# On Extensions of the Power Method for Normal Operators

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ABSTRACT

In a recent work by Sidi and Bridger some old and some new extensions of the power method have been considered, and some of these extensions have been shown to produce estimates of several dominant eigenvalues of an arbitrary square matrix. In the present work we continue the analysis of two versions of one of these extensions, called the MPE extension, as they are applied to normal matrices. We show that the convergence rate of these methods for normal matrices is twice that for nonnormal matrices. We also give precise asymptotic bounds on the errors of the estimates obtained for the eigenvalues. Further deflation-type extensions of the power method for normal matrices are suggested and analyzed for their convergence. All the results are stated and proved in the general setting of inner-product spaces.

## 1. INTRODUCTION

In a recent work by Sidi and Bridger [1] some old and some new extensions of the power method have been considered. It has been shown for some of these extensions that they enable one to estimate several dominant eigenvalues of an arbitrary square matrix. In the present work we continue this analysis for one of these extensions, namely, the *minimal polynomial extrapolation* (MPE) extension, as it is employed in estimating the dominant eigenvalues of a *normal* matrix. (MPE is a method used in accelerating the convergence of vector sequences. For information and references pertaining to MPE and other similar methods, see [1] and the references therein.)

To proceed, we give a brief description of two versions of the MPE extension of the power method in the notation and general setting of [1].




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Let $\mathbf{B}$ be an inner-product space over the field of complex numbers, and let $(x, y)$ be the inner product associated with $\mathbf{B}$. The homogeneity property of the inner product is such that for $\alpha$ and $\beta$ complex numbers, and $x$ and $y$ vectors in $\mathbf{B}$, $(\alpha x, \beta y) = \bar{\alpha}\beta(x, y)$. Let also $\|x\| = \sqrt{(x, x)}$ be the norm associated with $\mathbf{B}$.

Let $x_0, x_1, x_2, \ldots,$ be a sequence of vectors in $\mathbf{B}$, and assume that $x_m$ has an asymptotic expansion of the form

$$x_m \sim \sum_{j=1}^{\infty} \left[ \sum_{i=0}^{p_j} y_{ji} \binom{m}{i} \right] \lambda_j^m \quad \text{as} \quad m \to \infty, \tag{1.1}$$

where

$$\lambda_i \neq 0, \quad i = 1, 2, \ldots, \quad \text{and} \quad \lambda_i \neq \lambda_j \quad \text{if} \quad i \neq j, \tag{1.2}$$

and

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots, \tag{1.3}$$

and the vectors $y_{ji}$, $0 \leqslant i \leqslant p_j$, $j = 1, 2, \ldots,$ are linearly independent. Here $\binom{m}{i}$ are the binomial coefficients. We further assume that there can be only a finite number of $\lambda_j$'s having the same modulus. We agree to order the $\lambda_i$ so that if $|\lambda_j| = |\lambda_{j+1}|$ for some $j$, then $p_j \geqslant p_{j+1}$. The interpretation of (1.1) is that for any positive integer $N$ there exist a positive constant $K$ and a positive integer $m_0$ that depend only on $N$, such that for every $m \geqslant m_0$,

$$\left\| x_m - \sum_{j=1}^{N-1} \left[ \sum_{i=0}^{p_j} y_{ji} \binom{m}{i} \right] \lambda_j^m \right\| \leqslant K |\lambda_N|^m m^{p_N}. \tag{1.4}$$

Special cases of vector sequences $x_m$, $m = 0, 1, 2, \ldots,$ of the form described above arise naturally from the iterative process $x_{j+1} = A x_j$, $j = 0, 1, \ldots,$ where $A$ is an $M \times M$ complex matrix and $x_0$ is an arbitrary $M$-dimensional complex column vector. For this case the $\lambda_j$ are nonzero eigenvalues of the matrix $A$, and the vectors $y_{ji}$, $0 \leqslant i \leqslant p_j$, are some linearly independent combinations of the eigenvectors and principal vectors corresponding to $\lambda_j$. For these special cases (1.1) takes on a simpler form, in the sense that the infinite sum on the right-hand side of (1.1) is replaced by a finite sum and (1.1) reduces from being an asymptotic equivalence to being an equality. For more details on this see [1, Sections 2 and 6].

The extensions of the power method that have been suggested in [1] are all meant to produce approximations to $\lambda_1, \lambda_2, \ldots$, and they are based on knowledge of the vectors $x_i$ *only*. In all the extensions the approximations to the largest $\lambda_j$ are obtained as the zeros of a polynomial

$$P^{(n,k)}(\lambda) = \sum_{i=0}^{k} c_i^{(n,k)} \lambda^i, \tag{1.5}$$

whose coefficients are determined from the $x_i$. All the extensions differ from one another in the way the $c_i^{(n,k)}$ are determined. For the MPE extension $c_i^{(n,k)} \equiv c_i$ are determined from either

$$\sum_{j=0}^{k} (x_{n+i}, x_{n+j}) c_j = 0, \quad 0 \leqslant i \leqslant k-1, \qquad c_k = 1, \tag{1.6a}$$

or

$$\sum_{j=0}^{k} (x_{n+i+1}, x_{n+j}) c_j = 0, \quad 0 \leqslant i \leqslant k-1, \qquad c_k = 1. \tag{1.6b}$$

The extension through (1.6a), which we shall call the MPE1 extension, is a general version of one suggested in Wilkinson [3, pp. 583–584], while the extension through (1.6b), which we shall call the MPE2 extension, is new. Furthermore, the $c_i$ in (1.6a) and (1.6b) are obtained from the solutions to the minimization problems

$$\underset{c_0, \ldots, c_{k-1}}{\text{minimize}} \left\| \sum_{i=0}^{k-1} c_i x_{n+i} + x_{n+k} \right\| \tag{1.7a}$$

and

$$\underset{d_1, \ldots, d_k}{\text{minimize}} \left\| x_n + \sum_{i=1}^{k} d_i x_{n+i} \right\|,$$

$$\text{and set} \quad c_i = d_i / d_k, \quad i = 0, 1, \ldots, k, \quad \text{where} \quad d_0 = 1, \tag{1.7b}$$

respectively.

For $k=1$ the zero of $P^{(n,1)}(\lambda)$, as determined from (1.6a), is given by the expression $(x_n, x_{n+1})/(x_n, x_n)$. When $x_{n+1}=Ax_n$, where $A$ is a linear operator on **B**, this expression is the well-known Rayleigh quotient.

The key result of [1, Theorem 6.1] pertaining to the MPE extensions of the power method is as follows:

THEOREM 1.1. *Let the sequence* $x_0, x_1, x_2, \ldots,$ *be as described by* (1.1)–(1.4) *above. Let the positive integer t be such that*

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_t| > |\lambda_{t+1}|, \tag{1.8}$$

*and let*

$$k = \sum_{j=1}^{t} (p_j + 1). \tag{1.9}$$

*Define*

$$\phi(\lambda) = \prod_{j=1}^{t} (\lambda - \lambda_j)^{p_j+1} = \sum_{i=0}^{k} \delta_i \lambda^i, \tag{1.10}$$

*and let* $P^{(n,k)}(\lambda)$ *in* (1.5) *be as obtained through* (1.6a) (MPE1 *extension*) *or through* (1.6b) (MPE2 *extension*). *Then*

$$c_i^{(n,k)} = \delta_i + O\left( n^{\alpha} \left| \frac{\lambda_{t+1}}{\lambda_t} \right|^n \right) \qquad as \quad n \to \infty, \tag{1.11}$$

*where* $\alpha$ *is some nonnegative integer* (*for* $p_j = 0,\ j = 1, 2, \ldots,\ \alpha = 0$); *thus*

$$\lim_{n \to \infty} P^{(n,k)}(\lambda) = \phi(\lambda). \tag{1.12}$$

The immediate implication of (1.11) is, obviously, that the zeros $\lambda_i(n)$, $i = 1, \ldots, k$, of $P^{(n,k)}(\lambda)$ for $n \to \infty$, tend to $\lambda_1, \ldots, \lambda_t$, with their respective multiplicities.

The result given in (1.11) can be improved considerably for the case in which $p_j = 0$ and the vectors $y_{j0}$, $j = 1, 2, \ldots$, form an orthogonal sequence. In fact, as will be shown in the next section, for this case the rate of convergence of the $c_i^{(n,k)}$ to the corresponding $\delta_i$ is twice as large as that shown by (1.11). This case is of particular interest in that vector sequences with the properties above arise from the iterative process $x_{j+1} = Ax_j$, $j = 0, 1, 2, \dots$, where $A$ is an $M \times M$ complex normal matrix and $x_0$ is an arbitrary $M$-dimensional complex column vector. The inner product suitable for this case is, of course, $(x, y) = x^*y$, where $x$ and $y$ are $M$-dimensional complex column vectors, and $x^*$ denotes the hermitian conjugate of $x$. In the next section we shall also give precise asymptotic rates of convergence of the $\lambda_s(n)$ to the corresponding $\lambda_s$, again for the same case. All this is done in Theorem 2.1, which is one of the main results of this work. Theorem 2.5, which is the second main result, treats the case in which $k$ takes on values other than those considered in Theorem 2.1, extending and modifying the latter considerably.

In Section 3 we treat the general problem discussed in the beginning of the present section, assuming this time that some of the $\lambda_j$'s are known with their corresponding multiplicities $p_j + 1$. We propose for this case deflation-type extensions of the power method that produce approximations to the unknown dominant $\lambda_j$'s.

## 2. MAIN RESULTS

THEOREM 2.1. *Let $p_j = 0$, $j = 1, 2, \dots$, and denote $y_j \equiv y_{j0}$ for short in* (1.1). *Let the positive integer $k$ be such that*

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_k| > |\lambda_{k+1}| \geqslant \cdots . \tag{2.1}$$

*Define*

$$\phi(\lambda) = \prod_{j=1}^{k} (\lambda - \lambda_j) = \sum_{i=0}^{k} \delta_i \lambda^i. \tag{2.2}$$

*If, in addition,*

$$(y_i, y_j) = z_i \delta_{ij} \tag{2.3}$$

*(obviously $z_j > 0$, $j = 1, 2, \dots$), then for the* MPE1 *and* MPE2 *extensions described in Section* 1 *we have*

$$c_i^{(n,k)} = \delta_i + O\left( \left| \frac{\lambda_{k+1}}{\lambda_k} \right|^{2n} \right) \qquad as \quad n \to \infty; \tag{2.4}$$

*thus*

$$\lim_{n\to\infty} P^{(n,k)}(\lambda) = \phi(\lambda). \tag{2.5}$$

*If we let* $\lambda_1(n),\ldots,\lambda_k(n)$ *be the zeros of* $P^{(n,k)}(\lambda)$ *that tend to* $\lambda_1,\ldots,\lambda_k$ *respectively, then*

$$\lambda_s(n) = \lambda_s + O\left(\left|\frac{\lambda_{k+1}}{\lambda_s}\right|^{2n}\right) \quad \text{as } n\to\infty, \qquad 1\leqslant s\leqslant k. \tag{2.6}$$

*The result in* (2.6) *can be refined considerably as follows: Let* $r$ *be that positive integer for which*

$$|\lambda_{k+1}| = \cdots = |\lambda_{k+r}| > |\lambda_{k+r+1}|. \tag{2.7}$$

*Then*

$$\lambda_s(n) \sim \lambda_s + \left[\sum_{i=1}^{r} e_{[s;k+i]}\frac{z_{k+i}}{z_s}\left(\prod_{\substack{j=1\\ j\neq s}}^{k}\left|\frac{\lambda_{k+i}-\lambda_j}{\lambda_s-\lambda_j}\right|^2\right)(\lambda_{k+i}-\lambda_s)\right]\left|\frac{\lambda_{k+1}}{\lambda_s}\right|^{2n}$$

$$\text{as} \quad n\to\infty, \tag{2.8}$$

*where* $e_{[s;k+i]}=1$ *for the* MPE1 *extension or* $e_{[s;k+i]}=\bar{\lambda}_{k+i}/\bar{\lambda}_s$ *for the* MPE2 *extension.*

NOTE.

(1) On comparing (2.4) with (1.11), we see that $c_i^{(n,k)}$ in Theorem 2.1 converge to the corresponding $\delta_i$ at twice the speed of those in Theorem 1.1, in general. This, roughly speaking, suggests that the zeros of the polynomials $P^{(n,k)}(\lambda)$ in Theorem 2.1 should tend to the corresponding zeros of $\phi(\lambda)$ at twice the speed of those in Theorem 1.1, in general. Practically speaking this is indeed so. Precise results for the general case that was treated in [1] and is described in Section 1 of the present work, analogous to those in (2.6) and (2.8), will be given in a future publication.

(2) When $\lambda_j>0$ for $1\leqslant j\leqslant k+1$ and $r=1$ in (2.7) and hence in (2.8)—which is the case, for example, when the matrix $A$ of Section 1 is

hermitian positive semidefinite—(2.8) shows that for each $s$, $1 \leqslant s \leqslant k$, $\lambda_s(n)$ tends to $\lambda_s$ monotonically from below.

To prove this theorem we shall make use of Lemmas 2.2 and 2.3 below, which are of interest in themselves. We shall also use the shorthand notation

$$\sum_j, \qquad \sum_{j_1<j_2<\cdots<j_k}, \quad \text{and} \quad \alpha_n \sim \beta_n,$$

to mean respectively

$$\sum_{j=1}^{\infty}, \qquad \sum_{j_1=1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \cdots \sum_{j_k=j_{k-1}+1}^{\infty}, \quad \text{and} \quad \alpha_n \sim \beta_n \quad \text{as } n \to \infty.$$

LEMMA 2.2. *Let $\sigma_1, \sigma_2, \ldots,$ and $\mu_1, \mu_2, \ldots,$ be two sequences of nonzero complex numbers such that $\sigma_i$ are distinct and*

$$|\mu_1| \geqslant |\mu_2| \geqslant |\mu_3| \geqslant \cdots, \tag{2.9}$$

*and assume that there can be only a finite number of $\mu_j$'s having the same modulus. Let $H_n(\sigma)$, a polynomial in $\sigma$ of degree $k$, be defined by the determinant*

$$H_n(\sigma) = \begin{vmatrix} 1 & \sigma & \cdots & \sigma^k \\ u_{1,0}^{(n)} & u_{1,1}^{(n)} & \cdots & u_{1,k}^{(n)} \\ u_{2,0}^{(n)} & u_{2,1}^{(n)} & \cdots & u_{2,k}^{(n)} \\ \vdots & \vdots & & \vdots \\ u_{k,0}^{(n)} & u_{k,1}^{(n)} & \cdots & u_{k,k}^{(n)} \end{vmatrix}, \tag{2.10}$$

*where $u_{p,q}^{(n)}$ satisfy*

$$u_{p,q}^{(n)} \sim \sum_{j=1}^{\infty} z_{j,p}\sigma_j^q\mu_j^n \qquad as \quad n \to \infty, \tag{2.11}$$

*with $z_{j,p}$ some scalars. Define*

$$Z_{i_1,i_2,\ldots,i_k}=\begin{vmatrix} z_{i_1,1} & z_{i_1,2} & \cdots & z_{i_1,k} \\ z_{i_2,1} & z_{i_2,2} & \cdots & z_{i_2,k} \\ \vdots & \vdots & & \vdots \\ z_{i_k,1} & z_{i_k,2} & \cdots & z_{i_k,k} \end{vmatrix}, \tag{2.12}$$

*with $i_1, i_2, \ldots,$ being positive integers. Then we have*

$$H_n(\sigma) \sim \sum_{j_1<j_2<\cdots<j_k} Z_{j_1,j_2,\ldots,j_k} V(\sigma, \sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_k}) \left( \prod_{p=1}^{k} \mu_{j_p}^n \right), \tag{2.13}$$

*where $V(a_1, a_2, \ldots, a_r)$ is the Vandermonde determinant*

$$V(a_1,a_2,\ldots,a_r)=\begin{vmatrix} 1 & a_1 & \cdots & a_1^{r-1} \\ 1 & a_2 & \cdots & a_2^{r-1} \\ \vdots & \vdots & & \vdots \\ 1 & a_r & \cdots & a_r^{r-1} \end{vmatrix} = \prod_{1\leqslant i<j\leqslant r}(a_j-a_i). \tag{2.14}$$

(*If the summation in* (2.11) *is finite, and* $\sim$ *is replaced by* $=$, *then the multiple summation in* (2.13) *is finite, and* $\sim$ *is replaced by* $=$ *there too.*)

*Proof.* Substituting (2.11) in (2.10), we obtain

$$H_n(\sigma) \sim \begin{vmatrix} 1 & \sigma & \cdots & \sigma^k \\ \sum\limits_{j_1} z_{j_1,1}\sigma_{j_1}^0\mu_{j_1}^n & \sum\limits_{j_1} z_{j_1,1}\sigma_{j_1}^1\mu_{j_1}^n & \cdots & \sum\limits_{j_1} z_{j_1,1}\sigma_{j_1}^k\mu_{j_1}^n \\ \sum\limits_{j_2} z_{j_2,2}\sigma_{j_2}^0\mu_{j_2}^n & \sum\limits_{j_2} z_{j_2,2}\sigma_{j_2}^1\mu_{j_2}^n & \cdots & \sum\limits_{j_2} z_{j_2,2}\sigma_{j_2}^k\mu_{j_2}^n \\ \vdots & \vdots & & \vdots \\ \sum\limits_{j_k} z_{j_k,k}\sigma_{j_k}^0\mu_{j_k}^n & \sum\limits_{j_k} z_{j_k,k}\sigma_{j_k}^1\mu_{j_k}^n & \cdots & \sum\limits_{j_k} z_{j_k,k}\sigma_{j_k}^k\mu_{j_k}^n \end{vmatrix}. \tag{2.15}$$

Using the multilinearity property of determinants, and removing common factors from each row, we can express (2.15) in the form

$$H_n(\sigma) \sim \sum_{j_1}\sum_{j_2}\cdots\sum_{j_k}\left(\prod_{p=1}^{k} z_{j_p,p}\right)\left(\prod_{p=1}^{k}\mu_{j_p}^n\right)V(\sigma,\sigma_{j_1},\sigma_{j_2},\dots,\sigma_{j_k}). \quad (2.16)$$

Since $(\prod_{p=1}^{k}\mu_{j_p}^n)V(\sigma,\sigma_{j_1},\sigma_{j_2},\dots,\sigma_{j_k})$ is odd under an interchange of any two of the indices $j_1,\dots,j_k$, the lemma in [2, Appendix] (see also [1, Lemma 5.3]) applies, and (2.13) follows. ■

LEMMA 2.3. *If in Lemma 2.2 we also assume that*

$$|\mu_k| > |\mu_{k+1}| \quad (2.17)$$

*and*

$$Z_{1,2,\dots,k} \neq 0, \quad (2.18)$$

*then for $\sigma \neq \sigma_i$, $1 \leqslant i \leqslant k$, the dominant behavior of $H_n(\sigma)$ is given by*

$$H_n(\sigma) = (-1)^k Z_{1,2,\dots,k}V(\sigma_1,\dots,\sigma_k)\left(\prod_{j=1}^{k}\mu_j^n\right)\left[\prod_{j=1}^{k}(\sigma-\sigma_j) + O\left(\left|\frac{\mu_{k+1}}{\mu_k}\right|^n\right)\right]$$

$$as \quad n \to \infty. \quad (2.19)$$

*This implies that $H_n(\sigma)$ has $k$ zeros that tend to $\sigma_1,\sigma_2,\dots,\sigma_k$ as $n\to\infty$. If we denote these zeros by $\sigma_1(n),\sigma_2(n),\dots,\sigma_k(n)$ respectively, then*

$$\sigma_s(n) = \sigma_s + O\left(\left|\frac{\mu_{k+1}}{\mu_s}\right|^n\right) \quad as \ n\to\infty, \qquad 1\leqslant s\leqslant k. \quad (2.20)$$

*Let $r$ be that positive integer for which*

$$|\mu_{k+1}| = \cdots = |\mu_{k+r}| > |\mu_{k+r+1}|, \quad (2.21)$$

*and for* $1 \leqslant s \leqslant k$ *denote*

$$Z_{1,2,\ldots,s-1,s+1,\ldots,k,k+i} = Z_{[s;k+i]}. \tag{2.22}$$

*Then precisely*

$$\sigma_s(n) \sim \sigma_s + (-1)^{k-s} \sum_{i=1}^{r} \frac{Z_{[s;k+i]}}{Z_{1,2,\ldots,k}} \left( \prod_{\substack{j=1 \\ j \neq s}}^{k} \frac{\sigma_{k+i} - \sigma_j}{\sigma_s - \sigma_j} \right) (\sigma_{k+i} - \sigma_s) \left( \frac{\mu_{k+i}}{\mu_s} \right)^n$$

$$as \quad n \to \infty. \tag{2.23}$$

*Proof.* By (2.17) and (2.18) the dominant term in (2.13) is that for which $j_1 = 1$, $j_2 = 2, \ldots,$ $j_k = k$, this term being of order $|\mu_1 \mu_2 \cdots \mu_k|^n$. The next dominant terms are of order $|\mu_1 \mu_2 \cdots \mu_{k-1} \mu_{k+1}|^n$ by the assumption that there can be only a finite number of $\mu_j$'s with modulus $|\mu_{k+1}|$. By this and by (2.14), (2.19) now follows. For the rest of the proof we proceed as follows: If we let $\sigma = \sigma_s$ in (2.13), then all the terms having $j_p = s$ for any one of $j_p$, $1 \leqslant p \leqslant k$, vanish, since this implies that the corresponding Vandermonde determinant $V(\sigma_s, \sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_k})$ has two identical rows. Consequently, (2.13) becomes

$$H_n(\sigma_s) \sim \sum Z_{j_1, j_2, \ldots, j_k} V(\sigma_s, \sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_k}) \left( \prod_{p=1}^{k} \mu_{j_p}^n \right). \tag{2.24}$$



and $d_n$ is the cofactor of $\lambda^k$ in the first row of the determinant expression for $H_n(\lambda)$.

Now by (2.30a), (2.30b), and (2.3), $u_{p,q}^{(n)}$ has an asymptotic expansion of the form (2.11) with $z_{j,p}$, $\sigma_j$, and $\mu_j$ there given as

$$z_{j,p} = z_j \bar{\lambda}_j^{p-1} \quad \text{for MPE1 extension}, \tag{2.31a}$$

$$z_{j,p} = z_j \bar{\lambda}_j^{p} \quad \text{for MPE2 extension}, \tag{2.31b}$$

and

$$\sigma_j = \lambda_j \quad \text{and} \quad \mu_j = |\lambda_j|^2. \tag{2.32}$$

Now by (2.19)

$$H_n'(\sigma_s) = \frac{d}{d\sigma} H_n(\sigma)\bigg|_{\sigma=\sigma_s}$$

$$\sim (-1)^k Z_{1,2,\ldots,k} V(\sigma_1,\ldots,\sigma_k)\left(\prod_{\substack{j=1\\ j\neq s}}^{k}(\sigma_s-\sigma_j)\right)\left(\prod_{j=1}^{k}\mu_j^n\right). \quad (2.26)$$

By the Taylor theorem with remainder we have

$$0 = H_n(\sigma_s(n)) = H_n(\sigma_s) + H_n'(\tilde{\sigma}_s(n))[\sigma_s(n)-\sigma_s], \quad (2.27)$$

where $\tilde{\sigma}_s(n)$ is along the line segment joining $\sigma_s$ and $\sigma_s(n)$. Consequently,

$$\sigma_s(n) - \sigma_s \sim -\frac{H_n(\sigma_s)}{H_n'(\sigma_s)}. \quad (2.28)$$

Substituting (2.25) and (2.26) in (2.28), and invoking (2.14), we obtain (2.23) and hence (2.20). ■

We note that the result of Lemma 2.2 and the technique used for proving it are similar to and extend those of [1] and Sidi, Ford, and Smith [2].

*Proof of Theorem 2.1.* As is shown in [1], the polynomial $P^{(n,k)}(\lambda)$ has the determinant representation

$$P^{(n,k)}(\lambda) = \frac{H_n(\lambda)}{d_n}, \quad (2.29)$$

where $H_n(\lambda)$ is as in (2.10) with

$$u_{p,q}^{(n)} = (x_{n+p-1}, x_{n+q}) \qquad \text{for MPE1 extension} \quad (2.30a)$$

or

$$u_{p,q}^{(n)} = (x_{n+p}, x_{n+q}) \qquad \text{for MPE2 extension,} \quad (2.30b)$$

and $d_n$ is the cofactor of $\lambda^k$ in the first row of the determinant expression for $H_n(\lambda)$.

Now by (2.30a), (2.30b), and (2.3), $u_{p,q}^{(n)}$ has an asymptotic expansion of the form (2.11) with $z_{j,p}$, $\sigma_j$, and $\mu_j$ there given as

$$z_{j,p} = z_j \bar{\lambda}_j^{p-1} \quad \text{for MPE1 extension,} \tag{2.31a}$$

$$z_{j,p} = z_j \bar{\lambda}_j^{p} \quad \text{for MPE2 extension,} \tag{2.31b}$$

and

$$\sigma_j = \lambda_j \quad \text{and} \quad \mu_j = |\lambda_j|^2. \tag{2.32}$$

Therefore, Lemma 2.2 applies and (2.13) holds.

Substituting (2.31a) or (2.31b) in the determinant expression for $Z_{j_1,\ldots,j_k}$ given in (2.12), we obtain

$$Z_{j_1,j_2,\ldots,j_k} = E_{j_1,\ldots,j_k}\left(\prod_{p=1}^{k} z_{j_p}\right) V\left(\bar{\lambda}_{j_1}, \bar{\lambda}_{j_2}, \ldots, \bar{\lambda}_{j_k}\right), \tag{2.33}$$

where

$$E_{j_1,\ldots,j_k} = \begin{cases} 1 & \text{for MPE1 extension,} \\ \prod_{p=1}^{k} \bar{\lambda}_{j_p} & \text{for MPE2 extension.} \end{cases} \tag{2.34}$$

Thus $Z_{j_1,\ldots,j_k} \neq 0$ for distinct $j_p$, $1 \leqslant p \leqslant k$. Consequently, Lemma 2.3 applies. Invoking (2.32)–(2.34) in (2.19), (2.20), and (2.23), we obtain (2.4) and (2.5), (2.6), and (2.8) respectively. This completes the proof. ∎

As is seen from the proof of Lemma 2.3, an important requirement that makes (2.19), (2.20), and (2.23) possible is (2.17). In the absence of (2.17), i.e., when $|\mu_k| = |\mu_{k+1}|$, the results of Lemma 2.3 need to be modified considerably. This is done in detail in Lemma 2.4 below.

LEMMA 2.4. *In Lemma* 2.2 *let*

$$|\mu_1| \geqslant \cdots \geqslant |\mu_t| > |\mu_{t+1}| = \cdots = |\mu_{t+r}| > |\mu_{t+r+1}| \geqslant \cdots \tag{2.35}$$

*for some $t \geqslant 1$ and $r \geqslant 2$, and let*

$$t+1 \leqslant k < t+r. \tag{2.36}$$

*Denote the coefficient of $\sigma^i$ in $H_n(\sigma)$ by $d_{n,i}$, and denote also*

$$B(j_{t+1},\dots,j_k) = (-1)^k Z_{1,2,\dots,t,j_{t+1},\dots,j_k} V(\sigma_1,\dots,\sigma_t,\sigma_{j_{t+1}},\dots,\sigma_{j_k}). \tag{2.37}$$

(1) *When $\mu_{t+1},\dots,\mu_{t+r}$ are not all the same, assume that*

$$R(n;\sigma) = \sum_{t+1 \leqslant j_{t+1} < \cdots < j_k \leqslant t+r} B(j_{t+1},\dots,j_k) \times \left[\prod_{p=t+1}^{k}(\sigma-\sigma_{j_p})\right]\left(\prod_{p=t+1}^{k}\frac{\mu_{j_p}}{|\mu_{j_p}|}\right)^n \not\equiv 0 \tag{2.38}$$

*for some integer $n$. Note that $R(n;\sigma)$ is a polynomial in $\sigma$ of degree at most $k-t$. Then there exists a subsequence $\{R(n_l;\sigma)\}_{l=0}^{\infty}$ that converges to a polynomial in $\sigma$ of degree $q$, $q$ being some integer satisfying $0 \leqslant q \leqslant k-t$. Let us denote the zeros of this limit polynomial by $\sigma_1',\dots,\sigma_q'$. Obviously $q$ and $\sigma_1',\dots,\sigma_q'$ depend on the $z_{j,p}$ and $\sigma_j$. Thus the subsequence $\{H_{n_l}(\sigma)/d_{n_l,t+q}\}_{l=0}^{\infty}$ converges to the polynomial $\hat{Q}(\sigma) = [\prod_{j=1}^{t}(\sigma-\sigma_j)][\prod_{j=1}^{q}(\sigma-\sigma_j')]$. Actually, if we denote*

$$\eta_t = \max\left(\left|\frac{\mu_{t+1}}{\mu_t}\right|, \left|\frac{\mu_{t+r+1}}{\mu_{t+1}}\right|\right), \tag{2.39}$$

*then*

$$\frac{H_n(\sigma)}{d_{n,t+q}} = \hat{Q}(\sigma) + O(\eta_t^n) \qquad \text{as} \quad n \to \infty, \tag{2.40}$$

*for this subsequence. Consequently, if $\sigma_s \notin \{\sigma_1',\dots,\sigma_q'\}$ for any $s$, $1 \leqslant s \leqslant t$, then the zero $\sigma_s(n)$ of $H_n(\sigma)$ that tends to $\sigma_s$ satisfies*

$$\sigma_s(n) = \sigma_s + O\left(\left|\frac{\mu_{t+1}}{\mu_s}\right|^n\right) \qquad \text{as} \quad n \to \infty \tag{2.41}$$

*for the subsequence in consideration.*

(2) *When* $\mu_{t+1} = \cdots = \mu_{t+r}$, *assume that*

$$T(\sigma) = \sum_{t+1 \leqslant j_{t+1} < \cdots < j_k \leqslant t+r} B(j_{t+1}, \ldots, j_k)\left[\prod_{p=t+1}^{k} (\sigma - \sigma_{j_p})\right] \not\equiv 0. \tag{2.42}$$

*Note that* $T(\sigma)$ *is a polynomial in* $\sigma$ *of degree* $q$ *for some integer* $q$ *satisfying* $0 \leqslant q \leqslant k - t$. *Let* $\sigma_1', \ldots, \sigma_q'$ *be the zeros of* $T(\sigma)$. *Obviously* $q$ *and* $\sigma_1', \ldots, \sigma_q'$ *depend on* $z_{j,p}$ *and* $\sigma_j$. *Then the sequence* $\{H_n(\sigma)/d_{n,t+q}\}_{n=0}^{\infty}$ *converges, and its limit is the polynomial* $\hat{Q}(\sigma) = [\prod_{j=1}^{t}(\sigma - \sigma_j)][\prod_{j=1}^{q}(\sigma - \sigma_j')]$. (2.40) *holds in this case too. Also, if* $\sigma_s \notin \{\sigma_1', \ldots, \sigma_q'\}$ *for any* $s$, $1 \leqslant s \leqslant t$, *then the zero* $\sigma_s(n)$ *of* $H_n(\sigma)$ *that tends to* $\sigma_s$ *satisfies* (2.41). *Both* (2.40) *and* (2.41) *are satisfied for the whole sequence.*

*Proof.* For $\sigma \neq \sigma_i$, $1 \leqslant i \leqslant t$, the dominant part of $H_n(\sigma)$ is the sum of those terms in the summation in (2.13) having the indices

$$j_1 = 1, \qquad j_2 = 2, \ldots, \qquad j_t = t, \qquad t+1 \leqslant j_{t+1} < j_{t+2} < \cdots < j_k \leqslant t+r,$$

namely

$$H_n(\sigma) \sim \left\{\left[\prod_{j=1}^{t} (\sigma - \sigma_j)\right] R(n; \sigma) + U(n; \sigma)\right\}\left(\prod_{j=1}^{t} \mu_j^n\right) |\mu_{t+1}|^{(k-t)n}$$
$$\text{as} \quad n \to \infty. \tag{2.43}$$

A careful analysis of $U(n; \sigma)$ reveals that its asymptotic behavior is determined by the terms in $H_n(\sigma)$ that have the indices

$$j_1 = 1,\ j_2 = 2, \ldots,\ j_t = t,\ t+1 \leqslant j_{t+1} < \cdots < j_{k-1} \leqslant t+r,\ j_k = t+r+1,$$

and

$$j_1 = 1, \qquad j_2 = 2, \ldots, \qquad j_{t-1} = t-1, \qquad t+1 \leqslant j_t < \cdots < j_k \leqslant t+r,$$

and other terms that are at most as large as these. Consequently,

$$U(n;\sigma)=O(\eta_t^n) \qquad \text{as} \quad n\to\infty. \tag{2.44}$$

For the proof of part (1) observe that $R(n;\sigma)$, which is a polynomial in $\sigma$ of degree at most $k-t$, is also a trigonometric sum in $n$, since $(\prod_{p=t+1}^{k}\mu_{j_p}/|\mu_{j_p}|)^n$ is of the form $e^{in\phi}$, $\phi$ real. Therefore, the sequence $\{R(n;\sigma)\}_{n=0}^{\infty}$ of polynomials in $\sigma$ is bounded, and this implies that it has a convergent subsequence with $\hat{R}(\sigma)$, a polynomial in $\sigma$, as its limit, and $\hat{R}(\sigma)\not\equiv 0$. [If every convergent subsequence of $\{R(n;\sigma)\}_{n=0}^{\infty}$ had zero as its limit, then this would imply that $\lim_{n\to\infty}R(n;\sigma)\equiv 0$, which in turn would imply that $R(n;\sigma)\equiv 0$ for all $n$, contradicting the assumption that $R(n;\sigma)\not\equiv 0$ for some integer $n$.] Let $q$ be the degree of $\hat{R}(\sigma)$, $0\leqslant q\leqslant k-t$, and let $\sigma_1',\ldots,\sigma_q'$ be its zeros. Combining this and (2.44) in (2.43), we see that (2.40) holds for this subsequence.

For the proof of (2.41) we observe that (2.28) is true for the $\sigma_s(n)$ of the present lemma, the asymptotic relation there holding for the subsequence of the previous paragraph. Now from (2.24) and (2.35) it follows that

$$H_n(\sigma_s)=O\big(|\mu_1\mu_2\cdots\mu_t\mu_{t+1}^{k-t+1}\mu_s^{-1}|^n\big) \qquad \text{as} \quad n\to\infty. \tag{2.45}$$

From (2.43) it also follows that

$$H_n'(\sigma_s)\sim\left[\prod_{\substack{j=1\\ j\neq s}}^{t}(\sigma_s-\sigma_j)\right]\hat{R}(\sigma_s)\left(\prod_{j=1}^{t}\mu_j^n\right)|\mu_{t+1}|^{(k-t)n} \qquad \text{as} \quad n\to\infty \tag{2.46}$$

for the subsequence of the previous paragraph. Combining these in (2.28), we see that (2.41) holds for this subsequence. This completes the proof of part (1).

For part (2), $\mu_{t+1}=\cdots=\mu_{t+r}$; thus (2.43) [by (2.44)] reduces to

$$H_n(\sigma)=\left\{\left[\prod_{j=1}^{t}(\sigma-\sigma_j)\right]T(\sigma)+O(\eta_t^n)\right\}\left(\prod_{j=1}^{t}\mu_j^n\right)\mu_{t+1}^{(k-t)n} \qquad \text{as} \quad n\to\infty. \tag{2.47}$$

The proof of part (2) can now be completed like that of part (1). ■

We can now use part (2) of Lemma 2.4 to tackle the problem of eigenvalue estimation by MPE extensions of the power method for normal matrices in case $|\lambda_k| > |\lambda_{k+1}|$ in (2.1) in Theorem 2.1 is not satisfied.

THEOREM 2.5. *In Theorem* 2.1 *let*

$$|\lambda_1| \geqslant \cdots \geqslant |\lambda_t| > |\lambda_{t+1}| = \cdots = |\lambda_{t+r}| > |\lambda_{t+r+1}| \geqslant \cdots \tag{2.48}$$

*for some $t \geqslant 1$ and $r \geqslant 2$, and let*

$$t+1 \leqslant k < t+r. \tag{2.49}$$

*Let also*

$$\begin{aligned} S(\lambda) = &\sum_{t+1 \leqslant j_{t+1} < \cdots < j_k \leqslant t+r} E_{j_{t+1},\ldots,j_k}\left(\sum_{p=t+1}^{k} z_{j_p}\right) \\ &\times \left|V(\lambda_1,\ldots,\lambda_t,\lambda_{j_{t+1}},\ldots,\lambda_{j_k})\right|^2 \left[\prod_{p=t+1}^{k}(\lambda-\lambda_{j_p})\right] \\ &\not\equiv 0, \end{aligned} \tag{2.50}$$

*with $E_{j_1,\ldots,j_k}$ as defined in* (2.34). *$S(\lambda)$ is a polynomial in $\sigma$ of degree $q$ for some integer $q$ satisfying $0 \leqslant q \leqslant k-t$.* [*From* (2.3) *and* (2.34) *it is clear that for the* MPE1 *extension the coefficient of $\lambda^{k-t}$ in $S(\lambda)$ is positive so that $q = k-t$ exactly for this extension.*] *Denote its zeros by $\lambda'_1,\ldots,\lambda'_q$. Of course, $q$ and $\lambda'_1,\ldots,\lambda'_q$ depend on the $z_j$ and $\lambda_j$. Then in both of the* MPE *extensions* (*defined with $c_{t+q}^{(n,k)} = 1$ instead of $c_k^{(n,k)} = 1$ in* (1.6a) *and* (1.6b))

$$\frac{P^{(n,k)}(\lambda)}{c_{t+q}^{(n,k)}} = \left[\prod_{j=1}^{t}(\lambda-\lambda_j)\right]\left[\prod_{j=1}^{q}(\lambda-\lambda'_j)\right] + O(\varepsilon_t^{2n}) \qquad as \quad n \to \infty, \tag{2.51}$$

*where*

$$\varepsilon_t = \max\left(\left|\frac{\lambda_{t+1}}{\lambda_t}\right|, \left|\frac{\lambda_{t+r-1}}{\lambda_{t+1}}\right|\right). \tag{2.52}$$

*Consequently, if $\lambda_s \notin \{\lambda'_1,\ldots,\lambda'_q\}$ for any $s$, $1 \leqslant s \leqslant t$, then the zero $\lambda_s(n)$*

*of* $P^{(n,k)}(\lambda)$ *that tends to* $\lambda_s$ *satisfies*

$$\lambda_s(n) = \lambda_s + O\left(\left|\frac{\lambda_{t+1}}{\lambda_s}\right|^{2n}\right) \qquad as \quad n \to \infty. \tag{2.53}$$

*Proof.* By (2.32) and (2.48) we see that $\mu_{t+1} = \cdots = \mu_{t+r} = |\lambda_{t+1}|^2$, so that part (2) of Lemma 2.4 applies with $T(\lambda) = E_{1,\ldots,t}(\prod_{j=1}^{t} z_j)S(\lambda)$. We leave out the details. ∎

## 3. FURTHER DEVELOPMENTS

Let us assume now that some of the $\lambda_i$ in the expansion (1.1) are known along with their corresponding multiplicities $p_i+1$. Without loss of generality we can take them to be $\lambda_1, \ldots, \lambda_h$. We now propose other extensions of the power method that, in effect, serve as deflation methods and provide approximations to $\lambda_{h+1}, \lambda_{h+2}, \ldots$. These extensions are identical in form to those of [1], with the difference that the vector sequence $\{x_j\}_{j=0}^{\infty}$ is now replaced by another, $\{\tilde{x}_j\}_{j=0}^{\infty}$, where the vectors $\tilde{x}_j$ are related to the $x_i$ linearly through

$$\tilde{x}_m = \sum_{s=0}^{v} \alpha_s x_{m+s}, \qquad v = \sum_{i=1}^{h} (p_i+1), \tag{3.1}$$

and $\alpha_0, \alpha_1, \ldots, \alpha_v$ are the coefficients of the polynomial

$$\psi(\lambda) = \sum_{s=0}^{v} \alpha_s \lambda^s = \prod_{i=1}^{h} (\lambda - \lambda_i)^{p_i+1}. \tag{3.2}$$

[For example, for deflation-type MPE extensions we replace the $x_j$ in (1.6)–(1.7) by the $\tilde{x}_j$.] This approach is justified rigorously in the next paragraph.

Invoking (1.1) in (3.1), after appropriate interchanges of the summations, we obtain

$$\tilde{x}_m \sim \sum_{j=1}^{\infty} \left\{ \sum_{i=0}^{p_j} y_{ji} \frac{\lambda_j^{i-m}}{i!} \frac{d^i}{d\lambda^i} [\psi(\lambda)\lambda^m] \Big|_{\lambda=\lambda_j} \right\} \lambda_j^m \qquad \text{as} \quad m \to \infty. \tag{3.3}$$

But $(d^i/d\lambda^i)[\psi(\lambda)\lambda^m]|_{\lambda=\lambda_j} = 0$ for $0 \leqslant i \leqslant p_j$, $1 \leqslant j \leqslant h$, by (3.2). This means that the summation index $j$ in (3.3) runs from $h+1$ to infinity. Note also that the summation on $i$ in (3.3) is a polynomial of degree at most $p_j$ (in fact, exactly $p_j$, as will be seen shortly) in $m$, with vector coefficients. As a consequence of this, we can re-express (3.3) in the form

$$\tilde{x}_m \sim \sum_{j=h+1}^{\infty} \left[ \sum_{i=0}^{p_j} \hat{y}_{ji} \binom{m}{i} \right] \lambda_j^m \quad \text{as} \quad m \to \infty, \tag{3.4}$$

where $\hat{y}_{ji}$ are given by

$$\hat{y}_{ji} = \sum_{q=0}^{p_j - i} y_{j,q+i} \frac{\lambda_j^q}{q!} \psi^{(q)}(\lambda_j), \qquad 0 \leqslant i \leqslant p_j. \tag{3.5}$$

As can be seen from (3.5), $\hat{y}_{ji}$ is a linear combination of $y_{ji}, y_{j,i+1}, \dots, y_{jp_j}$, and the coefficient of $y_{ji}$ in this combination is $\psi(\lambda_j)$, which is nonzero for all $j \geqslant h+1$. Consequently, the vectors $\hat{y}_{ji}$, $0 \leqslant i \leqslant p_j$, $j \geqslant h+1$, form a linearly independent set. From this we conclude that with $\lambda_1, \lambda_2, \dots,$ replaced by $\lambda_{h+1}, \lambda_{h+2}, \dots,$ *all* results pertaining to the various extensions of the power method in [1] and in Section 2 of the present work hold.

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