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# ON RATES OF ACCELERATION OF EXTRAPOLATION METHODS FOR OSCILLATORY INFINITE INTEGRALS

AVRAM SIDI

*Computer Science Department, Technion – Israel Institute of Technology, Haifa 32000, Israel.*

**Abstract.**

The purpose of this work is to complement and expand our knowledge of the convergence theory of some extrapolation methods for the accurate computation of oscillatory infinite integrals. Specifically, we analyze in detail the convergence properties of the $W$- and $\bar{D}$-transformations of the author as they are applied to three integrals, all with totally different behavior at infinity. The results of the analysis suggest different convergence and acceleration of convergence behavior for the above mentioned transformations on the different integrals, and they improve considerably those that can be obtained from the existing convergence theories.

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## 1. Introduction.

In a recent work [3] a nonlinear extrapolation method, the $D$-transformation, was proposed, and this method proved to be very useful in accelerating the convergence of infinite integrals $\int_a^\infty f(t)dt$, $a \geq 0$, of different kinds. The $D$-transformation was analyzed for its convergence properties in [4] within the framework of the generalized Richardson extrapolation process. Two modifications of the $D$-transformation for oscillatory infinite integrals were proposed in [6], which were denoted the $\bar{D}$-and $\tilde{D}$-transformations. Another modification, the $W$-transformation, useful for "very oscillatory" infinite integrals was given in [7], and this modification was recently extended in [9] to divergent oscillatory infinite integrals that are defined in the sense of (Abel) summability. The advantage of these modifications over the $D$-transformation is that they can achieve a given level of accuracy with considerably less computing than the $D$-transformation.

It is not the purpose of this work to go into the details of the transformations above, as this has already been done at great length in the appropriate papers. We

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shall be content with a brief description of extrapolation methods for infinite integrals, which will render the developments of this paper easier to follow.

All the methods of extrapolation mentioned in the first paragraph are based on the assumption that the integral $\int_x^\infty f(t)dt$ has a well structured asymptotic expansion as $x \to \infty$. For example, in [3] it is shown that for a very large family of functions $f(t)$ (denoted $\mathbf{B}^{(m)}$) that are integrable at infinity, asymptotic expansions of the form

$$(1.1)\quad \int_a^\infty f(t)dt - \int_a^x f(t)dt = \int_x^\infty f(t)dt \sim \sum_{k=0}^{m-1} x^{\rho_k} f^{(k)}(x) \sum_{i=0}^{\infty} \beta_{ki} x^{-i} \quad \text{as } x \to \infty,$$

exist, where $\rho_k \le k+1$ are some integers that depend on $f(x)$, and the $\beta_{ki}$ are coefficients that are independent of $x$.

Once the existence of such an asymptotic expansion has been established, the integral $\int_a^\infty f(t)dt$ can be approximated by the $D$-transformation of [3] as follows. Pick a sequence of points $a < x_0 < x_1 < x_2 < \dots$, such that $\lim_{l\to\infty} x_l = \infty$, and compute the finite integrals $\int_a^{x_l} f(t)dt,\ \ l = 0, 1, 2, \dots$, by employing appropriate numerical quadrature formulas. For given nonnegative integers $j$ and $n_k,\ k = 0, 1, \dots, m-1$, define the approximation $D_n^{(m,j)}$ with $n \equiv (n_0, n_1, \dots, n_{m-1})$ to be the solution to the linear system of equations

$$(1.2)\qquad D_n^{(m,j)} - \int_a^{x_l} f(t)dt = \sum_{k=0}^{m-1} x_l^{\rho_k} f^{(k)}(x_l) \sum_{i=0}^{n_k} \bar{\beta}_{ki} x_l^{-i}, \quad j \le l \le j+N,$$

where $N = \sum_{k=0}^{m-1} (n_k + 1)$ and $\bar{\beta}_{ki}$ are the remaining $N$ unknowns. The $D$-transformation can be made more user-friendly by replacing $\rho_k$ in (1.2) by $k+1,\ 0 \le k \le m-1$, as suggested in the review paper [10], without affecting its accuracy very much. We note that the solution of (1.2) for $D_n^{(m,j)}$ can be achieved recursively and very efficiently by using the $W$-algorithm of [8] for $m = 1$, and the $W^{(m)}$-algorithm of [1] for all other values of $m$.

The limiting process in which the $n_k$ are all fixed and $j \to \infty$ is called *Process I*, whereas that in which $j$ is held fixed and the $n_k \to \infty$ simultaneously is called *Process II*. Both numerical experiments and the theoretical results of [4], [5], [6], [7], and [9] suggest that Process II has more powerful convergence properties than Process I.

Now by picking the $x_l$ in a suitable manner we can attain a given level of accuracy in $D_n^{(m,j)}$, the approximation to $\int_a^\infty f(t)dt$, with less computational effort than required for arbitrary $x_l$. For instance, the $\bar{D}$-transformation of [6] achieves this for oscillatory integrals by picking $x_l$ to be the consecutive zeros of some of $f^{(k)}(x)$,

$0 \leq k \leq m-1$. This eliminates a large number of the unknowns $\bar{\beta}_{ki}$, thus reducing the number of equations, and hence the number of the finite integrals $\int_a^{x_l} f(t)dt$, substantially. The philosophy behind the $\tilde{\bar{D}}$-transformation of [6] and the $W$-transformation of [7] and [9] is very similar. Consequently, the $W$- or $W^{(m)}$-algorithms can again be used for implementing the $\bar{D}$-, $\tilde{\bar{D}}$-, and $W$-transformations.

Our aim in the present work is to contribute, even in a small way, to our understanding of the convergence properties of these transformations for Process II. Although all our numerical experience suggests that, when applied properly, the methods above have excellent convergence and convergence acceleration properties for Process II, the analysis of Process II in [6], [7], and [9] is not complete. The reason for this is that the assumptions made in this analysis concerning the analytic properties of the integrands $f(t)$ are of a rather general nature. Nevertheless, the existing theoretical results on Process II for the $W$-transformation as applied to very oscillatory convergent or divergent integrals are quite useful in that 1) they show convergence in all cases, and 2) in cases where the integrand does not decay exponentially at infinity (i.e., $f(t) = O(t^\gamma)$ as $t \to \infty$ for some $\gamma$) they show very meaningful acceleration of convergence. Using an approach that was employed in [5, Section 4] for analyzing the $T$-transformation of Levin [2], we shall treat the convergence of Process II for the $W$- and $\bar{D}$-transformations, as they are applied to the following three test problems, the first two of which are Fourier cosine integrals:

EXAMPLE 1: $$\int_0^\infty t^\gamma \cos \pi t \, dt = -\frac{\Gamma(\gamma+1)}{\pi^{\gamma+1}} \sin \frac{\gamma\pi}{2}, \quad \gamma > -1.$$

This integral exists in the ordinary sense only for $\gamma < 0$. For $\gamma \geq 0$, however, it does not exist in the ordinary sense since it does not converge at infinity, in which case it exists in the (Abel) summability sense (see [9]). For all $\gamma > -1$ this integral is purely oscillatory and behaves like $x^\gamma$ at infinity. We shall analyze the convergence of Process II for the $W$-transformation on this integral.

EXAMPLE 2: $$\int_0^\infty e^{-ct^2/2} \cos \pi t \, dt = \tfrac{1}{2}(\pi/c)^{\frac{1}{2}} \exp(-\pi^2/c), \quad c > 0.$$

This integral is also purely oscillatory, but, unlike that in Example 1, it decays exponentially at infinity. We shall analyze the convergence of Process II for the $W$-transformation on this integral.

EXAMPLE 3: $$\int_0^\infty (\sin \pi t/\pi t)^2 dt = \tfrac{1}{2}.$$

Unlike those in Examples 1 and 2, this integral is not purely oscillatory as its integrand is nonnegative. In fact, it is the sum of two convergent integrals, one of which is purely oscillatory and the other, monotonic, and it behaves like $x^{-1}$ at infinity. We shall analyze the convergence of Process II for the $\bar{D}$-transformation on this integral.

We note at this point that the results that we derive on convergence and rate of acceleration remain the same when, in Examples 1 and 2, $\cos \pi t$ is replaced by $\sin \pi t$.

Note the different nature of the three examples above. Below we shall use a unified approach to their treatment. However, as will become clear, the convergence results for each of these examples seem to have different characteristics, which are not obtained from the convergence analyses available in [6], [7], and [9]. In fact, the analysis of the $W$-transformation for Process II, as presented in [7] and [9] shows convergence acceleration for Example 1, only convergence for Example 2, whereas the analysis of the $\bar{D}$-transformation as presented in [6] is inconclusive for Example 3 even as far as convergence is concerned. Even when they do show convergence acceleration, the above mentioned analyses do not provide us with precise rates of acceleration in all cases, this being due to their general nature.

The purpose of this work is to obtain more precise information on convergence and rates of acceleration for the three examples above. It will be shown below that a different kind of convergence acceleration is achieved for each example under consideration.

It is hoped that the approach of the present work will stimulate further research into the convergence and acceleration questions for the different extrapolation procedures. In view of the results of this work, one interesting topic could be the classification of those integrals for which rates of acceleration are of the same kind.

## 2. Derivation of error expressions

We first write down the equations defining the necessary approximations to the integrals involved. Next, we obtain compact error expressions (involving Laplace transforms) that we analyze in the next section. Most of the details are left out; therefore, we advise the reader to refer to the original papers. We treat each example separately.

Here is some notation that we shall be using throughout.

$$I[f] = \int_0^\infty f(t)dt, \qquad F(x) = \int_0^x f(t)dt,$$
$$R(x) = I[f] - F(x) = \int_x^\infty f(t)dt. \tag{2.1}$$

We assume, of course, that $I[f]$ and $R(x)$ exist in the summability sense when the integrals involved do not converge at infinity.

The Laplace transform of a function $g(\xi)$ will be denoted by

$$\mathscr{L}[g(\xi); x] = \int_0^\infty e^{-x\xi} g(\xi) d\xi \tag{2.2}$$

when this integral exists.

Below we shall also be using the notation $\mathbf{A}^{(s)}$, $\mathbf{B}^{(m)}$, $\mathbf{B}_c$, and $\mathbf{B}_d$, though very briefly. Since we shall not need this notation in the analysis to follow, we refer the reader to [3], [5], [6], [7], and [9] for the precise definitions of $\mathbf{A}^{(s)}$, $\mathbf{B}^{(m)}$, $\mathbf{B}_c$, and $\mathbf{B}_d$.

EXAMPLE 1: In the notation of [9], the integrand $f(x) = x^\gamma \cos \pi x$ is in the family $\mathbf{B}_c$ when $I[f]$ converges (i.e., when $\gamma < 0$), and in the family $\mathbf{B}_d$ when $I[f]$ diverges (i.e., when $\gamma \geq 0$), and has

$$\theta(x) = \pi x = \bar{\theta}(x), \qquad \phi(x) = 0 = \bar{\phi}(x), \qquad h(x) = x^\gamma. \tag{2.3}$$

Thus, as shown in [9], for *all* $\gamma > -1$

$$R(x) = x^\gamma [(\cos \pi x) b_1(x) + (\sin \pi x) b_2(x)], \tag{2.4}$$

where $b_1(x)$ and $b_2(x)$ are in the family $\mathbf{A}^{(0)}$.

Picking $x_l$, $l = 0, 1, \ldots$, to be consecutive zeros of $\sin \bar{\theta}(x) = \sin \pi x$, i.e., $x_l = l + 1$, $l = 0, 1, \ldots$, the approximations $W_n^{(j)}$ to $I[f]$ are defined through the set of linear equations

$$W_n^{(j)} = F(x_l) + \psi(x_l) \sum_{i=0}^{n} \beta_i / x_l^i, \qquad j \leq l \leq j + n + 1, \tag{2.5}$$

where $W_n^{(j)}$ and $\beta_i$ are unknowns, and

$$\psi(x_l) = (-1)^l x_l^\gamma, \qquad l = 0, 1, \ldots. \tag{2.6}$$

An explicit expression for $R(x)$ involving Laplace transforms can be obtained as follows. Assume first that $\gamma < 0$ so that the integral representation for $R(x)$ in (2.1) exists in the ordinary sense. Making the change of variable of integration $\pi t = x(\pi + \tau)$, $R(x)$ can now be expressed as

$$R(x) = (x/\pi)^{\gamma+1} \operatorname{Re}\left\{ e^{i\pi x} \int_0^\infty (\pi + \tau)^\gamma e^{ix\tau} d\tau \right\}. \tag{2.7}$$

Viewing the integral $S(x) = \int_0^\infty (\pi + \tau)^\gamma e^{ix\tau} d\tau$ as a contour integral in the complex $z$-plane with $z = \tau + i\xi$, and rotating the path of integration by 90°, we obtain

$$S(x) = i \int_0^\infty (\pi + i\xi)^\gamma e^{-x\xi} d\xi. \tag{2.8}$$

The integral in (2.8) exists for *all* $\gamma > -1$ when $x > 0$ and is analytic in $\gamma$.

Consequently, by analytic continuation,

$$R(x) = x^{\gamma+1}\,\mathrm{Re}\,\{e^{i\pi x}\,\mathscr{L}[\omega(\xi); x]\}, \tag{2.9}$$

where

$$\omega(\xi) = i(\pi + i\xi)^{\gamma}/\pi^{\gamma+1}, \tag{2.10}$$

for *all* $\gamma > -1$.

EXAMPLE 2: The integrand $f(x) = e^{-cx^2/2}\cos \pi x$ is in the family $\mathbf{B}_c$ and has

$$\theta(x) = \pi x = \bar{\theta}(x), \qquad \phi(x) = -cx^2/2 = \bar{\phi}(x), \qquad h(x) = 1. \tag{2.11}$$

Thus

$$R(x) = x^{-1}e^{-cx^2/2}\,[(\cos \pi x)\,b_1(x) + (\sin \pi x)\,b_2(x)], \tag{2.12}$$

where $b_1(x)$ and $b_2(x)$ are in the family $\mathbf{A}^{(0)}$

Picking $x_l$, $l = 0, 1, \ldots,$ to be consecutive zeros of $\sin\bar{\theta}(x) = \sin \pi x$, i.e., $x_l = l + 1$, $l = 0, 1, \ldots,$ the approximations $W_n^{(j)}$ to $I[f]$ are defined through the set of linear equations (2.5) with

$$\psi(x_l) = (-1)^l x_l^{-1} e^{-cx_l^2/2}, \qquad l = 0, 1, \ldots. \tag{2.13}$$

Making the change of variable of integration $t = x + \xi/c$ in the integral representation of $R(x)$ in (2.1), we obtain

$$R(x) = e^{-cx^2/2}\,\mathrm{Re}\,\{e^{i\pi x}\,\mathscr{L}[\omega(\xi); x]\}, \tag{2.14}$$

where

$$\omega(\xi) = c^{-1}\exp[c^{-1}(i\pi\xi - \xi^2/2)]. \tag{2.15}$$

EXAMPLE 3: In the notation of [3] and [6] the integrand $f(x) = (\sin \pi x/\pi x)^2$ is in the family $\mathbf{B}^{(3)}$, and as is shown in [3, Example 4.5] (see also [6, Example 3]),

$$R(x) = xf(x)\beta_0(x) + f'(x)\beta_1(x) + xf''(x)\beta_2(x), \tag{2.16}$$

with $\beta_0(x)$, $\beta_1(x)$, and $\beta_2(x)$ in $\mathbf{A}^{(0)}$.

For the $\bar{D}$-transformation we pick $x_l = l + 1$, $l = 0, 1, \ldots,$ so that $f(x_l) = f'(x_l) = 0$. It is easy to show that $f''(x_l) = (2\cos 2\pi x_l)/x_l^2 = 2/x_l^2$. Thus the approximations $\bar{D}_n^{(j)}$ to $I[f]$, which we denote by $W_n^{(j)}$ for the sake of uniformity, are defined through the set of linear equations in (2.5) with

$$\psi(x_l) = 2/x_l, \qquad l = 0, 1, \ldots. \tag{2.17}$$

In order to express $R(x)$ in terms of Laplace transforms we proceed as follows: First,

$$R(x) = \int_x^\infty \frac{1 - \cos 2\pi t}{2\pi^2 t^2}\,dt = \frac{1}{2\pi^2 x} - \int_x^\infty \frac{\cos 2\pi t}{2\pi^2 t^2}\,dt. \tag{2.18}$$

Next, making the change of integration variable $2\pi t = x(2\pi + \tau)$ in the integral on the right hand side of (2.18), and then proceeding as in Example 1, we obtain

$$R(x) = 1/(2\pi^2 x) + (2/x)\,\mathrm{Re}\,\{e^{2\pi i x}\,\mathscr{L}[\omega(\xi); x]\} \quad \text{where} \tag{2.19}$$

$$\omega(\xi) = 1/(2\pi i(2\pi + i\xi)^2). \tag{2.20}$$

We now let $l + 1 = L$, $j + 1 = J$, and $n + 1 = N$ in (2.5). Consequently, for all three examples, $x_l = L$, thus (2.5) can be written as

$$W_n^{(j)} = F(L) + \psi(L) \sum_{i=0}^{N-1} \beta_i / L^i, \qquad J \le L \le J + N, \tag{2.21}$$

with $\psi(L) = \psi(x_l)$ as in (2.6), (2.13), and (2.17) for Examples 1, 2, and 3, respectively.

We now note that the equations in (2.21) are identical in form to those defining the $T$-transformation of [2]. Therefore, their solution for $W_n^{(j)}$ can be expressed as

$$W_n^{(j)} = \frac{\Delta^N (J^{N-1} F(J)/\psi(J))}{\Delta^N (J^{N-1}/\psi(J))}, \tag{2.22}$$

where $\Delta$ is the forward difference operator operating on $J$; see, for example, [5, equations (1.1)-(1.3)]. Consequently, the error in $W_n^{(j)}$ can be expressed as

$$I[f] - W_n^{(j)} = \frac{\Delta^N (J^{N-1} R(J)/\psi(J))}{\Delta^N (J^{N-1}/\psi(J))}. \tag{2.23}$$

Now

$$R(J) = \begin{cases} (-1)^J J^{\gamma+1}\,\mathscr{L}[g(\xi); J] & \text{for example 1} \\ (-1)^J \exp(-cJ^2/2)\,\mathscr{L}[g(\xi); J] & \text{for example 2,} \\ 1/(2\pi^2 J) + (2/J)\,\mathscr{L}[g(\xi); J] & \text{for example 3} \end{cases} \tag{2.24}$$

where

$$g(\xi) = \mathrm{Re}\,\omega(\xi), \qquad \xi \ge 0, \text{ for all three examples.} \tag{2.25}$$

Recall that $\omega(\xi)$ is given by (2.10), (2.15), and (2.20) for Examples 1,2, and 3, respectively.

Combining $R(J)$ with the corresponding $\psi(J)$, (2.23) can be expressed as

$$|I[f] - W_n^{(j)}| = \frac{|\Delta^N (J^\nu\,\mathscr{L}[g(\xi); J])|}{|\Delta^N (J^{N-1}/\psi(J))|}, \quad \text{where} \tag{2.26}$$

$$\nu = \begin{cases} N & \text{for Examples 1 and 2} \\ N - 1 & \text{for Example 3} \end{cases}, \tag{2.27}$$

and we have used the fact that

$$\Delta^N (J^k) = 0 \quad \text{for } k = 0, 1, \ldots, N - 1, \tag{2.28}$$

in example 3. Finally, following the developments of [5, Section 4],

$$|I[f] - W_n^{(j)}| = \frac{|\mathscr{L}[(e^{-\xi} - 1)^N g^{(\nu)}(\xi); J]|}{|\Delta^N (J^{N-1}/\psi(J))|}. \tag{2.29}$$

The rate of acceleration of $W_n^{(j)}$ is defined to be the quotient

$$R_n^{(j)} = \left| \frac{I[f] - W_n^{(j)}}{I[f] - F(J + N)} \right| \tag{2.30}$$

for all three examples, since it is $F(1), F(2), \ldots, F(J + N)$ that go into the construction of the approximation $W_n^{(j)}$ to $I[f]$, and $F(J + N)$ is asymptotically the best member of this sequence in case $I[f]$ converges.

## 3. Analysis of $I[f] - W_n^{(j)}$ for process II

We now analyze $I[f] - W_n^{(j)}$ for Process II, i.e., for $j$ fixed and $n \to \infty$. We begin this by stating a variation of a result that was proved in [5] in the analysis of Process II for the $T$-transformation.

LEMMA. *Let $g(\xi)$ be analytic in the half strip $S(u) = \{\xi | \operatorname{Re} \xi \geq -u, |\operatorname{Im}\xi| \leq u\}$ for some $u > 0$, and let $g^{(p)}(\xi)$, for some fixed nonnegative integer $p$, be uniformly bounded in $S(u)$. Then for any positive integers $m$ and $N$, such that $p \leq m \leq N$, we have*

$$|\Delta^N (J^m \mathscr{L}[g(\xi); J])| \leq M \frac{(m - p)!}{u^{m-p}} \frac{N!}{J(J + 1)\ldots(J + N)}, \tag{3.1}$$

*where* $M = \sup_{\xi \in S(u)} |g^{(p)}(\xi)|$.

PROOF. The proof of (3.1) can be achieved by observing that $g^m(\xi) = \dfrac{d^{m-p}}{d\xi^{m-p}} g^{(p)}(\xi)$, and proceeding as in the proof of Lemma 4.1 in [5]. ■

EXAMPLE 1. We have $g(\xi) = \operatorname{Re}\{i(\pi + i\xi)^\gamma/\pi^{\gamma+1}\}$. There are two cases we may consider:

a) $\gamma = 0, 1, 2, \ldots.$
For this case $g(\xi)$ is a polynomial of degree $\leq \gamma$ in $\xi$, thus $J^\nu \mathscr{L}[g(\xi); J]$ is a polynomial of degree $\leq N - 1$ in $J$ when $N \geq \gamma + 1$. Consequently, by (2.26) and (2.28), $W_n^{(j)} = I[f]$ for $n \geq \gamma$.

b) $\gamma \neq 0, 1, 2, \ldots.$
For this case $g(\xi)$ has two branch points at $\xi = \pm i\pi$ in the complex $\xi$ plane, but is analytic in any half strip $S(u)$ with $u < \pi$. When $\gamma > 0$, $g(\xi)$ is not bounded for $\xi \to \infty$, but $g^{(p)}(\xi)$ is uniformly bounded in $S(u)$ for $\gamma < p < \gamma + 1$. Thus

$$|\Delta^N(J^\nu \mathscr{L}[g(\xi); J])| \leq M \frac{(\nu - p)!}{u^{\nu - p}} \frac{N!}{J(J+1)\dots(J+N)} \tag{3.2}$$

in the notation of the lemma above. When $\gamma < 0$, $g(\xi)$ is uniformely bounded in $S(u)$, thus (3.2) holds with $p = 0$ in this case. Recalling that $\nu = N$ for Example 1, and using Stirling's formula, we obtain

$$|\Delta^N(J^\nu \mathscr{L}[g(\xi); J])| \leq O(N! u^{-N} N^{-J-p}) \quad \text{as } N \to \infty. \tag{3.3}$$

For both $\gamma > 0$ and $\gamma < 0$ we have

$$|\Delta^N(J^{N-1}/\psi(J))| = \sum_{q=0}^{N} \binom{N}{q} (J+q)^{N-1}/(J+q)^\gamma > \binom{N}{\lfloor N/2 \rfloor} (J + \lfloor N/2 \rfloor)^{N-1-\gamma}. \tag{3.4}$$

By Stirling's formula the right hand side of this inequality becomes

$$\binom{N}{\lfloor N/2 \rfloor} (J + \lfloor N/2 \rfloor)^{N-1-\gamma} \sim C\, N! e^N N^{-\gamma-1} \quad \text{as } N \to \infty, \tag{3.5}$$

for some positive constant $C$. Combining all the above in (2.26), we obtain

$$I[f] - W_n^{(j)} = O((ue)^{-N} N^{-J-p+\gamma+1}) \quad \text{as } N \to \infty. \tag{3.6}$$

The rate of acceleration of the $W$-transformation for Example 1 thus satisfies

$$R_n^{(j)} = \left| \frac{I[f] - W_n^{(j)}}{I[f] - F(J+N)} \right| = O((ue)^{-N} N^{-J-p+1}) \quad \text{as } N \to \infty, \tag{3.7}$$

where $u = \pi - \delta$ for some $\delta > 0$ arbitrarily small, and $p$, an integer, such that $p = 0$ for $\gamma < 0$ and $\gamma < p < \gamma + 1$ for $\gamma > 0$.

EXAMPLE 2: In this case we have $g(\xi) = \operatorname{Re}\{c^{-1} \exp[c^{-1}(i\pi\xi - \xi^2/2)]\}$. Here $g(\xi)$ is an entire function, uniformly bounded in $S(u)$ for *any* $u > 0$. Thus, applying the lemma above, as in Example 1, we obtain

$$|\Delta^N(J^\nu \mathscr{L}[g(\xi); J])| = O(N! u^{-N} N^{-J}) \quad \text{as } N \to \infty. \tag{3.8}$$

Also

$$\begin{aligned} |\Delta^N(J^{N-1}/\psi(J))| &= \sum_{q=0}^{N} \binom{N}{q} (J+q)^N e^{c(J+q)^2/2} > (J+N)^N e^{c(J+N)^2/2} \\ &\sim C\, N!\, e^N N^{-1/2} e^{c(J+N)^2/2} \quad \text{as } N \to \infty, \end{aligned} \tag{3.9}$$

for some positive constant $C$, the last part of (3.9) following from Stirling's formula.

Combining (3.8) and (3.9) in (2.26), we obtain

$$I[f] - W_n^{(j)} = O((ue)^{-N} N^{-J+1/2} e^{-c(J+N)^2/2}) \quad \text{as } N \to \infty, \tag{3.10}$$

the rate of acceleration satisfying

$$R_n^{(j)} = \left| \frac{I[f] - W_n^{(j)}}{I[f] - F(J+N)} \right| = O((ue)^{-N} N^{-J+3/2}) \quad \text{as } N \to \infty. \tag{3.11}$$

Note that since $u$ for this example can be arbitrarily large, (3.11) implies that

$$R_n^{(j)} = o(e^{-\alpha N}) \quad \text{as } N \to \infty, \text{ for any } \alpha > 0. \tag{3.12}$$

This immediately suggests the question whether $R_n^{(j)} = O(\exp(-\alpha N^{1+\beta}))$ for some $\alpha > 0$ and $\beta > 0$, to which we do not have an answer yet.

EXAMPLE 3: We have $g(\xi) = \mathrm{Re}\{2\pi i(2\pi + i\xi)^2\}^{-1}$. Here $g(\xi)$ has two double poles at $\xi = \pm 2\pi i$, thus it is uniformly bounded in $S(u)$ with $u < 2\pi$. Consequently, applying the lemma, we have

$$|\Delta^N(J^\nu \mathscr{L}[g(\xi); J])| = O(N! u^{-N} N^{-J-1}) \quad \text{as } N \to \infty. \tag{3.13}$$

Also

$$|\Delta^N(J^{N-1}/\psi(J))| = |\Delta^N(J^N)|/2 = N!/2. \tag{3.14}$$

Combining (3.13) and (3.14) in (2.26), we finally obtain

$$I[f] - W_n^{(j)} = O(u^{-N} N^{-J-1}) \quad \text{as } N \to \infty. \tag{3.15}$$

The rate of acceleration for this case satisfies

$$R_n^{(j)} = \left| \frac{I[f] - W_n^{(j)}}{I[f] - F(J+N)} \right| = O(u^{-N} N^{-J}) \quad \text{as } N \to \infty. \tag{3.16}$$

where $u = 2\pi - \delta$ for $\delta > 0$ arbitrarily small.

Finally, the results stated in (3.6) and (3.7) for Example 1 and those in (3.10) and (3.11) for Example 2 remain valid when $\cos \pi t$ in $I[f]$ in both examples is replaced by $\sin \pi t$. To see this we observe that the only change in the analysis takes place in $g(\xi)$, with $g(\xi) = \mathrm{Re}\, \omega(\xi)$ in (2.25) replaced by $g(\xi) = \mathrm{Im}\, \omega(\xi)$ in both examples. The rest of the analysis and conclusions remain the same.

## 4. Concluding remarks.

We have analyzed the convergence and rate of acceleration for the $W$- and $\bar{D}$-transformations as they are applied to three oscillatory infinite integrals, having different behaviors at infinity. We have shown the following:

1) For the $W$-transformation, as applied to $I[f] = \int_0^\infty t^\gamma v(\pi t)dt$, $\gamma > -1$, where $v(x)$ stands for $\cos x$ or $\sin x$, $\limsup_{n \to \infty} (R_n^{(j)})^{1/n} \le (\pi e)^{-1} < 1$ as $n \to \infty$.

2) For the $W$-transformation, as applied to $I[f] = \int_0^\infty e^{-ct^2/2} v(\pi t)dt$, where again $v(x)$ stands for $\cos x$ or $\sin x$,

$$R_n^{(j)} = o(e^{-\alpha n}) \quad \text{as } n \to \infty, \text{ for all } \alpha > 0.$$

3) For the $\bar{D}$-transformation, as applied to $I[f] = \int_0^\infty (\sin \pi t/\pi t)^2 dt$,

$$\limsup_{n\to\infty} (R_n^{(j)})^{1/n} \le (2\pi)^{-1} < 1 \quad \text{as } n \to \infty.$$

All of these results are much stronger than the ones that can be obtained by applying the existing results in [6], [7], and [9] as has already been mentioned. Indeed, the information that can be extracted from the existing results is

$$I[f] - W_n^{(j)} = o(n^{-\mu}) \quad \text{as } n \to \infty, \text{ for all } \mu > 0,$$

for Examples 1 and 2, no such information being available for Example 3. From this it follows that we can obtain

$$R_n^{(j)} = o(n^{-\mu}) \quad \text{as } n \to \infty, \text{ for all } \mu > 0,$$

only for Example 1, and no information on the rate of acceleration for Example 2. In spite of the above, however, the results of the present work are still not the best possible. This is mainly due to our employing the inequality given in (3.1) in the Lemma to bound the numerators in the right hand side of (2.26). Efforts to improve the Lemma have not been successful so far. Further improvement in Example 1 can be obtained by using a better lower bound in (3.4), although we shall omit this here.

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