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Comparison of Some Numerical Quadrature Formulas for Weakly Singular Periodic Fredholm Integral Equations

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Abstract — Zusammenfassung

Comparison of Some Numerical Quadrature Formulas for Weakly Singular Periodic Fredholm Integral Equations. Several numerical quadrature formulas that are used in the quadrature method for the numerical solution of periodic Fredholm integral equations are analyzed and precise asymptotic expansions for their errors are derived. All of these formulas are based on the trapezoidal rule with equidistant abscissas. They are compared with respect to their computational cost, accuracy, and efficiency when used in conjunction with the Richardson extrapolation. On the basis of this comparison it is concluded that the formula developed in [4] is the most advantageous. A numerical example is appended.

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Vergleich von numerischen Quadraturformeln für schwach singuläre periodische Fredholm-Integralgleichungen. Eine Reihe numerischer Quadraturformeln, die bei der numerischen Lösung periodischer Fredholm-Integralgleichungen in Verwendung sind, werden analysiert und es werden für ihre Fehler genaue asymetrische Entwicklungen hergeleitet. Alle diese Formeln beruhen auf der Trapezregel mit gleichabständigen Abszissen. Sie werden bezüglich Rechenaufwand, Genauigkeit und Effizienz bei Ihrer Verwendung im Zusammenhang mit Richardson-Extrapolation verglichen. Es ergibt sich, daß die Formel aus [4] am vorteilhaftesten ist. Ein numerisches Beispiel ist beigefügt.

1. Introduction

Let $f(t)$ be the solution to the Fredholm integral equation

$$\omega f(t) + \int_a^b K(t,x) f(x)\, dx = g(t), \qquad a \le t \le b, \tag{1.1}$$

where $\omega = 0$ or $\omega = 1$ according to whether (1.1) is of first kind or of second kind respectively. We assume that

(1) $K(t,x)$ is periodic both in t and in x with period $T = b - a$,

(2) $g(t)$ and $f(t)$ are periodic in t with period $T = b - a$,

(3) $K(t,x)$ is continuously differentiable a sufficient number of times with respect to both t and x, except for $t = x$, but has a weak (integrable) singularity at $t = x$,

(4) $g(t)$ and $f(t)$ are continuously differentiable a sufficient number of times.

One of the methods for solving (1.1) numerically is *the quadrature method*, in which one first replaces the integral $\int_a^b K(t,x)f(x)\,dx$ by a numerical quadrature formula, whose abscissas are $x_j, j = 1, \ldots, n$, with $t = x_i$, $i = 1, \ldots, n$, then replaces the $f(x_j)$ by their approximations $\tilde{f}_j$, and finally solves the resulting system of linear equations for the $\tilde{f}_j$. Obviously the accuracy of the $\tilde{f}_j$ depends very strongly on the accuracy of the numerical quadrature formula used in approximating the integrals $\int_a^b K(t,x)f(x)\,dx$ for $t = x_i$, $i = 1, \ldots, n$.

The purpose of the present note is to analyze and compare in detail several numerical quadrature formulas that were specifically developed for use in the approximate solution of weakly singular Fredholm integral equations, and that are based directly or indirectly on the trapezoidal rule with equidistant abscissas. To the best of our knowledge the results of this work have not been given before.

Let n be a positive integer, and define $h = T/n$. Let the x_j in the numerical quadrature formulas be

$$x_j = a + jh, \qquad j = 0, 1, \ldots, n. \tag{1.2}$$

Let

$$I(t) = \int_a^b K(t,x)f(x)\,dx. \tag{1.3}$$

All the numerical quadrature formulas $I_n(x_i)$ for $I(x_i)$ that we discuss below are ultimately of the form

$$I_n(x_i) = h \sum_{\substack{j=1 \\ j \neq i}}^{n} K(x_i, x_j)f(x_j) + r_n(x_i)f(x_i) \equiv S_n(x_i) + r_n(x_i)f(x_i), \tag{1.4}$$

in which $r_n(x_i)$ varies according to the formula. It will be shown that the accuracy and nature of $I_n(x_i)$ depend heavily on what we pick for $r_n(x_i)$. We shall analyze $I_n(x_i)$ for the case in which $K(t,x)$ can be expressed in the form

$$K(t,x) = H_1(t,x)\log|t-x| + H_2(t,x), \tag{1.5}$$

where $H_1(t,x)$ and $H_2(t,x)$ are differentiable in t and x (including $t = x$) as many times as needed, but are not required to be periodic neither in t nor in x. Also $H_1(t,t) \not\equiv 0$. Although we limit ourselves to kernels with logarithmic singularity for $t = x$, the treatment of kernels with certain other singularities at $t = x$ does not present additional difficulties; see Sidi and Israeli [4].

The numerical quadrature formulas

(1) In the numerical quadrature formula $I_n(x_i) \equiv I_n^1(x_i) = S_n(x_i) + r_n^1(x_i)f(x_i)$ that appears in *the modified quadrature method* we have

$$r_n^1(x_i) = \int_a^b K(x_i,x)\,dx - h \sum_{\substack{j=1 \\ j \neq i}}^{n} K(x_i, x_j), \tag{1.6}$$

see, e.g. Baker [1, Chap. 5, Sect. 4]. This formula is obtained by first writing (with $t = x_i$)

$$I(t) = \int_a^b K(t,x)[f(x) - f(t)]\,dx + f(t)\int_a^b K(t,x)\,dx,$$

and then replacing the first of the integrals by a sum of the form $S_n(x_i)$, and the second by its exact value.

(2) Define $\phi(x) = \int_0^x \log\xi\, d\xi = x\log x - x$, and let $I_n(x_i) \equiv I_n^2(x_i) = S_n(x_i) + r_n^2(x_i)f(x_i)$, where

$$r_n^2(x_i) = H_1(x_i, x_i)u_n + hH_2(x_i, x_i), \tag{1.7}$$

with

$$u_n = \begin{cases} 2\phi\left(\dfrac{T}{2}\right) - h \displaystyle\sum_{\substack{k=-n/2\\ k\neq 0}}^{n/2}{}'' \log|kh|, & n \text{ even} \\ \phi\left(\dfrac{T-h}{2}\right) + \phi\left(\dfrac{T+h}{2}\right) - h \displaystyle\sum_{\substack{k=-(n-1)/2\\ k\neq 0}}^{(n+1)/2}{}'' \log|kh|, & n \text{ odd}. \end{cases} \tag{1.8}$$

Here and in the sequel $\sum_{i=\mu}^{\nu}{}'' \omega_i$ is taken to mean $\sum_{i=\mu+1}^{\nu-1} \omega_i + \frac{1}{2}(\omega_\mu + \omega_\nu)$. This formula is obtained by first writing (with $t = x_i$)

$$I(t) = \int_{a'}^{b'} [H_1(t,x)f(x) - H_1(t,t)f(t)]\log|t-x|\,dx + \int_{a'}^{b'} H_2(t,x)f(x)\,dx$$
$$+ H_1(t,t)f(t)\int_{a'}^{b'} \log|t-x|\,dx, \qquad b' - a' = T,$$

which is allowed by the periodicity of $K(t,x)f(x)$, and then replacing the first and second integrals by the sums $h\sum_{\substack{j=0\\ \xi_j\neq t}}^{n}{}'' [H_1(t,\xi_j)f(\xi_j) - H_1(t,t)f(t)]\log|t-\xi_j|$ and $h\sum_{j=0}^{n}{}'' H_2(t,\xi_j)f(\xi_j)$ respectively. Here, $\xi_j = a' + jh$, $j = 0, 1, \ldots$. The third integral is replaced by its exact value, namely, by $\phi(t - a') + \phi(b' - t)$. Now we pick $a' = t - T/2$ and $b' = t + T/2$ for even n and $a' = t - (T-h)/2$ and $b' = t + (T+h)/2$ for odd n. This implies that t is the midpoint of $[a', b']$ for even n, whereas for odd n, $t = (a' + b')/2 - h/2$. From this development it now becomes clear that u_n in (1.8) is

$$u_n = \int_{a'}^{b'} \log|x_i - x|\,dx - h\sum_{\substack{j=0\\ \xi_j\neq x_i}}^{n}{}'' \log|x_i - \xi_j| \tag{1.9}$$

for both even and odd n.

By replacing u_n in (1.7) by $u_n + \dfrac{h^2}{3T}$ we can improve the accuracy of $I_n^2(x_i)$ as we show in the next section.

(3) For the logarithmically singular kernel

$$K(t,x) = \log[(\xi(t) - \xi(x))^2 + (\eta(t) - \eta(x))^2]^{1/2}, \tag{1.10}$$

where $(\xi(t), \eta(t))$, $a \leq t \leq b$, is the parametric representation of a closed curve in the $\xi - \eta$ plane, Christiansen [2] has proposed two numerical quadrature formulas that we denote $I_n(x_i) \equiv I_n^{3,Q}(x_i) = S_n(x_i) + r_n^{3,Q}(x_i)f(x_i)$, $Q = I, II$, in which

$$r_n^{3,I}(x_i) = \int_{x_i-h/2}^{x_i+h/2} K(x_i, x)\,dx, \tag{1.11}$$

and, for n an even integer,

$$r_n^{3,II}(x_i) = h\left[\log s_i' + (n-1)\log\frac{T}{2} + \frac{1}{3n} - n - 2\sum_{k=1}^{n/2-1}\log(kh)\right], \tag{1.12}$$

where

$$s_i' = [\xi'(x_i)^2 + \eta'(x_i)^2]^{1/2}.$$

Note that $r_n^{3,I}(x_i)$ is independent of the explicit form of $K(t,x)$ given in (1.10). $r_n^{3,II}(x_i)$, however, does depend explicitly on $K(t,x)$ as given in (1.10). Furthermore, as will be shown in the next section, there is a very close connection between $r_n^{3,II}(x_i)$ and $r_n^2(x_i)$ for this particular $K(t,x)$. In fact, $I_n^{3,II}(x_i)$ can be obtained from $I_n^2(x_i)$ in a very simple way. Based on the formula $I_n^2(x_i)$ with odd n, it is also possible to extend $I_n^{3,II}(x_i)$ to odd n. The details will be given in the next section.

(4) In the numerical quadrature formula proposed recently by Sidi and Israeli [4], $I_n(x_i) \equiv I_n^4(x_i) = S_n(x_i) + r_n^4(x_i)f(x_i)$, where

$$r_n^4(x_i) = h\left[H_1(x_i, x_i)\log\left(\frac{h}{2\pi}\right) + H_2(x_i, x_i)\right]. \tag{1.13}$$

In [4, Theorem 7] it is also shown that, with the periodicity and differentiability assumptions of the first paragraph of this section,

$$I(x_i) - I_n^4(x_i) \sim \sum_{\mu=1}^{\infty} \alpha_\mu(x_i)h^{2\mu+1} \quad \text{as} \quad h \to 0\ (n \to \infty), \tag{1.14}$$

where

$$\alpha_\mu(x_i) = 2\frac{\zeta'(-2\mu)}{(2\mu)!}\frac{\partial^{2\mu}}{\partial x^{2\mu}}[H_1(x_i, x)f(x)]|_{x=x_i}, \tag{1.15}$$

thus giving

$$I(x_i) - I_n^4(x_i) = O(h^3) \quad \text{as} \quad h \to 0. \tag{1.16}$$

In (1.15) $\zeta'(s)$ denotes the derivative of $\zeta(s)$, the Riemann zeta function. Note that the $\alpha_\mu(x_i)$ are independent of the end points a and b.

Unlike $I_n^1(x_i)$ and $I_n^2(x_i)$, it seems that $I_n^4(x_i)$ can not be obtained by any kind of simple manipulation of $I(t)$ and (1.3). This should also be clear from the very special form of $r_n^4(x_i)$. In fact, $I_n^4(x_i)$ is obtained from a careful analysis of the appropriate Euler-Maclaurin expansion associated with $S_n(x_i)$.

In the next section we shall compare $I_n^p(x_i), p = 1, \ldots, 4$, with the help of (1.14)–(1.15) and the following result that follows from [4, Corollary to Theorem 6]:

Theorem 1: *Let* $G(x) = g(x)\log|t - x|$, $g(x)$ *being sufficiently smooth on* $[a, b]$, *and fix* t *such that* $t \in \{x_1, x_2, \ldots, x_{n-1}\}$ *for* $n = n_0, n_1, n_2, \ldots$. *Then*

$$\int_a^b G(x)\,dx - h\left[\sum_{\substack{j=1\\ x_j \neq t}}^{n-1} G(x_j) + \frac{1}{2}G(a) + \frac{1}{2}G(b) + g(t)\log\left(\frac{h}{2\pi}\right)\right]$$
$$\sim -\sum_{\mu=1}^{\infty} \frac{B_{2\mu}}{(2\mu)!}\left[G^{(2\mu-1)}(x)\Big|_{x=a}^{x=b}\right]h^{2\mu} + 2\sum_{\mu=1}^{\infty} \frac{\zeta'(-2\mu)}{(2\mu)!} g^{(2\mu)}(t) h^{2\mu+1} \quad \text{as} \quad h \to 0. \tag{1.17}$$

(Here, $h \to 0$ is taken to mean that $n \to \infty$ through the set $\{n_l\}_{l=0}^{\infty}$ so that t always remains an abscissa.)

B_ν in (1.17) are the Bernoulli numbers.

We note that both (1.14)–(1.15) and (1.17) have been obtained in [4] by employing [4, Theorem 3], which is a special case of a more general and fundamental result due to Navot [3].

The main results concerning the precise asymptotic expansions of the errors $I(x_i) - I_n^p(x_i)$ are given in (2.6) for $p = 1$, in (2.14) for $p = 2$, and in (2.19) and (2.23) for $p = 3$.

Based on the results of Section 2, it is concluded in Section 3 that $I_n^4(x_i)$ of [4] is the most advantageous of all the methods considered in this work. Section 4 includes some numerical results that support this conclusion.

Before we proceed, we note that for arbitrary t we can always write $I(t) = \int_{t-T}^{t} K(t, x) f(x)\,dx$ by the periodicity of the integrand. That is to say, we can always take $a = t - T$ and $b = t$ without affecting $I(t)$. This forces $x_n = t$, which means that t can always be made an abscissa. This enables us to use the numerical quadrature formulus $I_n^p(x_i)$ above with arbitrary t.

2. Analysis of the formulas

We now give a detailed analysis of the errors in the formulas $I_n^p(x_i)$, $p = 1, \ldots, 4$. To this effect, we first observe that for $p = 1, 2, 3$,

$$I(x_i) - I_n^p(x_i) = [I(x_i) - I_n^4(x_i)] - [I_n^p(x_i) - I_n^4(x_i)], \tag{2.1}$$

and since $I(x_i) - I_n^4(x_i)$ is known fully from (1.14) and (1.15), it is sufficient to analyze $I_n^p(x_i) - I_n^4(x_i)$. From (1.4), however, we realize that

$$I_n^p(x_i) - I_n^4(x_i) = [r_n^p(x_i) - r_n^4(x_i)]f(x_i) \equiv V_n^p(x_i)f(x_i), \tag{2.2}$$

so that it is sufficient to analyze $V_n^p(x_i)$.

(1) *Analysis of* $I_n^1(x_i)$

From (1.6) and (1.13) we have

$$V_n^1(x_i) = \int_a^b K(x_i, x)\,dx - h \sum_{\substack{j=1 \\ j\neq i}}^{n} K(x_i, x_j) - h\left[H_1(x_i, x_i)\log\left(\frac{h}{2\pi}\right) + H_2(x_i, x_i)\right]. \tag{2.3}$$

Invoking (1.14) and (1.15), with $f(x)$ there replaced by the constant function 1 (which is also periodic), we see that

$$V_n^1(x_i) \sim \sum_{\mu=1}^{\infty} \beta_\mu(x_i) h^{2\mu+1} \qquad \text{as} \qquad h \to 0\ (n \to \infty), \tag{2.4}$$

where

$$\beta_\mu(x_i) = 2\frac{\zeta'(-2\mu)}{(2\mu)!}\frac{\partial^{2\mu}}{\partial x^{2\mu}} H_1(x_i, x)|_{x=x_i}, \tag{2.5}$$

and are independent of the end points a and b. Consequently,

$$I(x_i) - I_n^1(x_i) \sim \sum_{\mu=1}^{\infty} [\alpha_\mu(x_i) - \beta_\mu(x_i) f(x_i)] h^{2\mu+1} \qquad \text{as} \qquad h \to 0. \tag{2.6}$$

So far we have assumed that the integrals $J_i = \int_a^b K(x_i, x)\,dx$ are computed exactly. Normally, however, the J_i are computed approximately, in which case (2.6) needs to be modified to accommodate this approximation procedure. If the error in the approximation of J_i is $O(h^m)$ for some m, $I(x_i) - I_n^1(x_i)$ will be $O(h^q)$, where $q = \min(m, 3)$. Furthermore, whether an expansion of the form $\sum_{\mu=1}^{\infty} \rho_\mu(x_i) h^{2\mu+1}$ exists for $I(x_i) - I_n^1(x_i)$ depends on how the J_i are approximated.

(2) *Analysis of* $I_n^2(x_i)$

From (1.7) and (1.13), in conjunction with (1.9), we have

$$V_n^2(x_i) = H_1(x_i, x_i) W_n, \tag{2.7}$$

where

$$W_n = \int_{a'}^{b'} \log|t - x|\,dx - h \sum_{\substack{j=0 \\ \xi_j \neq t}}^{n}{}'' \log|t - \xi_j| - h\log\left(\frac{h}{2\pi}\right), \qquad t = x_i. \tag{2.8}$$

Invoking Theorem 1 of the previous section with $g(x) \equiv 1$, we obtain

$$W_n \sim \sum_{\mu=1}^{\infty} \gamma_\mu(h) h^{2\mu} \qquad \text{as} \qquad h \to 0, \tag{2.9}$$

where

$$\begin{aligned}\gamma_\mu(h) &= -\frac{B_{2\mu}}{(2\mu)!}\left[\frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}} \log|t - x|\right]\Bigg|_{x=a'}^{x=b'} \\ &= -\frac{B_{2\mu}}{2\mu(2\mu - 1)}[(b' - t)^{-2\mu+1} + (t - a')^{-2\mu+1}] \neq 0.\end{aligned} \tag{2.10}$$

Now for even n, $b' - t = t - a' = T/2$, thus

$$\gamma_\mu(h) = \hat{\gamma}_\mu = -\frac{B_{2\mu}}{\mu(2\mu - 1)}\left(\frac{2}{T}\right)^{2\mu-1}. \tag{2.11}$$

For odd n, $b' - t = (T + h)/2$ and $t - a' = (T - h)/2$, thus

$$\gamma_\mu(h) = -\frac{B_{2\mu}}{2\mu(2\mu - 1)}\left[\left(\frac{T+h}{2}\right)^{-2\mu+1} + \left(\frac{T-h}{2}\right)^{-2\mu+1}\right]. \tag{2.12}$$

It is clear that if $\gamma_\mu(h)$ is expanded in powers of h, its expansion contains only even powers of h. Combining these with (2.9) we see that

$$W_n \sim \sum_{\mu=1}^{\infty} \hat{\gamma}_\mu h^{2\mu} \qquad \text{as} \qquad h \to 0, \tag{2.13}$$

where $\hat{\gamma}_\mu$ for even n is as given in (2.11), while for odd n, $\hat{\gamma}_\mu$ is slightly more complicated and will not be given here. As a result of (2.13), we have

$$I(x_i) - I_n^2(x_i) \sim \sum_{\mu=1}^{\infty} \alpha_\mu(x_i)h^{2\mu+1} - H_1(x_i, x_i)f(x_i) \sum_{\mu=1}^{\infty} \hat{\gamma}_\mu h^{2\mu} \qquad \text{as} \qquad h \to 0. \tag{2.14}$$

This implies that

$$I(x_i) - I_n^2(x_i) = O(h^2) \qquad \text{as} \qquad h \to 0. \tag{2.15}$$

Furthermore, from (2.14) and $\hat{\gamma}_1 = -\dfrac{2B_2}{T} = -\dfrac{1}{3T}$ both for even and odd n, we see that if we replace u_n in (1.7) by $u_n + \dfrac{h^2}{3T}$, the second sum in (2.14) becomes $O(h^4)$. This improves (2.15) in the sense that $I(x_i) - I_n^2(x_i) = O(h^3)$ now, and the coefficient of h^3 is the same as that in $I(x_i) - I_n^4(x_i)$, namely $\alpha_1(x_i)$. This implies that $I(x_i) - I_n^2(x_i) \sim I(x_i) - I_n^4(x_i)$ as $h \to 0$ for this case.

(3) *Analysis of* $I_n^3(x_i)$

We start with the analysis of $I_n^{3,I}(x_i)$ and assume first that $K(t,x)$ is of the general form given in (1.5). From (1.11) and (1.13) we have

$$V_n^{3,I}(x_i) = \int_{x_i-h/2}^{x_i+h/2} K(x_i, x)\,dx - h\left[H_1(x_i, x_i)\log\left(\frac{h}{2\pi}\right) + H_2(x_i, x_i)\right]. \tag{2.16}$$

Expanding the functions $H_1(x_i, x)$ and $H_2(x_i, x)$ about $x = x_i$, and substituting these expansions in $L_i = \int_{x_i-h/2}^{x_i+h/2} K(x_i, x)\,dx$, after some tedious manipulations we obtain

$$V_n^{3,I}(x_i) \sim hH_1(x_i, x_i)\log(\pi/e)$$
$$+ \sum_{\mu=1}^{\infty}\left[\delta_{\mu,2}(x_i) + \delta_{\mu,1}(x_i)\log\left(\frac{h}{2}\exp\left(-\frac{1}{2\mu+1}\right)\right)\right]h^{2\mu+1} \quad \text{as} \quad h \to 0, \tag{2.17}$$

where

$$\delta_{\mu,q}(x_i) = \frac{2^{-2\mu}}{(2\mu+1)!}\left[\frac{\partial^{2\mu}}{\partial x^{2\mu}} H_q(x_i, x)\right]\Bigg|_{x=x_i}, \qquad q = 1, 2. \tag{2.18}$$

Consequently,

$$\begin{aligned} I(x_i) &- I_n^{3,I}(x_i) \\ &\sim hH_1(x_i, x_i) f(x_i) \log(\pi/e) \\ &+ \sum_{\mu=1}^{\infty} \left\{\alpha_\mu(x_i) - \left[\delta_{\mu,2}(x_i) + \delta_{\mu,1}(x_i)\log\left(\frac{h}{2}\exp\left(-\frac{1}{2\mu+1}\right)\right)\right] f(x_i)\right\} h^{2\mu+1} \\ &\qquad \text{as} \quad h \to 0. \end{aligned} \tag{2.19}$$

This implies that

$$I(x_i) - I_n^{3,I}(x_i) = O(h) \qquad \text{as} \qquad h \to 0. \tag{2.20}$$

When we specialize (2.19) to the kernel $K(t, x)$ given in (1.10), we see that $H_1(t, x) \equiv 1$ as mentioned below, thus $\delta_{\mu,1}(x_i) = 0$, $\mu = 1, 2, \ldots$. This results in an expansion of the form

$$I(x_i) - I_n^{3,I}(x_i) \sim \sum_{\mu=0}^{\infty} \rho_\mu(x_i) h^{2\mu+1} \qquad \text{as} \qquad h \to 0.$$

In any case, we note that as for the formula $I_n^1(x_i)$, for $I_n^{3,I}(x_i)$ too the exact nature of the expansion for $I(x_i) - I_n^{3,I}(x_i)$ is likely to change if the integrals L_i are computed only approximately.

As for the formula $I_n^{3,II}(x_i)$, we proceed by showing that it can be obtained from $I_n^2(x_i)$. For the kernel $K(t, x)$ in (1.10), it is not difficult to see that $H_1(t, t) \equiv 1$ and $H_2(t, t) = \log[\xi'(t)^2 + \eta'(t)^2]^{1/2}$. Thus $H_2(x_i, x_i) = \log s_i'$, where s_i' is as defined following (1.12). Also n is even for $I_n^{3,II}(x_i)$. Thus, for this case (1.7) becomes

$$r_n^2(x_i) = 2\phi\left(\frac{T}{2}\right) - h \sum_{\substack{k=-n/2 \\ k \neq 0}}^{n/2}{}'' \log|kh| + h \log s_i'. \tag{2.21}$$

Comparing (2.21) with (1.12), we see that for this case

$$r_n^2(x_i) = r_n^{3,II}(x_i) - \frac{h}{3n} = r_n^{3,II}(x_i) - \frac{h^2}{3T}. \tag{2.22}$$

This, as explained following (2.15), results in

$$I(x_i) - I_n^{3,II}(x_i) \sim \sum_{\mu=1}^{\infty} \alpha_\mu(x_i) h^{2\mu+1} - f(x_i) \sum_{\mu=2}^{\infty} \hat{\gamma}_\mu h^{2\mu} \qquad \text{as} \qquad h \to 0, \tag{2.23}$$

which implies that

$$I(x_i) - I_n^{3,II}(x_i) = O(h^3) \qquad \text{as} \qquad h \to 0, \tag{2.24}$$

the coefficient of h^3 being the same as that in $I(x_i) - I_n^4(x_i)$, namely $\alpha_1(x_i)$. Thus $I(x_i) - I_n^{3,II}(x_i) \sim I(x_i) - I_n^4(x_i)$ as $h \to 0$.

We note here that $I_n^{3,I}(x_i)$ and $I_n^{3,II}(x_i)$ have been observed to produce errors of order h and h^3 respectively in the numerical solution of some integral equations of the kind mentioned in the introduction, see [2]. Our analyses above, the results in (2.20) and (2.24) in particular, provide the justification for this.

3. Discussion of results

We now compare the different formulas that were analyzed in Section 2 in view of their costs, errors, and possibilities of their being used in conjunction with Romberg-type integration.

We note again that the difference between the various methods stems from the different ways in which $r_n(x_i)$ are computed. For $I_n^1(x_i)$ and $I_n^{3,I}(x_i)$ the computation of $r_n^p(x_i)$ involves the determination of the integrals $J_i = \int_a^b K(x_i, x)\,dx$ and $L_i = \int_{x_i-h/2}^{x_i+h/2} K(x_i, x)\,dx$ with sufficient accuracy, and this increases the cost of $I_n^1(x_i)$ and $I_n^{3,I}(x_i)$.

As far as the accuracy of the methods is concerned, the error in $I_n^{3,I}(x_i)$ is $O(h)$, while the errors for $I_n^1(x_i)$, $I_n^{3,II}(x_i)$, and $I_n^4(x_i)$ are all $O(h^3)$. The error in $I_n^2(x_i)$ is $O(h^2)$, but with a minor change in $r_n^2(x_i)$, can be made $O(h^3)$.

Next, the errors in $I_n^{3,I}(x_i)$ (for the kernel in (1.10)), $I_n^1(x_i)$, and $I_n^4(x_i)$ have only the odd powers of h in their expansions, while those of $I_n^2(x_i)$ and $I_n^{3,II}(x_i)$ have both even and odd powers of h in their expansions. The existence of well defined asymptotic expansions for the errors in the different formulas $I_n^p(x_i)$ immediately suggests that the accuracy of these formulas can be improved by applying to them the Richardson extrapolation. This idea has been proposed and successfully implemented in [4] in conjunction with $I_n^4(x_i)$. The amount of work in setting up the appropriate Romberg-type integration procedures is directly proportional to the number of terms eliminated from the asymptotic expansions of $I(x_i) - I_n^p(x_i)$. In view of this we see that for a given amount of effort in setting up the Romberg-type integration formulas, more accuracy can be obtained for $I_n^1(x_i)$ and $I_n^4(x_i)$ than can be obtained for $I_n^{3,II}(x_i)$. For example, if we wish to eliminate ν terms from their asymptotic expansions, we will obtain $O(h^{2\nu+3})$ accuracy for the former, while $O(h^{\nu+3})$ accuracy is obtained for the latter.

When all the factors of cost, error, and efficiency of use with Richardson extrapolation are taken into account, it is seen that the formula $I_n^4(x_i)$ of [4] is the most advantageous of all the formulas considered here.

The numerical quadrature formula $I_n^4(x_i)$ has been employed in [4] in the solution of periodic Fredholm integral equations with logarithmically singular kernels resulting in errors of order h^3 in the numerical solutions, as suggested by (1.16). Romberg-type numerical quadrature formulas obtained by applying the Richardson extrapolation process to $I_n^4(x_i)$ through the expansion in (1.14) have been observed to produce errors of orders h^5, h^7, etc., in the numerical solutions of the same integral equations, thus implying the correctness of (1.14). These results are documented in detail in [4], where a brief account of the relevant extrapolation methods is also provided.

4. A numerical example

Consider the integral $I(t) = \int_0^{2\pi} K(t,x) f(x)\,dx$, with $K(t,x) = \log\left(2c \sin\frac{|t-x|}{2}\right)$ and $f(x) = \cos x$. We have $I(t) = -\pi\cos t$ for all $c > 0$. For the kernel $K(t,x)$ in consideration we have $H_1(t,t) \equiv 1$ and $H_2(t,t) \equiv \log c$. Also $K(t,x)$ is of the form given in (1.10) with $\xi(t) = c\cos t$ and $\eta(t) = c\sin t$, $0 \le t \le 2\pi$.

Below we present some numerical results obtained for $I(t)$ by employing the rules $I_n^{3,II}$ and I_n^4. As will become clear, these results verify the theoretical results of Section 2 and justify the conclusion of Section 3. In all our computations we have set $c = \sqrt{e}$ and $t = 0$.

Tables 1–3 give some of the results obtained for $I(t)$ above by the rules I_n^4 and $I_n^{3,II}$ with the Richardson extrapolation applied to the sequences $T_k = I_{2^k}^p(t)$, $k = 1, 2, \ldots$. A short description of the Richardson extrapolation and its generalizations can be found in [4, Appendix]. We shall only mention that the approximations resulting from the application of the Richardson extrapolation can be arranged in a table of the form

$$\begin{array}{lllll} T_0^{(1)} = T_1 & & & & \\ T_0^{(2)} = T_2 & T_1^{(1)} & & & \\ T_0^{(3)} = T_3 & T_1^{(2)} & T_2^{(1)} & & \\ T_0^{(4)} = T_4 & T_1^{(3)} & T_2^{(2)} & T_3^{(1)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{4.1}$$

Table 1. Comparison of the rules $I_n^4(t)$ and $I_n^{3,II}(t)$ as they are applied to $I(t)$. The last column contains $\alpha_1(t)h^3$, the dominant term in the asymptotic expansions of $I(t) - I_n^4(t)$ and $I(t) - I_n^{3,II}(t)$.

n	$I(t) - I_n^4(t)$	$I(t) - I_n^{3,II}(t)$	$\alpha_1(t)h^3$
2	1.2135795 + 00	1.1993050 + 00	9.4409328 − 01
4	1.2478648 − 01	1.2376241 − 01	1.1801166 − 01
8	1.4953355 − 02	1.4886343 − 02	1.4751458 − 02
16	1.8501693 − 03	1.8459270 − 03	1.8439322 − 03
32	2.3068588 − 04	2.3041986 − 04	2.3049152 − 04
64	2.8817510 − 05	2.8800870 − 05	2.8811440 − 05
128	3.6016197 − 06	3.6005795 − 06	3.6014301 − 06
256	4.5018468 − 07	4.5011967 − 07	4.5017876 − 07
512	5.6272530 − 08	5.6268466 − 08	5.6272345 − 08
1024	7.0340489 − 09	7.0337949 − 09	7.0340431 − 09
2048	8.7925557 − 10	8.7923970 − 10	8.7925539 − 10
4096	1.0990693 − 10	1.0990594 − 10	1.0990692 − 10
8192	1.3738366 − 11	1.3738304 − 11	1.3738365 − 11

Here each column converges to $I(t)$ more quickly than the ones preceding it, and the diagonals converge more quickly than the columns.

Table 1 contains the errors $e_n^p(t) = I(t) - I_n^p(t)$ and the dominant term in their asymptotic expansion for $n \to \infty$, namely, $\alpha_1(t)h^3 = \zeta'(-2)f''(t)\left(\frac{2\pi}{n}\right)^3$. Here, $\zeta'(-2) = -0.030448457\ldots$. The closeness of $e_n^4(t)$, $e_n^{3,II}(t)$, and $\alpha_1(t)h^3$ for large n is in accordance with the results of Section 2.

Table 2a gives the errors $I(t) - T_j^{(i)}$ for the rule I_n^4. From Table 2b it is seen that the second column behaves h^5, the third column like h^7, etc., as implied by the expansion in (1.14).

Table 3a gives the errors $I(t) - T_j^{(i)}$ for the rule $I_n^{3,II}$. From Table 3b it is seen that the second column behaves like h^4, the third column like h^5, etc., as implied by the expansion in (2.23).

Table 2a. Errors $e_j^{(i)} = I(t) - T_j^{(i)}$ in the Richardson extrapolation applied to the sequence $T_k = I_{2^k}^4(t)$, $k = 1, 2, \ldots$, for $I(t)$. $e_j^{(i)}$ appears in the same location as $T_j^{(i)}$ in the table in (4.1). The lack of improvement at the end of the last three columns is due to the finite precision arithmetic being used.

1.21 + 00					
1.25 − 01	−3.08 − 02				
1.50 − 02	−7.37 − 04	2.31 − 04			
1.85 − 03	−2.17 − 05	1.36 − 06	−4.48 − 07		
2.31 − 04	−6.69 − 07	9.99 − 09	−6.57 − 10	2.18 − 10	
2.88 − 05	−2.08 − 08	7.69 − 11	−1.20 − 12	7.99 − 14	−2.66 − 14
3.60 − 06	−6.50 − 10	5.98 − 13	−2.31 − 15	3.66 − 17	−2.44 − 18
4.50 − 07	−2.03 − 11	4.67 − 15	−4.50 − 18	1.76 − 20	−2.79 − 22
5.63 − 08	−6.35 − 13	3.65 − 17	−8.78 − 21	8.55 − 24	−3.35 − 26
7.03 − 09	−1.98 − 14	2.85 − 19	−1.71 − 23	4.17 − 27	−5.65 − 30
8.79 − 10	−6.20 − 16	2.23 − 21	−3.35 − 26	−9.08 − 31	−2.95 − 30
1.10 − 10	−1.94 − 17	1.74 − 23	−7.15 − 29	−6.07 − 30	−6.07 − 30
1.37 − 11	−6.05 − 19	1.36 − 25	−1.21 − 29	−1.20 − 29	−1.20 − 29

Table 2b. The ratios $v_j^{(i)} = e_j^{(i)}/e_j^{(i+1)}$ with $e_j^{(i)}$ as in Table 2a. $v_j^{(i)}$ appears in the same location as $e_j^{(i)}$ in Table 2a. The entries that are omitted are those corresponding to the $e_j^{(i)}$ that suffer from loss of significance due to the finite precision arithmetic being used.

9.7					
8.3	41.7				
8.1	33.9	169.7			
8.0	32.5	136.3	681.7		
8.0	32.1	130.0	545.9	2729.7	
8.0	32.0	128.5	520.1	2184.3	10921.7
8.0	32.0	128.1	514.0	2080.5	8737.9
8.0	32.0	128.0	512.5	2056.0	8321.8
8.0	32.0	128.0	512.1	2050.8	5927.7
8.0	32.0	128.0	512.0	−4591.8	
8.0	32.0	128.0	468.6		
8.0	32.0	128.0			

Table 3a. Errors $e_j^{(i)} = I(t) - T_j^{(i)}$ in the Richardson extrapolation applied to the sequence $T_k = I_{2^k}^{3,II}(t)$, $k = 1, 2, \ldots$, for $I(t)$. $e_j^{(i)}$ appears in the same location as $T_j^{(i)}$ in the table in (4.1).

1.20 + 00					
1.24 − 01	−2.99 − 02				
1.49 − 02	−6.67 − 04	1.28 − 03			
1.85 − 03	−1.70 − 05	2.64 − 05	−1.41 − 05		
2.30 − 04	−3.67 − 07	7.41 − 07	−8.54 − 08	1.37 − 07	
2.88 − 05	−1.84 − 09	2.25 − 08	−6.94 − 10	6.51 − 10	−4.21 − 10
3.60 − 06	5.38 − 10	6.97 − 10	−6.35 − 12	4.57 − 12	−5.22 − 13
4.50 − 07	5.40 − 11	2.17 − 11	−6.47 − 14	3.51 − 14	−5.58 − 16
5.63 − 08	4.01 − 12	6.78 − 13	−7.40 − 16	2.75 − 16	3.34 − 19
7.03 − 09	2.70 − 13	2.12 − 14	−9.44 − 18	2.15 − 18	5.79 − 21
8.79 − 10	1.75 − 14	6.61 − 16	−1.31 − 19	1.68 − 20	3.09 − 23
1.10 − 10	1.11 − 15	2.07 − 17	−1.92 − 21	1.32 − 22	1.37 − 25
1.37 − 11	7.03 − 17	6.49 − 19	−2.89 − 23	1.03 − 24	5.52 − 28

Table 3b. The ratios $v_j^{(i)} = e_j^{(i)}/e_j^{(i+1)}$ with $e_j^{(i)}$ as in Table 3a. $v_j^{(i)}$ appears in the same location as $e_j^{(i)}$ in Table 3a.

9.7					
8.3	44.8				
8.1	39.3	48.6			
8.0	46.3	35.6	164.9		
8.0	199.1	33.0	123.1	210.2	
8.0	−3.4	32.3	109.3	142.5	806.4
8.0	10.0	32.1	98.2	130.0	936.4
8.0	13.5	32.0	87.4	127.8	−1670.4
8.0	14.8	32.0	78.4	127.7	57.7
8.0	15.4	32.0	72.1	127.8	187.2
8.0	15.7	32.0	68.3	127.9	226.4
8.0	15.9	32.0	66.2	127.9	247.2

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