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Quantitative and constructive aspects of the generalized Koenig's and de Montessus's theorems for Padé approximants *

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Abstract: The generalized Koenig's theorem and de Montessus's theorem are two classical results concerning the convergence of the rows of the Padé table for meromorphic functions. Employing a technique that was recently developed for the analysis of vector extrapolation methods, refined versions of these theorems are proved in the present work. Specifically, complete expansions for the numerators and denominators of Padé approximants are derived. These expansions are then used to obtain (1) precise asymptotic rates of convergence of the poles of the Padé approximants to the corresponding poles, simple or multiple, of the meromorphic function in question, and (2) the precise asymptotic behavior of the error in the relevant Padé approximants. One important feature of the asymptotic results derived in this work is that these are expressed in terms of a very small number of parameters. Approximations of optimal accuracy to multiple poles and the principal parts of the corresponding Laurent expansions are also constructed. In addition, the convergence problem for the case in which the only singularities on the circle of meromorphy are poles is solved completely through the solution of a nonlinear integer programming problem.

Keywords: Padé approximants, meromorphic functions, generalized Koenig's theorem, de Montessus's theorem, row convergence, intermediate rows, generalized Dirichlet series.

1. Introduction

Suppose that we are given a power series $\Sigma_{i=0}^{\infty} c_i z^i$ representing a function $f(z)$, so that

$$f(z) = \sum_{i=0}^{\infty} c_i z^i. \tag{1.1}$$

The (m/k) Padé approximant associated with $f(z)$, if it exists, is defined to be the rational function

$$f_{mk}(z) = \frac{P_{mk}(z)}{Q_{mk}(z)} \equiv \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{i=0}^{k} b_i z^i}, \quad b_0 = 1, \tag{1.2}$$

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that satisfies

$$f_{mk}(z)=\sum_{i=0}^{m+k} c_i z^i + \mathrm{O}(z^{m+k+1}) \quad \text{as } z\to 0. \tag{1.3}$$

It then follows that $f_{mk}(z)$ has the representation

$$f_{mk}(z)=\frac{D_{mk}\left(z^k S_{m-k}(z),\ z^{k-1}S_{m-k+1}(z),\dots,\ z^0 S_m(z)\right)}{D_{mk}\left(z^k,\ z^{k-1},\dots,\ z^0\right)}, \tag{1.4}$$

where $D_{mk}(\sigma_0, \sigma_1,\dots,\sigma_k)$ is the determinant

$$D_{mk}(\sigma_0,\sigma_1,\dots,\sigma_k)=\begin{vmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_k \\ c_{m-k+1} & c_{m-k+2} & \cdots & c_{m+1} \\ c_{m-k+2} & c_{m-k+3} & \cdots & c_{m+2} \\ \vdots & \vdots & & \vdots \\ c_m & c_{m+1} & \cdots & c_{m+k} \end{vmatrix}, \tag{1.5}$$

and

$$S_j(z)=\sum_{i=0}^{j} c_i z^i, \quad j=0,1,2,\dots . \tag{1.6}$$

For details see, for example, [1,2].

There are two classical theorems concerning the convergence of the rows of the Padé table, i.e., of those $f_{mk}(z)$ with fixed k and increasing m, in case the function $f(z)$ is meromorphic in a neighborhood of $z=0$. These are the generalized Koenig's theorem and de Montessus's theorem. We state them below as Theorems 1.1 and 1.2.

Theorem 1.1 (generalized Koenig's theorem). *Let $f(z)$ be meromorphic in the disk $K=\{z: |z|<R\}$, and let it have exactly k poles, $z_1,\dots,z_k$, not necessarily distinct, in this disk. Let $|z_1|\leqslant\cdots\leqslant|z_k|<\sigma R<R$, and let $Q(z)=\prod_{j=1}^{k}(1-z/z_j)=\sum_{i=0}^{k}q_i z^i$, $q_0=1$. Write $Q_{mk}(z)=\sum_{i=0}^{k}q_i^{(m,k)}z^i$, $q_0^{(m,k)}=1$, where $Q_{mk}(z)$ is the denominator polynomial of $f_{mk}(z)$ as in* (1.2). *Then* (1) *$q_i^{(m,k)}=q_i+\mathrm{o}(\sigma^m)$ as $m\to\infty$, and* (2) *$Q_{mk}(z)=Q(z)+\mathrm{o}(\sigma^m)$ as $m\to\infty$.*

The case $k=1$, i.e., that of a single simple pole, of this theorem was given by Koenig [12]. The theorem also follows from a closely related theorem of Hadamard [7], and it was proved also in [3] and, more recently, in [6]. Proofs of Theorem 1.1 for the case of k simple poles can also be found in [4,8]. The technique employed in the present work, in fact, generalizes that used in [4] to account for multiple poles. An excellent source of information concerning the generalized Koenig's theorem is [9]. For Hadamard's theory of meromorphic functions and its consequences see also [5].

Theorem 1.2 (de Montessus's theorem). *With the notation and conditions of Theorem* 1.1, $\lim_{m\to\infty}f_{mk}(z)=f(z)$, *in fact*, $\limsup_{m\to\infty}|f(z)-f_{mk}(z)|^{1/m}\leqslant|z/R|$, *uniformly in every compact subset of the set $K\setminus\{z_1,\dots,z_k\}$.*

This theorem was originally proved by de Montessus de Ballore [13] and is a consequence of Hadamard's work [7]. Different proofs of it have been given in [1,2,11,16]. Another proof, for the special case of simple poles, has also been given in [4], and it also includes a refinement over all previous proofs. It is important to note that the proof given in [16] deals with the general rational interpolation problem, the Padé approximants being special cases of this. The techniques of [16] are similar to those used in [15] in the analysis of row convergence of Walsh arrays for meromorphic functions.

In Section 3, we shall restate and refine Theorems 1.1 and 1.2 as Theorems 3.1 and 3.3, respectively, allowing $f(z)$ to have additional poles in the annulus $\{z: |z_k| < |z| < R\}$. We shall supplement Theorems 3.1 and 3.3 with some new constructive results in Theorem 3.5. The proof of Theorem 3.1 is given in Section 4, and those of Theorems 3.3 and 3.5, in Section 5. We mention that the result of Theorem 3.3 of the present work is qualitatively in the spirit of some of the results given in a series of papers by Wilson [20–22], in which extensions of de Montessus's theorem to cover the Padé approximants $f_{m,k+1}(z), f_{m,k+2}(z),\dots$ are also given. In Section 6 of the present work we shall look at the same problem, and characterize those rows of the Padé table that converge to $f(z)$ through the solution to a nonlinear integer programming problem. Results analogous to those of [20–22] have been given for Walsh arrays in [15]. In this connection we also mention the related work by Parlett [14], in which the convergence of the basic QR algorithm has been analyzed in detail for a defective Hessenberg matrix. Multiple poles of a meromorphic function $f(z)$ correspond to defective eigenvalues of a matrix, and the convergence rate $O(s^{-1})$ derived in [14] for the basic QR algorithm seems to be analogous to that given in (6.4), Theorem 6.1 of the present work, for denominators of intermediate rows of the Padé table.

The obvious implication of Theorem 1.1 is that the zeros of $Q_{mk}(z)$ approach those of $Q(z)$ as $m \to \infty$. In Theorem 3.1 this observation is refined considerably in that results concerning the precise rates of convergence of the zeros of $Q_{mk}(z)$ to the corresponding zeros of $Q(z)$ are obtained. For example, under the conditions stated in Theorem 1.1, we show that if $\hat{z}$ is a pole of $f(z)$ of multiplicity ω, and if $\hat{z}_1(m),\dots,\hat{z}_\omega(m)$ are the corresponding zeros of $Q_{mk}(z)$ that tend to $\hat{z}$ as $m \to \infty$, then

$$(1)\qquad \limsup_{m\to\infty} |\hat{z}_i(m) - \hat{z}|^{1/m} \leqslant \left|\frac{\hat{z}}{R}\right|^{1/\omega}, \quad 1 \leqslant i \leqslant \omega.$$

It is clear that when $\omega > 1$, the convergence of the $\hat{z}_i(m)$ to $\hat{z}$ is not optimal. We can, however, use the $\hat{z}_i(m)$ to construct an approximation to $\hat{z}$ that has an optimal rate of convergence. In fact, we show that

$$(2)\qquad \limsup_{m\to\infty} \left|\frac{1}{\omega}\sum_{i=1}^{\omega} \hat{z}_i(m) - \hat{z}\right|^{1/m} \leqslant \left|\frac{\hat{z}}{R}\right|.$$

In addition, we show that the $(\omega-1)$st derivative with respect to ζ of the polynomial $\zeta^k Q_{mk}(\zeta^{-1})$ has exactly one zero, $1/\tilde{z}(m)$ say, that satisfies

$$(3)\qquad \limsup_{m\to\infty} |\tilde{z}(m) - \hat{z}|^{1/m} \leqslant \left|\frac{\hat{z}}{R}\right|.$$

In case there are additional poles $z_{k+1}, z_{k+2},\dots$ in the annulus $\{z: |z_k| < |z| < R\}$, Theorem 3.1 provides a more interesting version of (1). Roughly speaking, $\hat{z}_1(m),\dots,\hat{z}_\omega(m)$ are uniformly distributed over the boundary of a disk with center at $\hat{z}$ and radius that tends to zero at the rate of $m^\alpha |\hat{z}/z_{k+1}|^{m/\omega}$ as $m \to \infty$, where $|z_k| < |z_{k+1}| \leqslant |z_{k+2}| \leqslant \cdots < R$, α being a nonnegative integer. Theorem 3.1 also provides improved versions of (2) and (3).

Similarly, the presence of additional poles in the annulus $\{z : |z_k| < |z| < R\}$ gives rise to an interesting behavior of the error

$$f(z) - f_{mk}(z) = \frac{Q_{mk}(z)\, f(z) - P_{mk}(z)}{Q_{mk}(z)}$$

in de Montessus's theorem, which is borne out by Theorem 3.3. The asymptotic result of Theorem 3.3 reduces to that given in [4] for the special case in which all the poles are assumed to be simple and to satisfy $|z_k| < |z_{k+1}| < |z_{k+2}|$.

An important feature of the results stated in Theorems 3.1 and 3.3 is that these are quantitative in nature, and are expressed in terms of only a few parameters. This is made possible by the representation chosen for $f(z)$ in (4.1) and by Lemmas 4.1 and 5.1 concerning the dependence of c_m and $S_m(z)$ on m.

We note that the technique employed in the proof of Theorem 3.3 is identical to that used in the proof of Theorem 3.1. It involves the complete expansion of $Q_{mk}(z)$ and $Q_{mk}(z)f(z) - P_{mk}(z)$, and the analysis of the most dominant terms in these expansions. As is shown in Theorems 4.2 and 5.2, roughly speaking, these expansions are of the form $\sum_{i=1}^{\infty} \beta_i(m, z)\gamma_i^m$ for $Q_{mk}(z)$, and of the form $\sum_{i=1}^{\infty} \tilde{\alpha}_i(z)\tilde{\beta}_i(m)(\tilde{\gamma}_i z)^m$ for $Q_{mk}(z)f(z) - P_{mk}(z)$, in case $f(z)$ has an infinite number of poles. Here $\beta_i(m, z)$ and $\tilde{\beta}_i(m)$ are polynomials in m, and $\beta_i(m, z)$ are also polynomials in z. The γ_i and $\tilde{\gamma}_i$ can be very simply expressed in terms of the z_j, $j = 1, 2, \ldots$. The $\beta_i(m, z)$ and $\tilde{\beta}_i(m)$ are independent of m in case $f(z)$ has only simple poles.

The techniques employed in the proof of Theorems 3.1 and 3.3 of the present work are based in part on those of Sidi et al. [19], Sidi [17] and Sidi and Bridger [18] that were developed for the analysis of vector extrapolation methods.

Finally, in Theorem 3.5 we show how $f_{mk}(z)$ can be used in constructing an approximation to the principal part of the Laurent expansion of $f(z)$ about the pole $\hat{z}$. This approximation has an optimal rate of convergence, i.e., the error associated with it tends to zero like $O(|\hat{z}/R'|^m)$ as $m \to \infty$, where $R' = R - \epsilon$, $\epsilon > 0$ arbitrarily close to 0.

In Section 7 we show that Theorems 3.1 and 3.5 are applicable to the problem of determining some of the important parameters in generalized Dirichlet series.

We begin our treatment by giving some technical preliminaries in Section 2.

2. Technical preliminaries

The following lemma, whose proof can be found in [19], will be very useful.

Lemma 2.1. *Let $i_0, i_1, \ldots, i_k$ be positive integers, and assume that the scalars $v_{i_0,i_1,\ldots,i_k}$ are odd under an interchange of any two indices $i_0, i_1, \ldots, i_k$. Let σ_i, $i \geqslant 1$, and $t_{i,j}$, $i \geqslant 1$, $1 \leqslant j \leqslant k$, be scalars. Define*

$$I_{k,N} = \sum_{i_0=1}^{N} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \sigma_{i_0} \left(\prod_{p=1}^{k} t_{i_p,p} \right) v_{i_0,i_1,\ldots,i_k} \tag{2.1}$$

and

$$J_{k,N} = \sum_{1 \leqslant i_0 < i_1 < \cdots < i_k \leqslant N} \begin{vmatrix} \sigma_{i_0} & \sigma_{i_1} & \cdots & \sigma_{i_k} \\ t_{i_0,1} & t_{i_1,1} & \cdots & t_{i_k,1} \\ t_{i_0,2} & t_{i_1,2} & \cdots & t_{i_k,2} \\ \vdots & \vdots & & \vdots \\ t_{i_0,k} & t_{i_1,k} & \cdots & t_{i_k,k} \end{vmatrix} v_{i_0,i_1,\ldots,i_k}. \tag{2.2}$$

Then

$$I_{k,N} = J_{k,N}. \tag{2.3}$$

For Definitions 2.2 and 2.3 and Lemma 2.4 below see [18].

Definition 2.2. Let the nonnegative integers p_j, $j = 1, 2, \ldots,$ be given, and let ji be an ordered pair of integers, such that $j \geqslant 1$ and $0 \leqslant i \leqslant p_j$. For two such pairs ji and $j'i'$ we will write

$$ji < j'i' \quad \text{if } (j < j' \quad \text{or} \quad (j = j' \text{ and } i < i')), \tag{2.4}$$

$$ji = j'i' \quad \text{if } (j = j' \text{ and } i = i'), \tag{2.5}$$

$$ji \leqslant j'i' \quad \text{if either (2.4) or (2.5) holds.} \tag{2.6}$$

This implies the *lexicographic ordering* $10, 11, \ldots, 1p_1, 20, 21, \ldots, 2p_2, 30, 31\ldots$ of these pairs.

Definition 2.3. Let $\binom{n}{l}$ denote the binomial coefficients. Define

$$Y\left(\binom{n_1}{l_1}\lambda_1^{m_1}, \binom{n_2}{l_2}\lambda_2^{m_2}, \ldots, \binom{n_q}{l_q}\lambda_q^{m_q}, g_{q+1}, \ldots, g_M\right)$$

to be the $M \times M$ *determinant*, whose sth row is

$$\left(\binom{n_s}{l_s}\lambda_s^{m_s}, \binom{n_s+1}{l_s}\lambda_s^{m_s+1}, \ldots, \binom{n_s+M-1}{l_s}\lambda_s^{m_s+M-1}\right) \quad \text{for } 1 \leqslant s \leqslant q,$$

and

$$(g_{s,1}, g_{s,2}, \ldots, g_{s,M}) \quad \text{for } q+1 \leqslant s \leqslant M,$$

the $g_{i,j}$ being arbitrary.

Lemma 2.4. *Let $n_1, \ldots, n_q$ be arbitrary nonnegative integers, and let*

$$Y^{n_1,\ldots,n_q} = Y\left(\binom{n_1}{0}\lambda^{m_1}, \binom{n_2}{1}\lambda^{m_2}, \ldots, \binom{n_q}{q-1}\lambda^{m_q}, g_{q+1}, \ldots, g_M\right). \tag{2.7}$$

Then

$$Y^{n_1,\ldots,n_q} = Y^{0,\ldots,0}, \tag{2.8}$$

i.e., $Y^{n_1,\ldots,n_q}$ *is independent of* $n_1,\ldots,n_q$. *In particular, if* $p_1,\ldots,p_r$ *are nonnegative integers such that* $\sum_{i=1}^{r}(p_i+1)=M$, *then*

$$Y\left(\binom{n}{0}\lambda_1^0,\binom{n}{1}\lambda_1^0,\ldots,\binom{n}{p_1}\lambda_1^0,\ldots,\binom{n}{0}\lambda_r^0,\binom{n}{1}\lambda_r^0,\ldots,\binom{n}{p_r}\lambda_r^0\right)$$

$$=\left(\prod_{j=1}^{r}\lambda_j^{p_j(p_j+1)/2}\right)\tilde{Y}(\lambda_1,\,p_1;\,\lambda_2,\,p_2;\ldots;\,\lambda_r,\,p_r), \tag{2.9}$$

where

$$\tilde{Y}(\lambda_1,\,p_1;\ldots;\,\lambda_r,\,p_r)=\prod_{1\leqslant i<j\leqslant r}(\lambda_j-\lambda_i)^{(p_i+1)(p_j+1)}, \tag{2.10}$$

is the generalized Vandermonde determinant. (The ordinary Vandermonde determinant is obtained by setting $p_j=0,\ j=1,\ldots,r$.)

The following perturbation lemma seems to be of interest in itself.

Lemma 2.5. *Let the polynomial* $\Psi_\delta(\lambda)$ *be given by*

$$\Psi_\delta(\lambda)=\sum_{i=0}^{k}d_i(\delta)(\lambda-\hat{\lambda})^i,\quad \textit{some fixed } \hat{\lambda}, \tag{2.11}$$

such that

$$\lim_{\delta\to 0}\frac{d_i(\delta)}{d_\omega(\delta)}=\begin{cases}0 & \textit{for } 0\leqslant i\leqslant\omega-1,\\ \hat{d}_i & \textit{for } \omega+1\leqslant i\leqslant k,\end{cases} \tag{2.12}$$

with $\hat{d}_k\neq 0$.

(a) *Then* $\Psi_\delta(\lambda)$ *has* ω *zeros,* $\lambda_l(\delta)$, $1\leqslant l\leqslant\omega$, *that tend to* $\hat{\lambda}$ *as* $\delta\to 0$.

(b) *If, in addition,*

$$d_i(\delta)=\mathrm{O}(d_0(\delta))\quad \textit{as } \delta\to 0,\ \textit{for } 1\leqslant i\leqslant\omega-1, \tag{2.13}$$

then $\lambda_l(\delta)$, $1\leqslant l\leqslant\omega$, *can be ordered such that*

$$\lambda_l(\delta)\sim\hat{\lambda}+\left|\frac{d_0(\delta)}{d_\omega(\delta)}\right|^{1/\omega}\exp\left[\mathrm{i}\left(\frac{\theta(\delta)+2\pi(l-1)}{\omega}\right)\right]\quad \textit{as } \delta\to 0, \tag{2.14}$$

where

$$\theta(\delta)=\arg\left(\frac{-d_0(\delta)}{d_\omega(\delta)}\right), \tag{2.15}$$

and

$$\frac{1}{\omega}\sum_{l=1}^{\omega}\lambda_l(\delta)=\hat{\lambda}+\mathrm{O}\left(\frac{d_0(\delta)}{d_\omega(\delta)}\right)\quad \textit{as } \delta\to 0, \tag{2.16a}$$

$$\left[\frac{1}{\omega}\sum_{l=1}^{\omega}\lambda_l^{-1}(\delta)\right]^{-1}=\hat{\lambda}+\mathrm{O}\left(\frac{d_0(\delta)}{d_\omega(\delta)}\right)\quad \textit{as } \delta\to 0,\ \textit{for } \hat{\lambda}\neq 0. \tag{2.16b}$$

(c) *Furthermore, whether* (2.13) *is satisfied or not,* $\Psi_\delta^{(\omega-1)}(\lambda)$ *has a unique zero,* $\tilde{\lambda}(\delta)$, *that tends to* $\hat{\lambda}$, *which satisfies*

$$\tilde{\lambda}(\delta) \sim \hat{\lambda} - \frac{1}{\omega}\frac{d_{\omega-1}(\delta)}{d_\omega(\delta)} \quad \text{as } \delta \to 0. \tag{2.17}$$

Proof. The proof of part (a) follows from the fact that (2.11) and (2.12) imply

$$\lim_{\delta\to 0}\frac{\Psi_\delta(\lambda)}{d_\omega(\delta)} = (\lambda-\hat{\lambda})^\omega\left[1+\sum_{i=\omega+1}^{k}\hat{d}_i(\lambda-\hat{\lambda})^{i-\omega}\right]. \tag{2.18}$$

For the proof of part (b) we proceed as follows: from (2.11) and $\Psi_\delta(\lambda_l(\delta))=0$ we have

$$-(\lambda_l(\delta)-\hat{\lambda})^\omega = \frac{\sum_{i=0}^{\omega-1} d_i(\delta)(\lambda_l(\delta)-\hat{\lambda})^i}{\sum_{i=\omega}^{k} d_i(\delta)(\lambda_l(\delta)-\hat{\lambda})^{i-\omega}}. \tag{2.19}$$

By the fact that $\lambda_l(\delta)-\hat{\lambda}=o(1)$ as $\delta\to 0$, and the assumption in (2.13), it follows that the right-hand side of (2.19) is asymptotically equivalent to $d_0(\delta)/d_\omega(\delta)$ as $\delta\to 0$. From this and from (2.23) below we obtain (2.14) with (2.15).

Let us denote the remaining zeros of $\Psi_\delta(\lambda)$ by $\lambda_j(\delta)$, $\omega+1\leqslant j\leqslant k$. Then $\lim_{\delta\to 0}\lambda_j(\delta)$, $\omega+1\leqslant j\leqslant k$, being the zeros of the polynomial $1+\sum_{i=\omega+1}^{k}\hat{d}_i(\lambda-\hat{\lambda})^{i-\omega}$, are different from $\hat{\lambda}$. We now consider the polynomial $\phi_\delta(\tau)=\sum_{i=0}^{k}d_i(\delta)\tau^i$. The zeros of $\phi_\delta(\tau)$ are $\tau_j(\delta)\equiv\lambda_j(\delta)-\hat{\lambda}$, $1\leqslant j\leqslant k$. Thus $\lim_{\delta\to 0}\tau_j(\delta)=0$, $1\leqslant j\leqslant\omega$, and $\lim_{\delta\to 0}\tau_j(\delta)=t_j\neq 0$, $\omega+1\leqslant j\leqslant k$. Let $T_0(\delta;\ \tau_1,\ldots,\tau_k)=1$, and denote

$$T_p(\delta;\ \tau_1,\ldots,\tau_k) = \sum_{1\leqslant i_1<i_2<\cdots<i_p\leqslant k}\ \prod_{s=1}^{p}\tau_{i_s}(\delta), \quad p=1,\ldots,k, \tag{2.20}$$

i.e., $T_p(\delta;\ \tau_1,\ldots,\tau_k)$ is the sum of all possible products of p of the $\tau_j(\delta)$, $1\leqslant j\leqslant k$. It can be shown that

$$T_p(\delta;\ \tau_1,\ldots,\tau_k) = \sum_{i=\omega+p-k}^{\omega} T_i(\delta;\ \tau_1,\ldots,\tau_\omega)T_{p-i}(\delta;\ \tau_{\omega+1},\ldots,\tau_k), \quad k-\omega+1\leqslant p\leqslant k. \tag{2.21}$$

As is known

$$T_p(\delta;\ \tau_1,\ldots,\tau_k) = (-1)^p\frac{d_{k-p}(\delta)}{d_k(\delta)}, \quad 0\leqslant p\leqslant k. \tag{2.22}$$

Combining (2.21) and (2.22), and invoking (2.13), in the order $p=k,\ k-1,\ldots,k-\omega+1$, results in

$$T_q(\delta;\ \tau_1,\ldots,\tau_\omega) = O\left(\frac{d_0(\delta)}{d_\omega(\delta)}\right) \quad \text{as } \delta\to 0,\ 1\leqslant q\leqslant\omega. \tag{2.23}$$

Recalling that $\tau_j(\delta) = \lambda_j(\delta) - \hat{\lambda}$, $1 \leqslant j \leqslant k$, and employing (2.23) with $q = 1$, (2.16a) follows. Assume now that $\hat{\lambda} \neq 0$, and consider

$$L(\delta) = \frac{1}{\omega} \sum_{j=1}^{\omega} \lambda_j^{-1}(\delta) - \hat{\lambda}^{-1}. \tag{2.24}$$

After some tedious manipulations it can be shown that

$$L(\delta) = -\left(\omega\hat{\lambda} \prod_{j=1}^{\omega} \lambda_j(\delta)\right)^{-1} \sum_{s=1}^{\omega} sT_s(\delta;\, \tau_1, \ldots, \tau_\omega)\hat{\lambda}^{\omega-s}. \tag{2.25}$$

Employing (2.23) in (2.25), we obtain

$$L(\delta) = \mathrm{O}\left(\frac{d_0(\delta)}{d_\omega(\delta)}\right) \quad \text{as } \delta \to 0, \tag{2.26}$$

from which (2.16b) follows easily.

As for the proof of part (c), we first note that

$$\frac{\Psi_\delta^{(\omega-1)}(\lambda)}{(\omega-1)!} = \sum_{i=\omega-1}^{k} \binom{i}{\omega-1} d_i(\delta)(\lambda - \hat{\lambda})^{i-\omega+1}. \tag{2.27}$$

By (2.12)

$$\lim_{\delta\to 0} \frac{\Psi_\delta^{(\omega-1)}(\lambda)}{\omega! d_\omega(\delta)} = (\lambda - \hat{\lambda})\left[1 + \sum_{i=\omega+1}^{k} \frac{1}{\omega}\binom{i}{\omega-1}\hat{d}_i(\lambda - \hat{\lambda})^{i-\omega}\right]. \tag{2.28}$$

From this it follows that $\Psi_\delta^{(\omega-1)}(\lambda)$ has exactly one zero, $\tilde{\lambda}(\delta)$, that tends to $\hat{\lambda}$ as $\delta \to 0$. By (2.27) and $\Psi_\delta^{(\omega-1)}(\tilde{\lambda}(\delta)) = 0$ we have

$$-(\tilde{\lambda}(\delta) - \hat{\lambda}) = \frac{d_{\omega-1}(\delta)}{\sum_{i=\omega}^{k} \binom{i}{\omega-1} d_i(\delta)(\tilde{\lambda}(\delta) - \hat{\lambda})^{i-\omega}}. \tag{2.29}$$

(2.17) now follows from (2.29) in the same way (2.14) follows from (2.19). □

Note 2.6. (2.14) implies that, to lowest order, $\lambda_l(\delta)$, $1 \leqslant l \leqslant \omega$, are uniformly distributed over a circle with center $\hat{\lambda}$ and with radius that is shrinking to zero.

Corollary 2.7. *In* (2.12) *let*

$$\frac{d_i(\delta)}{d_\omega(\delta)} = \mathrm{O}(\delta) \quad \textit{as } \delta \to 0, \textit{ for } 0 \leqslant i \leqslant \omega - 1. \tag{2.30}$$

Then, whether (2.13) *is satisfied or not,*

$$\lambda_l(\delta) = \hat{\lambda} + \mathrm{O}(|\delta|^{1/\omega}) \quad \textit{as } \delta \to 0,\ 1 \leqslant l \leqslant \omega, \tag{2.31}$$

and

$$\frac{1}{\omega}\sum_{l=1}^{\omega} \lambda_l(\delta) = \hat{\lambda} + \mathrm{O}(\delta) \quad \textit{as } \delta \to 0, \tag{2.32a}$$

$$\left(\frac{1}{\omega}\sum_{l=1}^{\omega} \lambda_l^{-1}(\delta)\right)^{-1} = \hat{\lambda} + \mathrm{O}(\delta) \ \textit{as } \delta \to 0, \textit{ for } \hat{\lambda} \neq 0, \tag{2.32b}$$

and

$$\tilde{\lambda}(\delta) = \hat{\lambda} + O(\delta) \text{ as } \delta \to 0. \tag{2.33}$$

Proof. Left to the reader. □

For future use we rewrite (1.5) in the form

$$D_{mk}(\sigma_0, \ldots, \sigma_k) = \begin{vmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_k \\ z_{n,1} & z_{n+1,1} & \cdots & z_{n+k,1} \\ z_{n,2} & z_{n+1,2} & \cdots & z_{n+k,2} \\ \vdots & \vdots & & \vdots \\ z_{n,k} & z_{n+1,k} & \cdots & z_{n+k,k} \end{vmatrix} \tag{2.34}$$

with

$$n \equiv m - k + 1 \quad \text{and} \quad z_{h,s} \equiv c_{h+s-1}. \tag{2.35}$$

In the next sections we shall use both m and n interchangeably.

3. New statements of Theorems 1.1 and 1.2

Let the function $f(z)$ that has the power series representation given in (1.1) be meromorphic in the disk $K = \{z: |z| < R\}$. Let ζ_j^{-1}, $j = 1, 2, \ldots, \nu$, be the distinct poles of $f(z)$ in K, whose respective multiplicities are $p_j + 1 \equiv \omega_j$, $j = 1, 2, \ldots, \nu$. Let the ζ_j be ordered so that

$$0 < |\zeta_1^{-1}| \leqslant |\zeta_2^{-1}| \leqslant \cdots \leqslant |\zeta_\nu^{-1}| < \xi^{-1} < R, \tag{3.1a}$$

ξ being arbitrary otherwise, and

$$p_j \geqslant p_{j+1} \quad \text{when } |\zeta_j^{-1}| = |\zeta_{j+1}^{-1}|. \tag{3.1b}$$

Take t to be a positive integer, such that

$$\text{either } (t = \nu) \quad \text{or} \quad \left(t < \nu \text{ and } |\zeta_t^{-1}| < |\zeta_{t+1}^{-1}|\right). \tag{3.2}$$

When $\nu > t$, without loss of generality, we assume

$$|\zeta_{t+1}^{-1}| = \cdots = |\zeta_{t+r}^{-1}| < |\zeta_{t+r+1}^{-1}| \leqslant \cdots \leqslant |\zeta_\nu^{-1}| \tag{3.3a}$$

and

$$\bar{p} \equiv p_{t+1} = \cdots = p_{t+\mu} > p_{t+\mu+1} \geqslant \cdots \geqslant p_{t+r}. \tag{3.3b}$$

(When $t + r = \nu$ in (3.3a) or $\mu = r$ in (3.3b), equalities are assumed to prevail throughout.) By the assumption that ζ_j^{-1} is a pole of order ω_j, we also have

$$A_{jp_j} \equiv \lim_{z \to \zeta_j^{-1}} (1 - \zeta_j z)^{\omega_j} f(z) \neq 0, \quad j = 1, \ldots, \nu. \tag{3.4}$$

Define

$$\hat{v}_s = \sum_{j=1}^{s} (p_j + 1) = \sum_{j=1}^{s} \omega_j,$$

$$u_j = \tfrac{1}{2} p_j (p_j + 1), \quad j = 1, 2, \ldots, \qquad v_s = \sum_{j=1}^{s} u_j, \quad s = 1, 2, \ldots, \tag{3.5a}$$

and

$$Q(z) = \prod_{j=1}^{t} (1 - \zeta_j z)^{\omega_j} \quad \text{and} \quad \pi = \prod_{j=1}^{t} \zeta_j^{\omega_j}, \tag{3.5b}$$

and let

$$k = \hat{v}_t = \sum_{j=1}^{t} \omega_j \tag{3.5c}$$

in $f_{mk}(z)$, the (m/k) Padé approximant to $f(z)$. The assumptions and definitions above will be used throughout the remainder of this work. Only in Section 6 we shall use values of k other than that in (3.5c).

Theorem 3.1 (generalized Koenig's theorem). *Let the function $f(z)$ be as described above.*

(a) *Then for $z \neq \zeta_j^{-1}$, $1 \leqslant j \leqslant t$, $D_{mk}(z^k, z^{k-1}, \ldots, z^0) \equiv \hat{D}_{mk}(z)$ has the dominant asymptotic behavior*

$$\hat{D}_{mk}(z) = D_{mk}(z^k, z^{k-1}, \ldots, z^0) = W\pi^n [Q(z) + O(\epsilon(n))] \quad \text{as } n \to \infty, \tag{3.6}$$

where

$$W = (-1)^{k+v_t} \left(\prod_{j=1}^{t} A_{jp_j}^{\omega_j} \right) \left(\prod_{j=1}^{t} \zeta_j^{2u_j} \right) [\tilde{Y}(\zeta_1, p_1; \ldots; \zeta_t, p_t)]^2 \neq 0, \tag{3.7}$$

and

$$\epsilon(n) = \begin{cases} n^{\alpha} \left| \dfrac{\zeta_{t+1}}{\zeta_t} \right|^n & \text{if } \nu > t, \ \alpha \text{ some nonnegative integer}, \\ \left| \dfrac{\xi}{\zeta_t} \right|^n & \text{if } \nu = t. \end{cases} \tag{3.8}$$

Actually, $\alpha = \bar{p}$ if the poles whose moduli are $|\zeta_t^{-1}|$ are simple.

(b) *Part (a) above implies that $\hat{D}_{mk}(z)$ has $\omega_s = p_s + 1$ zeros $\zeta_{s0}^{-1}(n), \ldots, \zeta_{sp_s}^{-1}(n)$ that tend to ζ_s^{-1} as $n \to \infty$, for $1 \leqslant s \leqslant t$. Actually, $\zeta_{sl}(n)$, $0 \leqslant l \leqslant p_s$, $1 \leqslant s \leqslant t$, are the k zeros of the polynomial in ζ, $\tilde{D}_{mk}(\zeta) \equiv D_{mk}(\zeta^0, \zeta^1, \ldots, \zeta^k)$. Also the p_sth derivative of $\tilde{D}_{mk}(\zeta)$ with respect to ζ has exactly one zero, $\tilde{\zeta}_s(n)$ say, that tends to ζ_s as $n \to \infty$. For these we have*

$$\zeta_{sl}(n) = \zeta_s + O\left(\delta_s(n)^{1/\omega_s}\right) \quad \text{as } n \to \infty, \ 0 \leqslant l \leqslant p_s,$$

$$\frac{1}{\omega_s} \sum_{l=0}^{p_s} \zeta_{sl}(n) = \zeta_s + O(\delta_s(n)) \quad \text{as } n \to \infty,$$

$$\tilde{\zeta}_s(n) = \zeta_s + O(\delta_s(n)) \quad \text{as } n \to \infty, \tag{3.9}$$

where

$$\delta_s(n)=\begin{cases} n^{\bar{p}}\left|\dfrac{\zeta_{t+1}}{\zeta_s}\right|^n & \text{if } \nu>t,\\ \left|\dfrac{\xi}{\zeta_s}\right|^n & \text{if } \nu=t.\end{cases} \tag{3.10}$$

Also (3.9) *remains true when all the* ζ*'s there are replaced by their reciprocals.*

When $\nu>t$ *the first of the results in* (3.9) *can be made more quantitative and interesting in nature. In this case let*

$$N_s(n)=\frac{1}{\bar{p}!}\sum_{h=t+1}^{t+\mu}\frac{A_{h\bar{p}}}{A_{sp_s}}\zeta_s^{-p_s}\left[\prod_{\substack{i=1\\ i\neq s}}^{t}\left(\frac{\zeta_h-\zeta_i}{\zeta_s-\zeta_i}\right)^{2\omega_i}\right](\zeta_h-\zeta_s)^{2p_s+1}\exp\left(in\arg\left(\frac{\zeta_h}{\zeta_s}\right)\right). \tag{3.11}$$

Now the sequence $\{N_s(n)\}$ *has a convergent subsequence* $\{N_s(n_q)\}$ *with a nonzero limit* $\hat{N}_s$. *The* $\zeta_{sl}(n)$ *then satisfy*

$$\zeta_{sl}(n_q)\sim\zeta_s+\hat{N}_{sl}\delta_s(n_q)^{1/\omega_s}\quad\text{as } q\to\infty,\ 0\leqslant l\leqslant p_s, \tag{3.12}$$

where $\hat{N}_{sl}$ *are the* ω_s*th roots of* $\hat{N}_s$. *From* (3.12) *it follows that*

$$\zeta_{sl}^{-1}(n_q)\sim\zeta_s^{-1}-\zeta_s^{-2}\hat{N}_{sl}\delta_s(n_q)^{1/\omega_s}\quad\text{as } q\to\infty.$$

Needless to say, for $\omega_s=1$, $\zeta_{s0}(n)=\zeta_s+N_s(n)\delta_s(n)+\mathrm{o}(\delta_s(n))$ *as* $n\to\infty$.

Note 3.2. Roughly speaking, the results in Theorem 3.1 imply that

(1) if ζ_s^{-1} and $\zeta_{s'}^{-1}$ have the same multiplicity and $|\zeta_s^{-1}|<|\zeta_{s'}^{-1}|$, then the $\zeta_{sl}^{-1}(n)$, the approximations to ζ_s^{-1}, will have more accuracy than the $\zeta_{s'l}^{-1}(n)$, the approximations to $\zeta_{s'}^{-1}$;
(2) if $|\zeta_s^{-1}|=|\zeta_{s'}^{-1}|$ and ζ_s^{-1} has multiplicity smaller than that of $\zeta_{s'}^{-1}$, then $\zeta_{sl}^{-1}(n)$ will have more accuracy than $\zeta_{s'l}^{-1}(n)$;
(3) in general, the accuracy of the $\zeta_{jl}^{-1}(n)$ for fixed j increases with increasing k;
(4) if $\omega_s>1$, $\hat{\zeta}_s(n)$, the average of the $\zeta_{sl}(n)$, $0\leqslant l\leqslant p_s$, converges to ζ_s, ω_s times more quickly than the individual $\zeta_{sl}(n)$, i.e., $\hat{\zeta}_s(n)$ converges to ζ_s at the rate $\zeta_{s0}(n)$ would converge to ζ_s if ζ_s were a simple zero of $\tilde{D}_{mk}(\zeta)$.

Theorem 3.3 (de Montessus's theorem). *Let the function* $f(z)$ *be as in Theorem* 3.1. *Then for* $q=0,1,2,\ldots$ $\lim_{m\to\infty}f_{mk}^{(q)}(z)=f^{(q)}(z)$ *uniformly in any compact subset of* $\{z:|z|<\rho\}\setminus\{\zeta_1^{-1},\ldots,\zeta_t^{-1}\}$, *where* $\rho=|\zeta_{t+1}^{-1}|$ *when* $\nu>t$, *and* $\rho=R$ *when* $\nu=t$. *Actually, for all* $q=0,1,2,\ldots$

$$f^{(q)}(z)-f_{mk}^{(q)}(z)=\begin{cases}\mathrm{O}(n^{\bar{p}}|\zeta_{t+1}z|^n) & \text{as } n\to\infty \quad \text{if } \nu>t,\\ \mathrm{O}(|\xi z|^n) & \text{as } n\to\infty \quad \text{if } \nu=t,\end{cases} \tag{3.13}$$

uniformly in any compact subset of $K\setminus\{\zeta_1^{-1},\ldots,\zeta_\nu^{-1}\}$. *Furthermore, when* $\nu>t$, $f(z)-f_{mk}(z)$ *has dominant asymptotic behavior*

$$f(z)-f_{mk}(z)=\frac{n^{\bar{p}}}{\bar{p}!}\sum_{h=t+1}^{t+\mu}\phi_h(z)(\zeta_h z)^{n+2k}+\mathrm{o}(n^{\bar{p}}|\zeta_{t+1}z|^n)\quad\text{as } n\to\infty, \tag{3.14}$$

uniformly in any compact subset of $K \backslash \{\zeta_1^{-1}, \ldots, \zeta_\nu^{-1}\}$, *where*

$$\phi_h(z) = \frac{A_{hp_h}}{1 - \zeta_h z} \left[\frac{Q(\zeta_h^{-1})}{Q(z)} \right]^2. \tag{3.15}$$

The $o(n^{\bar{p}} |\zeta_{t+1} z|^n)$ *term in* (3.14) *is, in fact,* $O(n^{\bar{p}-1} |\zeta_{t+1} z|^n)$ *when* $\bar{p} > 0$, $O(n^{p_{t+r+1}} |\zeta_{t+r+1} z|^n)$ *when* $\bar{p} = 0$ $(\mu = r)$ *and* $t + r < \nu$, *and* $o(|\xi z|^n)$ *when* $\bar{p} = 0$ $(\mu = r)$ *and* $t + r = \nu$.

Note 3.4. As will become clear in the next sections, the proofs for the case in which $f(z)$ has only simple poles is not very difficult technically. The presence of the multiple poles, however, complicates every stage of the proofs considerably.

Theorem 3.5 below shows how one can use only $f_{mk}(z)$ and the $\zeta_{sl}(n)$ to construct good approximations to the coefficients H_{sq}, $0 \leqslant q \leqslant p_s$, of the principal part of the Laurent expansion of $f(z)$ about ζ_s^{-1},

$$f(z) = \sum_{q=0}^{p_s} \frac{H_{sq}}{(z - \zeta_s^{-1})^{q+1}} + \tilde{f}_s(z), \quad \tilde{f}_s(z) \text{ analytic at } \zeta_s^{-1}. \tag{3.16}$$

Theorem 3.5. *Define* $\hat{z}_s(n)$ *by either*

$$\hat{z}_s(n) = \frac{1}{\omega_s} \sum_{l=0}^{p_s} \zeta_{sl}^{-1}(n) \tag{3.17a}$$

or

$$\hat{z}_s(n) = \left[\frac{1}{\omega_s} \sum_{l=0}^{p_s} \zeta_{sl}(n) \right]^{-1}. \tag{3.17b}$$

Let $H_{sq,l}(n)$ *be the residue of the rational function* $(z - \hat{z}_s(n))^q f_{mk}(z)$ *at* $\zeta_{sl}^{-1}(n)$, *and let*

$$\Lambda_s = \begin{cases} \left| \dfrac{\zeta_{t+1}}{\zeta_s} \right| & \text{if } \nu > t, \\ |\zeta_s R|^{-1} & \text{if } \nu = t. \end{cases} \tag{3.18}$$

Then

$$\limsup_{n \to \infty} \left| \sum_{l=0}^{p_s} H_{sq,l}(n) - H_{sq} \right|^{1/n} \leqslant \Lambda_s, \quad 0 \leqslant q \leqslant p_s. \tag{3.19}$$

4. Proof of Theorem 3.1

We begin by observing that the function $f(z)$ has the representation

$$f(z) = \sum_{j=1}^{\nu} \sum_{i=0}^{p_j} \frac{A_{ji}}{(1 - \zeta_j z)^{i+1}} + g(z), \tag{4.1}$$

where A_{jp_j} are as defined in (3.4), and $g(z)$ is analytic in the disk K. Then $g(z)$ has the power series representation

$$g(z)=\sum_{i=0}^{\infty} d_i z^i, \quad \text{convergent for } z\in K. \tag{4.2}$$

Of course, when $g(z)$ is a polynomial, i.e., the series in (4.2) has a finite number of terms, $f(z)$ is a rational function. Note that $H_{ji}(-\zeta_j)^{i+1}=A_{ji}$ with H_{ji} as defined prior to the statement of Theorem 3.5.

Lemma 4.1. *Define*

$$\tilde{A}_{jl}=\sum_{i=l}^{p_j} A_{ji}\binom{i}{i-l} \quad \text{and} \quad \hat{A}_{ji,s}=\sum_{l=i}^{p_j} \tilde{A}_{jl}\binom{s-1}{l-i}\zeta_j^{s-1}. \tag{4.3}$$

Consequently,

$$\hat{A}_{jp_j,s}=\tilde{A}_{jp_j}\zeta_j^{s-1}=A_{jp_j}\zeta_j^{s-1}\neq 0. \tag{4.4}$$

Then the coefficient c_m in $f(z)=\sum_{i=0}^{\infty}c_i z^i$ has the expansion

$$c_m=\sum_{j=1}^{\nu}\left[\sum_{l=0}^{p_j}\tilde{A}_{jl}\binom{m}{l}\right]\zeta_j^m+\tilde{A}(m,\xi)\xi^m, \tag{4.5}$$

where

$$d_m=\tilde{A}(m,\xi)\xi^m, \quad |\tilde{A}(m,\xi)|\leqslant M(\xi)\equiv \max_{|z|=\xi^{-1}}|g(z)|. \tag{4.6}$$

Consequently, $z_{n,s}$ defined in (2.35) *has the expansion*

$$z_{n,s}=\sum_{j=0}^{\nu}\left[\sum_{i=0}^{p_j}\hat{A}_{ji,s}\binom{n}{i}\right]\zeta_j^n+\hat{A}_s(n,\xi)\xi^n, \tag{4.7}$$

where

$$\hat{A}_s(n,\xi)=\tilde{A}(n+s-1,\xi)\xi^{s-1}. \tag{4.8}$$

Proof. (4.6) follows from the Cauchy inequalities for analytic functions. The rest of the proof can be accomplished by expanding $f(z)-g(z)$ about $z=0$ and then employing the well-known identity

$$\binom{\alpha+\beta}{q}=\sum_{i=0}^{q}\binom{\alpha}{i}\binom{\beta}{q-i}. \tag{4.9}$$

The details are left to the interested reader. □

Theorem 4.2. *Let $\zeta\equiv 1/z$, and denote*

$$\tilde{D}_{mk}(\zeta)\equiv D_{mk}(\zeta^0,\zeta^1,\ldots,\zeta^k)=z^{-k}\hat{D}_{mk}(z). \tag{4.10}$$

Denote also

$$Z_{j_1l_1,j_2l_2,\dots,j_kl_k}=\begin{vmatrix}\hat{A}_{j_1l_1,1} & \hat{A}_{j_2l_2,1} & \cdots & \hat{A}_{j_kl_k,1}\\ \hat{A}_{j_1l_1,2} & \hat{A}_{j_2l_2,2} & \cdots & \hat{A}_{j_kl_k,2}\\ \vdots & \vdots & & \vdots\\ \hat{A}_{j_1l_1,k} & \hat{A}_{j_2l_2,k} & \cdots & \hat{A}_{j_kl_k,k}\end{vmatrix}. \tag{4.11}$$

Then $\tilde{D}_{mk}(\zeta)$ *has the expansion*

$$\begin{aligned}\tilde{D}_{mk}(\zeta)=\Bigg\{&\sum_{10\leqslant j_1l_1<j_2l_2<\cdots<j_kl_k\leqslant \nu p_\nu} Z_{j_1l_1,j_2l_2,\dots,j_kl_k}\\ &\times Y\left(\binom{0}{0}\zeta^0, \binom{n}{l_1}\zeta_{j_1}^0, \binom{n}{l_2}\zeta_{j_2}^0,\dots,\binom{n}{l_k}\zeta_{j_k}^0\right)\left(\prod_{h=1}^{k}\zeta_{j_h}^n\right)\Bigg\}\\ &+E_{mk}(\zeta,\xi),\end{aligned} \tag{4.12}$$

where $E_{mk}(\zeta,\xi)$ *is the only term in this expansion that depends on* $g(z)$ *and thus on* ξ. *When* $g(z)$ *is a polynomial* $E_{mk}(\zeta,\xi)\equiv 0$ *for* $m\geqslant k+\deg g(z)$. *The exact form of* $E_{mk}(\zeta,\xi)$ *will be described through an example at the end of the proof of this theorem.*

Proof. For the sake of simplicity we shall first assume that $g(z)$ in (4.1) is a polynomial. This implies that $\tilde{A}(m,\xi)=0$ in (4.5) for $m>\deg g(z)$ and $\hat{A}_s(n,\xi)=0$ in (4.7) for $n+s-1>\deg g(z)$. Substituting now (4.7) in (2.34) with $\sigma_j=\zeta^j$, $j=0,\dots,k$, there, we obtain

$$\tilde{D}_{mk}(\zeta)=\begin{vmatrix}\zeta^0 & \zeta^1 & \cdots & \zeta^k\\ \sum_{j_1l_1}\hat{A}_{j_1l_1,1}\binom{n}{l_1}\zeta_{j_1}^n & \sum_{j_1l_1}\hat{A}_{j_1l_1,1}\binom{n+1}{l_1}\zeta_{j_1}^{n+1} & \cdots & \sum_{j_1l_1}\hat{A}_{j_1l_1,1}\binom{n+k}{l_1}\zeta_{j_1}^{n+k}\\ \sum_{j_2l_2}\hat{A}_{j_2l_2,2}\binom{n}{l_2}\zeta_{j_2}^n & \sum_{j_2l_2}\hat{A}_{j_2l_2,2}\binom{n+1}{l_2}\zeta_{j_2}^{n+1} & \cdots & \sum_{j_2l_2}\hat{A}_{j_2l_2,2}\binom{n+k}{l_2}\zeta_{j_2}^{n+k}\\ \vdots & \vdots & & \vdots\\ \sum_{j_kl_k}\hat{A}_{j_kl_k,k}\binom{n}{l_k}\zeta_{j_k}^n & \sum_{j_kl_k}\hat{A}_{j_kl_k,k}\binom{n+1}{l_k}\zeta_{j_k}^{n+1} & \cdots & \sum_{j_kl_k}\hat{A}_{j_kl_k,k}\binom{n+k}{l_k}\zeta_{j_k}^{n+k}\end{vmatrix}, \tag{4.13}$$

where Σ_{jl} denotes $\Sigma_{j=1}^{\nu}\Sigma_{l=0}^{p_j}$. Using the multilinearity property of determinants and Definition 2.3, (4.13) can be expressed as

$$\tilde{D}_{mk}(\zeta)=\sum_{j_1l_1}\cdots\sum_{j_kl_k}\left(\prod_{s=1}^{k}\hat{A}_{j_sl_s,s}\right)\left(\prod_{s=1}^{k}\zeta_{j_s}^n\right)Y\left(\binom{0}{0}\zeta^0, \binom{n}{l_1}\zeta_{j_1}^0, \binom{n}{l_2}\zeta_{j_2}^0,\dots,\binom{n}{l_k}\zeta_{j_k}^0\right). \tag{4.14}$$

Since the product

$$\left(\prod_{s=1}^{k}\zeta_{j_s}^{n}\right)Y\left(\binom{0}{0}\zeta^0,\binom{n}{l_1}\zeta_{j_1}^0,\ldots,\binom{n}{l_k}\zeta_{j_k}^0\right)$$

is odd under an interchange of any two of the pairs of indices $j_1l_1,\ldots,j_kl_k$, we can apply Lemma 2.1 to the multiple sum of (4.14). This results in (4.12) with $E_{mk}(\zeta,\xi)\equiv 0$.

(4.12) with $E_{mk}(\zeta,\xi)\not\equiv 0$ is obtained by adding the term $\hat{A}_s(n+i,\xi)\xi^{n+i}$ to the corresponding term

$$\sum_{j_s l_s}\hat{A}_{j_s l_s,s}\binom{n+i}{l_s}\zeta_{j_s}^{n+i}$$

in (4.13). A cursory look at the resulting determinant shows that $E_{mk}(\zeta,\xi)$ too is a sum of determinants similar in form to the Y determinants in the multiple sum of (4.12). To make this point absolutely clear we shall work out the example in which $\nu=3$, $p_1=1$, $p_2=0$, p_3 being arbitrary. Pick $t=2$ so that $k=(p_1+1)+(p_2+1)=3$. Then

$$\begin{aligned}
E_{m3}(\zeta,\xi)={}&\sum_{10\leqslant j_1l_1<j_2l_2\leqslant 3p_3} Z^{(1,2)}_{j_1l_1,j_2l_2}Y\left(\binom{0}{0}\zeta^0,\binom{n}{l_1}\zeta_{j_1}^0,\binom{n}{l_2}\zeta_{j_2}^0,\hat{A}_3(n,\xi)\xi^0\right)(\zeta_{j_1}\zeta_{j_2}\xi)^n\\
&+\sum_{10\leqslant j_1l_1<j_3l_3\leqslant 3p_3} Z^{(1,3)}_{j_1l_1,j_3l_3}Y\left(\binom{0}{0}\zeta^0,\binom{n}{l_1}\zeta_{j_1}^0,\hat{A}_2(n,\xi)\xi^0,\binom{n}{l_3}\zeta_{j_3}^0\right)(\zeta_{j_1}\zeta_{j_3}\xi)^n\\
&+\sum_{10\leqslant j_2l_2<j_3l_3\leqslant 3p_3} Z^{(2,3)}_{j_2l_2,j_3l_3}Y\left(\binom{0}{0}\zeta^0,\hat{A}_1(n,\xi)\xi^0,\binom{n}{l_2}\zeta_{j_2}^0,\binom{n}{l_3}\zeta_{j_3}^0\right)(\zeta_{j_2}\zeta_{j_3}\xi)^n\\
&+\sum_{10\leqslant j_1l_1\leqslant 3p_3} Z^{(1)}_{j_1l_1}Y\left(\binom{0}{0}\zeta^0,\binom{n}{l_1}\zeta_{j_1}^0,\hat{A}_2(n,\xi)\xi^0,\hat{A}_3(n,\xi)\xi^0\right)(\zeta_{j_1}\xi^2)^n\\
&+\sum_{10\leqslant j_2l_2\leqslant 3p_3} Z^{(2)}_{j_2l_2}Y\left(\binom{0}{0}\zeta^0,\hat{A}_1(n,\xi)\xi^0,\binom{n}{l_2}\zeta_{j_2}^0,\hat{A}_3(n,\xi)\xi^0\right)(\zeta_{j_2}\xi^2)^n\\
&+\sum_{10\leqslant j_3l_3\leqslant 3p_3} Z^{(3)}_{j_3l_3}Y\left(\binom{0}{0}\zeta^0,\hat{A}_1(n,\xi)\xi^0,\hat{A}_2(n,\xi)\xi^0,\binom{n}{l_3}\zeta_{j_3}^0\right)(\zeta_{j_3}\xi^2)^n\\
&+Y\left(\binom{0}{0}\zeta^0,\hat{A}_1(n,\xi)\xi^0,\hat{A}_2(n,\xi)\xi^0,\hat{A}_3(n,\xi)\xi^0\right)(\xi^3)^n,
\end{aligned}$$

where

$$Z^{(s)}_{jl}=\hat{A}_{jl,s}\quad\text{and}\quad Z^{(s,s')}_{jl,j'l'}=\begin{vmatrix}\hat{A}_{jl,s} & \hat{A}_{j'l',s}\\ \hat{A}_{jl,s'} & \hat{A}_{j'l',s'}\end{vmatrix}. \qquad \square$$

Note that in case $f(z)$ is a rational function with only simple poles $\zeta_1^{-1},\ldots,\zeta_\nu^{-1}$, $\nu>k$, which we write in the form $f(z)=\sum_{j=1}^{\nu}A_j/(1-\zeta_jz)+g(z)$, the result in (4.12) reduces to that given in [4], and it reads

$$\hat{D}_{mk}(z)=(-1)^k\sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant\nu}\left(\prod_{s=1}^{k}A_{j_s}\right)\left[V(\zeta_{j_1},\ldots,\zeta_{j_k})\right]^2\left[\prod_{s=1}^{k}(1-\zeta_{j_s}z)\right]\left(\prod_{s=1}^{k}\zeta_{j_s}\right)^n, \tag{4.12$'$}$$

for $m \geqslant k + \deg g(z)$. Here $V(\lambda_1, \ldots, \lambda_k) = \tilde{Y}(\lambda_1, 0; \ldots; \lambda_k, 0)$ is the ordinary Vandermonde determinant. We have used the fact that

$$Z_{j_1 l_1, \ldots, j_k l_k} = \left(\prod_{s=1}^{k} A_{j_s} \right) V(\zeta_{j_1}, \ldots, \zeta_{j_k}) \quad \text{for } p_{j_s} = 0,\ 1 \leqslant s \leqslant k,$$

which follows from (4.11).

Proof of Theorem 3.1. We begin by observing that the determinants Y in the expansion (4.12) are polynomials in n. Therefore, the asymptotic behavior of the multiple sum in (4.12) is determined by the products $(\prod_{h=1}^{k} \zeta_{j_h}^{n})$. Thus, the most dominant term in this sum is that for which

$$j_1 l_1,\ j_2 l_2, \ldots, j_k l_k = 10,\ 11, \ldots, 1p_1,\ 20, \ldots, 2p_2, \ldots, tp_t,$$

and it is given by

$$Z_{10,11,\ldots,tp_t} Y\left(\binom{0}{0} \zeta^0, \binom{n}{0} \zeta_1^0, \binom{n}{1} \zeta_1^0, \ldots, \binom{n}{p_1} \zeta_1^0, \ldots, \binom{n}{0} \zeta_t^0, \ldots, \binom{n}{p_t} \zeta_t^0 \right) \pi^n, \tag{4.15}$$

with π as defined in (3.5b). Substituting the expression given for $\hat{A}_{ji,s}$ in (4.3) in the determinant representation of $Z_{10,\ldots,tp_t}$, and performing elementary column transformations, and finally invoking (4.4) (see also Appendix A), we obtain

$$Z_{10,\ldots,tp_t} = (-1)^{v_t} \left(\prod_{j=1}^{t} A_{jp_j}^{\omega_j} \right) \left(\prod_{j=1}^{t} \zeta_j^{u_j} \right) \tilde{Y}(\zeta_1, p_1; \ldots; \zeta_t, p_t) \neq 0. \tag{4.16}$$

Substituting (4.16) in (4.15), and applying Lemma 2.4 to the Y determinant in (4.15), we obtain the dominant term given in (3.6) and (3.7).

When $\nu = t$ the multiple sum in (4.12) contains only one term, namely, the dominant term discussed in the previous paragraph.

When $\nu > t$ the next dominant term in this multiple sum is the sum of those terms with indices

$$\begin{aligned} j_1 l_1, \ldots, j_k l_k &= 10, \ldots, 1p_1, \ldots, s0, \ldots, s(i-1),\ s(i+1), \ldots, sp_s, \ldots, t0, \ldots, tp_t,\ jl \\ &\equiv [si;\ jl], \end{aligned} \tag{4.17}$$

$0 \leqslant l \leqslant p_j$, $t+1 \leqslant j \leqslant t+r$, cf. (3.3a), and $0 \leqslant i \leqslant p_s$, all s for which $|\zeta_s| = |\zeta_t|$. It is easy to see that all these terms are of order $n^{\gamma} \pi^n (\zeta_{t+1}/\zeta_t)^n$ for $n \to \infty$, for some nonnegative integers γ. Now let α be the maximum of the γ's.

A careful analysis of the terms that form $E_{mk}(\zeta, \xi)$ shows that they grow at most like $n^{\beta} \pi^n (\xi/\zeta_t)^n$ for $n \to \infty$, where β is a nonnegative integer. But since ξ is arbitrary, we can set $\beta = 0$ by choosing a slightly larger value for ξ.

Combining all the above, we obtain the $O(\epsilon(n))$ term in (3.6). The validity of the claim that $\alpha = \bar{p}$ if all poles having modulus equal to $|\zeta_t^{-1}|$ are simple will become clear through the discussion that leads from (4.22) to (4.24) below. This completes the proof of part (a) of Theorem 3.1.

The proof of (3.9) and (3.10) is much more involved and requires a careful analysis of $\tilde{D}_{mk}(\zeta)$ in the neighborhood of ζ_s, $1 \leqslant s \leqslant t$. We start by noting that

$$\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}\zeta^i}\tilde{D}_{mk}(\zeta)$$
$$= \left\{ \sum_{10 \leqslant j_1 l_1 < \cdots < j_k l_k \leqslant \nu p_\nu} Z_{j_1 l_1, \ldots, j_k l_k} \zeta^{-i} Y\left(\binom{0}{i} \zeta^0, \binom{n}{l_1} \zeta_{j_1}^0, \ldots, \binom{n}{l_k} \zeta_{j_k}^0 \right) \left(\prod_{h=1}^{k} \zeta_{j_h}^n \right) \right\}$$
$$+ \frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}\zeta^i} E_{mk}(\zeta, \xi). \tag{4.18}$$

Let us now substitute $\zeta = \zeta_s$ in (4.18), and determine the precise asymptotic nature of $(1/i!)(\mathrm{d}^i/\mathrm{d}\zeta^i)\tilde{D}_{mk}(\zeta_s)$. Our main aim is to establish first

$$\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}\zeta^i}\tilde{D}_{mk}(\zeta_s) = \begin{cases} \mathrm{O}(\pi^n \delta_s(n)) & \text{as } n \to \infty \quad \text{for } 0 \leqslant i \leqslant p_s, \\ \mathrm{O}(\pi^n) & \text{as } n \to \infty \quad \text{for } p_s + 1 \leqslant i \leqslant k, \end{cases} \tag{4.19}$$

where $\delta_s(n)$ is as defined in (3.10), and

$$\frac{1}{\omega_s!}\frac{\mathrm{d}^{\omega_s}}{\mathrm{d}\zeta^{\omega_s}}\tilde{D}_{mk}(\zeta_s) \sim \Gamma_s \pi^n \quad \text{as } n \to \infty \text{ for some } \Gamma_s \neq 0. \tag{4.20}$$

When $\zeta = \zeta_s$ the term with the indices $j_1 l_1, \ldots, j_k l_k = 10, 11, \ldots, t p_t$ in the multiple sum on the right-hand side of (4.18) vanishes for $0 \leqslant i \leqslant p_s$, and is most dominant for $i = \omega_s = p_s + 1$ and $i = k$, and possibly most dominant for $\omega_s + 1 \leqslant i \leqslant k - 1$. To see this we first analyze the Y determinant of this term, namely,

$$Y\left(\binom{0}{i} \zeta_s^0, \binom{n}{0} \zeta_1^0, \ldots, \binom{n}{p_t} \zeta_t^0 \right),$$

which, by Lemma 2.4, is equal to

$$Y\left(\binom{0}{i} \zeta_s^0, \binom{0}{0} \zeta_1^0, \ldots, \binom{0}{p_t} \zeta_t^0 \right).$$

For $0 \leqslant i \leqslant p_s$ this last Y determinant has two identical rows and thus vanishes. For $i = p_s + 1$, however, it is equal to

$$(-1)^{\hat{\sigma}_s} Y\left(\binom{0}{0} \zeta_1^0, \ldots, \binom{0}{p_1} \zeta_1^0, \ldots, \binom{0}{0} \zeta_s^0, \ldots, \binom{0}{p_s} \zeta_s^0, \binom{0}{p_s + 1} \zeta_s^0, \binom{0}{0} \zeta_{s+1}^0, \ldots, \binom{0}{p_t} \zeta_t^0 \right).$$

By (2.9) and (2.10) and (4.16), we see that for $i = p_s + 1$ the most dominant term in the multiple sum on the right-hand side of (4.18) is

$$C_{p_s+1}(n) = (-1)^{\hat{\sigma}_s} Z_{10, \ldots, t p_t} \zeta_s^{-\omega_s}$$
$$\times Y\left(\binom{0}{0} \zeta_1^0, \ldots, \binom{0}{p_1} \zeta_1^0, \ldots, \binom{0}{0} \zeta_s^0, \ldots, \binom{0}{p_s} \zeta_s^0, \binom{0}{p_s + 1} \zeta_s^0, \binom{0}{0} \zeta_{s+1}^0, \ldots, \binom{0}{p_t} \zeta_t^0 \right) \pi^n, \tag{4.21}$$

and this term is nonzero. For $i=k$ the left-hand side of (4.18) is simply the coefficient of ζ^k in $\tilde{D}_{mk}(\zeta)$, which, by (3.6) and (3.7), is exactly $O(\pi^n)$ as $n\to\infty$. For $p_s+2\leqslant i\leqslant k-1$, we see easily that the multiple sum in (4.18) is of the order of π^n at most, and this observation is sufficient for our purposes.

We now go back to the case $0\leqslant i\leqslant p_s$. When $\nu=t$ the multiple sum in (4.18) contains only one term, and this term vanishes as has been shown above. When $\nu>t$ the most dominant term in this multiple sum is the sum of the terms whose indices are $j_1l_1,\ldots,j_kl_k=[si;\ jl]$, cf. (4.17), $0\leqslant l\leqslant p_j$, $t+1\leqslant j\leqslant t+r$, i.e.,

$$C_i(n)=(-1)^{i+\hat{v}_{s-1}}\zeta_s^{-i}\pi^n\sum_{j=t+1}^{t+r}\sum_{l=0}^{p_j}Z_{[si;\,jl]}Y\left(\binom{0}{0}\zeta_1^0,\ldots,\binom{0}{p_t}\zeta_t^0,\binom{n}{l}\zeta_j^0\right)\left(\frac{\zeta_j}{\zeta_s}\right)^n. \tag{4.22}$$

By the fact that $\binom{n}{l}=n^l/l!+O(n^{l-1})$ as $n\to\infty$, it follows that

$$\begin{aligned}&Y\left(\binom{0}{0}\zeta_1^0,\ldots,\binom{0}{p_t}\zeta_t^0,\binom{n}{l}\zeta_j^0\right)\\&\quad=Y\left(\binom{0}{0}\zeta_1^0,\ldots,\binom{0}{p_t}\zeta_t^0,\binom{0}{0}\zeta_j^0\right)\frac{n^l}{l!}+O(n^{l-1})\quad\text{as } n\to\infty.\end{aligned} \tag{4.23}$$

Thus $C_i(n)$, for $0\leqslant i\leqslant p_s$, has the dominant asymptotic behavior

$$\begin{aligned}&C_i(n)\sim(-1)^{i+\hat{v}_{s-1}}\zeta_s^{-i}\frac{n^{\bar{p}}}{\bar{p}!}\pi^n\sum_{j=t+1}^{t+\mu}Z_{[si;\,j\bar{p}]}Y\left(\binom{0}{0}\zeta_1^0,\ldots,\binom{0}{p_t}\zeta_t^0,\binom{0}{0}\zeta_j^0\right)\left(\frac{\zeta_j}{\zeta_s}\right)^n\\&\quad\text{as } n\to\infty,\end{aligned} \tag{4.24}$$

cf. (3.3b), i.e., $C_i(n)=O(\pi^n\delta_s(n))$ as $n\to\infty$.

We finally analyze the contribution of $(1/i!)(\mathrm{d}^i/\mathrm{d}\zeta^i)E_{mk}(\zeta_s)$. A careful analysis, similar to the one for the multiple sum in (4.18), shows this term to be at most of order $n^{\beta_1}\pi^n(\xi/\zeta_s)^n$ as $n\to\infty$ for $0\leqslant i\leqslant p_s$, and of order $n^{\beta_2}\pi^n(\xi/\zeta_t)^n$ as $n\to\infty$ for $p_s+1\leqslant i\leqslant k$. Here β_1 and β_2 are some nonnegative integers. Since ξ is arbitrary, we can set $\beta_1=\beta_2=0$ by choosing a slightly larger value for ξ. Thus this term is of order $\pi^n(\xi/\zeta_s)^n$ as $n\to\infty$ for $0\leqslant i\leqslant p_s$, and of order $\pi^n(\xi/\zeta_t)^n$ as $n\to\infty$ for $p_s+1\leqslant i\leqslant k$.

Combining all the above, we see that (4.19) is true. We furthermore obtain

$$\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}\zeta^i}\tilde{D}_{mk}(\zeta_s)\sim C_i(n)\quad\text{as } n\to\infty \text{ when } \nu>t,\ 0\leqslant i\leqslant p_s, \tag{4.25}$$

with $C_i(n)$ as given in (4.24). We also obtain

$$\frac{1}{\omega_s!}\frac{\mathrm{d}^{\omega_s}}{\mathrm{d}\zeta^{\omega_s}}\tilde{D}_{mk}(\zeta_s)\sim C_{\omega_s}(n)\quad\text{as } n\to\infty, \tag{4.26}$$

with $C_{\omega_s}(n)$ as given in (4.21), and this shows that (4.20) holds, providing Γ_s at the same time. Thus Corollary 2.7 applies with $d_i(\delta)=(1/i!)(\mathrm{d}^i/\mathrm{d}\zeta^i)\tilde{D}_{mk}(\zeta_s)$ and $\delta=\delta_s(n)$, and this proves (3.9) and (3.10). (3.11) and (3.12) follow from part (b) of Lemma 2.5 and the fact that $-d_0(\delta)/d_{\omega_s}(\delta)\sim N_s(n)$, as $n\to\infty$. The last asymptotic relation follows from (4.16),

$$\begin{aligned}Z_{[s0;\,hp_h]}&=(-1)^{v_t-p_s}\left(\prod_{j=1}^{t}A_{jp_j}^{\omega_j}\right)\frac{A_{hp_h}}{A_{sp_s}}\left(\prod_{j=1}^{t}\zeta_j^{u_j}\right)\frac{1}{\zeta_s^{p_s}}\\&\quad\times\tilde{Y}(\zeta_1,\ p_1;\ldots;\ \zeta_{s-1},\ p_{s-1};\ \zeta_s,\ p_s-1;\ \zeta_{s+1},\ p_{s+1};\ldots;\ \zeta_t,\ p_t;\ \zeta_h,\ 0),\end{aligned} \tag{4.27}$$

and (2.10) in conjunction with (4.25) and (4.26). The proof of (4.27) is very similar to that of (4.16). The only thing that remains to be shown is that the sequence $\{N_s(n)\}$ has a convergent subsequence $\{N_s(n_q)\}$ with a nonzero limit $\hat{N}_s$. First, we note that $N_s(n)$ is a trigonometric sum of the form $\Gamma(n)=\sum_{j=-J}^{J}\alpha_j \exp(\mathrm{i}\beta_j n)$ with distinct real β_j, $0\leqslant\beta_j<2\pi$, and $\alpha_j\neq 0$, $-J\leqslant j\leqslant J$. It can be shown that $\lim_{n\to\infty}\Gamma(n)=0$ if and only if $\alpha_j=0$, $-J\leqslant j\leqslant J$. Therefore, if not all α_j are zero, then the sequence $\{\Gamma(n)\}$ has a convergent subsequence with a nonzero limit $\hat{\Gamma}$, for otherwise all convergent subsequences of $\{\Gamma(n)\}$ would have a zero limit, implying that $\lim_{n\to\infty}\Gamma(n)=0$, which is impossible. This then proves the assertion above. Since now $\hat{N}_s\neq 0$, ultimately all members of the subsequence $\{N_s(n_q)\}$ are bounded away from zero, and this guarantees that $N_s(n_q)$ is the most dominant term in the asymptotic behavior of $(\zeta_{sl}(n_q)-\zeta_s)^{\omega_s}$ as $q\to\infty$.

This completes the proof of Theorem 3.1. □

Note that in all of the dominant terms described in Theorem 3.1, of all A_{ji}, $0\leqslant i\leqslant p_j$, $j=1,2,\ldots$, only A_{jp_j}, $1\leqslant j\leqslant t+\mu$, are present.

5. Proofs of Theorems 3.3 and 3.5

In order to achieve the proof of Theorem 3.3 we need some additional auxiliary results.

We start by noting that the numerator of $f_{mk}(z)$ in (1.4) can be replaced by

$$D\big(z^k S_q(z),\ z^{k-1}S_{q+1}(z),\ldots,\ z^0 S_{q+k}(z)\big)\quad\text{with any } q,\ m-k\leqslant q\leqslant m, \tag{5.1}$$

by (1.3). We choose to take $q=m-k+1=n$. This is not essential, but it simplifies things a little.

Next, the error $f(z)-f_{mk}(z)$ has the determinant representation

$$\begin{aligned} f(z)-f_{mk}(z) &= \frac{D_{mk}\big(z^k[f(z)-S_n(z)],\ z^{k-1}[f(z)-S_{n+1}(z)],\ldots,\ z^0[f(z)-S_{n+k}(z)]\big)}{D_{mk}(z^k,\ z^{k-1},\ldots,\ z^0)} \\ &\equiv \frac{\hat{F}_{mk}(z)}{\hat{D}_{mk}(z)}. \end{aligned} \tag{5.2}$$

Lemma 5.1. *Define*

$$\hat{B}_{jl}(z)=\sum_{i=l}^{p_j}A_{ji}\sum_{s=l}^{i}\binom{i+1}{s-l}\left(\frac{\zeta_j z}{1-\zeta_j z}\right)^{i-s+1},\quad z\neq\zeta_j^{-1},\ 1\leqslant j\leqslant\nu. \tag{5.3}$$

$\hat{B}_{jl}(z)$ can be expressed in terms of the $\tilde{A}_{ji}$ in the form

$$\begin{aligned} &\hat{B}_{jl}(z)=\tilde{A}_{jl}U_j+\sum_{q=l+1}^{p_j}\tilde{A}_{jq}(U_j+1)U_j^{q-l},\quad 0\leqslant l\leqslant p_j-1,\quad U_j\equiv\frac{\zeta_j z}{1-\zeta_j z},\\ &\hat{B}_{jp_j}(z)=\tilde{A}_{jp_j}U_j=A_{jp_j}U_j\neq 0. \end{aligned} \tag{5.4}$$

Then $f(z) - S_n(z)$ *has the expansion*

$$f(z) - S_n(z) = \sum_{j=1}^{\nu}\left[\sum_{l=0}^{p_j} \hat{B}_{jl}(z)\binom{n}{l}\right](\zeta_j z)^n + \hat{B}(n, \xi, z)(\xi z)^n, \tag{5.5}$$

where

$$\hat{B}(n, \xi, z)(\xi z)^n = g(z) - \sum_{i=0}^{n} d_i z^i,\ |\hat{B}(n, \xi, z)| \leqslant M(\xi)\frac{|\xi z|}{1 - |\xi z|},\quad |z| < \xi^{-1}, \tag{5.6}$$

with $M(\xi)$ *as defined in* (4.6).

Proof. (5.6) follows by bounding the right-hand side of

$$g(z) - \sum_{i=0}^{n} d_i z^i = \frac{1}{2\pi i}\int_{|\lambda| = \xi^{-1}} g(\lambda)\frac{\left(\frac{z}{\lambda}\right)^{n+1}}{\lambda - z}\,d\lambda. \tag{5.7}$$

The proof of (5.5) is based on the observation that

$$\begin{aligned}(1-z)^{-i-1} &= \frac{1}{i!}\frac{d^i}{dz^i}(1-z)^{-1} = \frac{1}{i!}\frac{d^i}{dz^i}\left[\sum_{s=0}^{n+i} z^s + \frac{z^{n+i+1}}{1-z}\right] \\ &= \sum_{s=0}^{n}\binom{s+i}{i}z^s + \left[\sum_{s=0}^{i}\binom{n+i+1}{s}\left(\frac{z}{1-z}\right)^{i-s+1}\right]z^n.\end{aligned} \tag{5.8}$$

Now, apply the identity in (4.9) to $\binom{n+i+1}{s}$ in (5.8) with $\alpha = n$ and $\beta = i + 1$. Next, replace z in (5.8) by $\zeta_j z$, multiply both sides of the resulting equality by A_{ji} and sum over j and i, $0 \leqslant i \leqslant p_j$, $1 \leqslant j \leqslant \nu$. By appropriate interchange of summations (5.5) now follows. To prove (5.4) we start by replacing the summation index s in (5.3) by the index q, where $i - s = q$. The binomial coefficient $\binom{i+1}{s-l}$ then becomes $\binom{i+1}{i-q-l}$. Now interchange the summations over i and q, and invoke the identity

$$\binom{i+1}{i-q-l} = \binom{i}{i-q-l} + \binom{i}{i-q-l-1}.$$

(5.4) follows from this by recalling (4.3). The details are left to the reader. □

Theorem 5.2. *Denote*

$$X_{j_0 l_0, j_1 l_1, \ldots, j_k l_k}(z) = \begin{vmatrix} \hat{B}_{j_0 l_0}(z) & \hat{B}_{j_1 l_1}(z) & \cdots & \hat{B}_{j_k l_k}(z) \\ \hat{A}_{j_0 l_0, 1} & \hat{A}_{j_1 l_1, 1} & \cdots & \hat{A}_{j_k l_k, 1} \\ \hat{A}_{j_0 l_0, 2} & \hat{A}_{j_1 l_1, 2} & \cdots & \hat{A}_{j_k l_k, 2} \\ \vdots & \vdots & & \vdots \\ \hat{A}_{j_0 l_0, k} & \hat{A}_{j_1 l_1, k} & \cdots & \hat{A}_{j_k l_k, k} \end{vmatrix}. \tag{5.9}$$

Then $\hat{F}_{mk}(z)$, the numerator determinant on the right-hand side of (5.2), *has the expansion*

$$\hat{F}_{mk}(z)=z^{n+k}\Bigg\{\sum_{10\leqslant j_0l_0<j_1l_1<\cdots<j_kl_k\leqslant \nu p_\nu} X_{j_0l_0,j_1l_1,\ldots,j_kl_k}(z)$$
$$\times Y\left(\binom{n}{l_0}\zeta_{j_0}^0,\binom{n}{l_1}\zeta_{j_1}^0,\ldots,\binom{n}{l_k}\zeta_{j_k}^0\right)\left(\prod_{h=0}^{k}\zeta_{j_h}^n\right)\Bigg\}+G_{mk}(z,\xi), \tag{5.10}$$

where $G_{mk}(z,\xi)$ is the only term in this expansion that depends on $g(z)$ and thus on ξ. When $g(z)$ is a polynomial $G_{mk}(z,\xi)\equiv 0$ for $m\geqslant \deg g(z)$. The exact form of $G_{mk}(z,\xi)$ will be described through the example considered in the proof of Theorem 4.2. *Note that the multiple sum in* (5.10) *is empty when $\nu=t$.*

Proof. For the sake of simplicity, again we shall first assume that $g(z)$ is a polynomial. This implies that $\hat{B}(n,\xi,z)\equiv 0$ for $n>\deg g(z)$ in (5.5), as well as $\hat{A}_s(n,\xi)=0$ for $n+s\geqslant \deg g(z)$ in (4.7). Substituting now (4.7) and (5.5) in the determinant representation of $\hat{F}_{mk}(z)$, and assuming that $\nu>t$, we obtain

$$\hat{F}_{mk}(z)=\begin{vmatrix} z^k\sum_{j_0l_0}\hat{B}_{j_0l_0}(z)\binom{n}{l_0}(\zeta_{j_0}z)^n & z^{k-1}\sum_{j_0l_0}\hat{B}_{j_0l_0}(z)\binom{n+1}{l_0}(\zeta_{j_0}z)^{n+1} & \cdots & z^0\sum_{j_0l_0}\hat{B}_{j_0l_0}(z)\binom{n+k}{l_0}(\zeta_{j_0}z)^{n+k} \\ \sum_{j_1l_1}\hat{A}_{j_1l_1,1}\binom{n}{l_1}\zeta_{j_1}^n & \sum_{j_1l_1}\hat{A}_{j_1l_1,1}\binom{n+1}{l_1}\zeta_{j_1}^{n+1} & \cdots & \sum_{j_1l_1}\hat{A}_{j_1l_1,1}\binom{n+k}{l_1}\zeta_{j_1}^{n+k} \\ \vdots & \vdots & & \vdots \\ \sum_{j_kl_k}\hat{A}_{j_kl_k,k}\binom{n}{l_k}\zeta_{j_k}^n & \sum_{j_kl_k}\hat{A}_{j_kl_k,k}\binom{n+1}{l_k}\zeta_{j_k}^{n+1} & \cdots & \sum_{j_kl_k}\hat{A}_{j_kl_k,k}\binom{n+k}{l_k}\zeta_{j_k}^{n+k} \end{vmatrix}, \tag{5.11}$$

where, as before, Σ_{jl} denotes $\Sigma_{j=1}^{\nu}\Sigma_{l=0}^{p_j}$. Using the multilinearity property of determinants, (5.11) can be expressed as

$$\hat{F}_{mk}(z)=z^{n+k}\left\{\sum_{j_0l_0}\cdots\sum_{j_kl_k}\hat{B}_{j_0l_0}(z)\left(\prod_{s=1}^{k}\hat{A}_{j_sl_s,s}\right)\left(\prod_{s=0}^{k}\zeta_{j_s}^n\right)Y\left(\binom{n}{l_0}\zeta_{j_0}^0,\binom{n}{l_1}\zeta_{j_1}^0,\ldots,\binom{n}{l_k}\zeta_{j_k}^0\right)\right\}. \tag{5.12}$$

(5.10) with $G_{mk}(z,\xi)\equiv 0$ now follows from (5.12) by employing Lemma 2.1.

The result with $G_{mk}(z,\xi)\not\equiv 0$ is obtained by adding $\hat{B}(n+i,\xi,z)(\xi z)^{n+i}$ and $\hat{A}(n+i,\xi)\xi^{n+i}$ to the appropriate entries in (5.11). Again $G_{mk}(z,\xi)$ is a sum of determinants similar in form to the Y determinants in (5.10). For the example considered in the proof of Theorem 4.2 we have

$$z^{-n-3}G_{m3}(z,\xi)$$
$$=\sum_{10\leqslant j_1l_1<j_2l_2<j_3l_3\leqslant 3p_3} X^{(1,2,3)}_{j_1l_1,j_2l_2,j_3l_3}(z)Y\left(\hat{B}(n,\xi,z)\xi^0,\binom{n}{l_1}\zeta_{j_1}^0,\binom{n}{l_2}\zeta_{j_2}^0,\binom{n}{l_3}\zeta_{j_3}^0\right)(\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\xi)^n$$
$$+\sum_{10\leqslant j_0l_0<j_2l_2<j_3l_3\leqslant 3p_3} X^{(0,2,3)}_{j_0l_0,j_2l_2,j_3l_3}(z)Y\left(\binom{n}{l_0}\zeta_{j_0}^0,\hat{A}_1(n,\xi)\xi^0,\binom{n}{l_2}\zeta_{j_2}^0,\binom{n}{l_3}\zeta_{j_3}^0\right)(\zeta_{j_0}\zeta_{j_2}\zeta_{j_3}\xi)^n$$

$$+ \sum_{10 \leqslant j_0 l_0 < j_1 l_1 < j_3 l_3 \leqslant 3p_3} X^{(0,1,3)}_{j_0 l_0, j_1 l_1, j_3 l_3}(z) Y\left(\binom{n}{l_0}\zeta^0_{j_0}, \binom{n}{l_1}\zeta^0_{j_1}, \hat{A}_2(n, \xi)\xi^0, \binom{n}{l_3}\zeta^0_{j_3}\right)(\zeta_{j_0}\zeta_{j_1}\zeta_{j_3}\xi)^n$$

$$+ \sum_{10 \leqslant j_0 l_0 < j_1 l_1 < j_2 l_2 \leqslant 3p_3} X^{(0,1,2)}_{j_0 l_0, j_1 l_1, j_2 l_2}(z) Y\left(\binom{n}{l_0}\zeta^0_{j_0}, \binom{n}{l_1}\zeta^0_{j_1}, \binom{n}{l_2}\zeta^0_{j_2}, \hat{A}_3(n, \xi)\xi^0\right)(\zeta_{j_0}\zeta_{j_1}\zeta_{j_2}\xi)^n$$

$$+ \cdots + Y\left(\hat{B}(n, \xi, z)\xi^0, \hat{A}_1(n, \xi)\xi^0, \hat{A}_2(n, \xi)\xi^0, \hat{A}_3(n, \xi)\xi^0\right)\xi^4,$$

where

$$X^{(1,2,3)}_{jl, j'l', j''l''}(z) = Z^{(1,2,3)}_{jl, j'l', j''l''},$$

$$X^{(0,1,2)}_{jl, j'l', j''l''}(z) = \begin{vmatrix} \hat{B}_{jl}(z) & \hat{B}_{j'l'}(z) & \hat{B}_{j''l''}(z) \\ \hat{A}_{jl,1} & \hat{A}_{j'l',1} & \hat{A}_{j''l'',1} \\ \hat{A}_{jl,2} & \hat{A}_{j'l',2} & \hat{A}_{j''l'',2} \end{vmatrix}, \text{ etc.} \qquad \square$$

In case $f(z)$ is a rational function with only simple poles $\zeta_1^{-1}, \ldots, \zeta_\nu^{-1}$, $\nu > k$, which we write in the form $f(z) = \sum_{j=1}^{\nu} A_j/(1 - \zeta_j z) + g(z)$, the result in (5.10) reduces to

$$\hat{F}_{mk}(z) = (-1)^k z^{2k} \sum_{1 \leqslant j_0 < j_1 < \cdots < j_k \leqslant \nu} \left(\prod_{s=0}^{k} A_{j_s}\right) \frac{\left[V(\zeta_{j_0}, \zeta_{j_1}, \ldots, \zeta_{j_k})\right]^2}{\prod_{s=0}^{k}(1 - \zeta_{j_s} z)} \left(z \prod_{s=0}^{k} \zeta_{j_s}\right)^n \quad (5.10')$$

for $m \geqslant k + \deg g(z)$. This result follows from the fact that $X_{j_0 l_0, j_1 l_1, \ldots, j_k l_k}(z)$ in (5.10) for this case is simply $T(\zeta_{j_0}, \zeta_{j_1}, \ldots, \zeta_{j_k}; z)$ defined and analyzed in Appendix A to this work, cf. (A.11)–(A.13). Combining now (4.12′) with (5.10′), we have a complete expansion of $f(z) - f_{mk}(z) = \hat{F}_{mk}(z)/\hat{D}_{mk}(z)$ for this case.

Proof of Theorem 3.3. With the preliminary result in Theorem 5.2 the proof of Theorem 3.3 is almost identical to that of Theorem 3.1.

We first treat the case in which $\nu > t$. For this case the most dominant part of the multiple sum on the right-hand side of (5.10) is the sum of those terms with indices

$$j_0 l_0, j_1 l_1, \ldots, j_k l_k = 10, 11, \ldots, 1p_1, 20, \ldots, 2p_2, \ldots, t0, \ldots, tp_t, jl,$$

$0 \leqslant l \leqslant p_j$, $t + 1 \leqslant j \leqslant t + r$. By (4.23), a term with the indices above is of order $n^l \pi^n |\zeta_{t+1}|^n$. Therefore, the most dominant behavior of the sum above is determined by those terms with $t + 1 \leqslant j \leqslant t + \mu$, $l = \bar{p}$, cf. the discussion covering (4.22)–(4.24). Consequently, the most dominant part of $\hat{F}_{mk}(z) - G_{mk}(z, \xi)$ is given by

$$H(n, z) = \frac{n^{\bar{p}}}{\bar{p}!} \pi^n z^{n+k} \sum_{j=t+1}^{t+\mu} X_{10,11,\ldots,tp_t,j\bar{p}}(z) Y\left(\binom{0}{0}\zeta^0_1, \ldots, \binom{0}{p_t}\zeta^0_t, \binom{0}{0}\zeta^0_j\right)\zeta^n_j. \quad (5.13)$$

As stated in Theorem 5.2, the multiple sum is empty when $\nu = t$. (This is consistent with the fact that $f_{mk}(z) \equiv f(z)$ for all m sufficiently large, when $f(z)$ is a rational function with degree of denominator equal to k.)

A careful analysis of the terms that form $G_{mk}(z, \xi)$ shows that they grow at most like $\pi^n(\xi z)^n$ for $n \to \infty$.

Combining all the above, we see that when $\nu > t$, $H(n, z)$ is the dominant part of $\hat{F}_{mk}(z)$ and is of order $(\pi\zeta_{t+1}z)^n$ for $n \to \infty$, and when $\nu = t$, $\hat{F}_{mk}(z)$ is of order $(\pi\xi z)^n$ for $n \to \infty$. This, combined with (3.6), results in (3.13) with $q = 0$. (3.14) and (3.15) follow by combining (5.13) and (3.6) and invoking (3.7) and (A.18) from Appendix A.

The assertions about uniform convergence of $f_{mk}(z)$ and uniformness of (3.13) with $q = 0$ and of (3.14) are seen to be valid by the facts that $\hat{B}_{ji}(z)$ are uniformly bounded in any compact subset of $K\backslash\{\zeta_1^{-1},\dots,\zeta_\nu^{-1}\}$ and that $\hat{B}(n, \xi, z)$ is uniformly bounded in any compact subset of K and for all n, cf. (5.6).

Finally, the assertions about uniform convergence of $f_{mk}^{(q)}(z)$, and uniformness of (3.13) with $q = 1, 2,\dots$ can be seen to be valid by a cursory analysis of

$$f^{(q)}(z) - f_{mk}^{(q)}(z) = \frac{\mathrm{d}^q}{\mathrm{d}z^q}\frac{\hat{F}_{mk}(z)}{\hat{D}_{mk}(z)} \tag{5.14}$$

in the light of Theorems 4.2 and 5.2. The details are left to the reader.

This completes the proof of Theorem 3.3. □

Proof of Theorem 3.5. For $1 \leqslant s \leqslant t$ let $K_s = \{z: |z - \zeta_s^{-1}| \leqslant \gamma\}$, with γ chosen sufficiently small to ensure that K_s contains none of the ζ_j^{-1} with $j \neq s$. By Theorem 3.1, for n sufficiently large, K_s contains only $\zeta_{s0}^{-1}(n),\dots,\zeta_{sp_s}^{-1}(n)$. Let ∂K_s denote the boundary of K_s traversed in the counterclockwise direction.

From Theorem 3.3

$$\max_{z\in\partial K_s} |f(z) - f_{mk}(z)| = \mathrm{O}(\tilde{\delta}_s(n)) \quad \text{as } n \to \infty, \tag{5.15}$$

where

$$\tilde{\delta}_s(n) = \begin{cases} n^{\bar{p}}\left[\,|\zeta_{t+1}|(|\zeta_s|^{-1} + \gamma)\right]^n & \text{if } \nu > t, \\ \left[\xi(|\zeta_s|^{-1} + \gamma)\right]^n & \text{if } \nu = t. \end{cases} \tag{5.16}$$

For the proof of (3.19), we begin by noting that

$$\begin{aligned} H_{sq} &= \frac{1}{2\pi\mathrm{i}}\int_{\partial K_s}(z - \zeta_s^{-1})^q f(z)\,\mathrm{d}z, \\ \sum_{l=0}^{p_s} H_{sq,l}(n) &= \frac{1}{2\pi\mathrm{i}}\int_{\partial K_s}(z - \hat{z}_s(n))^q f_{mk}(z)\,\mathrm{d}z, \end{aligned} \tag{5.17}$$

for n sufficiently large. Now

$$\begin{aligned} \Delta_{mk}(z) &= (z - \hat{z}_s(n))^q f_{mk}(z) - (z - \zeta_s^{-1})^q f(z) \\ &= (z - \hat{z}_s(n))^q[f_{mk}(z) - f(z)] + \left[(z - \hat{z}_s(n))^q - (z - \zeta_s^{-1})^q\right]f(z). \end{aligned} \tag{5.18}$$

The first term on the right-hand side of (5.18) is $O(\tilde{\delta}_s(n))$ as $n \to \infty$ uniformly on ∂K_s. Also,

$$(z - \hat{z}_s(n))^q - (z - \zeta_s^{-1})^q = (\zeta_s^{-1} - \hat{z}_s(n)) \sum_{i=0}^{q-1} (z - \hat{z}_s(n))^i (z - \zeta_s^{-1})^{q-i-1} = O(\delta_s(n))$$

$$\text{as } n \to \infty, \tag{5.19}$$

since $\zeta_s^{-1} - \hat{z}_s(n) = O(\delta_s(n))$ as $n \to \infty$ by (3.9). Consequently

$$\max_{z \in \partial K_s} |\Delta_{mk}(z)| = O(\tilde{\delta}_s(n)) \quad \text{as } n \to \infty. \tag{5.20}$$

Taking now the difference between the two equalities in (5.17), and using (5.20), and finally recalling that γ and ξ^{-1} can be taken arbitrarily close to 0 and R respectively, (3.19) follows. □

We note that the technique employed in the proof of Theorem 3.5 can be used in obtaining

$$\limsup_{n \to \infty} |\hat{z}_s(n) - \zeta_s^{-1}|^{1/n} \leqslant \Lambda_s,$$

which is a weaker version of $\hat{z}_s(n) - \zeta_s^{-1} = O(\delta_s(n))$ as $n \to \infty$ with $\hat{z}_s(n)$ as in (3.17). In this case observe that, for K_s sufficiently small,

$$-\omega_s \zeta_s^{\pm 1} = \frac{1}{2\pi i} \int_{\partial K_s} z^{\mp 1} \frac{f'(z)}{f(z)} \, dz \quad \text{and} \quad -\sum_{l=0}^{p_s} \zeta_{sl}^{\pm 1}(n) = \frac{1}{2\pi i} \int_{\partial K_s} z^{\mp 1} \frac{f'_{mk}(z)}{f_{mk}(z)} \, dz.$$

Now continue as in the proof of Theorem 3.5.

6. Analysis of intermediate rows

In the previous sections we took k to be the total number of poles of $f(z)$, counted according to their multiplicities, that are contained in an open disk with its center at the origin. In this section we shall consider intermediate values of k. Specifically, we shall assume that $\nu > t$, $\sum_{j=t+1}^{t+r} \omega_j > 1$, and consider $\sum_{j=1}^{t} \omega_j < k < \sum_{j=t+1}^{t+r} \omega_j$. (Here we can also assume $t = 0$ and replace $\sum_{j=1}^{t} \omega_j$ by 0 in this case.) The problem with intermediate values of k is that not for all such k do the rows $\{f_{mk}(z)\}_{m=0}^{\infty}$ have to converge. In this respect we recall the result of [11], which, roughly speaking, states that there is a subsequence of $\{f_{mk}(z)\}_{m=0}^{\infty}$ that converges to $f(z)$ uniformly in any compact subset of $\{z : |z| < |\zeta_{t+1}^{-1}|\} \setminus \{\zeta_1^{-1}, \ldots, \zeta_t^{-1}, z'_1, \ldots, z'_s\}$, where $z'_1, \ldots, z'_s$ are limit points of the poles of those $f_{mk}(z)$ in this subsequence.

As can be seen from the proofs of Theorems 3.1 and 3.3, whether $\lim_{m \to \infty} f_{mk}(z)$ exists or not, depends on whether $\lim_{m \to \infty} Q_{mk}(z)$ exists or not. Therefore, we shall concentrate on the question for what intermediate values of k $\lim_{m \to \infty} Q_{mk}(z)$ exists. The main result of this section is stated in the next theorem.

Theorem 6.1. *Let $f(z)$ be as in Theorem 3.1, and let k be as in the first paragraph of this section. Denote*

$$\tau = k - \sum_{j=1}^{t} \omega_j. \tag{6.1}$$

Designate by IP(τ) *the nonlinear integer programming problem*

$$\underset{\sigma_{t+1},\dots,\sigma_{t+r}}{\text{maximize}} \sum_{j=t+1}^{t+r} \left(\omega_j\sigma_j - \sigma_j^2\right)$$

$$\text{subject to} \sum_{j=t+1}^{t+r} \sigma_j = \tau \quad \text{and } 0 \leqslant \sigma_j \leqslant \omega_j,\ t+1 \leqslant j \leqslant t+r. \tag{6.2}$$

Then $\lim_{m\to\infty} Q_{mk}(z)$ *exists if* IP(τ) *has a unique solution. When it exists, denote the unique solution of* IP(τ) *by* σ_j^*, $t+1 \leqslant j \leqslant t+r$, *and let*

$$\beta^* = \sum_{j=t+1}^{t+r} \left(\omega_j\sigma_j^* - \sigma_j^{*2}\right). \tag{6.3}$$

Then

$$\hat{D}_{mk}(z) = W' n^{\beta^*}\left(\pi \prod_{j=t+1}^{t+r} \zeta_j^{\sigma_j^*}\right)^n \left[Q(z) \prod_{j=t+1}^{t+r} (1-\zeta_j z)^{\sigma_j^*} + O(n^{-1})\right] \text{ as } n\to\infty, \tag{6.4}$$

where

$$W' = (-1)^{k+\kappa}\left(\prod_{j=1}^{t} A_{jp_j}^{\omega_j}\right)\left(\prod_{j=t+1}^{t+r} A_{jp_j}^{\sigma_j^*}\right)\left(\prod_{j=1}^{t} \zeta_j^{2u_j}\right)\left(\prod_{j=t+1}^{t+r} \zeta_j^{2u_j^*}\right)$$
$$\times\left[\tilde{Y}(\zeta_1, p_1;\dots;\zeta_t, p_t;\zeta_{t+1}, \sigma_{t+1}^*-1;\dots;\zeta_{t+r}, \sigma_{t+r}^*-1)\right]^2$$
$$\times \prod_{\substack{i=1\\ \sigma_{t+i}^*\neq 0}}^{r} \prod_{j=1}^{\sigma_{t+i}^*} \frac{(j-1)!}{(p_{t+i}-\sigma_{t+i}^*+j)!} \neq 0 \tag{6.5}$$

with $\kappa = v_t + \sum_{j=t+1}^{t+r} u_j^*$, $u_j^* = \frac{1}{2}\sigma_j^*(\sigma_j^*-1)$, $t+1 \leqslant j \leqslant t+r$, i.e., $\hat{D}_{mk}(z)$ *has exactly* ω_j *zeros that tend to* ζ_j^{-1}, $1 \leqslant j \leqslant t$, *and exactly* σ_j^* *zeros that tend to* ζ_j^{-1}, $t+1 \leqslant j \leqslant t+r$, *and* $\hat{D}_{mk}(z) = O(n^{\beta^*}\pi^n |\zeta_{t+1}|^{\tau n})$ *as* $n\to\infty$. *(In* (6.5) *and below whenever* $\sigma_{t+i}^* = 0$ *for some* i, $1 \leqslant i \leqslant r$, *all reference to* ζ_{t+i} *is to be deleted.)*

Denote by $\bar{\sigma}_j$, $t+1 \leqslant j \leqslant t+r$, *any solution—not necessarily unique—of* IP($\tau+1$), *and let*

$$\bar{\beta} = \sum_{j=t+1}^{t+r} \left(\omega_j\bar{\sigma}_j - \bar{\sigma}_j^2\right). \tag{6.6}$$

(Note that $\bar{\beta}$ *is the same for all solutions of* IP($\tau+1$).) *Let* $\zeta_{sl}(n)$, $0 \leqslant l \leqslant p_s$, *and* $\tilde{\zeta}_s(n)$, $1 \leqslant s \leqslant t$, *be exactly as defined in Theorem* 3.1. *Then* (3.9) *holds with* $\delta_s(n) = n^{\bar{\beta}-\beta^*}|\zeta_{t+1}/\zeta_s|^n$ *now. Also, if we let* $\zeta_{sl}^*(n)$, $1 \leqslant l \leqslant \sigma_s^*$, *be the approximations to* ζ_s, $t+1 \leqslant s \leqslant t+r$, *then*

$$\frac{1}{\sigma_s^*}\sum_{l=1}^{\sigma_s^*} \zeta_{sl}^*(n) = \zeta_s + O(n^{-1}) \quad \text{as } n\to\infty,\ t+1 \leqslant s \leqslant t+r. \tag{6.7}$$

Again (3.9) *with the new* $\delta_s(n)$, *and* (6.7), *hold also when the* ζ'*s there are replaced by their reciprocals.*

Finally, $f_{mk}(z)$ *converges to* $f(z)$ *uniformly in any compact subset of* $\{z: |z| < |\zeta_{t+1}^{-1}|\} \setminus \{\zeta_1^{-1},\ldots,\zeta_t^{-1}\}$. *Actually,*

$$f(z) - f_{mk}(z) = O\left(n^{\bar{\beta}-\beta^*}|\zeta_{t+1}z|^n\right) \quad \text{as } n \to \infty, \tag{6.8}$$

uniformly in any compact subset of $K\setminus\{\zeta_1^{-1},\ldots,\zeta_\nu^{-1}\}$. *In addition, Theorem* 3.5 *holds with no changes for this* k.

Note 6.2. An algorithm for the solution of IP(τ) has been given in [14]. A different algorithm has recently been proposed by Kaminski and Sidi [10].

Let σ_j, $t+1 \leqslant j \leqslant t+r$, be a solution of IP($\tau$).

(1) $\sigma_j' = \omega_j - \sigma_j$, $t+1 \leqslant j \leqslant t+r$, is a solution of IP($\tau'$) with $\tau' = \sum_{j=t+1}^{t+r}\omega_j - \tau$.

(2) If $\omega_{j'} = \omega_{j''}$ for some j', j'', $t+1 \leqslant j', j'' \leqslant t+r$, and if $\sigma_{j'} = \delta_1$, $\sigma_{j''} = \delta_2$ in a solution to IP(τ), $\delta_1 \neq \delta_2$, then there is another solution to IP(τ) with $\sigma_{j'} = \delta_2$, $\sigma_{j''} = \delta_1$. Consequently, a solution to IP(τ) cannot be unique unless $\sigma_{j'} = \sigma_{j''}$. One implication of this is that for $\omega_{t+1} = \cdots = \omega_{t+r} = \bar{\omega} > 1$, IP($\tau$) has a unique solution only for $\tau = qr$, $q = 1,\ldots,\bar{\omega}-1$, and in this solution $\sigma_j^* = q$, $t+1 \leqslant j \leqslant t+r$. For $\omega_{t+1} = \cdots = \omega_{t+r} = 1$ no unique solution to IP(τ) exists with $1 \leqslant \tau \leqslant r-1$. Another implication is that for $\omega_{t+1} = \cdots = \omega_{t+\mu} > \omega_{t+\mu+1} \geqslant \cdots \geqslant \omega_{t+r}$, $\mu < r$, no unique solution to IP(τ) exists for $\tau = 1,\ldots,\mu-1$, and a unique solution exists for $\tau = \mu$, this solution being $\sigma_{t+1}^* = \cdots = \sigma_{t+\mu}^* = 1$, $\sigma_j^* = 0$, $t+\mu+1 \leqslant j \leqslant t+r$.

(3) A unique solution to IP(τ) exists when ω_j, $t+1 \leqslant j \leqslant t+r$, are all even or all odd, and $\tau = qr + \frac{1}{2}\sum_{j=t+1}^{t+r}(\omega_j - \omega_{t+r})$, $0 \leqslant q \leqslant \omega_{t+r}$. This solution is given by $\sigma_j^* = q + \frac{1}{2}(\omega_j - \omega_{t+r})$, $t+1 \leqslant j \leqslant t+r$.

(4) Obviously, when $r = 1$ a unique solution to IP(τ) exists for all possible τ. When $r = 2$ and $\omega_1 + \omega_2$ is odd a unique solution to IP(τ) exists for all possible τ, as shown in [10].

Proof of Theorem 6.1. A careful analysis of the terms in the expansion of $\tilde{D}_{mk}(\zeta)$ in Theorem 4.2 reveals that the dominant ones are those having the indices

$$j_1l_1,\ldots,j_kl_k = 10,\ 11,\ldots,\ tp_t,\ q_1i_1,\ q_2i_2,\ldots,q_\tau i_\tau \equiv \{q_1i_1,\ldots,q_\tau i_\tau\} \tag{6.9}$$

subject to the condition

$$(t+1)0 \leqslant q_1i_1 < q_2i_2 < \cdots < q_\tau i_\tau \leqslant (t+r)p_{t+r}. \tag{6.10}$$

Each one of these terms has the form

$$Z_{\{q_1i_1,\ldots,q_\tau i_\tau\}}\pi^n\left(\prod_{h=1}^{\tau}\zeta_{q_h}\right)^n$$
$$\times Y\left(\binom{0}{0}\zeta^0,\binom{0}{0}\zeta_1^0,\ldots,\binom{0}{p_1}\zeta_1^0,\ldots,\binom{0}{0}\zeta_t^0,\ldots,\binom{0}{p_t}\zeta_t^0,\binom{n}{i_1}\zeta_{q_1}^0,\ldots,\binom{n}{i_\tau}\zeta_{q_\tau}^0\right), \tag{6.11}$$

so that it is of order $n^\beta|\pi\zeta_{t+1}^\tau|^n$ for $n \to \infty$, for some nonnegative integer β. Now we can write

$$q_1i_1,\ldots,q_\tau i_\tau = (t+1)l_{11},\ (t+1)l_{12},\ldots,(t+1)l_{1\sigma_{t+1}},\ (t+2)l_{21},\ldots,(t+2)l_{2\sigma_{t+2}},\ldots,$$
$$(t+r)l_{r1},\ldots,(t+r)l_{r\sigma_{t+r}}, \tag{6.12}$$

for some integers l_{ij} and σ_{t+i} satisfying

$$0 \leqslant l_{i1} < l_{i2} < \cdots < l_{i\sigma_{t+i}} \leqslant p_{t+i}, \quad 1 \leqslant i \leqslant r,$$
$$0 \leqslant \sigma_{t+i} \leqslant \omega_{t+i}, \quad 1 \leqslant i \leqslant r, \quad \sum_{j=t+1}^{t+r} \sigma_j = \tau. \tag{6.13}$$

Of course, if $\sigma_{t+i} = 0$, for some i, then $(t+i)l_{ij}$ are absent from (6.12), and all reference to ζ_{t+i} below is to be deleted. By Appendix B, (6.12), and (2.9) and (2.10), the term in (6.11) is exactly

$$Y\left(\begin{pmatrix}0\\0\end{pmatrix}\zeta^0, \begin{pmatrix}0\\0\end{pmatrix}\zeta_1^0, \ldots, \begin{pmatrix}0\\p_t\end{pmatrix}\zeta_t^0, \begin{pmatrix}0\\0\end{pmatrix}\zeta_{t+1}^0, \ldots, \begin{pmatrix}0\\\sigma_{t+1}-1\end{pmatrix}\zeta_{t+1}^0, \ldots, \begin{pmatrix}0\\0\end{pmatrix}\zeta_{t+r}^0, \ldots, \begin{pmatrix}0\\\sigma_{t+r}-1\end{pmatrix}\zeta_{t+r}^0\right)$$
$$\times n^{\beta}\left(\pi \prod_{j=t+1}^{t+r} \zeta_j^{\sigma_j}\right)^n \left[Z_{\{q_1 i_1, \ldots, q_\tau i_\tau\}}\left(\prod_{i=1}^{r} C(l_{i1}, \ldots, l_{i\sigma_{t+i}})\right) + \mathrm{O}(n^{-1})\right] \tag{6.14}$$

with

$$\beta = \sum_{i=1}^{r}\left(\sum_{j=1}^{\sigma_{t+i}} l_{ij} - \tfrac{1}{2}\sigma_{t+i}(\sigma_{t+i}-1)\right). \tag{6.15}$$

With the σ_{t+i} fixed for the moment, the largest value of β is obtained when the l_{ij} take their maximum values consistent with (6.13), i.e., when $l_{ij} = p_{t+i} - \sigma_{t+i} + j$, $1 \leqslant j \leqslant \sigma_{t+i}$. For these l_{ij} we have

$$\beta = \sum_{i=t+1}^{t+r} \left(\omega_i \sigma_i - \sigma_i^2\right) \equiv \beta(\sigma) \tag{6.16}$$

and

$$\prod_{i=1}^{r} C(l_{i1}, \ldots, l_{i\sigma_{t+i}}) = \prod_{\substack{i=1\\ \sigma_{t+i} \neq 0}}^{r} \prod_{j=1}^{\sigma_{t+i}} \frac{(j-1)!}{(p_{t+i} - \sigma_{t+i} + j)!} \neq 0 \tag{6.17}$$

by (B.9), and

$$Z_{\{q_1 i_1, \ldots, q_\tau i_\tau\}} = (-1)^{\phi'}\left(\prod_{j=1}^{t} A_{jp_j}^{\omega_j}\right)\left(\prod_{j=t+1}^{t+r} A_{jp_j}^{\sigma_j}\right)\left(\prod_{j=1}^{t} \zeta_j^{u_j}\right)\left(\prod_{j=t+1}^{t+r} \zeta_j^{u_j'}\right)$$
$$\times \tilde{Y}(\zeta_1, p_1; \ldots; \zeta_t, p_t; \zeta_{t+1}, \sigma_{t+1}-1; \ldots; \zeta_{t+r}, \sigma_{t+r}-1) \neq 0, \tag{6.18}$$

where

$$\phi' = v_t + \sum_{j=t+1}^{t+r} u_j', \quad u_j' = \tfrac{1}{2}\sigma_j(\sigma_j - 1). \tag{6.19}$$

So far we have shown that the dominant part of $\hat{D}_{mk}(z)$ for $n \to \infty$ is

$$R(n, z) = \sum_{\substack{\sigma_{t+1}, \ldots, \sigma_{t+r}\\ 0 \leqslant \sigma_i \leqslant \omega_i, t+1 \leqslant i \leqslant t+r,\\ \sum_{i=t+1}^{t+r} \sigma_i = \tau}} \left[\hat{C}(\sigma, z) + \mathrm{O}(n^{-1})\right] n^{\beta(\sigma)}\left(\pi \prod_{j=t+1}^{t+r} \zeta_j^{\sigma_j}\right)^n, \tag{6.20}$$

none of $\hat{C}(\sigma, z) \equiv \hat{C}(\sigma_{t+1}, \ldots, \sigma_{t+r}, z)$ in this summation being zero. The dominant behavior of

$R(n, z)$ is determined by the maximum value of $\beta(\sigma)$ in (6.16), since $|\pi\prod_{j=t+1}^{t+r}\zeta_j^{\sigma_j}| = |\pi\zeta_{t+1}^{\tau}|$ for all the terms of the sum in (6.20). If IP(τ) has a unique solution σ_j^*, $t+1 \leqslant j \leqslant t+r$, then $\beta(\sigma^*) = \beta^*$, and

$$R(n, z) = \hat{C}(\sigma^*, z) n^{\beta^*}\left(\pi \prod_{j=t+1}^{t+r} \zeta_j^{\sigma_j^*}\right)^n \left[1 + O(n^{-1})\right] \quad \text{as } n \to \infty, \tag{6.21}$$

and this proves (6.4) with (6.5).

The rest of the proof can be accomplished exactly like those of Theorems 3.1, 3.3 and 3.5. □

Note 6.3. When IP(τ) does not have a unique solution, then the dominant part of $R(n, z)$ in (6.20) has at least two terms of order $n^{\beta^*}(\pi|\zeta_{t+1}|^{\tau})^n$, thus $R(n, z) = n^{\beta^*}|\pi\zeta_{t+1}^{\tau}|^n Q(z)[U(n, z) + O(n^{-1})]$ as $n \to \infty$, where $U(n, z)$ is a polynomial in z and a trigonometric sum in n. Now $U(n, z) \not\equiv 0$, in general, although we do not have a rigorous proof of this at present. That $U(n, z) \not\equiv 0$ can be shown in some special cases like that in which $\omega_j = 1$, $t+1 \leqslant j \leqslant t+r$. From this we conclude that when IP(τ) does not have a unique solution and $U(n, z) \not\equiv 0$, $\lim_{m\to\infty} Q_{mk}(z)$ does not exist, only a subsequence of $\{Q_{mk}(z)\}_{m=0}^{\infty}$ converges, the limit polynomial being of the form $Q(z)\prod_{j=1}^{\tau}(1 - \zeta_j' z)$ for some ζ_j', $1 \leqslant j \leqslant \tau$, not necessarily distinct. For this subsequence $\hat{D}_{mk}(z) = O(n^{\beta^*}\pi^n|\zeta_{t+1}|^{\tau n})$ as $n \to \infty$, and consequently (6.8) holds (for the same subsequence) uniformly in any compact subset of $K\setminus\{\zeta_1^{-1}, \ldots, \zeta_\nu^{-1}, 1/\zeta_1', \ldots, 1/\zeta_\tau'\}$. Under the assumption that $U(n, z) \not\equiv 0$ always when IP(τ) does not have a unique solution, we see that $\{Q_{mk}(z)\}_{m=0}^{\infty}$ and hence $\{f_{mk}(z)\}_{m=0}^{\infty}$ converge if and only if IP(τ) has a unique solution, and this is a stronger result than Theorem 6.1. As mentioned above, however, no rigorous proof of $U(n, z) \not\equiv 0$ exists so far.

7. Application to generalized Dirichlet series

Let c_m, $m = 0, 1, 2, \ldots$ be a sequence of real or complex scalars. Assume that c_m has the asymptotic expansion

$$c_m \sim \sum_{j=1}^{\infty} P_j(m)\zeta_j^m \quad \text{as } m \to \infty, \tag{7.1}$$

where $P_j(m)$ are polynomials in m of degree $p_j \equiv \omega_j - 1$, which we choose to write as

$$P_j(m) = \sum_{l=0}^{p_j} \tilde{A}_{jl}\binom{m}{l}, \quad \tilde{A}_{jp_j} \neq 0, \tag{7.2}$$

and ζ_j are distinct nonzero scalars ordered such that

$$|\zeta_1| \geqslant |\zeta_2| \geqslant |\zeta_3| \geqslant \cdots. \tag{7.3}$$

In addition, assume that there can be only a finite number of ζ_j having the same modulus. Without loss of generality, assume that

$$p_j \geqslant p_{j+1} \quad \text{if } |\zeta_j| = |\zeta_{j+1}|. \tag{7.4}$$

Note that the infinite series $\sum_{j=1}^{\infty} P_j(m)\zeta_j^m$ in (7.1) is a generalized Dirichlet series. The interpretation of (7.1) is as follows: for any positive integer ν there exist a positive constant C and a positive integer m_0 that depend only on ν, such that for every $m \geqslant m_0$,

$$\left| c_m - \sum_{j=1}^{\nu} P_j(m)\zeta_j^m \right| \leqslant C\, m^{p_{\nu+1}} |\zeta_{\nu+1}|^m. \tag{7.5}$$

As a consequence, if we write

$$c_m = \sum_{j=1}^{\nu} P_j(m)\zeta_j^m + \tilde{A}(m, \xi)\xi^m, \quad \xi = |\zeta_{\nu+1}| + \epsilon, \ \epsilon > 0 \text{ arbitrarily close to } 0, \tag{7.6}$$

and take (7.6) as the definition of $\tilde{A}(m, \xi)$, then for all $m = 0, 1, \ldots,$

$$|\tilde{A}(m, \xi)| \leqslant M(\xi), \quad \text{some } M(\xi) > 0. \tag{7.7}$$

When we compare (7.6), (7.2) and (7.7) with (4.5) and (4.6) we realize that they are identical in form. This implies that, with c_m as given above, the infinite series $\sum_{i=0}^{\infty} c_i z^i$ represents a function $f(z)$ that is meromorphic in any compact subset of the complex plane. Thus, Theorems 3.1 and 3.5 provide us with a method, by which we can obtain approximations to the largest of the ζ_j and the corresponding $\tilde{A}_{jl}$ from the knowledge of the c_m only. For convenience we restate Theorems 3.1 and 3.5 for generalized Dirichlet series. The notation is exactly as before.

Theorem 7.1. *Let the sequence c_m, $m = 0, 1, 2, \ldots$, be as described above. Then there exist positive integers t, r, and μ, for which*

$$|\zeta_1| \geqslant \cdots \geqslant |\zeta_t| > |\zeta_{t+1}| = \cdots = |\zeta_{t+r}| > |\zeta_{t+r+1}| \geqslant \cdots, \tag{7.8}$$

and

$$\bar{p} \equiv p_{t+1} = \cdots = p_{t+\mu} > p_{t+\mu+1} \geqslant \cdots \geqslant p_{t+r}. \tag{7.9}$$

As before, $\mu = r$ implies that equalities prevail throughout (7.9). *Pick $k = \sum_{j=1}^{t}(p_j + 1) = \sum_{j=1}^{t}\omega_j$.*

(a) *For $\zeta \neq \zeta_j$, $1 \leqslant j \leqslant t$, the determinant $\tilde{D}_{mk}(\zeta) \equiv D_{mk}(\zeta^0, \zeta^1, \ldots, \zeta^k)$, which is a polynomial in ζ of degree at most k, satisfies*

$$\tilde{D}_{mk}(\zeta) = W\pi^n \left[\prod_{j=1}^{k} (\zeta - \zeta_j)^{\omega_j} + O\left(n^{\alpha} \left| \frac{\zeta_{t+1}}{\zeta_t} \right|^n \right) \right] \quad \text{as } n \to \infty, \tag{7.10}$$

α some nonnegative integer, $n \equiv m - k + 1$.

Actually, $\alpha = \bar{p} = p_{t+1}$ if the ζ_j whose moduli are $|\zeta_t|$ are all simple.

(b) *Let $\zeta_{s0}(n), \ldots, \zeta_{sp_s}(n)$ be the zeros of $\tilde{D}_{mk}(\zeta)$ that converge to ζ_s as $n \to \infty$. Then, for $1 \leqslant s \leqslant t$,*

$$\zeta_{sl}(n) = \zeta_s + O\left(\left[n^{\bar{p}} \left| \frac{\zeta_{t+1}}{\zeta_s} \right|^n \right]^{1/\omega_s} \right) \quad \text{as } n \to \infty, \ 0 \leqslant l \leqslant p_s. \tag{7.11}$$

In fact, if we let

$$N_s(n) = \frac{1}{\bar{p}!} \sum_{h=t+1}^{t+\mu} \frac{\tilde{A}_{h\bar{p}}}{\tilde{A}_{sp_s}} \zeta_s^{-p_s} \left[\prod_{\substack{i=1 \\ i \neq s}}^{t} \left(\frac{\zeta_h - \zeta_i}{\zeta_s - \zeta_i} \right)^{2\omega_i} \right] (\zeta_h - \zeta_s)^{2p_s+1} \exp\left(in \arg\left(\frac{\zeta_h}{\zeta_s} \right) \right), \tag{7.12}$$

then the sequence $\{N_s(n)\}$ *has a convergent subsequence* $\{N_s(n_q)\}$ *with a nonzero limit* $\hat{N}_s$. *The* $\zeta_{sl}(n)$ *then satisfy*

$$\zeta_{sl}(n_q) \sim \zeta_s + \hat{N}_{sl}\left(n_q^{\bar{p}}\left|\frac{\zeta_{t+1}}{\zeta_s}\right|^{n_q}\right)^{1/\omega_s} \quad \text{as } q \to \infty,\ 0 \leqslant l \leqslant p_s, \tag{7.13}$$

where $\hat{N}_{sl}$, $0 \leqslant l \leqslant p_s$, *are the* ω_s*th roots of* $\hat{N}_s$. *Furthermore,*

$$\frac{1}{\omega_s}\sum_{l=0}^{p_s} \zeta_{sl}(n) = \zeta_s + \mathrm{O}\left(n^{\bar{p}}\left|\frac{\zeta_{t+1}}{\zeta_s}\right|^{n}\right) \quad \text{as } n \to \infty. \tag{7.14}$$

Also the p_s*th derivative of* $\tilde{D}_{mk}(\zeta)$ *has exactly one zero* $\tilde{\zeta}_s(n)$ *that converges to* ζ_s *as* $n \to \infty$, *and satisfies*

$$\tilde{\zeta}_s(n) = \zeta_s + \mathrm{O}\left(n^{\bar{p}}\left|\frac{\zeta_{t+1}}{\zeta_s}\right|^{n}\right) \quad \text{as } n \to \infty. \tag{7.15}$$

(c) *Let the* $H_{sq,l}(n)$ *be as constructed in Theorem* 3.5, *and let*

$$H_{sq}(n) = \sum_{l=0}^{p_s} H_{sq,l}(n), \quad 0 \leqslant q \leqslant p_s, \tag{7.16}$$

and

$$\tilde{A}_{sl}(n) = \sum_{i=l}^{p_s}\binom{i}{i-l} H_{si}(n)\left(-\hat{z}_s(n)\right)^{-i-1}, \quad 0 \leqslant l \leqslant p_s, \tag{7.17}$$

cf. (4.3). *Then* $\tilde{A}_{sl}(n)$ *is an approximation to* $\tilde{A}_{sl}$ *satisfying*

$$\limsup_{n\to\infty}\left|\tilde{A}_{sl}(n) - \tilde{A}_{sl}\right|^{1/n} \leqslant \left|\frac{\zeta_{t+1}}{\zeta_s}\right|, \quad 0 \leqslant l \leqslant p_s. \tag{7.18}$$

Theorem 6.1 can similarly be restated for generalized Dirichlet series. We leave the details to the interested reader.

Appendix A. Analysis of $X_{10,11,\ldots,tp_t,hp_h}(z)$

From (5.9) $X_{10,11,\ldots,tp_t,hp_h}(z) \equiv X_h(z)$ has the columnwise partition

$$X_h(z) = \det\left[x_{10}(z)\,|\,x_{11}(z)\,|\ldots|\,x_{1p_1}(z)\,|\ldots|\,x_{t0}(z)\,|\ldots|\,x_{tp_t}(z)\,|\,x_{hp_h}(z)\right], \tag{A.1}$$

where

$$x_{si}(z) = \left(\hat{B}_{si}(z),\ \hat{A}_{si,1},\ \hat{A}_{si,2},\ldots,\hat{A}_{si,k}\right)^{\mathrm{T}}. \tag{A.2}$$

Substitute now in (A.2) the representations of $\hat{B}_{si}(z)$ and $\hat{A}_{si,q}$ in terms of the $\tilde{A}_{jl}$ from (5.4) and (4.3), respectively. Note that both $\hat{B}_{si}(z)$ and $\hat{A}_{si,q}$ depend only on $\tilde{A}_{sl}$, $i \leqslant l \leqslant p_s$.

Let us now concentrate on the columns $x_{s0}(z), x_{s1}(z),\ldots, x_{sp_s}(z)$. First, we have

$$x_{sp_s}(z) = \tilde{A}_{sp_s}\left(\frac{\zeta_s z}{1-\zeta_s z}, 1, \zeta_s, \zeta_s^2,\ldots,\zeta_s^{k-1}\right)^{\mathrm{T}} \equiv x_{sp_s}^{(0)}(z), \tag{A.3}$$

thus we can factor out $\tilde{A}_{sp_s}$ from the $x_{sp_s}(z)$ column. We can now use the $x_{sp_s}(z)$ column to eliminate $\tilde{A}_{sl}$ from the $x_{sl}(z)$ column to obtain the new column $x_{sl}^{(1)}(z)$ for $l=0, 1, \ldots, p_s-1$. The resulting new $x_{s,p_s-1}(z)$ column is

$$x_{s,p_s-1}^{(1)}(z)=\tilde{A}_{sp_s}\left(U_s^2+U_s, 0, \binom{1}{1}\zeta_s, \binom{2}{1}\zeta_s^2, \ldots, \binom{k-1}{1}\zeta_s^{k-1}\right)^{\mathrm{T}}, \quad U_s \equiv \frac{\zeta_s z}{1-\zeta_s z}, \tag{A.4}$$

thus we can factor out $\tilde{A}_{sp_s}$ from this column too. We can now use the $x_{s,p_s-1}^{(1)}(z)$ column to eliminate $\tilde{A}_{s,l+1}$ from the $x_{sl}^{(1)}$ column to obtain the new column $x_{sl}^{(2)}(z)$, for $l=0, 1, \ldots, p_s-2$. Continuing in this way, we obtain

$$X_h(z)=\det\left[x_{10}(z) \mid \ldots \mid x_{s0}^{(p_s)}(z) \mid x_{s1}^{(p_s-1)}(z) \mid \ldots \mid x_{s,p_s-1}^{(1)}(z) \mid x_{sp_s}^{(0)}(z) \mid \right.$$
$$\left. \ldots \mid x_{tp_t}(z) \mid x_{hp_h}(z)\right], \tag{A.5}$$

where, for $1 \leqslant i \leqslant p_s$,

$$x_{s,p_s-i}^{(i)}(z)=\tilde{A}_{sp_s}\left(U_s^i(U_s+1), 0, \ldots, 0, \binom{i}{i}\zeta_s^i, \binom{i+1}{i}\zeta_s^{i+1}, \ldots, \binom{k-1}{i}\zeta_s^{k-1}\right)^{\mathrm{T}}. \tag{A.6}$$

Now, it is easy to verify that

$$U_s^i(U_s+1)=\frac{(\zeta_s z)^i}{(1-\zeta_s z)^{i+1}}=\zeta_s^i \frac{1}{i!} \frac{\mathrm{d}^i}{\mathrm{d}\zeta^i} \frac{\zeta z}{1-\zeta z}\bigg|_{\zeta=\zeta_s}, \quad i=1, 2, \ldots . \tag{A.7}$$

Combining (A.3), (A.6) and (A.7), we obtain, for $0 \leqslant i \leqslant p_s$,

$$x_{s,p_s-i}^{(i)}(z)=\tilde{A}_{sp_s}\zeta_s^i \frac{1}{i!} \frac{\mathrm{d}^i}{\mathrm{d}\zeta^i}\left(\frac{\zeta z}{1-\zeta z}, 1, \zeta, \zeta^2, \ldots, \zeta^{k-1}\right)^{\mathrm{T}}\bigg|_{\zeta=\zeta_s} \equiv \tilde{A}_{sp_s}\zeta_s^i q_{si}(z). \tag{A.8}$$

Let us now order the columns $q_{si}(z)$ such that they appear in the order $q_{s0}(z), q_{s1}(z), \ldots, q_{sp_s}(z)$. This requires $\sum_{i=0}^{p_s} i=\frac{1}{2}p_s(p_s+1)=u_s$ column interchanges.

Performing similar elementary column transformations on the remaining columns of $X_h(z)$, we finally obtain

$$X_h(z)=(-1)^{v_t}\left(\prod_{j=1}^{t} A_{jp_j}^{\omega_j}\right) A_{hp_h}\left(\prod_{j=1}^{t} \zeta_j^{u_j}\right) Q_h(z), \tag{A.9}$$

with

$$Q_h(z)=\det\left[q_{10}(z) \mid \ldots \mid q_{1p_1}(z) \mid \ldots \mid q_{t0}(z) \mid \ldots \mid q_{tp_t}(z) \mid q_{h0}(z)\right]. \tag{A.10}$$

We note that the proofs of (4.16) for $Z_{10,\ldots,tp_t}$ and of (4.27) for $Z_{[s0;hp_h]}$ can be achieved by going through exactly the same steps that lead to (A.9) and (A.10).

Surprisingly, the determinant expression for $Q_h(z)$ that is given in (A.10) can be simplified further.

Consider first the determinant

$$T(\lambda_1, \lambda_2, \ldots, \lambda_\mu; z) = \begin{vmatrix} \dfrac{\lambda_1 z}{1-\lambda_1 z} & \cdots & \dfrac{\lambda_\mu z}{1-\lambda_\mu z} \\ 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_\mu \\ \vdots & & \vdots \\ \lambda_1^{\mu-2} & \cdots & \lambda_\mu^{\mu-2} \end{vmatrix}. \tag{A.11}$$

Adding the 2nd row to the 1st in this determinant, we see that the 1st row becomes $((1-\lambda_1 z)^{-1}, (1-\lambda_2 z)^{-1}, \ldots, (1-\lambda_\mu z)^{-1})$. Now factoring out $(1-\lambda_j z)^{-1}$ from the jth column, $j = 1, \ldots, \mu$, we obtain

$$\left[\prod_{j=1}^{\mu}(1-\lambda_j z)\right] T(\lambda_1, \ldots, \lambda_\mu; z) = \begin{vmatrix} 1 & \cdots & 1 \\ 1-\lambda_1 z & \cdots & 1-\lambda_\mu z \\ \lambda_1 - \lambda_1^2 z & \cdots & \lambda_\mu - \lambda_\mu^2 z \\ \vdots & & \vdots \\ \lambda_1^{\mu-2} - \lambda_1^{\mu-1} z & \cdots & \lambda_\mu^{\mu-2} - \lambda_\mu^{\mu-1} z \end{vmatrix}. \tag{A.12}$$

Subtract the 1st row from the 2nd, and factor out $(-z)$ to obtain the new 2nd row $(\lambda_1, \lambda_2, \ldots, \lambda_\mu)$. Next, subtract the new 2nd row from the 3rd, and factor out $(-z)$ to obtain the new 3rd row $(\lambda_1^2, \lambda_2^2, \ldots, \lambda_\mu^2)$. Continuing in this way, we finally obtain

$$T(\lambda_1, \ldots, \lambda_\mu; z) = (-1)^{\mu-1} z^{\mu-1} \frac{V(\lambda_1, \lambda_2, \ldots, \lambda_\mu)}{\prod_{j=1}^{\mu}(1-\lambda_j z)}, \tag{A.13}$$

where $V(\lambda_1, \ldots, \lambda_\mu) = \tilde{Y}(\lambda_1, 0; \ldots; \lambda_\mu, 0)$ is the ordinary Vandermonde determinant.

We now go back to (A.10). Since $q_{si}(z) = (1/i!)(\partial^i/\partial \zeta_s^i) q_{s0}(z)$ by (A.8), we can write

$$Q_h(z) = \left(\prod_{j=1}^{t} \prod_{i=1}^{p_j} \frac{1}{i!} \frac{\partial^i}{\partial \zeta_{ji}^i}\right) T(\zeta; z)\Bigg|_{\zeta_{ji}=\zeta_j, 1 \leqslant i \leqslant p_j, 1 \leqslant j \leqslant t} \tag{A.14}$$

where

$$\zeta \equiv \zeta_1, \zeta_{11}, \ldots, \zeta_{1p_1}, \zeta_2, \zeta_{21}, \ldots, \zeta_{2p_2}, \ldots, \zeta_t, \zeta_{t1}, \ldots, \zeta_{tp_t}, \zeta_h. \tag{A.15}$$

By the fact that

$$\left(\prod_{j=1}^{t} \prod_{i=1}^{p_j'} \frac{1}{i!} \frac{\partial^i}{\partial \zeta_{ji}^i}\right) V(\zeta)\Bigg|_{\zeta_{ji}=\zeta_j, 1 \leqslant i \leqslant p_j, 1 \leqslant j \leqslant t}$$

$$= \begin{cases} \tilde{Y}(\zeta_1, p_1; \ldots; \zeta_t, p_t; \zeta_h, 0) & \text{if } p_j' = p_j, \quad 1 \leqslant j \leqslant t, \\ 0 & \text{if } p_j' < p_j \quad \text{for some } j, \ 1 \leqslant j \leqslant t, \end{cases} \tag{A.16}$$

(A.14) reduces to

$$Q_h(z) = (-1)^k z^k \frac{\tilde{Y}(\zeta_1, p_1; \ldots; \zeta_t, p_t; \zeta_h, 0)}{\left[\prod_{j=1}^{t} (1-\zeta_j z)^{\omega_j}\right](1-\zeta_h z)}. \tag{A.17}$$

Substituting (A.17) in (A.9), and employing (2.10), we finally obtain

$$X_h(z) = (-1)^{k+v_t} \left(\prod_{j=1}^{t} A_{jp_j}^{\omega_j} \right) \left(\prod_{j=1}^{t} \zeta_j^{u_j} \right)$$

$$\times \tilde{Y}(\zeta_1, p_1; \ldots; \zeta_t, p_t) \frac{A_{hp_h}}{1-\zeta_h z} \frac{\prod_{j=1}^{t} \left(1 - \frac{\zeta_j}{\zeta_h}\right)^{\omega_j}}{\prod_{j=1}^{t} (1-\zeta_j z)^{\omega_j}} (\zeta_h z)^k. \tag{A.18}$$

Appendix B. Analysis of $Y(\binom{n}{l_1}\lambda^0, \binom{n}{l_2}\lambda^0, \ldots, \binom{n}{l_q}\lambda^0, g_{q+1}, \ldots, g_M)$

We start by observing that for $1 \leqslant h \leqslant q$ the hth row of the determinant $Y(\binom{n}{l_1}\lambda^0, \ldots, \binom{n}{l_q}\lambda^0, g_{q+1}, \ldots, g_M) \equiv y(n)$ is given as the row vector

$$R_h = \left(\binom{n}{l_h}\lambda^0, \binom{n+1}{l_h}\lambda^1, \binom{n+2}{l_h}\lambda^2, \ldots, \binom{n+M-1}{l_h}\lambda^{M-1} \right). \tag{B.1}$$

By the identity

$$\binom{n+j}{l_h} = \sum_{i_h=0}^{l_h} \binom{n}{l_h - i_h} \binom{j}{i_h},$$

(B.1) can be expressed as

$$R_h = \sum_{i_h=0}^{l_h} \binom{n}{l_h - i_h} \left(\binom{0}{i_h}\lambda^0, \binom{1}{i_h}\lambda^1, \binom{2}{i_h}\lambda^2, \ldots, \binom{M-1}{i_h}\lambda^{M-1} \right). \tag{B.2}$$

Consequently, $y(n)$ has the expansion

$$y(n) = \sum_{i_1=0}^{l_1} \cdots \sum_{i_q=0}^{l_q} \left[\prod_{h=1}^{q} \binom{n}{l_h - i_h} \right] Y\left(\binom{0}{i_1}\lambda^0, \ldots, \binom{0}{i_q}\lambda^0, g_{q+1}, \ldots, g_M \right). \tag{B.3}$$

We note that the upper limits $l_1, \ldots, l_q$ of the multiple sum in (B.3) can all be replaced by L, where

$$L = \max\{l_1, l_2, \ldots, l_q\}, \tag{B.4}$$

since $\binom{n}{i} = 0$ for $i < 0$. We also note that the determinant $Y(\binom{0}{i_1}\lambda^0, \ldots, \binom{0}{i_q}\lambda^0, g_{q+1}, \ldots, g_M)$ in (B.3) is odd under an interchange of any two of the indices $i_1, \ldots, i_q$. Consequently, we can apply Lemma 2.1 to the multiple sum in (B.3). As a result we obtain

$$y(n) = \sum_{0 \leqslant i_1 < i_2 < \cdots < i_q \leqslant L} Y\left(\binom{0}{i_1}\lambda^0, \ldots, \binom{0}{i_q}\lambda^0, g_{q+1}, \ldots, g_M \right) C_{i_1, \ldots, i_q}^{l_1, \ldots, l_q}(n), \tag{B.5}$$

where

$$C_{i_1,\ldots,i_q}^{l_1,\ldots,l_q}(n) = \begin{vmatrix} \binom{n}{l_1 - i_1} & \binom{n}{l_2 - i_1} & \cdots & \binom{n}{l_q - i_1} \\ \binom{n}{l_1 - i_2} & \binom{n}{l_2 - i_2} & \cdots & \binom{n}{l_q - i_2} \\ \vdots & \vdots & & \vdots \\ \binom{n}{l_1 - i_q} & \binom{n}{l_2 - i_q} & \cdots & \binom{n}{l_q - i_q} \end{vmatrix}. \tag{B.6}$$

Now the determinant $C_{i_1,\ldots,i_q}^{l_1,\ldots,l_q}(n)$ is the sum of the products

$$(\pm)\binom{n}{l_1 - i_{h_1}}\binom{n}{l_2 - i_{h_2}}\cdots\binom{n}{l_q - i_{h_q}},$$

$\{h_1,\ldots,h_q\}$ being a permutation of $\{1,\ldots,q\}$. Thus each of these products is a polynomial in n of degree $d = \Sigma_{j=1}^q (l_j - i_{h_j}) = \Sigma_{j=1}^q l_j - \Sigma_{j=1}^q i_j$. As a result $C_{i_1,\ldots,i_q}^{l_1,\ldots,l_q}(n)$ is a polynomial in n of degree at most d. The maximum value of d is attained when $i_1, i_2,\ldots, i_q$ take on their smallest possible values, and these are $i_1 = 0$, $i_2 = 1,\ldots, i_q = q-1$. In this case $d = \Sigma_{j=1}^q l_j - \frac{1}{2}q(q-1) \equiv d^*$. Consequently, by the fact that $\binom{n}{j} = n^j/j! + O(n^{j-1})$, it follows that $y(n)$ is a polynomial in n of degree at most d^*, and that

$$y(n) = C(l_1,\ldots,l_q) Y\left(\binom{0}{0}\lambda^0, \binom{0}{1}\lambda^0,\ldots,\binom{0}{q-1}\lambda^0, g_{q+1},\ldots, g_M\right) n^{d^*} + O(n^{d^*-1}), \tag{B.7}$$

where

$$C(l_1,\ldots,l_q) = \begin{vmatrix} \frac{1}{l_1!} & \cdots & \frac{1}{l_q!} \\ \frac{1}{(l_1-1)!} & \cdots & \frac{1}{(l_q-1)!} \\ \vdots & & \vdots \\ \frac{1}{(l_1-q+1)!} & \cdots & \frac{1}{(l_q-q+1)!} \end{vmatrix}. \tag{B.8}$$

Note that the first q rows of the Y determinant in (B.7) are linearly independent. Thus the coefficient of n^{d^*} cannot vanish on account of these q rows. Also note that when $l_1, l_2,\ldots, l_q$ take on consecutive values, $C(l_1,\ldots,l_q)$ in (B.7) is nonzero provided $l_1 \geqslant 0$. In fact,

$$C(l_1,\ldots,l_q) = \prod_{i=1}^{q} \frac{(i-1)!}{(p-q+i)!}, \quad \text{for } l_i = p-q+i,\ 1 \leqslant i \leqslant q. \tag{B.9}$$

$$d^* = (p+1)q - q^2,$$

For $C(l_1,\ldots,l_q)$ in (B.9), see, for example, [2, pp. 11–12].

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