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On a Generalization of the Richardson Extrapolation Process

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Summary. A convergence result for a generalized Richardson extrapolation process is improved upon considerably and additional results of interest are proved. An application of practical importance is also given. Finally, some known results concerning the convergence of Levin's T-transformation are reconsidered in light of the results of the present work.

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1 A Generalized Richardson Extrapolation

Let $A(y)$ be a scalar function of a discrete or continuous variable y, defined for $0<y\leqq b<\infty$. Let there exist constants A and α_k, $k=1, 2, \ldots$, and functions $\phi_k(y)$, $k=1, 2, \ldots$, which form an asymptotic sequence in the sense that

$$\phi_{k+1}(y)=o(\phi_k(y)) \quad \text{as} \quad y\to 0+, \tag{1.1}$$

and assume that $A(y)$ has the asymptotic expansion

$$A(y)\sim A+\sum_{k=1}^{\infty}\alpha_k\,\phi_k(y) \quad \text{as} \quad y\to 0+. \tag{1.2}$$

Here $A(y)$ and $\phi_k(y)$, $k=1, 2, \ldots$, are assumed to be known for $0<y\leqq b$, but A and α_k, $k=1, 2, \ldots$, are unknown. The problem is to approximate A, which, in many cases is $\lim_{y\to 0+} A(y)$ when the latter exists. (When $\lim_{y\to 0+} A(y)$ does not exist, A is said to be the antilimit of $A(y)$ as $y\to 0+$.)

Pick a decreasing sequence y_l, $l=0, 1, 2, \ldots$, in $(0, b]$, such that $\lim_{l\to\infty} y_l=0$. Then, for each pair (j, p) of nonnegative integers, the solution for A_p^j of the system of linear equations

$$A(y_l)=A_p^j+\sum_{k=1}^{p}\bar{\alpha}_k\,\phi_k(y_l), \quad j\leqq l\leqq j+p, \tag{1.3}$$

is taken as an approximation to A. We note that the equations in (1.3) are obtained by truncating the infinite sum in (1.2) at the term $\alpha_p\,\phi_p(y)$, replacing A and $\alpha_1, \ldots, \alpha_p$ by A_p^j and $\bar{\alpha}_1, \ldots, \bar{\alpha}_p$ respectively, treating the latter as unknowns ($p+1$ in number), and collocating at the points $y_j, y_{j+1}, \ldots, y_{j+p}$ to obtain $p+1$ linear equations for these unknowns. The process described by (1.3) is known as a generalized Richardson extrapolation process.

The general setting of (1.1)–(1.3) above can be found in Hart et al. (1968, p. 39). By Cramer's rule it can be shown that A_p^j has a determinant representation. Levin (1973) seems to be the first to point this out explicitly. This representation may be used effectively in deriving algorithms for computing the A_p^j, and also in analyzing them. As pointed out also by Brezinski (1980), many of the known convergence acceleration methods are directly or indirectly related to the general setting described above.

It seems that Schneider (1975) was the first to give a recursive algorithm for the implementation of the extrapolation process defined through (1.3) when the $\phi_k(y_l)$ have no particular structure. Using different techniques, the same algorithm was later derived by Håvie (1979), and then by Brezinski (1980), who called it the E-algorithm. Håvie (1979) also gave simplified versions of the E-algorithm for some special cases.

Recently Ford and Sidi (1987) derived a different recursive algorithm for implementing the same extrapolation process, and their algorithm turns out to be computationally more economical than the E-algorithm. Indeed, the E-algorithm and the Ford-Sidi algorithm require approximately the same number of additions, but the number of multiplications required by the E-algorithm is practically twice that required by the Ford-Sidi algorithm. The steps of the Ford-Sidi algorithm are summarized below:

Denote an arbitrary sequence $b(l)$, $l=0, 1, 2, \ldots$, by b. The sequence $1, 1, 1, \ldots$, will be denoted by I. Define the sequences g_k, $k=1, 2, \ldots$, and a by $g_k(l)=\phi_k(y_l)$, $k=1, 2, \ldots$, and $a(l)=A(y_l)$, $l=0, 1, 2, \ldots$.

(1) For $b=a$, I, and $b=g_k$, $k=2, 3, \ldots$, set $\psi_0^j(b)=b(j)/g_1(j)$, $j=0, 1, \ldots$.

(2) For $p=1, 2, \ldots$, let $D_p^j=\psi_{p-1}^{j+1}(g_{p+1})-\psi_{p-1}^j(g_{p+1})$ and $\psi_p^j(b)=[\psi_{p-1}^{j+1}(b)-\psi_{p-1}^j(b)]/D_p^j$ with $b=a$, I, and $b=g_k$, $k\geqq p+1$, $j=0, 1, \ldots$.

(3) Set $A_p^j=\psi_p^j(a)/\psi_p^j(I)$, all j, $p\geqq 0$.

For details see Ford and Sidi (1987).

In the problem that is solved by the classical Richardson extrapolation process $\phi_i(y)=y^i$, $i=1, 2, \ldots$, and this problem arises, for example, from finite difference approximations of derivatives, Euler-Maclaurin expansions for the trapezoidal rule approximation of finite range integrals of smooth functions, etc. Bulirsch and Stoer (1964) consider the case in which $\phi_i(y)=y^{\gamma_i}$, $i=1, 2, \ldots$, with arbitrary real γ_i and $y_l=y_0\,\rho^l$, $0<\rho<1$, and give a thorough convergence analysis for it.

A further generalization of the classical Richardson extrapolation process designated GREP was formulated in Sidi (1979a). In the same paper general convergence results for GREP and several practical examples that are covered by the framework of GREP are given. We will only mention that the D-transformation for infinite integrals and the d-transformation for infinite series of Levin and Sidi (1981) are covered by the framework of GREP. These are some of the most effective convergence acceleration methods for dealing with infinite integrals and series of various kinds, and their scope is much larger than most other methods of acceleration. The paper of Ford and Sidi (1987) gives a very efficient recursive algorithm for implementing a commonly occurring form of GREP. For further details on this and related matters and for more references see also the recent survey of Sidi (1988).

It is known in some cases and is observed numerically in many others that, for appropriate choices of y_l, A_p^j converges to A for $j\to\infty$ with p fixed or for $p\to\infty$ with j fixed. Furthermore, for some cases of interest it can be shown that

$$A_p^j - A = O(\phi_{p+1}(y_j)) \quad \text{as} \quad j\to\infty,\ p \text{ fixed}, \tag{1.4}$$

while better results can be obtained for others. We elaborate on this point in some detail in Sect. 4 of the present work. Under the conditions

$$\lim_{l\to\infty} \frac{\phi_k(y_{l+1})}{\phi_k(y_l)} = b_k \neq 1, \qquad k=1,2,\ldots, \tag{1.5}$$

and

$$b_j \neq b_k \qquad \text{for } j\neq k, \tag{1.6}$$

Brezinski (1980) shows that

$$\lim_{j\to\infty} \frac{A_p^j - A}{A_{p-1}^j - A} = 0, \qquad p=1,2,\ldots, \tag{1.7}$$

with $A_0^j = A(y_j)$, $j=0,1,\ldots$. This result says that in case $A=\lim_{y\to 0} A(y)$, A_p^j converges to A as $j\to\infty$ more quickly than A_{p-1}^j, but it does not give any information of the form (1.4), or of some other form, concerning the nature or rate of convergence. A result of the form (1.4), under the conditions (1.5) and (1.6), but with stringent conditions on the series $\sum_{k=1}^{\infty} \alpha_k\,\phi_k(y)$ in (1.2), was subsequently given by Wimp (1981, pp. 189–190).

The purpose of the present work is to present a detailed analysis for A_p^j for $j\to\infty$ with p fixed, under the conditions (1.5) and (1.6), and with *no* additional conditions on (1.2). In fact, in the next section we show that (1.4) holds under these conditions. In addition we show that this process is a stable one. We also show that the $\bar{\alpha}_k$ in (1.3) are valid approximations to the corresponding α_k in (1.2). We give precise rates of convergence of $\bar{\alpha}_k$ to α_k for $j\to\infty$ with p fixed. The techniques employed in our proofs are based on those of Sidi (1979a) and Wimp (1981). In Sect. 3 we apply the results of Sect. 2 to the trapezoidal rule approximation of integrals with end point singularities. Finally, in Sect. 4

we discuss some convergence results for the T-transformations of Levin (1973) in the light of the developments of Sect. 2. These results are taken from Sidi (1979b) and Sidi (1980).

For simplicity of notation, from here on we let $A(y_l)=a(l)$ and $\phi_k(y_l)=g_k(l)$, so that (1.1), (1.2), and (1.3) become

$$g_{k+1}(n)=o(g_k(n)) \quad \text{as} \quad n\to\infty, \tag{1.8}$$

$$a(n)\sim A+\sum_{k=1}^{\infty}\alpha_k\, g_k(n) \quad \text{as} \quad n\to\infty, \tag{1.9}$$

and

$$a(l)=A_p^j+\sum_{k=1}^{p}\bar{\alpha}_k\, g_k(l), \qquad j\leqq l\leqq j+p, \tag{1.10}$$

respectively.

The solution of (1.10) for A_p^j can be expressed as the quotient of two determinants in the form

$$A_p^j=f_p^j(a)/f_p^j(I), \tag{1.11}$$

where, for any sequence $b(l)$, $l=0, 1, \ldots,$

$$f_p^j(b)=\begin{vmatrix} g_1(j) & \ldots & g_p(j) & b(j)\\ g_1(j+1) & \ldots & g_p(j+1) & b(j+1)\\ \vdots & & \vdots & \vdots\\ g_1(j+p) & \ldots & g_p(j+p) & b(j+p)\end{vmatrix}, \tag{1.12}$$

and the sequence $I(l)$, $l=0, 1, \ldots$, is defined such that $I(l)=1$ for all l.

2 Theory

We assume that (1.5) and (1.6) hold. In terms of the $g_k(l)$, (1.5) is expressed as

$$\lim_{l\to\infty}\frac{g_k(l+1)}{g_k(l)}=b_k\neq 1, \qquad k=1, 2, \ldots. \tag{2.1}$$

Lemma 2.1. *Define the polynomial $H_p^j(\lambda)$ by*

$$H_p^j(\lambda)=\begin{vmatrix} g_1(j) & \ldots & g_p(j) & 1\\ g_1(j+1) & \ldots & g_p(j+1) & \lambda\\ \vdots & & \vdots & \vdots\\ g_1(j+p) & \ldots & g_p(j+p) & \lambda^p\end{vmatrix}. \tag{2.2}$$

Then

$$\lim_{j\to\infty} \frac{H_p^j(\lambda)}{\prod_{i=1}^{p} g_i(j)} = V(b_1, b_2, \ldots, b_p, \lambda), \tag{2.3}$$

where $V(\xi_1, \ldots, \xi_k)$ *is the Vandermonde determinant defined by*

$$V(\xi_1, \ldots, \xi_k) = \begin{vmatrix} 1 & \ldots & 1 \\ \xi_1 & \ldots & \xi_k \\ \vdots & & \vdots \\ \xi_1^{k-1} & \ldots & \xi_k^{k-1} \end{vmatrix} = \prod_{1\leq i<j\leq k} (\xi_j - \xi_i). \tag{2.4}$$

Proof. Dividing the ith column of $H_p^j(\lambda)$ by $g_i(j)$, $i=1, \ldots, p$, and letting

$$\tilde{g}_i^j(s) = \frac{g_i(j+s)}{g_i(j)}, \qquad s=0, 1, \ldots, \tag{2.5}$$

we obtain

$$\frac{H_p^j(\lambda)}{\prod_{i=1}^{p} g_i(j)} = \begin{vmatrix} 1 & \ldots & 1 & 1 \\ \tilde{g}_1^j(1) & \ldots & \tilde{g}_p^j(1) & \lambda \\ \vdots & & \vdots & \vdots \\ \tilde{g}_1^j(p) & \ldots & \tilde{g}_p^j(p) & \lambda^p \end{vmatrix}. \tag{2.6}$$

The result follows from

$$\lim_{j\to\infty} \tilde{g}_i^j(s) = b_i^s, \qquad s=0, 1, \ldots, \quad i=1, 2, \ldots, \tag{2.7}$$

which in turn follows from

$$\tilde{g}_i^j(s) = \frac{g_i(j+s)}{g_i(j+s-1)} \cdot \frac{g_i(j+s-1)}{g_i(j+s-2)} \cdots \frac{g_i(j+1)}{g_i(j)} \tag{2.8}$$

and (2.1). □

Corollary.

$$\lim_{j\to\infty} \frac{f_p^j(I)}{\prod_{i=1}^{p} g_i(j)} = V(b_1, \ldots, b_p, 1) \neq 0. \tag{2.9}$$

Proof. (2.9) follows from the fact that $f_p^j(I) = H_p^j(1)$ and from the assumptions $b_i \neq b_k$ for $i \neq k$ and $b_i \neq 1$ for all i. □

Theorem 2.2. *For p fixed A_p^j satisfies precisely*

$$A_p^j - A \sim \alpha_{p+1}\left[\prod_{i=1}^{p}\left(\frac{b_{p+1}-b_i}{1-b_i}\right)\right] g_{p+1}(j) \quad \text{as} \quad j\to\infty. \tag{2.10}$$

Proof. Letting

$$r(l) = a(l) - A, \qquad l = 0, 1, \dots, \tag{2.11}$$

we have

$$A_p^j - A = f_p^j(r)/f_p^j(I). \tag{2.12}$$

Now if we define $\varepsilon_{p+1}(n)$ by

$$r(n) = \sum_{k=1}^{p} \alpha_k\, g_k(n) + \alpha_{p+1}\, g_{p+1}(n)[1+\varepsilon_{p+1}(n)], \qquad n = 0, 1, \dots, \tag{2.13}$$

then (1.8) and (1.9) imply that

$$\lim_{n\to\infty} \varepsilon_{p+1}(n) = 0. \tag{2.14}$$

Substituting (2.13) in the determinant representation of $f_p^j(r)$, we have

$$\begin{aligned} f_p^j(r) &= \sum_{k=1}^{p} \alpha_k\, f_p^j(g_k) + \alpha_{p+1}\, f_p^j(g_{p+1}[1+\varepsilon_{p+1}]) \\ &= \alpha_{p+1}\, f_p^j(g_{p+1}[1+\varepsilon_{p+1}]), \end{aligned} \tag{2.15}$$

the last equality following from the fact that $f_p^j(g_k) = 0$, $k = 1, \dots, p$. Dividing now the ith column in $f_p^j(g_{p+1}[1+\varepsilon_{p+1}])$ by $g_i(j)$, $i = 1, \dots, p+1$, letting $j\to\infty$, and employing (2.7) and (2.14), we obtain

$$\lim_{j\to\infty} \frac{f_p^j(r)}{\prod_{i=1}^{p+1} g_i(j)} = \alpha_{p+1}\, V(b_1, \dots, b_p, b_{p+1}). \tag{2.16}$$

Substituting (2.16) and (2.9) in (2.12), and employing (2.4), (2.10) follows. □

Note. Theorem 2.2 shows that under the conditions stated in (1.8) and (1.9) and in (2.1) and (1.6), A_p^j tends to A more quickly than A_{p-1}^j for $j\to\infty$, i.e.,

$$\frac{A_p^j - A}{A_{p-1}^j - A} = O\left(\frac{g_{p+1}(j)}{g_p(j)}\right) = o(1) \quad \text{as} \quad j\to\infty, \tag{2.17}$$

which contains (1.7) that was given in Brezinski (1980, Theorem 7). The result of Theorem 2.2 has also been obtained by Wimp (1981, pp. 189–190) under the additional assumptions that (1.8) holds uniformly in k and $|\alpha_k| < \lambda^k$, $k = 1, 2, \dots$, for some λ, in which case $a(n)$ has the *convergent* expansion $a(n) = A$

$+\sum_{k=1}^{\infty} \alpha_k g_k(n)$ and this expansion converges *absolutely*, and *uniformly* in n. Thus our result improves that of Wimp considerably. It should be emphasized that in case $\lim_{n\to\infty} a(n)$ does not exist, then $\lim_{n\to\infty} g_k(n)$, at least for $k=1$, does not exist. In this case $A_p^j \to A$ as $j\to\infty$ provided $\lim_{n\to\infty} g_{p+1}(n)=0$, otherwise $\lim_{j\to\infty} A_p^j$ does not exist, although (2.17) always holds.

We now give a convergence result on the $\bar{\alpha}_k \equiv \alpha_{p,k}^j$ that appear in (1.10).

Theorem 2.3. *For* $k=1, \ldots, p$, $\alpha_{p,k}^j \to \alpha_k$ *as* $j\to\infty$ *with* p *fixed; in fact*

$$\text{(2.18)}\quad \alpha_{p,k}^j - \alpha_k \sim \alpha_{p+1}\left(\frac{b_{p+1}-1}{b_k-1}\right)\left[\prod_{\substack{i=1\\ i\neq k}}^{p}\left(\frac{b_{p+1}-b_i}{b_k-b_i}\right)\right]\frac{g_{p+1}(j)}{g_k(j)} \quad as \quad j\to\infty.$$

Proof. Solving (1.10) for $\bar{\alpha}_k$ by Cramer's rule, we obtain

$$\text{(2.19)}\quad \alpha_{p,k}^j f_p^j(I) = \begin{vmatrix} g_1(j) & \cdots & g_{k-1}(j) & a(j) & g_{k+1}(j) & \cdots & g_p(j) & 1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ g_1(j+p) & \cdots & g_{k-1}(j+p) & a(j+p) & g_{k+1}(j+p) & \cdots & g_p(j+p) & 1 \end{vmatrix}.$$

Substituting

$$a(n) = A + \sum_{k=1}^{p} \alpha_k g_k(n) + \alpha_{p+1} g_{p+1}(n)[1+\varepsilon_{p+1}(n)], \qquad n=0, 1, \ldots,$$

cf., (2.13) and (2.14), on the right hand side of (2.19), and expanding the resulting determinant with respect to its kth column, and dividing both sides by $f_p^j(I)$, we obtain

$$\text{(2.20)}\qquad \alpha_{p,k}^j = \alpha_k + \alpha_{p+1}\, U_{p,k}^j / f_p^j(I),$$

where $U_{p,k}^j$ is the determinant obtained by replacing $a(n)$, $n=j, \ldots, j+p$, on the right hand side of (2.19) by $g_{p+1}(n)[1+\varepsilon_{p+1}(n)]$, $n=j, \ldots, j+p$, respectively. The rest of the proof can now be completed by dividing the ith column of $U_{p,k}^j$ by $g_i(j)$, $i=1, \ldots, p$, $i\neq k$, and its kth column by $g_{p+1}(j)$, and by letting $j\to\infty$, recalling at the same time (2.7) and (2.3), and (2.4). □

Note. The result in (2.18) provides a precise rate of convergence for $\alpha_{p,k}^j$, to α_k, $k=1, \ldots, p$, for $j\to\infty$. Also for $1\leqq k_1 < k_2 \leqq p$, (2.18) implies

$$\text{(2.21)}\qquad \frac{\alpha_{p,k_1}^j - \alpha_{k_1}}{\alpha_{p,k_2}^j - \alpha_{k_2}} = O\left(\frac{g_{k_2}(j)}{g_{k_1}(j)}\right) = o(1) \quad \text{as} \quad j\to\infty,$$

which means that $\alpha_{p,1}^j$ converges more quickly than $\alpha_{p,2}^j$, which in turn converges more quickly than $\alpha_{p,3}^j$, etc. It should be emphasized that as $j\to\infty$ $\alpha_{p,k}^j \to \alpha_k$, $1\leqq k\leqq p$, always. (Recall the statement at the end of the note following the proof of Theorem 2.2 concerning $\lim_{j\to\infty} A_p^j$.)

As is obvious from (1.11) and as has been observed by many authors, A_p^j can be expressed in the form

(2.22)
$$A_p^j = \sum_{i=0}^{p} \gamma_{p,i}^j \, a(j+i),$$

where $\gamma_{p,i}^j$ are coefficients that depend on $g_1, \ldots, g_p$. Actually $\gamma_{p,i}^j$ is the cofactor of $b(j+i)$ in (1.12) divided by $f_p^j(I)$. It is also obvious that

(2.23)
$$\sum_{i=0}^{p} \gamma_{p,i}^j = 1.$$

Theorem 2.4 below and its corollary show, among other things, that the extrapolation process in which p is held fixed and $j \to \infty$ is stable under the conditions assumed in this section, in the sense that $\sup_j \sum_{i=0}^{p} |\gamma_{p,i}^j| < \infty$.

Theorem 2.4. *For p fixed* $\lim_{j\to\infty} \gamma_{p,i}^j = \tilde{\gamma}_i$, $i = 0, 1, \ldots, p$, *exist, and*

(2.24)
$$\lim_{j\to\infty} \sum_{i=0}^{p} \gamma_{p,i}^j \lambda^i = \sum_{i=0}^{p} \tilde{\gamma}_i \lambda^i = \prod_{i=1}^{p} \left(\frac{\lambda - b_i}{1 - b_i} \right).$$

Consequently

(2.25)
$$\sum_{i=0}^{p} |\gamma_{p,i}^j| \sim \sum_{i=0}^{p} |\tilde{\gamma}_i| < \infty \quad as \quad j \to \infty,$$

implying the stability of A_p^j *for all j sufficiently large.*

Proof. We observe that

(2.26)
$$\sum_{i=0}^{p} \gamma_{p,i}^j \lambda^i = \frac{H_p^j(\lambda)}{f_p^j(I)} = \frac{H_p^j(\lambda)}{H_p^j(1)}.$$

The proof can now be completed by invoking (2.3) and (2.4). □

Corollary. *If* $b_k > 0$, $k = 1, \ldots, p$, *then* $\gamma_{p,i+1}^j \gamma_{p,i}^j < 0$, $i = 0, 1, \ldots, p-1$, *for all j sufficiently large, and* $\sum_{i=0}^{p} |\gamma_{p,i}^j| \sim \prod_{i=1}^{p} \left| \frac{1+b_i}{1-b_i} \right|$ *as* $j \to \infty$. *If* $b_k < 0$, $k = 1, \ldots, p$, *then* $\gamma_{p,i}^j > 0$, $i = 0, 1, \ldots, p$, *for all j sufficiently large, hence* $\sum_{i=0}^{p} |\gamma_{p,i}^j| \sim 1$ *as* $j \to \infty$.

Proof. Left to the reader. □

Note. Through (2.22) A_p^j with fixed p can be viewed as a summability method, which has a banded summability matrix that contains at most $p+1$ nonzero elements in each row. Invoking (2.23) and Theorem 2.4, it is seen that this matrix satisfies all three conditions of the Silverman-Toeplitz theorem (see Powell and Shah (1972, pp. 23–27)) that guarantee that the summability method in question is regular.

3 An Example

Let $f(x)$ be infinitely differentiable on $[0, 1]$ and let $-1<s<0$ be fixed. Let $h=1/n$, where n is a positive integer. Define the trapezoidal rule approximation $T(h)$ to $I=\int_0^1 F(x)\,dx$ with $F(x)=x^s \log x\, f(x)$ by

$$T(h)=h\sum_{i=1}^{n-1} F(ih)+\frac{h}{2}F(1). \tag{3.1}$$

Then by a result due to Navot (1962)

$$\begin{aligned} T(h)\sim I &+\sum_{i=1}^{\infty}\frac{B_{2i}}{(2i)!}F^{(2i-1)}(1)\,h^{2i} \\ &+\sum_{i=0}^{\infty}[\zeta(-s-i)\log h-\zeta'(-s-i)]\frac{f^{(i)}(0)}{i!}h^{i+s+1} \quad \text{as} \quad h\to 0, \end{aligned} \tag{3.2}$$

where B_k are Bernoulli numbers and $\zeta(t)$ is the Riemann zeta function and $\zeta'(t)=d\zeta/dt$.

We note that, in general, the asymptotic expansion in (3.2) is divergent. This may happen, for example, when $f(x)$ is infinitely differentiable but not analytic on the interval $[0, 1]$.

Obviously, in (3.2) h is the discrete variable corresponding to y in the first paragraph of Sect. 1, and $T(h)$ and I are $A(y)$ and A respectively. Consequently,

$$\phi_k(h)=\begin{cases}[\zeta(-s-i+1)\log h-\zeta'(-s-i+1)]\,h^{i+s}, \\ \qquad i=k-[k/3],\ k=1, 2, 4, 5, 7, 8, \ldots, \\ h^{2k/3}, \\ \qquad k=3, 6, 9, \ldots. \end{cases} \tag{3.3}$$

It is clear that (1.1) is satisfied.

Letting now $y_l=h_l=h_0\rho^l$, $0<\rho<1$, we can see that

$$\begin{aligned} b_k&=\lim_{n\to\infty}\frac{g_k(n+1)}{g_k(n)}=\lim_{n\to\infty}\frac{\phi_k(h_{n+1})}{\phi_k(h_n)} \\ &=\begin{cases}\rho^{i+s}, & i=k-[k/3],\ k=1, 2, 4, 5, 7, 8, \ldots, \\ \rho^{2k/3}, & k=3, 6, 9, \ldots, \end{cases} \end{aligned} \tag{3.4}$$

i.e., both (1.5) (hence (2.1)) and (1.6) are satisfied. Consequently, when I is approximated by A_p^j as defined through (1.10), with $g_k(n)=\phi_k(h_0\rho^n)$ and $\phi_k(h)$ as given in (3.3), Theorems 2.2–2.4 hold. From (3.3) we notice that the zeta function and its derivative are needed for constructing the $\phi_k(h)$. The computation of $\zeta(-s-i)$, $i=0, 1, \ldots$, can be achieved by computing $\zeta(z)$ for $z>1$, and then using the reflection formula relating $\zeta(1-z)$ to $\zeta(z)$. Once $\zeta(z)$ can be computed for all real z, $\zeta'(-s-i)$, $i=0, 1, \ldots$, can be approximated with high accuracy by using an appropriate numerical differentiation formula.

Other examples involving integrals of functions $F(x)$ having more than one singularity can also be considered. This is left to the reader.

4 The T-Transformation Revisited

In the introduction we mentioned that only (1.4) can be shown to hold in some cases, while better results can be obtained for others. We now elaborate on this point by considering the T-transformation of Levin (1973), of which the t- and u-transformations are two special cases. The t- and u-transformations have proved to be very efficient for some classes of infinite sequences. For detailed convergence analyses of these methods see Sidi (1979b) and Sidi (1980).

For a given sequence $A_1, A_2, A_3, \ldots$, the T-transformation is defined by

$$(4.1) \qquad T_{k,n}=\frac{\sum_{j=0}^{k}(-1)^j\binom{k}{j}(n+j)^{k-1}A_{n+j}/R_{n+j}}{\sum_{j=0}^{k}(-1)^k\binom{k}{j}(n+j)^{k-1}/R_{n+j}}, \qquad k\geqq 0,\ n\geqq 1,$$

for some sequence $R_1, R_2, R_3, \ldots$. We recall that for the t-transformation one takes $R_1=A_1$, $R_r=A_r-A_{r-1}$, $r=2, 3, \ldots$, whereas for the u-transformation $R_1=A_1$, $R_r=r(A_r-A_{r-1})$, $r=2, 3, \ldots$. We note that the T-transformation is one of the simplest forms of GREP. In addition, it is also a generalized Richardson extrapolation process of the form considered in the present work, for which

(1) $A(y)\leftrightarrow A_r$, thus $y\leftrightarrow r^{-1}$, hence y is a discrete variable that takes on the values $1, 1/2, 1/3, \ldots$.

(2) $\phi_i(y)\leftrightarrow R_r r^{-i+1}$, $i=1, 2, \ldots$.

(3) $A_p^j\leftrightarrow T_{p,j}$ for $j\geqq 1$, with $y_l=1/l$, $l=1, 2, \ldots$. (A_0, R_0, and y_0 are simply not defined in this case, and this is possible as they are not needed for A_p^j, $j\geqq 1$, see (1.3).)

The following results for the T-transformation are true:

Theorem 4.1. *Let the sequence $A_1, A_2, \ldots$, be such that*

$$(4.2) \qquad A_r=A+R_r f(r), \qquad r=1, 2, \ldots,$$

where $f(r)$ and R_r have Poincaré-type asymptotic expansions of the forms

$$(4.3) \qquad f(r)\sim\sum_{i=0}^{\infty}\beta_i/r^i \quad as \quad r\to\infty, \qquad \beta_0\neq 0,$$

and

$$(4.4)\qquad R_r \sim \sum_{i=0}^{\infty} \delta_i / r^{\sigma+i} \quad \text{as} \quad r\to\infty, \quad \sigma>0, \quad \delta_0 \neq 0.$$

Then, when $\beta_p \neq 0$,

$$(4.5)\qquad A_p^j - A \sim \frac{\delta_0 \beta_p}{\binom{-\sigma}{p}} j^{-p-\sigma} \quad \text{as} \quad j\to\infty,$$

and

$$(4.6)\qquad \alpha_{p,k}^j - \beta_{k-1} = O(j^{-p+k-1}) \quad \text{as} \quad j\to\infty, \quad 1\leqq k\leqq p,$$

and

$$(4.7)\qquad \lim_{j\to\infty} \sum_{i=0}^{p} |\gamma_{p,i}^j| = \infty.$$

Note that under the conditions $\beta_0 \neq 0$ and $\delta_0 \neq 0$ stated in (4.3) and (4.4), $\sigma>0$ is necessary and sufficient for $\lim_{r\to\infty} A_r = A$ to hold. It is clear that A_p^j in Theorem 4.1 satisfies (1.4) exactly since (4.5) with (4.4) imply that

$$(4.5)'\qquad A_p^j - A \sim \frac{\beta_p}{\binom{-\sigma}{p}} \phi_{p+1}(y_j) \quad \text{as} \quad j\to\infty.$$

Consequently, (2.17) is also satisfied. Similarly, (4.6) with (4.4) imply that

$$(4.6)'\qquad \alpha_{p,k}^j - \beta_{k-1} = O\left(\frac{\phi_{p+1}(y_j)}{\phi_k(y_j)}\right) \quad \text{as} \quad j\to\infty, \quad 1\leqq k\leqq p,$$

which, qualitatively speaking, is the result in (2.18). The result in (4.7), however, is the opposite of that given in Theorem 2.4. We note that for the case in consideration

$$(4.8)\qquad \lim_{l\to\infty} \frac{\phi_k(y_{l+1})}{\phi_k(y_l)} = 1, \qquad k=1,2,\ldots,$$

so that neither (1.5) nor (1.6) is satisfied.

An example of sequences satisfying the conditions in Theorem 4.1 is one for which $A_r = \sum_{i=1}^{r} a_i$, $r=1,2,\ldots$, with $r a_r \sim \sum_{i=0}^{\infty} \gamma_i r^{-i-\sigma}$ as $r\to\infty$, $\sigma>0$. Here $R_r = r a_r$ is appropriate, thus the T-transformation reduces to the u-transformation. Sequences of this form are said to be logarithmically converging.

For Theorem 4.1 see Sidi (1979b, Theorems 4.2, 4.3, and 5.2).

Theorem 4.2. *In Theorem* 4.1 *let* R_r *have the expansion*

$$R_r \sim z^r \sum_{i=0}^{\infty} \delta_i/r^{i+\sigma} \quad as \quad r\to\infty, \qquad z \neq 1,\ \delta_0 \neq 0, \tag{4.9}$$

everything else remaining the same. Then, when $\beta_p \neq 0$,

$$A_p^j - A \sim \frac{\delta_0 \beta_p p!}{(1-1/z)^p} z^j j^{-2p-\sigma} \quad as \quad j\to\infty. \tag{4.10}$$

(4.6) *holds also for this case, and*

$$\lim_{j\to\infty} \sum_{i=0}^{p} |\gamma_{p,i}^j| = \left|\frac{1+1/|z|}{1-1/z}\right|^p. \tag{4.11}$$

Note that $\lim_{r\to\infty} A_r$ exists and is equal to A if and only if $|z|<1$, in which case $\lim_{j\to\infty} A_p^j = A$ always. When $|z|=1$ but $z\neq 1$, $\lim_{r\to\infty} A_r$ exists and is equal to A if and only if $\sigma>0$. Also $\lim_{j\to\infty} A_p^j = A$ for this case if $2p+\sigma>0$. When $|z|>1$, neither $\lim_{r\to\infty} A_r$ nor $\lim_{j\to\infty} A_p^j$, $p=1, 2, \ldots$, exist. In any case (4.10) with (4.9) imply that

$$A_p^j - A \sim \frac{\beta_p p!}{(1-1/z)^p} \phi_{p+1}(y_j)\, y_j^p \quad \text{as} \quad j\to\infty, \tag{4.10$'$}$$

which is a better result than the one in (1.4). (2.17) for this case is improved to

$$\frac{A_p^j - A}{A_{p-1}^j - A} = O\left(\left[\frac{g_{p+1}(j)}{g_p(j)}\right]^2\right) \quad \text{as} \quad j\to\infty. \tag{4.12}$$

(4.6)$'$ is seen to hold true for this case too. We note that for the case in consideration

$$\lim_{l\to\infty} \frac{\phi_k(y_{l+1})}{\phi_k(y_l)} = z \neq 1, \qquad k=1, 2, \ldots, \tag{4.13}$$

so that (1.5) is satisfied, but (1.6) is not.

An example of sequences satisfying the conditions in Theorem 4.2 is one for which $A_r = \sum_{i=1}^{r} a_i z^i$, $r=1, 2, \ldots$, with $a_r \sim \sum_{i=0}^{\infty} \gamma_i r^{-i-\sigma}$ as $r\to\infty$. Here $R_r = a_r z^r$ is appropriate, thus the T-transformation reduces to the t-transformation. Sequences of this form, when they converge, are said to be linearly converging.

For (4.10) and (4.6) in Theorem 4.2 see Sidi (1980, Theorems 3.1 and 3.2). The result in (4.11) is new and can be obtained by using the techniques of Sidi (1980).

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