vartheta(x)$ we have $m=2$, $\bar{\vartheta}(x)=x^2+2x$, and $\gamma=3$. The x_l are taken to be consecutive zeros of $\sin(\bar{\vartheta}(x))$. Hence $x_l=-1+\sqrt{1+(l+1)\pi}$ and $\psi(x_l)=(-1)^l x_l^2$, $l=0,1,2,\ldots$. The W-transformation with these x_l's and $\psi(x_l)$'s is applied to the real and imaginary parts of this integral, namely to the integrals $I_1=\int_0^\infty f_1(x)\,\mathrm{d}x$ and $I_2=\int_0^\infty f_2(x)\,\mathrm{d}x$ respectively, where $f_1(x)=\cos(\vartheta(x))\vartheta(x)\vartheta'(x)$ and $f_2(x)=\sin(\vartheta(x))\vartheta(x)\vartheta'(x)$. Note that application of the W-algorithm to the original (complex) integral $I=\int_0^\infty f(x)\,\mathrm{d}x$ produces exactly the same approximations for I_1 and I_2. That is to say, if we denote the approximations to the integrals I, I_1, and I_2 obtained by using the W-transformation by $W_n^{(j)}[f]$, $W_n^{(j)}[f_1]$, and $W_n^{(j)}[f_2]$ respectively, then $W_n^{(j)}[f]=W_n^{(j)}[f_1]+\mathrm{i}W_n^{(j)}[f_2]$. The numerical results for I_1 and I_2 are given in Table 1.

Before giving Examples 3.2 and 3.3 we would like to recall that for any ν and for $\vartheta\in A^{(m)}$, $m>0$ an integer, the Bessel functions $J_\nu(\vartheta(x))$ and $Y_\nu(\vartheta(x))$ are expressible in the form $\eta_1(x)\cos(\bar{\vartheta}(x))+\eta_2(x)\sin(\bar{\vartheta}(x))$, where $\eta_1,\eta_2\in A^{(-m/2)}$ and $\bar{\vartheta}(x)$ is as defined in (2.20). (See the example following Lemma 2.1 in [7].)

Table 1
$W_n^{(0)}$ for the real and imaginary parts $I_1 = -1$ and $I_2 = 0$ in Example 3.1

n	$W_n^{(0)}[f_1]$	$W_n^{(0)}[f_2]$
0	−0.1374706860366143 D01	0.6388917964269695 D00
1	−0.1094314526324539 D01	0.1413054892128269 D-02
2	−0.9968903998998289 D00	−0.1720446087780822 D-02
3	−0.1000203035968197 D01	0.2569127360275714 D-03
4	−0.9999952144834945 D00	−0.1952371029084969 D-04
5	−0.9999997183890984 D00	0.8446174719796831 D-06
6	−0.1000000030821927 D01	−0.1454733791697413 D-07
7	−0.9999999987003277 D00	−0.6274491325060319 D-09
8	−0.1000000000020748 D01	0.5386569040970118 D-10
9	−0.1000000000000778 D01	−0.1691551103268190 D-11
10	−0.1000000000000004 D01	0.5829608211821276 D-13
11	−0.1000000000000069 D01	0.4664849221110266 D-13
12	−0.1000000000000071 D01	0.4023335702912757 D-13

Example 3.2.

$$I_p = \int_0^\infty x^{2p} J_0(x)\,\mathrm{d}x\ .$$

By what has been said in the previous paragraph, for $p \geq \frac{1}{4}$, $f(x) = x^{2p}J_0(x)$ is in $\bar{B}_d$ with $m = 1$, $\vartheta(x) = x$ and $\gamma = 2p - \frac{1}{2}$. The x_l are chosen to be consecutive zeros of $\sin x$. Thus, $x_l = (l+1)\pi$ and $\psi(x_l) = (-1)^l x_l^{2p-1/2}$, $l = 0, 1, \ldots$. I_p was computed using the W-transformation for $p = 1$ and $p = 2$, for which, $I_1 = -1$ and $I_2 = 9$. The results of the computations for I_1 and I_2 are given in Table 2.

Table 2
$W_n^{(0)}$ for the integrals I_1 and I_2 in Example 3.2

n	$W_n^{(0)}$ for I_1	$W_n^{(0)}$ for I_2
0	−0.1653236227584530 D01	−0.1260894930754135 D02
1	−0.1029587932399560 D01	0.9420238026602777 D01
2	−0.9999473138596609 D00	0.1057006408650254 D02
3	−0.9999657260248673 D00	0.9046401056465052 D01
4	−0.1000002112607400 D01	0.8999889833220464 D01
5	−0.9999999817655246 D00	0.8999976953565624 D01
6	−0.9999999943779554 D00	0.9000001410221530 D01
7	−0.1000000000379949 D01	0.8999999969624580 D01
8	−0.9999999999961305 D00	0.8999999997889087 D01
9	−0.9999999999991695 D00	0.9000000000180987 D01
10	−0.1000000000000193 D01	0.8999999999963019 D01
11	−0.1000000000000113 D01	0.8999999999981573 D01
12	−0.1000000000000101 D01	0.9000000000001093 D01

Table 3
$W_n^{(0)}$ for $R = 0.1$, 1, 10 for the integral in Example 3.3

n	$W_n^{(0)}$ for $R = 0.1$	$W_n^{(0)}$ for $R = 1$	$W_n^{(0)}$ for $R = 10$
0	−0.2236867216169354 D02	−0.1477911294169409 D01	0.2607119122278270 D01
1	−0.1992978697914259 D02	−0.1205852663700003 D01	−0.2679131159825249 D00
2	−0.1996627934186043 D02	−0.1213270089319083 D01	0.7335054996255592 D00
3	−0.1996634143370108 D02	−0.1213056953892470 D01	0.6760342012727790 D00
4	−0.1996631270004846 D02	−0.1213061138406477 D01	0.6642368870807564 D00
5	−0.1996631330075435 D02	−0.1213061347309263 D01	0.6656988431306132 D00
6	−0.1996631331740060 D02	−0.1213061318078334 D01	0.6657192260583116 D00
7	−0.1996631331628912 D02	−0.1213061319419371 D01	0.6657087863481486 D00
8	−0.1996631331629833 D02	−0.1213061319430080 D01	0.6657091362908930 D00
9	−0.1996631331629941 D02	−0.1213061319424945 D01	0.6657091660364351 D00
10	−0.1996631331629937 D02	−0.1213061319425223 D01	0.6657091634853331 D00
11	−0.1996631331629936 D02	−0.1213061319425223 D01	0.6657091635030230 D00
12	−0.1996631331629936 D02	−0.1213061319425221 D01	0.6657091635098611 D00
13	−0.1996631331629935 D02	−0.1213061319425220 D01	0.6657091635094694 D00
14	−0.1996631331629934 D02	−0.1213061319425219 D01	0.6657091635094813 D00
15	−0.1996631331629934 D02	−0.1213061319425220 D01	0.6657091635094880 D00

Example 3.3. The integral in the first paragraph of Section 1 with $\alpha = 0$, namely

$$I = \int_0^\infty \frac{2x}{cy} \left[(c^2 + x^2)J_0(x) - xJ_1(x)\right] \mathrm{d}x .$$

Here, $c, y \in A^{(1)}$, and, by what has been said prior to Example 3.2, $(c^2 + x^2)J_0(x) - xJ_1(x)$ is of the form $\omega_1(x)\cos x + \omega_2(x)\sin x$ with $\omega_1, \omega_2 \in A^{(3/2)}$ so that $m = 1$ and $\bar{\vartheta}(x) = x$. As in Example 3.2, $x_l = (l+1)\pi$, thus $\psi(x_l) = (-1)^l x_l^{1/2}$, $l = 0, 1, \ldots$. Table 3 contains the numerical results obtained for I with $R = 0.1$, 1, and 10. For the sake of completeness we mention that, for $\alpha \neq 0$, the integrand of this integral is in B_c with $\bar{\vartheta}(x) = x\cos\alpha$. The W-transformation can be applied to it with $x_l = (l+1)\pi\sec\alpha$ and $\psi(x_l) = (-1)^l x_l^{1/2}\exp(-x_l \sin\alpha)$, $l = 0, 1, \ldots$, see [7].

4. Concluding remarks

In this work we have shown that divergent infinite oscillatory integrals of functions in the set B_d can be computed very efficiently by using the W-transformation. The W-transformation was originally designed to accelerate the convergence of a class of convergent infinite oscillatory integrals of functions in the set B_c. The sets B_c and B_d are complementary in the sense that if a function $f(x)$ is in B_d, then $f(x)/x^p$, for some positive integer p, is in B_c.

The subject of computation of divergent integrals does not seem to have received much attention with the exception of a few recent works like [1] and [3]. In both of these works, based on numerical testing, it is concluded that if the oscillatory integral $\int_a^\infty f(x)\,\mathrm{d}x$ is expressed as an infinite series $\Sigma_{j=0}^\infty u_j$, where $u_j = \int_{x_j}^{x_{j+1}} f(x)\,\mathrm{d}x$, and $x_0 = a$ and x_j, $j = 1, 2, \ldots$, are the consecutive zeros of $f(x)$ greater than a, then application of convergence acceleration methods to $\Sigma_{j=0}^{\infty} u_j$ may produce good results for $\int_a^\infty f(x)\,dx$ even when this is a divergent integral defined in the Abel summability sense. In fact [1] demonstrates the use of the Euler and iterated Shanks [5] transformations like e_1, e_1^2, e_1^3, etc., and [3] demonstrates the use of the higher order Shanks transformations (or the ϵ-algorithm [9]) and the Levin [4] transformations. All the integrals, convergent or divergent, dealt with in both [1] and [3] have integrands in B_c or B_d.

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