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Analysis

STRONG ASYMPTOTICS FOR POLYNOMIALS BIORTHOGONAL TO POWERS OF LOG X

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ABSTRACT: The polynomials

$$D_k(z) := \sum_{j=0}^{k} \binom{k}{j} (j+1)^k (-z)^j, \; k \geq 0,$$

were introduced by the second author in the context of convergence acceleration schemes and numerical integration. They are characterized by biorthogonality to powers of log x:

$$\int_0^1 (\log x)^j D_k(x)\, dx = 0, \; j < k.$$

We show that uniformly in compact subsets of $\mathbb{C}\backslash[0,1]$, there is the strong asymptotic

$$D_k(z) = k!\, e^w\, (2\pi k w)^{-1/2}\, (-ze^w)^k\, (1 + o(1)), \; k \to \infty,$$

where w is the unique root of the equation $ze^w(1-w) = 1$ in the region $\mathscr{A} := \{w = a + ib\colon a \geq 0,\ b \in (-\pi,\pi) \text{ and } 0 < |w-1|^2 \leq (b/\sin b)^2\}$. Moreover, we give pointwise asymptotics on (0,1) and deduce the asymptotic zero distribution.

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1. INTRODUCTION AND RESULTS

The polynomials

$$(1.1) \qquad D_k(z) := \sum_{j=0}^{k} \binom{k}{j} (j+1)^k (-z)^j, \; k \geq 0,$$

arise in several contexts in numerical analysis: In investigating the T–transformation, a standard method used to accelerate convergence of sequences; in construction of interpolatory integration rules; and in related rational interpolation [2], [11], [12], [13], [14]. They have many interesting features: They are *biorthogonal to powers of log x* on [0,1]:

$$(1.2) \qquad \int_0^1 (\log x)^j D_k(x)\, dx = 0, \; j = 0, 1, 2, \ldots, k-1.$$

It is easily deduced from this relation that D_k has k simple zeros in (0,1). There is a *Rodrigues type formula*:

$$(1.3) \qquad e^u D_k(e^u) = \left(\frac{d}{du}\right)^k [e^u(1-e^u)^k],$$

and via Cauchy's integral formula for derivatives, this leads to a *Schläfli–type integral formula*:

$$(1.4) \qquad e^u D_k(e^u) = \frac{k!}{2\pi i} \int_C \frac{e^t}{t-u} \left(\frac{1-e^t}{t-u}\right)^k dt.$$

Here C is a simple closed curve encircling u. So they have properties that mirror those of the classical orthogonal polynomials [16]. For these and other properties of D_k, see [11], [12], [13].

In this paper, we use (1.4) and the method of steepest descent to derive asymptotics for $D_k(z)$ as $k \to \infty$. We feel that these asymptotics have intrinsic interest, inasmuch as the $\{D_k\}_{k=0}^{\infty}$ are a special class of biorthogonal polynomials, that admit analysis. There is a vast literature on strong asymptotics of orthogonal polynomials [7], [8], [16] and on their zero distribution [15], but apparently nothing on biorthogonal polynomials of the form of the $\{D_k\}$. Furthermore, these asymptotics have some implications for the numerical integration rules based on the zeros of $\{D_k(z)\}_{k=0}^{\infty}$, and also for rational interpolation and convergence acceleration.

In our results, a special region and conformal map play a principal role. We define

(1.5) $\Gamma := \{w = a + ib\colon a \geq 0,\ b \in (-\pi,\pi) \text{ and } a = 1 - b \cot b\}$

$= \{w = a + ib\colon a \geq 0,\ b \in (-\pi,\pi) \text{ and } |w - 1|^2 = (b/\sin b)^2\}.$

We define $\mathscr{A}$ to be the inside of Γ, punctured at 1. More formally,

(1.6) $\mathscr{A} := \{w = a + ib\colon a \geq 0,\ b \in (-\pi,\pi) \text{ and } 0 < |w - 1|^2 < (b/\sin b)^2\}.$

So $\mathscr{A}$ is an open doubly connected set, containing $(0,\infty)\backslash\{1\}$, and contained inside the half–strip $\{z\colon |\mathrm{Im}\ z| < \pi,\ \mathrm{Re}\ z > 0\}$. $\mathscr{A}$ is pictured below.

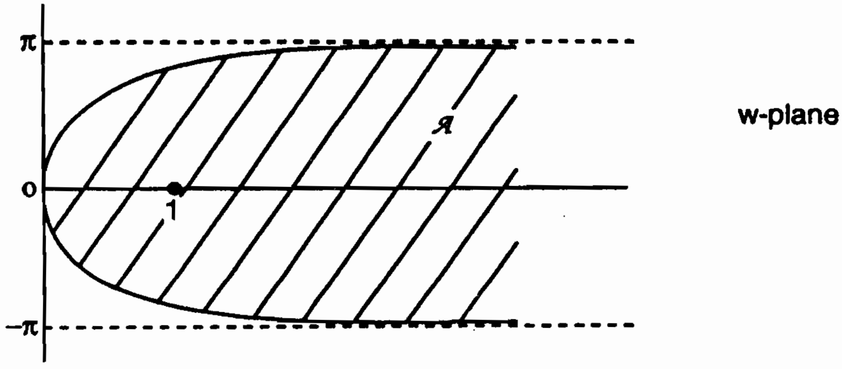

Fig. 1. The region $\mathscr{A}$

Theorem 1.1 Define

(1.7) $\Psi(w) := 1/(e^w(1 - w)),\ w \in \overline{\mathscr{A}}.$

(a) Ψ maps $\mathscr{A}$ conformally onto $\mathbb{C}\backslash[0,1]$ and maps $\mathscr{A} \cup \{1\}$ conformally onto $\overline{\mathbb{C}}\backslash[0,1]$.

(b) Ψ maps the "upper–half" of Γ, namely $\Gamma_+ := \Gamma \cap \{z\colon \mathrm{Im}\ z > 0\}$ one–one onto $(0,1)$ and maps the "lower–half" of Γ, namely, $\Gamma_- := \{z\colon \mathrm{Im}\ z < 0\}$ one–one onto $(0,1)$.

A more detailed version of Theorem 1.1 is given in Section 2. In the sequel, we shall let $\phi\colon \mathbb{C}\backslash[0,1] \to \mathscr{A}$ denote the inverse map of Ψ, that is $\phi(\Psi(w)) = w,\ w \in \mathscr{A}$. Moreover, note that

(1.8) $z = \Psi(w) \Leftrightarrow w = \phi(z) \Leftrightarrow ze^w(1 - w) = 1.$

Following is our strong asymptotic for z off [0,1]:

Theorem 1.2 We have as $k \to \infty$, uniformly in compact subsets of $\mathbb{C}\backslash[0,1]$,

$$D_k(z) = \frac{k!\ e^{\phi(z)}}{\sqrt{2\pi k}\ \phi(z)} (-ze^{\phi(z)})^k\ (1 + o(1)),\ k \to \infty. \tag{1.9}$$

(The branch of the square root is the principal one).

An alternative way to formulate (1.9) is

$$D_k(z) = \frac{k!\ e^w}{\sqrt{2\pi k}\ w}(-ze^w)^k\ (1 + o(1)) = \frac{k!\ e^w}{\sqrt{2\pi k}\ w}(w-1)^{-k}\ (1 + o(1)),$$

where $w = \phi(z)$ is the root of (1.7) in $\mathcal{A}$.

In discussing the zero distribution of $\{D_k\}_{k=0}^{\infty}$, it is convenient to use the monic polynomials formed from the $\{D_k\}$, namely

$$\hat{D}_k(z) := D_k(z)\ (-1)^k/(k+1)^k = \prod_{j=1}^{k} (z - x_{jk}),\ k \geq 1. \tag{1.10}$$

Associated with $\hat{D}_k(z)$ is the *counting measure*

$$\mu_k := \frac{1}{k}\sum_{j=1}^{k} \delta_{x_{jk}}, \tag{1.11}$$

where δ_x denotes a Dirac delta (or unit mass) at x. Thus μ_k is a measure supported in [0,1], of total mass one, and assigning mass 1/k to each of the zeros of D_k. Note that for $S \subset [0,1]$,

$$\mu_k(S) = \int_S d\mu_k = \frac{1}{k} \cdot \text{total number of zeros of } D_k \text{ in } S,$$

so $\mu_k(S)$ counts the proportion of zeros of D_k in S. The behaviour of μ_k as $k \to \infty$, describes the *zero distribution*, or asymptotic behaviour of zeros of $\{D_k\}_{k=0}^{\infty}$.

Recall the notion of *weak convergence* [6]: We write

$$\mu_k \xrightarrow{\ \ *\ \ } \mu,\ k \to \infty,$$

if

$$\lim_{k\to\infty} \int_0^1 f\, d\mu_k = \int_0^1 f\, d\mu \text{ for every continuous } f:[0,1] \to \mathbb{R}.$$

Equivalently for every $0 < a < b < 1$,

$$\mu_k[a,b] = \text{proportion of zeros of } D_k \text{ in } [a,b]$$

$$\to \int_a^b d\mu,\ k \to \infty.$$

See [15] for an extensive coverage of zero distribution of orthogonal polynomials for weights on finite intervals. Following is our result on zero distribution:

<u>**Theorem 1.3**</u> Define

$$h(b) := \frac{\sin\ b}{b} e^{b \cot b - 1},\ b \in [0,\pi]. \tag{1.12}$$

Then h decreases from $h(0) = 1$ to $h(\pi) = 0$ as b increases from 0 to π. Let $h^{[-1]}: [0,1] \to [0,\pi]$ denote the inverse function of h, and define

$$d\mu(x) := \mu'(x)\ dx := -\frac{dx}{\pi h'(h^{[-1]}(x))},\ x \in (0,1). \tag{1.13}$$

Then $d\mu$ is a positive measure on $[0,1]$ with

$$\int_0^1 d\mu = 1. \tag{1.14}$$

Moreover,

$$\mu_k \xrightarrow{\ *\ } \mu,\ k \to \infty, \tag{1.15}$$

and

$$\mu'(x) = \frac{1}{\pi\ \sqrt{2(1-x)}}(1 + o(1)),\ x \to 1-, \tag{1.16}$$

$$\mu'(x) = \frac{1}{x\ |\log x|^2}(1 + o(1)),\ x \to 0+. \tag{1.17}$$

Finally, uniformly for z in compact subsets of $\mathbb{C}\backslash[0,1]$,

$$\lim_{k\to\infty} |\hat{D}_k(z)|^{1/k} = \exp(\int_0^1 \log|z-t|\ d\mu(t)). \tag{1.18}$$

We remark that in [1], necessary conditions were given on the zero distribution of points in sequences of interpolatory integration rules for their convergence on all continuous functions. When transferred to $[0,1]$, these necessary conditions require that

$$\mu'(x) \geq \frac{1}{2}\frac{1}{\pi\ \sqrt{x(1-x)}},\ \text{a.e. } x \in (0,1).$$

This inequality is consistent with (1.16) and (1.17). So the convergence of the interpolatory integration rules [11] based on the zeros of the $\{D_k\}_{k=0}^{\infty}$ remains an interesting unsolved problem.

Our final result deals with pointwise asymptotics of $D_k(x)$ on (0,1):

Theorem 1.4 Let $h:[0,\pi] \to [0,1]$ be defined by (1.12) and let $h^{[-1]}: [0,1] \to [0,\pi]$ be its inverse function. Then for $x \in (0,1)$, as $k \to \infty$,

$$D_k(x) = k!\left(\frac{2}{k\pi}\right)^{1/2} (-xe^a)^k \left\{\frac{e^a}{|a+ib|^{1/2}} \cos[(k+1)b - \tfrac{1}{2}\arctan(\tfrac{b}{a})] + o(1)\right\}, \tag{1.19}$$

where

$$b := b(x) := h^{[-1]}(x);\ a := a(x) := 1 - h^{[-1]}(x) \cot h^{[-1]}(x). \tag{1.20}$$

The asymptotic holds uniformly for x in compact subsets of (0,1).

Our proofs of the asymptotics in Theorems 1.2 and 1.4 depend on the method of steepest descent and the representation (1.4). In Section 2, we take the first steps by investigating the conformal map Ψ that defines the location of the critical points of the integral (1.4). We could only give an explicit form of a suitable contour C for some ranges of $z = e^u$ and for other z needed to use a continuity argument, and a study of components defined by the level curves of certain functions, to prove the existence of a suitable contour. So in Section 3, we investigate the level curves (order stars) and components defined by the function whose kth power appears in (1.4). In Theorems 3.6 and 3.8, we prove the existence of a suitable contour for the purposes of Theorem 1.2 and 1.4 respectively. In Section 4, we use this to prove Theorems 1.2 and 1.4, and then in Section 5, we deduce the zero distribution of Theorem 1.3.

2. THE CONFORMAL MAP

We begin with a discussion of the mapping of the boundary Γ of $\mathscr{A}$ onto [0,1] by Ψ. (Recall that Γ was defined by (1.5)).

<u>**Lemma 2.1**</u> (a) Let $x \in (0,1)$. The equation

(2.1) $$x = \Psi(w) \Leftrightarrow xe^{w}(1-w) = 1,\ w \in \overline{\mathcal{A}},$$

has exactly two roots $w = a \pm ib$. These are determined by the conditions

$$a > 0;\ b \in (0,\pi);$$

(2.2) $$a = 1 - b \cot b;$$

(2.3) $$x = \frac{\sin b}{b} e^{b \cot b - 1} =: h(b).$$

Moreover, $a + ib$ lies in

(2.4) $$\Gamma_+ := \Gamma \cap \{z: \operatorname{Im} z > 0\},$$

and $a - ib$ lies in

(2.5) $$\Gamma_- := \Gamma \cap \{z: \operatorname{Im} z < 0\}.$$

(b) As x increases from 0 to 1, $b = b(x)$ decreases from π to 0 and $a = a(x)$ decreases from ∞ to 0. Moreover, $a = a(x)$ increases as $b = b(x)$ increases in $(0,\pi)$. Finally, $h(b)$ decreases from $h(0) = 1$ to $h(\pi) = 0$, as b increases from 0 to π.

(c) We can write

$$\Gamma_{\pm} = \{w = a + ib: a > 0,\ b \in (0,\pi) \text{ and } |w-1|^2 = \frac{b^2}{\sin^2 b}\}.$$

<u>**Proof**</u> (a), (b) We write $w = a + ib$ ($a \geq 0$, $b \in (0,\pi)$) and substitute into (2.1). Taking real and imaginary parts gives

$$xe^{a}\,[(\cos b)\,(1-a) + b \sin b] = 1;$$

$$(\sin b)\,(1-a) - b \cos b = 0.$$

The second of these equations gives (2.2). Moreover, solving for $1 - a$ from the second and then substituting into the first gives (2.3). Thus if $w = a + ib$ satisfies (2.1) with $a \geq 0$, $b \in (0,\pi)$, then it necessarily lies on Γ_+. It is clear that then $\overline{w} = a - ib$ lies on Γ_-. Next we show that for $x \in (0,1)$, there is a unique $b \in (0,\pi)$ for which $x = h(b)$, that is (2.3) holds. A simple calculation shows that

(2.6) $$h'(b) = \frac{e^{b \cot b - 1}}{b}\left[2 \cos b - \frac{\sin b}{b} - \frac{b}{\sin b}\right].$$

Here if $b \in (0,\pi)$, $t := \frac{\sin b}{b} \in (0,1)$, so using the inequality

$$2 \cos b \leq 2 < t + 1/t,$$

we see that

(2.7) $\qquad h'(b) < 0,\ b \in (0,\pi).$

Thus h(b) decreases from $h(0) = 1$ to $h(\pi) = 0$, as b increases from 0 to π, and (2.3) has a unique solution b for a given x. Finally, from (2.2),

$$\frac{da}{db} = \frac{b - \sin b \cos b}{\sin^2 b} = \frac{b - \frac{1}{2}\sin 2b}{\sin^2 b} > 0,$$

by the inequality $\sin u < u$, $u \in (0,\infty)$.

(c) This is immediate from (2.2). □

We remark that for $w = a \pm ib \in \Gamma$, (c) shows that

$$|w-1|^2 = \left(\frac{b}{\sin b}\right)^2 \ge 1,$$

with equality only for $b = 0$. So the punctured open ball $\{w: 0 < |w - 1| < 1\}$ is contained inside $\mathscr{A}$.

<u>**Proof of Theorem 1.1**</u> We note first that the proof of Theorem 1.1(b) is contained in Lemma 2.1. So we turn to the proof of Theorem 1.1(a). We let

$$g(w) := 1/\Psi(w) = e^w(1-w),\ w \in \mathscr{A} \cup \{1\}.$$

It suffices to show that g maps $\mathscr{A} \cup \{1\}$ conformally onto $\mathbb{C}\backslash[1,\infty)$. Let $r > 0$ and consider

$$\mathscr{G} := \{c: |c| \le r\}\backslash[1,\infty).$$

We fix $B > 1$ such that

(2.8) $\qquad e^B(B-1) > r,$

and form a truncated subregion of $\mathscr{A} \cup \{1\}$, namely

$$\mathscr{A}_B := (\mathscr{A} \cup \{1\}) \cap \{w: \operatorname{Re} w < B\}.$$

We let Γ_B denote the boundary of $\mathscr{A}_B$, positively oriented. Then Γ_B consists of that part of Γ with real part $< B$, together with a vertical line segment cutting the real line at B.

We shall apply the principal of the argument to the curve Γ_B. Note that if $w \in \Gamma_B \cap \Gamma$, then Lemma 2.1 shows that $g(w) = 1/\Psi(w) = e^w(1 - w) \in [1,\infty)$. Furthermore if $w \in \Gamma_B\backslash\Gamma$, then $w = B + is$, some $s \in \mathbb{R}$, so

$$|g(w)| = e^B((B-1)^2 + s^2)^{1/2} \ge e^B(B-1) > r,$$

by (2.8). Thus the function $g'(w)/(g(w)-c)$ is continuous for $(w,c) \in \Gamma_B \times \mathcal{G}$, and so

$$H(c) := \frac{1}{2\pi i} \int_{\Gamma_B} \frac{g'(w)}{g(w)-c}\, dw$$

is continuous for $c \in \mathcal{G}$. Since $\mathcal{G}$ is connected, so is $H(\mathcal{G})$. But $H(c)$ is integer valued, namely is the total multiplicity of roots of the equation $g(w) = c$ with $w \in \mathcal{A}_B$. Thus $H(\mathcal{G})$ is a connected set of integers, so consists of a single integer. Now $0 \in \mathcal{G}$, and the equation

$$g(w) = e^w(1-w) = 0$$

has exactly one root $w = 1 \in \mathcal{A}_B$, so we deduce $H(\mathcal{G}) = H(\{0\}) = \{1\}$.

It follows that given $c \in \mathbb{C}\backslash[1,\infty)$, the equation $g(w) = c$ has exactly one root in $\mathcal{A}_B$ for all B large enough. We have proved that g maps an open subset of $\mathcal{A} \cup \{1\}$, call it $\mathcal{U}$ say, in a one–to–one fashion onto $\mathbb{C}\backslash[1,\infty)$. We show finally that $\mathcal{U} = \mathcal{A} \cup \{1\}$. If this is not the case, then we can choose $z \in \mathcal{A} \cup \{1\} \setminus \mathcal{U}$. Now if $g(z) \in (1,\infty)$, then

$$x := 1/g(z) \in (0,1) \Leftrightarrow 1 = x\, g(z) = xe^z(1-z)$$

and Lemma 2.1 implies that $z \in \Gamma$, a contradiction. Similarly if $g(z) = 1$, $x := 1/g(z) = 1$, and the proof of Lemma 2.1 shows that $z = 0$, again a contradiction. Finally, if $g(z) \in \mathbb{C}\backslash[1,\infty)$, then we can find, by hypothesis, $w \in \mathcal{U}$, such that $g(w) = g(z) = c$, say. Then the equation $g(u) = c$ has two distinct roots w and z in $\mathcal{A} \cup \{1\}$. This contradiction shows that necessarily $\mathcal{U} = \mathcal{A} \cup \{1\}$, and so g maps $\mathcal{A} \cup \{1\}$ conformally onto $\mathbb{C}\backslash[1,\infty)$. □

Note that

$$\Psi'(w) = we^{-w}(1-w)^{-2} > 0,\ w \in (0,1) \cup (1,\infty).$$

Also if we restrict w to be real, then

$$\lim_{w\to 0} \Psi(w) = 1;\ \lim_{w\to 1-} \Psi(w) = \infty;$$

$$\lim_{w\to 1+} \Psi(w) = -\infty;\ \lim_{w\to \infty} \Psi(w) = 0;$$

So we have:

Proposition 2.2 (a) As w increases from 0 to 1, $\Psi(w)$ increases from 1 to ∞ (and conversely); (b) As w increases from 1 to ∞, $\Psi(w)$ increases from $-\infty$ to 0 (and conversely).

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3. LEVEL CURVES AND COMPONENTS (ORDER STARS)

In this section, we analyze the function whose kth power appears in our Schläfli type integral formula. But first, we note that by expanding $(1-e^u)^k$ via the binomial theorem, and then multiplying by e^u and differentiating k times, it is easy to verify that the D_k defined by (1.1) indeed satisfy (1.3) and hence (1.4). The biorthogonality (1.2) is an easy consequence of (1.3) and successive integrations by parts. Let us reformulate (1.4), by setting $t = u + v$ there, so that

$$D_k(e^u) = \frac{k!}{2\pi i}\int_\gamma \frac{e^v}{v}\left(\frac{1-e^{u+v}}{v}\right)^k dv, \tag{3.1}$$

where now γ is a positively oriented simple closed contour encircling 0.

Recall that if

$$z = e^u\text{: Re } u \in \mathbb{R} \text{ and Im } u \in (-\pi,\pi], \tag{3.2}$$

then we can uniquely define for $z \in \mathbb{C}\backslash[0,1]$, a $w \in \mathscr{A}$ by the equation

$$w = \phi(z) \Leftrightarrow z = \Psi(w) \Leftrightarrow ze^w(1-w) = 1. \tag{3.3}$$

Moreover, if $x \in (0,1)$, there is a unique $w \in \overline{\mathscr{A}}$ such that

$$\text{Im } w > 0 \text{ and } x = \Psi(w) \Leftrightarrow xe^w(1-w) = 1. \tag{3.4}$$

Then also ($\overline{w}$ denotes the conjugate of w)

$$x = \Psi(\overline{w}) \Leftrightarrow xe^{\overline{w}}(1-\overline{w}) = 1. \tag{3.5}$$

In the sequel we assume that $w = w(z)$ or $w = w(x)$ is defined this way.

The critical points of the function

$$H(v,z) := \frac{1-ze^v}{v} = \frac{1-e^{u+v}}{v}, \tag{3.6}$$

whose kth power appears in our integral (3.1) determine the behaviour of the integral (3.1), according to the classical method of steepest descent [9]. Below we show that for a given $z \in \mathbb{C}\backslash[0,1]$, $w = \phi(z)$ is the unique critical point lying in $\mathscr{A}$, and the integral behaves approximately like $H(w,z)^k$. Accordingly we define a normalized function

(3.7) $$G(v,z) := \frac{H(v,z)}{H(w,z)},\ w = \phi(z).$$

It is obviously easier to deal with an explicit contour γ, and we found that we could use the circular contour

$$\gamma = \{w\, e^{i\theta}: \theta \in [-\pi,\pi]\},$$

at least for $|z| \geq 3.4$. However, this choice of contour does not work for z close to [0,1], and so an alternative approach is required. To prove the existence of a suitable contour γ for all z, we consider the level curves

(3.8) $$\mathscr{U}(z) := \{v \in \mathbb{C}: |G(v,z)| = 1\},$$

and the components that they define

(3.9) $$\mathscr{B}(z) := \{v \in \mathbb{C}: |G(v,z)| < 1\},$$

(3.10) $$\mathscr{D}(z) := \{v \in \mathbb{C}: |G(v,z)| > 1\}.$$

Some of our analysis is reminiscent of the theory of *order stars* [4], [5], but we could not directly apply results from there: Much of our emphasis is on continuity of $\mathscr{U}(z)$ in z. We first prove that our components of $\mathscr{B}(z)$ and $\mathscr{D}(z)$ have suitable behaviour for $|z|$ large enough, and use a continuity/connectedness argument to deduce this behaviour for all z.

Note that as the level curve of an analytic function, $\mathscr{U}(z)$ (regarded as a curve in the plane) is locally rectifiable, and non self–intersecting except at points where $G'(v,z) = 0$ (the derivative is always with respect to v, and z is always fixed during the differentiation). These points are called *multiple,* more specifically, *double, triple, ... points* of the level curve, according to the multiplicity of the zero of $G'(v,z)$.

We begin by studying the critical points of $G(v,z)$.

<u>Lemma 3.1</u> Fix $z \in \mathbb{C}\backslash\{0,1\}$ and let u be defined by (3.2). Let $H(v,z)$ and $G(v,z)$ be defined by (3.6) and (3.7) respectively.

(a) If $v \in \mathbb{C}$, then

(3.11) $$G'(v,z) = 0 \Leftrightarrow H'(v,z) = 0 \Leftrightarrow z = e^u = \Psi(v).$$

In particular, $v = w$ satisfies these equations.

(b) If $v \in \mathbb{C}$ satisfies $G'(v,z) = 0$, then

$$H(v,z) = \frac{1 - ze^{v}}{v} = \frac{-1}{1 - v} = -ze^{v}. \tag{3.12}$$

In particular, this is true for $v = w$.

(c) If $v \in \mathbb{C}$ satisfies

$$G'(v,z) = 0; \ |G(v,z)| = 1, \tag{3.13}$$

then $v = w$ or $v = \overline{w}$. Moreover, if $z \in \mathbb{C}\backslash[0,1]$, then $v = w$ and so $v = w$ is the only (at least) double point of the curve $\mathcal{U}(z)$. If $z \in (0,1)$, then $v = w$ and $v = \overline{w}$ are the only (at least) double points of the curve $\mathcal{U}(z)$.

(d) The zeros of $G(v,z)$ have the form $v = -u + 2k\pi i$, k an integer, and the only pole of $G(v,z)$ is $v = 0$.

<u>**Proof**</u> (a) We see that

$$H'(v,z) = -v^{-2}\{1 - ze^{v}(1 - v)\} = -v^{-2}\{1 - z/\Psi(v)\},$$

so (3.11) follows. Of course by definition of w, $z = \Psi(w)$, so w satisfies these equations.

(b) We see that

$$H'(v,z) = 0 \Longrightarrow ze^{v} = 1/(1 - v)$$

$$\Longrightarrow H(v,z) = (1 - 1/(1-v))/v = -1/(1 - v).$$

Moreover, then

$$ze^{v} - zve^{v} = 1$$

$$\Longrightarrow 1 - ze^{v} = -vze^{v} \Longrightarrow H(v,z) = -ze^{v}.$$

(c) Now

$$|G(v,z)| = 1 \Longrightarrow |H(v,z)| = |H(w,z)|,$$

and if also $G'(v,z) = 0$, then as also $H'(w,z) = 0$, (b) shows that

$$|v - 1| = |w - 1|; \ |e^{v}| = |e^{w}|.$$

Then $\operatorname{Re} v = \operatorname{Re} w$, and both lie on the same circle centre 1, so necessarily $v = w$ or $v = \overline{w}$.

Now if $z \in (0,1)$, then recall from Lemma 2.1 that w and $\overline{w}$ are distinct and satisfy $z = \Psi(w) = \Psi(\overline{w})$, and moreover $H(\overline{w},z) = \overline{H(w,z)}$, so w and $\overline{w}$ are distinct double points of $\mathcal{U}(z)$.

Finally if $z \in \mathbb{C}\backslash[0,1]$, then we already know that $w = \phi(z)$ satisfies $z = \Psi(w)$ and so $G'(w,z) = 0$, and also $|G(w,z)| = 1$. We must show that $\overline{w}$ cannot be yet another double point of $\mathscr{U}(z)$. Obviously if $\overline{w}$ is a distinct double point, then $w \neq \overline{w}$. But in that case, we would have $G'(\overline{w},z) = 0$, so (a) gives $z = \Psi(w) = \Psi(\overline{w})$ and both w and $\overline{w}$ lie in $\mathscr{A}$, contradicting that Ψ is one–to–one in $\mathscr{A}$. So for such z, w is the unique double point of the curve $\mathscr{U}(z)$.

(d) is immediate from (3.6) and (3.7). □

Next, we recall the shape of the components near the double point w:

Lemma 3.2 Let $z \in \mathbb{C}\backslash\{0,1\}$. Then for small enough $|v - w|$,

$$G(v,z) = 1 + \frac{(v - w)^2}{2w} + O(v - w)^3. \tag{3.14}$$

Moreover, if $\sqrt{\ }$ denotes the principal branch of the $\sqrt{\ }$, then for $s \in \mathbb{R}$,

$$v = w + \sqrt{2w}\, s \Longrightarrow G(v,z) = 1 + s^2 + O(s^3),\ s \to 0; \tag{3.15}$$

$$v = w + \sqrt{2w}\, is \Longrightarrow G(v,z) = 1 - s^2 + O(s^3),\ s \to 0; \tag{3.16}$$

Consequently 2 sectors of $\mathscr{D}(z)$, say $\mathscr{F}_1(z)$ and $\mathscr{F}_2(z)$, and two sectors of $\mathscr{B}(z)$, say $\mathscr{E}_1(z)$ and $\mathscr{E}_2(z)$, touch at w.

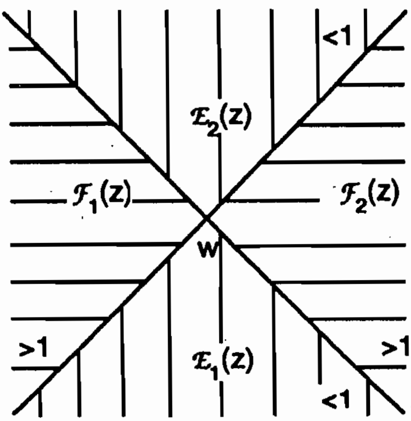


Fig. 2. Sectors of $\mathscr{B}(z)$ and $\mathscr{D}(z)$ near w

Proof Now

$$G(v,z) = \frac{H(v,z)}{H(w,z)} = \frac{1 - ze^{v}}{1 - ze^{w}}\,\frac{w}{v}$$

$$= [1 + \frac{ze^{w}}{1 - ze^{w}}(1 - e^{v-w})]\,\frac{w}{v}$$

$$= [1 - \frac{1}{w}(1 - e^{v-w})]\,\frac{w}{v},$$

by (3.12). By substitution of the Maclaurin series for e^{v-w}, we deduce that

$$G(v,z) = 1 + \frac{1}{2v}(v - w)^2 + \frac{1}{v}\sum_{j=3}^{\infty}\frac{(v - w)^j}{j!}. \tag{3.17}$$

Now (3.14) follows for $|v - w| < 1$. Then also (3.15) and (3.16) are immediate. □

Next, we turn our attention to unbounded components of $\mathscr{B}(z)$ and $\mathscr{D}(z)$.

Lemma 3.3 For $z \in \mathbb{C}\backslash\{0,1\}$, there is exactly one unbounded component of $\mathscr{B}(z)$ and exactly one unbounded component of $\mathscr{D}(z)$, say $\mathscr{B}_u(z)$ and $\mathscr{D}_u(z)$ respectively. Given $\epsilon > 0$, there exists $V_0 > 0$ and c (depending on z) such that $|v| \geq V_0$ and

$$\operatorname{Re} v - \ln|v| \leq \epsilon - c \Longrightarrow v \in \mathscr{B}_u(z);$$

$$\operatorname{Re} v - \ln|v| \geq \epsilon - c \Longrightarrow v \in \mathscr{D}_u(z).$$

Proof This is reminiscent of much more general results on asymptotic values and paths of entire/ meromorphic functions [3]. Let

$$\chi(v) := \operatorname{Re} v - \ln|v| + \ln|z| - \ln|H(w,z)|.$$

Then

$$|G(v,z)| \leq \frac{1 + |z|\,e^{\operatorname{Re} v}}{|v|\,|H(w,z)|} = e^{\chi(v)} + \frac{1}{|v|\,|H(w,z)|}$$

and similarly

$$|G(v,z)| \geq e^{\chi(v)} - \frac{1}{|v|\,|H(w,z)|}. \tag{3.18}$$

Thus given $\epsilon > 0$, there exists V_0 such that

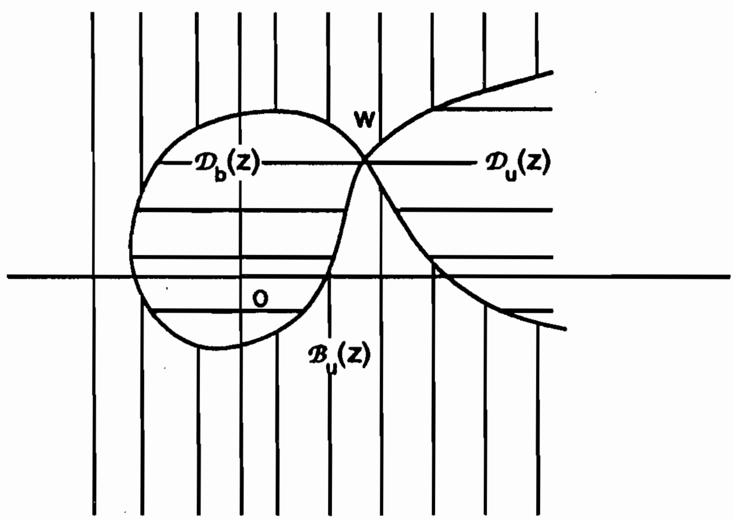
$$|v| \geq V_0 \text{ and } \chi(v) \geq \epsilon \Longrightarrow |G(v,z)| \geq e^{\epsilon/2} > 1;$$

$$|v| \geq V_0 \text{ and } \chi(v) \leq -\epsilon \Longrightarrow |G(v,z)| \leq e^{-\epsilon/2} < 1.$$

Essentially this says that the path $\chi(v) = 0$ forms the boundary for large $|v|$ of unbounded components of $\mathscr{B}(z)$ and $\mathscr{D}(z)$ respectively. Finally, as the maximum–modulus principle shows that each bounded component of $\mathscr{B}(z)$ contains a zero of $G(v,z)$ (of the form $-u + 2k\pi i$) and each bounded component of $D(z)$ contains a pole of $G(v,z)$ (namely 0) an easy application of Rouche's Theorem shows that $G(v,z)$ has the same number of zeros/ poles as $\frac{ze^v}{vH(w,z)}$ in closed regions near the curve $\chi(v) = 0$, and as there are none such zeros / poles for large $|v|$, we are done. □

By combining Lemmas 3.2 and 3.3, we can investigate the bounded components of $\mathscr{B}(z)$ and $\mathscr{D}(z)$ for large enough $|z|$. In the sequel, ∂A denotes the boundary of open $A \subset \mathbb{C}$.

<u>**Lemma 3.4**</u> For $|z| \geq e$, there is exactly one unbounded component $\mathscr{B}_u(z)$ of $\mathscr{B}(z)$ and $\mathscr{D}_u(z)$ of $\mathscr{D}(z)$, and moreover exactly one bounded component of $\mathscr{D}(z)$ and no bounded components of $\mathscr{B}(z)$. The bounded component of $\mathscr{D}(z)$, which we denote by $\mathscr{D}_b(z)$, contains 0. Furthermore, $\partial D_b(z)$ touches $\partial\mathscr{D}_u(z)$ and $\partial\mathscr{B}_u(z)$ at the unique double point w of the curve $\mathscr{U}(z)$.



<u>**Fig. 3. The Components of $\mathscr{B}(z)$ and $\mathscr{D}(z)$**</u>

Proof We begin by showing that all zeros of G(v,z) (recall z is fixed, we are regarding this as a function of v) lie in $\mathscr{B}_u(z)$ at least for $|z| \geq e$. Consider the vertical line $\{-u + iy: y \in \mathbb{R}\}$, which contains all the zeros of G(v,z). We show that this lies in a component of $\mathscr{B}(z)$, necessarily the unbounded one. Now (recall (3.6), (3.7) and that $z = e^u$)

$$|G(-u+iy,z)| = \left|\frac{1 - e^{iy}}{-u + iy}\right| / |H(w,z)|$$

$$\leq \frac{2}{|\mathrm{Re}\ u|} / |e^{u+w}| \qquad \text{(by (3.12))}$$

$$< \frac{2}{|\mathrm{Re}\ u| e^{\mathrm{Re}\ u}} = \frac{2}{|\ln |z|| \ |z|},$$

as Re w > 0 for $z \in \mathbb{C}\backslash\{0\}$. Thus

$$\sup_{y\in\mathbb{R}} |G(-u+iy,z)| < 1,$$

provided $|\ln |z|| \ |z| \geq 2$, which is certainly true for $|z| \geq e$. So for such z, there is no bounded component of $\mathscr{B}(z)$.

To prove that there is exactly one bounded component of $\mathscr{D}(z)$, refer to Figure 2, and recall that there are two sectors $\mathscr{S}_1(z)$ and $\mathscr{S}_2(z)$ of $\mathscr{D}(z)$ and two sectors $\mathscr{E}_1(z)$, $\mathscr{E}_2(z)$ of $\mathscr{B}(z)$ that touch at w. We claim that $\mathscr{S}_1(z)$ and $\mathscr{S}_2(z)$ cannot belong to the same component of $\mathscr{D}(z)$. If they did, then we could find a closed Jordan curve starting at w, passing through $\mathscr{S}_1(z)$ and then through $\mathscr{S}_2(z)$ to w, and lying wholly in one component of $\mathscr{D}(z)$. This curve would then enclose either sector $\mathscr{E}_1(z)$ or $\mathscr{E}_2(z)$, and so enclose a bounded component of $\mathscr{B}(z)$, contradicting that there are none. So $\mathscr{S}_1(z)$ and $\mathscr{S}_2(z)$ lie in different components of $\mathscr{D}(z)$, and so there is at least at one bounded component of $\mathscr{D}(z)$ containing one of $\mathscr{S}_1(z)$, $\mathscr{S}_2(z)$. This component must necessarily contain a pole of G(v,z), and there is exactly one pole at 0, so there is exactly one bounded component of $\mathscr{D}(z)$ containing 0, and one of the sectors $\mathscr{S}_1(z)$, $\mathscr{S}_2(z)$. □

To prove that the main details of Figure 3 persist for $|z| < e$, $z \in \mathbb{C}\backslash[0,1]$, we are forced to use a continuity argument. We almost show that the level curve $\mathscr{U}(z)$, when restricted to any bounded ball centre 0, is continuous in z with respect to Haussdorff distance between sets. We define for $a \in \mathbb{C}$, $B \subset \mathbb{C}$,

$$\mathrm{dist}(a,B) := \inf\{|a-b|: b \in B\}.$$

Lemma 3.5 Assume either that $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}\backslash[0,1]$ and $z_0 \in \mathbb{C}\backslash[0,1]$ or that $\{z_n\}_{n=1}^{\infty} \subset (0,1)$ and $z_0 \in (0,1)$. Moreover, assume that $z_n \to z_0$ as $n \to \infty$. Let $s > r > 0$. Let

$$U_n := \mathcal{U}(z_n) \cap \{v: |v| \le r\},\ n \ge 1; \tag{3.19}$$

$$\hat{U}_n := \mathcal{U}(z_n) \cap \{v: |v| < s\},\ n \ge 1; \tag{3.20}$$

and

$$U_0 := \mathcal{U}(z_0) \cap \{v: |v| \le r\}. \tag{3.21}$$

Then

$$\sup_{v \in U_n} \mathrm{dist}(v, U_0) \to 0,\ n \to \infty. \tag{3.22}$$

Moreover,

$$\sup_{v \in U_0} \mathrm{dist}(v, \hat{U}_n) \to 0,\ n \to \infty. \tag{3.23}$$

Proof This is elementary, but we could not find a reference, so include the details. We first prove (3.22). Let $\mathcal{T}$ be an infinite sequence of positive integers, and let

$$v_n \in U_n,\ n \in \mathcal{T}. \tag{3.24}$$

Choose a subsequence of $\mathcal{T}$, say $\mathcal{S}$, such that

$$v_n \to v_0,\ n \to \infty,\ n \in \mathcal{S}.$$

Note that $|v_0| \le r$. By continuity of $G(v,z)$ in both v and z, (for the relevant range of z, according as z_0 is inside $(0,1)$ or in $\mathbb{C}\backslash[0,1]$)

$$|G(v_0,z_0)| = \lim_{n\to\infty} |G(v_n,z_n)| = 1.$$

(Recall $v_n \in \mathcal{U}(z_n)$). So $v_0 \in U_0$. Then

$$\mathrm{dist}(v_n,U_0) \le |v_n - v_0| \to 0,\ n \to \infty,\ n \in \mathcal{S}.$$

Thus under (3.24), we have shown that

$$\liminf_{\substack{n\to\infty \\ n \in \mathcal{T}}} \mathrm{dist}(v_n,U_0) = 0. \tag{3.25}$$

Finally, if (3.22) is not true, we can find $\delta > 0$, a subsequence $\mathcal{T}$ of integers, and $v_n \in U_n$,

$n \in \mathscr{T}$, such that

$$\operatorname{dist}(v_n, U_0) \geq \delta, \; n \in \mathscr{T},$$

contradicting (3.25).

We turn to the proof of (3.23). Let $v_0 \in U_0$. We first show that

$$\operatorname{dist}(v_0, \hat{U}_n) \to 0, \; n \to \infty. \tag{3.26}$$

Now we have $|G(v_0,z_0)| = 1$, say $G(v_0,z) = \eta$ where $|\eta| = 1$. Now by continuity of $G(\cdot,\cdot)$ in both variables (again with suitable restrictions on the range of each variable),

$$\lim_{n\to\infty} (G(v,z_n) - \eta) = G(v,z_0) - \eta,$$

uniformly for v in an open set containing v_0. Since v_0 is a zero of the non–constant analytic function $G(v,z_0) - \eta$, Hurwitz' Theorem shows that there exists v_n such that

$$G(v_n,z_n) - \eta = 0$$

and

$$v_n \to v_0, \; n \to \infty.$$

Then $v_n \in \mathscr{U}(z_n)$ and for n large enough, $|v_n| < s$, so $v_n \in \hat{U}_n$, n large enough. Thus

$$\operatorname{dist}(v_0, \hat{U}_n) \leq |v_0 - v_n| \to 0, \; n \to \infty,$$

so we have (3.26). Finally, if (3.23) is not true, then $\exists\, \delta > 0$, $v_m \in U_0$ and $n_m \geq 1$, $m \geq 1$, such that $n_m \to \infty$, $m \to \infty$, and

$$\operatorname{dist}(v_m,\hat{U}_{n_m}) \geq \delta, \; m \geq 1.$$

By passing to a subsequence, if necessary, we may assume that $v_m \to v_0$, $m \to \infty$. Then $v_0 \in U_0$ (the latter set is closed) so

$$\delta \leq \operatorname{dist}(v_m,\hat{U}_{n_m}) \leq |v_m - v_0| + \operatorname{dist}(v_0,\hat{U}_{n_m}) \to 0, \; m \to \infty,$$

by (3.26), a contradiction. □

Now we can prove the required behaviour of components of $\mathscr{B}(z)$ and $\mathscr{D}(z)$ for all $z \in \mathbb{C}\backslash[0,1]$, not just for $|z| \geq e$. We cannot however obtain as strong a conclusion as in Lemma 3.4.

Theorem 3.6 For $z \in \mathbb{C}\backslash[0,1]$, there is exactly one unbounded component of $\mathscr{B}(z)$ and $\mathscr{D}(z)$, which we denote by $\mathscr{B}_u(z)$ and $\mathscr{D}_u(z)$ respectively. Moreover, there is exactly one bounded component of $\mathscr{D}(z)$, which we denote by $\mathscr{D}_b(z)$. The latter contains 0 and its boundary contains $w = \phi(z)$. We can find a smooth closed Jordan curve γ_z passing through w, encircling $\mathscr{D}_b(z)$ and so 0, and such that $\gamma_z\backslash\{w\}$ lies in $\mathscr{B}(z)$.

Remark We are not ruling out the possibility that there are one or more bounded components of $\mathscr{B}(z)$, but the important fact is that we can find the contour γ_z surrounding $\mathscr{D}_b(z)$ and lying in $\mathscr{B}(z)$ except at w. This is pictured below in Figure 4.

Proof Recall that for fixed z, G(v,z) has one pole at $v = 0$. So there is at most one bounded component of $\mathscr{D}(z)$, and so either one or two components of $\mathscr{D}(z)$ in all. We denote by $\mathscr{M}$ the set of $z \in \mathbb{C}\backslash[0,1]$ for which there is one component of $\mathscr{D}(z)$, and by $\mathscr{N}$ the set of $z \in \mathbb{C}\backslash[0,1]$ for which there are two components of $\mathscr{D}(z)$. We show that both $\mathscr{M}$ and $\mathscr{N}$ are open. Since their union is $\mathbb{C}\backslash[0,1]$ and the latter set is connected, one of them must be empty. But $\mathscr{N}$ contains all z with $|z| \geq e$, so necessarily $\mathscr{N} = \mathbb{C}\backslash[0,1]$ and there is then one bounded component of $\mathscr{D}(z)$ containing 0 for all $z \in \mathbb{C}\backslash[0,1]$. The reader will find it helpful to refer to Figures 2 and 3.

***$\mathscr{M}$* is open:** Let $z_1 \in \mathscr{M}$. Choose $v_1 \in (0,\infty)$ such that v_1 lies in the unbounded component of $\mathscr{D}(z_1)$, and moreover the interval $[v_1,\infty)$ lies in $\mathscr{D}(z_1)$ and $|G(v_1,z_1)| > 2$. (See the right–hand–side of (3.18) to convince yourself that this is possible, and note that the right–hand–side is increasing in v, for large enough real v). Since $z_1 \in \mathscr{M}$, $\mathscr{D}(z_1)$ consists of one component, so we can find a path σ from 0 to v_1 lying wholly in $\mathscr{D}(z_1)$. Then

$$\inf_{v\in\sigma} |G(v,z_1)| > 1.$$

By continuity of $G(\cdot,\cdot)$, if $\epsilon > 0$ is small enough,

$$|z-z_1| < \epsilon \Longrightarrow \inf_{v\in\sigma} |G(v,z)| > 1.$$

Moreover, the fact that $|G(v,z)|$ is bounded below for real v by a function that is increasing in v for large real v, and continuous in z (see (3.18)) shows that if ϵ is small enough, also

$$v \in [v_1,\infty) \Longrightarrow |G(v,z)| > 1 \Longrightarrow [v_1,\infty) \subset \mathscr{D}(z) \text{ for } |z - z_1| < \epsilon.$$

We have shown that for z close enough to z_1, σ is a path from 0 to the unbounded component of $\mathscr{D}(z)$ lying wholly in $\mathscr{D}(z)$. Since the only bounded component of $\mathscr{D}(z)$ would have to contain 0, it follows that $\mathscr{D}(z)$ has exactly one component, for $|z - z_1| < \epsilon$, and so such $z \in \mathscr{M}$.

__$\mathscr{N}$ is open:__ Let $z_1 \in \mathscr{N}$ and $w_1 = \phi(z_1)$. By hypothesis, there exist two components of $\mathscr{D}(z_1)$, say $\mathscr{D}_b(z_1)$ and $\mathscr{D}_u(z_1)$. Here $\mathscr{D}_u(z_1)$ is unbounded, and $\mathscr{D}_b(z_1)$ is bounded and contains 0. We consider two cases:

__I: The boundaries of $\mathscr{D}_u(z_1)$ and $\mathscr{D}_b(z_1)$ do not touch__ (We shall see just now that this is not possible). Then there is a positive distance between the boundaries of the two components, so we can find a closed Jordan curve, σ say, enclosing $\mathscr{D}_b(z_1)$ in its interior, and lying wholly in $\mathscr{B}(z_1)$. Then

$$\sup_{v\in\sigma} |G(v,z_1)| < 1,$$

so for some small enough $\epsilon > 0$,

$$|z - z_1| < \epsilon \Longrightarrow \sup_{v\in\sigma} |G(v,z)| < 1.$$

Then for such z, σ lies wholly in $\mathscr{B}(z)$ and encloses $0 \in \mathscr{D}(z)$, and so there is a bounded component of $\mathscr{D}(z)$ lying inside σ. So there is one bounded component and one unbounded component of $\mathscr{D}(z)$. Thus $|z - z_1| < \epsilon \Longrightarrow z \in \mathscr{N}$.

__II. The boundaries of $\mathscr{D}_u(z_1)$ and $\mathscr{D}_b(z_1)$ touch__ Now these can only touch at a point of $\mathscr{U}(z_1)$ that is at least a double point of $\mathscr{U}(z_1)$. By Lemma 3.1(c), this point must be $w_1 := \phi(z_1)$. We can then find a smooth closed Jordan curve σ enclosing $\mathscr{D}_b(z_1)$ (and so 0) and such that σ contains w_1, while $\sigma\backslash\{w_1\}$ is contained in $\mathscr{B}(z_1)$.

Recall now from Lemma 3.2 that for all $z \in \mathbb{C}\backslash[0,1]$, $w := \phi(z)$ is the unique double point of $\mathscr{U}(z)$ and that $w = \phi(z)$ is continuous in z. Moreover, Lemma 3.5 establishes the continuity of $\mathscr{U}(z)$ in z, when intersected with bounded balls centre 0.

It follows that we can find $a \in \sigma$, $b \in \sigma$, such that a lies in the sector $\mathscr{E}_1(z_1)$ of $\mathscr{B}(z_1)$ and such that b lies in the sector $\mathscr{E}_2(z_1)$ of $\mathscr{B}(z_1)$ (see Figure 2) and $\epsilon > 0$ such that for $|z - z_1| < \epsilon$, there is a smooth Jordan arc, τ_z say, passing from a to b, passing through $w = \phi(z)$ and such that $\tau_z \backslash \{w\}$ lies in $\mathscr{B}(z)$. (This can also be deduced from the inverse/implicit function theorems). The crucial thing is that a and b do not depend on z. Let σ_1 denote that part of σ omitting the arc of σ passing from a to b through w_1. Of course, σ_1 does not depend on z, only on our fixed z_1.

Now

$$\sup_{v \in \sigma_1} |G(v,z_1)| < 1,$$

so if ϵ above is small enough,

$$|z - z_1| < \epsilon \Rightarrow \sup_{v \in \sigma_1} |G(v,z)| < 1.$$

Then for such z, $\sigma_1 \cup \tau_z$ is a closed Jordan curve passing through $w = \phi(z)$, enclosing $0 \in \mathscr{D}(z)$ and such that $\sigma_1 \cup \tau_z \backslash \{w\}$ lies in $\mathscr{B}(z)$. This curve must then enclose a bounded component of $\mathscr{D}(z)$, and so there are two components of $\mathscr{D}(z)$. Thus $z \in \mathscr{N}$ for $|z - z_1| < \epsilon$.

So $\mathscr{M}$ and $\mathscr{N}$ are open, and their union is the connected set $\mathbb{C}\backslash[0,1]$. Since $\mathscr{N} \supset \{z\colon |z| \geq e\}$ is non–empty, $\mathscr{M}$ must be empty, and we have for all $z \in \mathbb{C}\backslash[0,1]$, that there is a single bounded component of $\mathscr{D}(z)$ containing 0, and an unbounded component of $\mathscr{D}(z)$.

It remains to show that the boundary $\partial\mathscr{D}_b(z)$ of $\mathscr{D}_b(z)$ contains w. We can proceed much as above: Let

$$\mathscr{M}_1 := \{z \in \mathbb{C}\backslash[0,1]\colon w = \phi(z) \notin \partial\mathscr{D}_b(z)\};$$

$$\mathscr{N}_1 := \{z \in \mathbb{C}\backslash[0,1]\colon w = \phi(z) \in \partial\mathscr{D}_b(z)\}.$$

Firstly for $z_1 \in \mathscr{M}_1$, the distance between $w_1 := \phi(z_1)$ and $\partial\mathscr{D}_b(z_1)$ is positive, and then continuity of $G(\cdot,\cdot)$ easily shows that the same is true in a neighbourhood of z_1. So $\mathscr{M}_1$ is open. For $z_1 \in \mathscr{N}_1$, we can find a path from 0 to $w_1 = \phi(z_1)$ such that $\sigma\backslash\{w_1\}$ lies in $\mathscr{D}_b(z_1)$. By splitting σ into a small piece near w_1, and a remaining large piece σ_1, we can use continuity of $G(\cdot,\cdot)$ to show that

$$\inf_{v\in\sigma_1} |G(v,z)| > 1 \text{ for all } |z - z_1| < \epsilon,$$

and use the behaviour of $G(v,z)$ near $w = \phi(z)$ to construct a small arc τ_z from the endpoint of σ_1 to w with $\tau_z\backslash\{w\}$ in $\mathscr{B}(z)$. Then $\sigma_1 \cup \tau_z$ is a path from 0 to w with $\sigma_1 \cup \tau_z \backslash \{w\}$ lying in $\mathscr{D}_b(z)$ necessarily. So $\mathscr{N}_1$ is also open and moreover from Lemma 3.4 contains $\{z: |z| \geq e\}$. Again we deduce that $\mathscr{N}_1 = \mathbb{C}\backslash[0,1]$. So for all $z \in \mathbb{C}\backslash[0,1]$, $\partial\mathscr{D}_b(z)$ contains $w = \phi(z)$.

Finally, we can find a closed Jordan curve, starting at w, passing through $\mathscr{E}_2(z)$, lying all the time very close to $\partial\mathscr{D}_b(z)$ but staying in $\mathscr{B}(z)$ until we reach the sector $\mathscr{E}_1(z)$ and then w. This Jordan curve can be taken as our γ_z. (Note that on one side of $\mathscr{U}(z)$ except at its double points, we are in $\mathscr{B}(z)$, and on the other side, we are in $\mathscr{D}(z)$. This is meaningful locally because $\mathscr{U}(z)$ is smooth). □

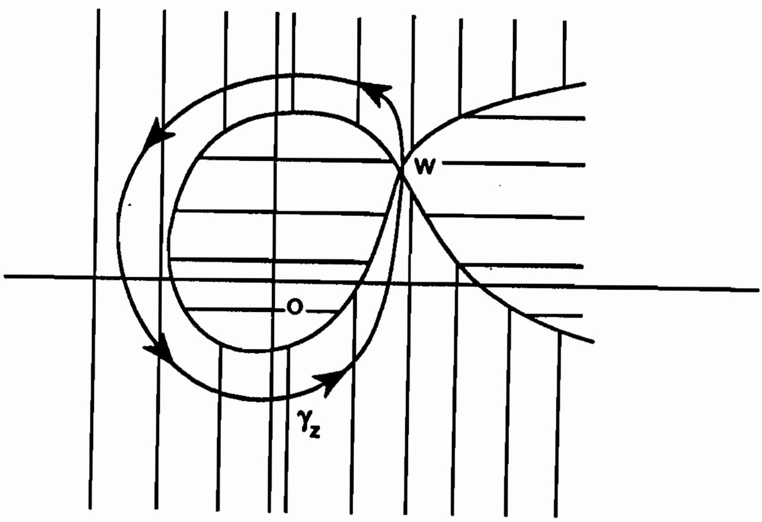


Fig. 4. The closed curve γ_z

We now turn to prove the existence of a suitable contour for $z = x \in (0,1)$. The proof is very similar to that for $z \in \mathbb{C}\backslash[0,1]$: We first deal with x not too close to 0, and then use a continuity / connectedness argument to extend this to all $x \in (0,1)$.

Lemma 3.7 Let $z = x \in [0.00076,1)$. Then there exists exactly one unbounded component of $\mathscr{B}(x)$, $\mathscr{D}(x)$, and exactly one bounded component of $\mathscr{B}(x)$, $\mathscr{D}(x)$. Denote these by $\mathscr{B}_u(x)$, $\mathscr{D}_u(x)$, $\mathscr{B}_b(x)$, $\mathscr{D}_b(x)$ respectively. The boundaries of all these four components touch at $w = w(x)$ and $\bar{w} = \bar{w}(x)$ determined by (3.4), (3.5), and these are the only double points of the curve $\mathscr{U}(x)$. Moreover, $\mathscr{D}_b(x)$ contains 0, and $\mathscr{B}_b(x)$ contains $u = -\log x$, and all 4 components are symmetric with respect to the real axis.

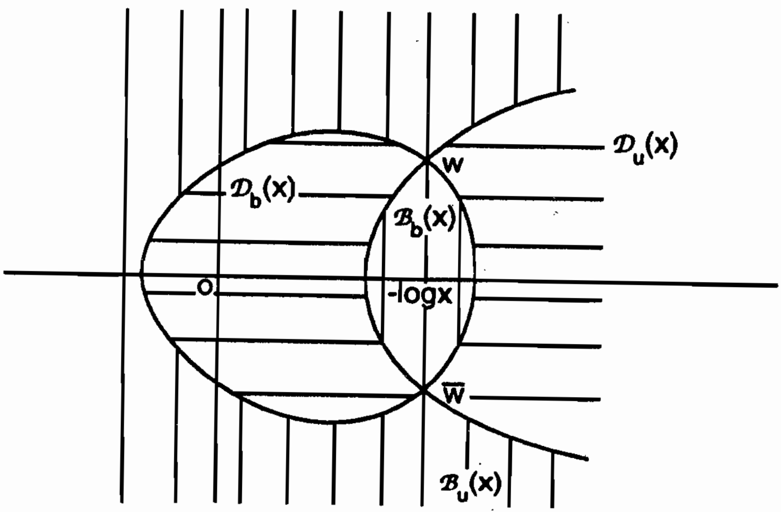


Fig. 5. The Components of $\mathscr{B}(x)$ and $\mathscr{D}(x)$

Proof We already know that there is exactly one unbounded component of $\mathscr{B}(x)$ and $\mathscr{D}(x)$. Moreover, the symmetry of the components with respect to the real axis follows as $H(v,x)$ is real on the real axis.

Recall that the zeros of $G(v,x)$ have the form $-u + 2k\pi i = -\log x + 2k\pi i$, k an integer.

We show that

$$(3.27) \qquad \sup_{|y|\geq 2\pi} |G(-u+iy,x)| < 1,$$

so that the rays that make up the set $\{-u + iy: |y| \geq 2\pi\}$ lie in the unbounded component of $\mathscr{B}(x)$. Write $x = h(b)$ as in (2.3) of Lemma 2.1. We have

$$|H(-u+iy,x)|^2 = \frac{|1 - e^{iy}|^2}{|-u + iy|^2}$$

$$= \frac{4\sin^2(y/2)}{u^2 + y^2}.$$

Also from Lemma 3.1(b) and then Lemma 2.1(c),

$$|H(w,x)|^2 = |w-1|^{-2} = (\frac{\sin b}{b})^2.$$

So,

$$|G(-u+iy,x)|^2 = \frac{4\sin^2(y/2)}{u^2 + y^2} (\frac{b}{\sin b})^2.$$

Recall that $u = -\log x = -\log h(b)$. Then for $y \geq 2\pi+2$,

$$(3.28) \qquad |G(-u+iy,x)|^2 \leq \frac{4}{(\log h(b))^2 + 4(\pi+1)^2} (\frac{b}{\sin b})^2 =: J(b),$$

say. Also using the inequality

$$\sin^2(y/2) = \sin^2((y-2\pi)/2) \leq (y-2\pi)^2/4,$$

we obtain for $y \in [2\pi, 2\pi+2]$,

$$|G(-u+iy,x)|^2 \leq \frac{(y - 2\pi)^2}{(\log h(b))^2 + y^2} (\frac{b}{\sin b})^2 =: \psi(y) (\frac{b}{\sin b})^2.$$

Here for $y \in [2\pi, 2\pi+2]$,

$$\psi'(y) = \frac{2(y - 2\pi)}{\{(\log h(b))^2 + y^2\}^2} \{(\log h(b))^2 + 2\pi y\} \geq 0,$$

so for such y,

$$|G(-u+iy,x)|^2 \leq \psi(2\pi+2) (\frac{b}{\sin b})^2 = J(b).$$

Thus

$$\sup_{|y|\geq 2\pi} |G(-u+iy,x)|^2 \leq J(b).$$

We claim that $J(b)$ defined by (3.28) is an increasing function of $b \in (0,\pi)$. For $h(b)$ is decreasing according to Lemma 2.1(b), and a differentiation shows that $(\frac{b}{\sin b})^2$ is increasing in this interval. A simple calculator evaluation of $h(b)$ and then $J(b)$ (recall h(b) is defined by (2.3)) for $b = \frac{5}{6}\pi$ shows that

$$J(\tfrac{5}{6}\pi) = 0.9115 \ldots < 1.$$

So $b \in (0,\frac{5}{6}\pi) \Rightarrow J(b) < 1$. But, by Lemma 2.1(b), this range of b corresponds to

$$x \in [h(\tfrac{5}{6}\pi), 1) = [0.000754\ldots,1).$$

In particular for $x \in [0.00076,1)$, we have (3.27) as $|G(-u+iy,x)|$ is even in y. So all but at most one of the zeros of $G(v,x)$ lie in the unbounded component of $\mathscr{B}(x)$, and hence there is at most one bounded component of $\mathscr{B}(x)$.

Next, note that from Lemma 3.2 and symmetry, we have the picture in Figure 6 for sectors of $\mathscr{B}(x)$ and $\mathscr{D}(x)$ near w, $\overline{w}$.

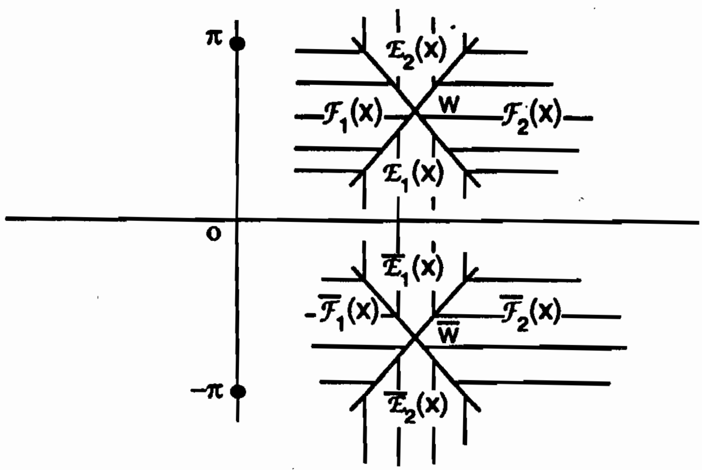


Fig. 6. Sectors of $\mathscr{B}(x)$ and $\mathscr{D}(x)$ near w and $\overline{w}$

We denote as in Lemma 3.2, the sectors of $\mathscr{B}(x)$ and $\mathscr{D}(x)$ near w, by $\mathscr{E}_1(x)$, $\mathscr{E}_2(x)$ and $\mathscr{F}_1(x)$, $\mathscr{F}_2(x)$. We claim that $\mathscr{F}_1(x)$ and $\mathscr{F}_2(x)$ lie in different components of $\mathscr{D}(x)$. If not, we could find a closed Jordan curve C_1 starting at w, passing through $\mathscr{F}_1(x)$ and then only through this component of $\mathscr{D}(x)$ and through $\mathscr{F}_2(x)$ to w. Because of the symmetry of the components w.r.t. the real axis, we can assume that C_1 lies in the upper–half–plane. Then

C_1 encloses either $\mathscr{E}_1(x)$ or $\mathscr{E}_2(x)$ and so encloses a bounded component of $\mathscr{B}(x)$ in the upper–half–plane. Then $\overline{C_1}$ encloses a distinct component of $\mathscr{B}(x)$ in the lower half–plane. So we obtain two bounded components of $\mathscr{B}(x)$, a contradiction to what we proved above.

So, $\mathscr{F}_1(x)$ and $\mathscr{F}_2(x)$ lie in different components of $\mathscr{D}(x)$ and the same is true of $\overline{\mathscr{F}}_1(x)$, $\overline{\mathscr{F}}_2(x)$. Since there can be only one bounded component of $\mathscr{D}(x)$, there must be one such component, containing 0, $\mathscr{F}_1(x)$ and $\overline{\mathscr{F}}_1(x)$ (or $\mathscr{F}_2(x)$ and $\overline{\mathscr{F}}_2(x)$). The boundary of this component then contains w and $\overline{w}$.

Next, we claim that $\mathscr{E}_1(x)$ and $\mathscr{E}_2(x)$ lie in different components of $\mathscr{B}(x)$. For if they lay in the same component of $\mathscr{B}(x)$, then we could find (as above) a closed Jordan curve lying wholly in the upper–half–plane, starting at w, passing through $\mathscr{E}_1(x)$ all the way through $\mathscr{B}(x)$, to $\mathscr{E}_2(x)$ and then ending at w. This then encloses a bounded component of $\mathscr{D}(x)$ in the upper–half–plane, and its reflection w.r.t the real axis gives another bounded component of $\mathscr{D}(x)$ in the lower–half–plane, contradicting that there is only one bounded component of $\mathscr{D}(x)$.

Thus there is a single bounded component of $\mathscr{B}(x)$, $\mathscr{B}_b(x)$ say, enclosing the zero u $= -\log x$ of G(v,x) and also containing either $\mathscr{E}_1(x)$ and $\overline{\mathscr{E}}_1(x)$ (or $\mathscr{E}_2(x)$ and $\overline{\mathscr{E}}_2(x)$). Then $\mathscr{B}_b(x)$ is adjacent to $\mathscr{D}_b(x)$ and the boundaries of all 4 components touch at w and $\overline{w}$. □

Now we can prove the required behaviour of components of $\mathscr{B}(x)$, $\mathscr{D}(x)$ for all $x \in (0,1)$, but we do not obtain as strong a conclusion as in Lemma 3.7:

Theorem 3.8 For $x \in (0,1)$, there is exactly one unbounded component of $\mathscr{B}(x)$ and $\mathscr{D}(x)$, which we denote by $\mathscr{B}_u(x)$ and $\mathscr{D}_u(x)$ respectively. Moreover, there is exactly one bounded component of $\mathscr{D}(x)$, which we denote by $\mathscr{D}_b(x)$. The latter contains 0 and its boundary contains the double points $w = w(x)$ and $\overline{w} = w(x)$ of $\mathscr{U}(x)$. We can find a smooth closed Jordan curve γ_x passing through w and $\overline{w}$, encircling $\mathscr{D}_b(x)$ and so 0, and such that $\gamma_x \setminus \{w, \overline{w}\}$ lies in $\mathscr{B}(x)$.

Remark We are not ruling out the possibility that there is more than one bounded component of $\mathscr{B}(x)$, but the important fact is that we can find the contour γ_x surrounding $\mathscr{D}_b(x)$ and lying in $\mathscr{B}(x)$ except at w, $\bar{w}$. This is pictured below in Figure 7.

Proof Recall that for fixed x, G(v,x) has one pole at v = 0. So there is at most one bounded component of $\mathscr{D}(x)$, and so either one or two components of $\mathscr{D}(x)$ in all. We denote by $\mathscr{M}$ the set of $x \in (0,1)$ for which there is one component of $\mathscr{D}(x)$, and by $\mathscr{N}$ the set of $x \in (0,1)$ for which there are two components of $\mathscr{D}(x)$. We show that both $\mathscr{M}$ and $\mathscr{N}$ are open sets in (0,1). Since their union is (0,1) and the latter set is connected, one of them must be empty. But $\mathscr{N}$ contains [0.00076,1), so necessarily $\mathscr{N} = (0,1)$ and there is then one bounded component of $\mathscr{D}(x)$ containing 0 for all $x \in (0,1)$. The reader will find it helpful to refer to Figures 5 and 6. Below we proceed much as in Theorem 3.6, so give less detail.

$\mathscr{M}$ is open: Let $x_1 \in \mathscr{M}$. Choose $v_1 \in (0,\infty)$ such that v_1 lies in the unbounded component of $\mathscr{D}(x_1)$, and moreover the interval $[v_1,\infty)$ lies in $\mathscr{D}(x_1)$ and $|G(v_1,x_1)| > 2$. (See the right–hand–side of (3.18) to convince yourself that this is possible, and note that the right–hand–side is increasing in v, for large enough real v). Since $x_1 \in \mathscr{M}$, $\mathscr{D}(x_1)$ consists of one component, so we can find a path σ from 0 to v_1 lying wholly in $\mathscr{D}(x_1)$. Then

$$\inf_{v\in\sigma} |G(v,x_1)| > 1.$$

By continuity of $G(\cdot,\cdot)$, if $\epsilon > 0$ is small enough,

$$|x - x_1| < \epsilon \Longrightarrow \inf_{v\in\sigma} |G(v,x)| > 1.$$

Moreover, the fact that $|G(v,x)|$ is bounded below for real v by a function that is increasing in v for large real v, and continuous in x (see (3.18)) shows that if ϵ is small enough, also

$$v \in [v_1,\infty) \Longrightarrow |G(v,x)| > 1 \Longrightarrow [v_1,\infty) \subset \mathscr{D}(x) \text{ for } |x - x_1| < \epsilon.$$

We have shown that for x close enough to x_1, σ is a path from 0 to the unbounded component of $\mathscr{D}(x)$ lying wholly in $\mathscr{D}(x)$. Since the only bounded component of $\mathscr{D}(x)$ would have to contain 0, it follows that $\mathscr{D}(x)$ has exactly one component, for $|x - x_1| < \epsilon$, and so such $x \in \mathscr{M}$.

$\mathscr{N}$ is open: Let $x_1 \in \mathscr{N}$ and $w_1 = w(x_1)$. By hypothesis, there exist two components of $\mathscr{D}(x_1)$, say $\mathscr{D}_b(x_1)$ and $\mathscr{D}_u(x_1)$. Here $\mathscr{D}_u(x_1)$ is unbounded, and $\mathscr{D}_b(x_1)$ is bounded and contains 0. We consider two cases:

I: The boundaries of $\mathscr{D}_u(x_1)$ and $\mathscr{D}_b(x_1)$ do not touch (We shall see just now that this is not possible). Then there is a positive distance between the boundaries of the two components, so we can find a closed Jordan curve, σ say, enclosing $\mathscr{D}_b(x_1)$ in its interior, and lying wholly in $\mathscr{B}(x_1)$. Then

$$\sup_{v\in\sigma} |G(v,x_1)| < 1$$

so for some small enough $\epsilon > 0$,

$$|x - x_1| < \epsilon \Longrightarrow \sup_{v\in\sigma} |G(v,x)| < 1.$$

Then for such x, σ lies wholly in $\mathscr{B}(x)$ and encloses $0 \in \mathscr{D}(x)$, and so there is a bounded component of $\mathscr{D}(x)$ lying inside σ. So there is one bounded component and one unbounded component of $\mathscr{D}(x)$. Thus $|x - x_1| < \epsilon \Longrightarrow x \in \mathscr{N}$.

II. The boundaries of $\mathscr{D}_u(x_1)$ and $\mathscr{D}_b(x_1)$ touch Now these can only touch at a point of $\mathscr{U}(x_1)$ that is at least a double point of $\mathscr{U}(x_1)$. By Lemma 3.1(c), this point must be w_1 and by symmetry of the components they also touch at $\overline{w}_1$. We can then find a smooth closed Jordan curve σ enclosing $\mathscr{D}_b(x_1)$ (and so 0) and such that σ contains w_1 and $\overline{w}_1$, while $\sigma\backslash\{w_1, \overline{w}_1\}$ is contained in $\mathscr{B}(x_1)$. Moroever, we may assume that σ is symmetric w.r.t the real axis.

Recall now from Lemma 3.2 that for all $x \in (0,1)$, $w := w(x)$ and $\overline{w}$ are the only double points of $\mathscr{U}(x)$ and that $w = w(x)$ is continuous in x. Moreover, Lemma 3.5 establishes the continuity of $\mathscr{U}(x)$ in x, when intersected with bounded balls centre 0.

It follows that we can find $a \in \sigma$, $b \in \sigma$, such that a lies in the sector $\mathscr{E}_1(x_1)$ of $\mathscr{B}(x_1)$ and such that b lies in the sector $\mathscr{E}_2(x_1)$ of $\mathscr{B}(x_1)$ (see Figure 6) and $\epsilon > 0$ such that for $|x - x_1| < \epsilon$, there is a smooth Jordan arc, τ_x say, passing from a to b, passing through $w = w(x)$ and such that $\tau_x\backslash\{w\}$ lies in $\mathscr{B}(x)$. (This can also be deduced from the inverse/

implicit function theorems). The crucial thing is that a and b do not depend on x. Let σ_1 denote that part of σ omitting the arcs of σ passing from a to b through w_1 and passing from $\bar{b}$ to $\bar{a}$ through $\bar{w}_1$. Of course, σ_1 does not depend on x either.

Now

$$\sup_{v\in\sigma_1} |G(v,x_1)| < 1,$$

so if ϵ above is small enough,

$$|x-x_1| < \epsilon \Longrightarrow \sup_{v\in\sigma_1} |G(v,x)| < 1.$$

Then for such x, $\sigma_1 \cup \tau_x \cup \bar{\tau}_x$ is a closed Jordan curve passing through $w = w(x)$ and $\bar{w}$, enclosing $0 \in \mathscr{D}(x)$ and such that $\sigma_1 \cup \tau_x \cup \bar{\tau}_x\backslash\{w, \bar{w}\}$ lies in $\mathscr{B}(x)$. This curve must then enclose a bounded component of $\mathscr{D}(x)$, and so there are two components of $\mathscr{D}(x)$. Thus $x \in \mathscr{N}$ for $|x-x_1| < \epsilon$.

So $\mathscr{M}$ and $\mathscr{N}$ are open, and their union is the connected set (0,1). Since $\mathscr{N}$ is non–empty, $\mathscr{M}$ must be empty, and we have for all $x \in (0,1)$, that there is a single bounded component of $\mathscr{D}(x)$ containing 0.

It remains to show that the boundary $\partial\mathscr{D}_b(x)$ of $\mathscr{D}_b(x)$ contains w and $\bar{w}$. We can proceed much as above: Let

$$\mathscr{M}_1 := \{x \in (0,1)\colon w = w(x) \notin \partial\mathscr{D}_b(x)\};$$
$$\mathscr{N}_1 := \{x \in (0,1)\colon w = w(x) \in \partial\mathscr{D}_b(x)\}.$$

Firstly for $x_1 \in \mathscr{M}_1$, the distance between $w_1 = w(x_1)$ and $\partial\mathscr{D}_b(x_1)$ is positive, and then continuity of G(v,x) easily shows that the same is true in a neighbourhood of x_1. So $\mathscr{M}_1$ is open. For $x_1 \in \mathscr{N}_1$, we can find a path from 0 to w_1 such that $\sigma\backslash\{w_1\}$ lies in $\mathscr{D}_b(x_1)$. By splitting σ into a small piece near w_1, and a remaining large piece σ_1, we can use continuity of $G(\cdot,\cdot)$ to show that

$$\inf_{v\in\sigma_1} |G(v,x)| > 1 \text{ for all } |x-x_1| < \epsilon,$$

and use the behaviour of G(v,x) near $w = w(x)$ to construct a small arc τ_x from the endpoint of σ_1 to w with $\tau_x\backslash\{w\}$ in $\mathscr{B}(x)$. Then $\sigma_1 \cup \tau_x$ is a path from 0 to w with $\sigma_1 \cup \tau_x \backslash \{w\}$ lying in $\mathscr{D}_b(x)$ necessarily. So $\mathscr{N}_1$ is also open and moreover from Lemma 3.7

contains [0.00076,1). Again we deduce that $\mathscr{N}_1 = (0,1)$. So for all $x \in (0,1)$, $\partial\mathscr{D}_b(x)$ contains $w = w(x)$ and also $\overline{w}$ by symmetry.

Finally, we can find a closed Jordan curve, starting at w, passing through $\mathscr{E}_2(x)$, lying all the time very close to $\partial\mathscr{D}_b(x)$ but staying in $\mathscr{B}(x)$ until we reach the sector $\overline{\mathscr{E}}_2(x)$ and then passing through $\overline{w}$ and then close to $\partial\mathscr{D}_b(x)$ through $\overline{\mathscr{E}}_1(x)$ to $\mathscr{E}_1(x)$ and ending at w. This Jordan curve can be taken as our γ_x. (Note that on one side of $\mathscr{U}(x)$ except at its double points, we are in $\mathscr{B}(x)$, and on the other side, we are in $\mathscr{D}(x)$. This is meaningful locally because $\mathscr{U}(x)$ is smooth). □

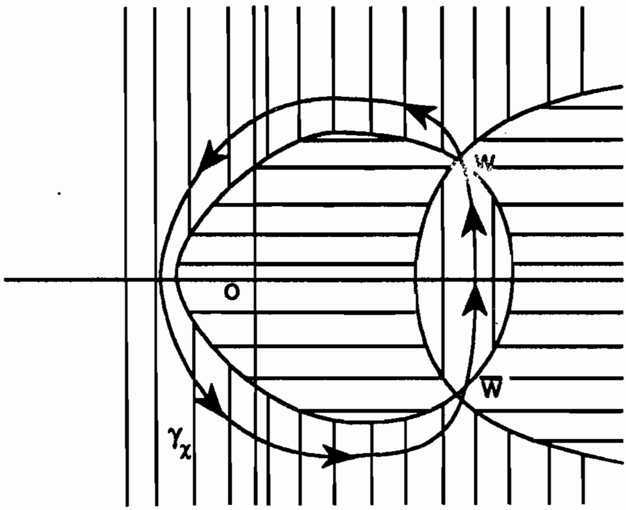


Fig. 7. The closed curve γ_x

4. PROOFS OF THEOREMS 1.2 AND 1.4

Most of the work towards the proof of Theorems 1.2 and 1.4 has already been completed in §3. All that we have to do is to analyze the integral in (3.1), but over a small arc near w or w and $\overline{w}$, according as z is in $\mathbb{C}\backslash[0,1]$ or $(0,1)$. We begin with a more precise version of Lemma 3.2:

Lemma 4.1 Let either $z \in \mathbb{C}\backslash[0,1]$ and $w = \phi(z)$ or $z \in (0,1)$ and w be defined by (3.4). Choose s_0 so small that

(4.1) $$s_0 < 2^{-5/2} \frac{\sqrt{|w|}}{1 + |w|}.$$

Then for $s \in [-s_0, s_0]$,

(4.2) $$G(w + i\sqrt{2w}\, s, z) = 1 - s^2 + \Delta(s),$$

where

(4.3) $$\Delta(s) \le |s|^3 \frac{2\sqrt{2}}{\sqrt{|w|}} (1 + |w|) < s^2/2.$$

Proof A little rearrangement of (3.17) shows that

$$G(v,z) = 1 + \frac{1}{2w}(v - w)^2 + \Delta$$

where

$$\Delta := -\frac{(v - w)^3}{2vw} + \frac{1}{v} \sum_{j=3}^{\infty} \frac{(v - w)^j}{j!}.$$

In particular, for $v = w + i\sqrt{2w}\, s$, we obtain (4.2), where

(4.4) $$\Delta(s) = \frac{i\sqrt{2w}\, s^3}{v} + \frac{1}{v} \sum_{j=3}^{\infty} \frac{(i\sqrt{2w}s)^j}{j!}.$$

Now for $|s| \le s_0$, (4.1) gives

$$|\sqrt{2w}\, s| \le \sqrt{2|w|}\, s_0 < 2^{-2} \frac{|w|}{1 + |w|} < 1,$$

and moreover,

$$|\sqrt{2w}\, s| \le \sqrt{2|w|}\, s_0 < 2^{-2} |w|,$$

so

$$|v| = |w + i\sqrt{2w}\, s| \ge |w|/2.$$

Then

$$|\Delta(s)| \le \frac{2}{|w|} \{\sqrt{2|w|}\, |s|^3 + (\sqrt{2|w|}\, |s|)^3 \sum_{j=3}^{\infty} \frac{1}{j!} \}$$

$$\le \frac{2\sqrt{2}}{\sqrt{|w|}} |s|^3 (1 + |w|)$$

$$\le (s^2/2) \frac{2^{5/2} s_0}{\sqrt{|w|}} (1 + |w|) < s^2/2,$$

by (4.1). □

Lemma 4.2 With the hypotheses of Lemma 4.1, let τ_z denote the path $\{w + i\sqrt{2w}\, s\colon s \in [-s_0,s_0]\}$. Then

$$(4.5)\qquad I := \frac{k!}{2\pi i}\int_{\tau_z}\frac{e^v}{v}\,H(v,z)^k\,dv$$
$$= k!\,e^w\,(2\pi kw)^{-1/2}\,H(w,z)^k\,(1+o(1)).$$

Proof We see that

$$I = \frac{k!}{2\pi i}\,H(w,z)^k\int_{\tau_z}\frac{e^v}{v}\,G(v,z)^k\,dv$$
$$= \frac{k!}{2\pi}\,H(w,z)^k\sqrt{2w}\,e^w\int_{-s_0}^{s_0}\frac{e^{i\sqrt{2w}\,s}}{w + i\sqrt{2w}\,s}\,G(w+i\sqrt{2w}s,\,z)^k\,ds.$$

Now by Lemma 4.1,

$$G(w+i\sqrt{2w}s,\,z)^k = e^{k\log(1-s^2+\Delta(s))} = e^{-ks^2+O(ks^3)}$$

and moreover, (4.3) shows that

$$|G(w+i\sqrt{2w}s,\,z)^k| \le e^{k\log(1-s^2/2)} \le e^{-ks^2/2}.$$

We thus see that most of the contribution to the integral I comes from the interval $|s| \le (\frac{2\log k}{k})^{1/2}$, and outside this interval, the integrand is $O(k^{-1})$. So

$$I =$$
$$\frac{k!}{2\pi}\,H(w,z)^k\sqrt{2w}\,\frac{e^w}{w}\,\{\int_{-(\frac{2\log k}{k})^{1/2}}^{(\frac{2\log k}{k})^{1/2}} e^{-ks^2}\,(1+O(\frac{\log^3 k}{k^{1/2}}))\,ds + O(k^{-1})\}$$

$$= \frac{k!}{\pi}\,H(w,z)^k\,\frac{e^w}{\sqrt{2w}}\,\{\frac{1}{\sqrt{k}}\int_{-(2\log k)^{1/2}}^{(2\log k)^{1/2}} e^{-t^2}\,dt + o(\frac{1}{\sqrt{k}})\}$$

$$= k!\,e^w\,(2\pi kw)^{-1/2}\,H(w,z)^k\,(1+o(1)).\qquad \square$$

Proof of Theorem 1.2 We know from (3.1) that for $z = e^u$,

$$D_k(z) = \frac{k!}{2\pi i} \int_\gamma \frac{e^v}{v} H(v,z)^k \, dv$$

where γ is a positively oriented simple closed curve encircling 0. Choose $\gamma = \gamma_z$, the contour of Theorem 3.6 (Recall Figure 4). We can deform γ_z a little so that that part of γ_z passing through w is the path τ_z of Lemma 4.2. Then

$$D_k(z) = \frac{k!}{2\pi i} \int_{\tau_z} \frac{e^v}{v} H(v,z)^k \, dv + \frac{k!}{2\pi i} H(w,z)^k \int_{\gamma_z \setminus \tau_z} \frac{e^v}{v} G(v,z)^k \, dv.$$

Since $\gamma_z \setminus \tau_z$ lies in $\mathscr{B}(z)$, at a positive distance from $\mathscr{U}(z)$, the second integral in the last right–hand–side is $O(\theta^k)$, $k \to \infty$, for some $0 < \theta < 1$. So Lemma 4.2 gives the result. The uniform convergence in compact sets follows as all the contours γ_z, τ_z etc. vary continuously with z, as do all our estimates. So we obtain uniformity first in small balls and then in arbitrary compact sets. □

__Proof of Theorem 1.4__ We may assume that our contour γ_x constructed in Theorem 3.8 is symmetric w.r.t. the real axis (See Figure 7) and so we can assume that γ_x contains both τ_x and $\bar{\tau}_x$, where τ_x is as in Lemma 4.2, and τ_x is the conjugate set of points, traversed so that γ_x is positively oriented. Thus

$$D_k(x) = k! \left[\frac{1}{2\pi i} \int_{\tau_x} + \frac{1}{2\pi i} \int_{\bar{\tau}_x} \right] \frac{e^v}{v} H(v,x)^k \, dv$$

$$+ \frac{k!}{2\pi i} H(w,x)^k \int_{\gamma_x \setminus \tau_x \cup \bar{\tau}_x} \frac{e^v}{v} G(v,x)^k \, dv =: I_1 + I_2.$$

Here $\gamma_x \setminus (\tau_x \cup \bar{\tau}_x)$ lies in $\mathscr{B}(x)$, at a positive distance from $\mathscr{U}(x)$, so for some $0 < \theta < 1$, and for large enough k,

$$|I_2| \le k! \, |H(w,x)|^k \, \theta^k.$$

Moreover, since the integrand $\frac{e^v}{v} H(v,x)^k$ is real for real v, it is easily seen that the 2 integrals in I_1 (with the factor $\frac{1}{2\pi i}$ included) are conjugates of one another, so that

$$D_k(x) = k! \, 2 \, \mathrm{Re} \left[\frac{1}{2\pi i} \int_{\tau_x} \frac{e^v}{v} H(v,x)^k \, dv \right] + O(k! \, |H(w,x)|^k \, \theta^k)$$

$$= k! \left(\frac{2}{\pi k}\right)^{1/2} \left\{ \mathrm{Re} \left[\frac{e^w}{\sqrt{w}} H(w,x)^k \right] + o(H(w,x)^k) \right\},$$

by Lemma 4.2. Finally, recall from (3.12) and then Lemma 2.1 that

$$H(w,x)^k = (-xe^w)^k = (-xe^a)^k e^{ibk},$$

where

$$w = a + ib;\ b = h^{[-1]}(x);$$

and

$$a = 1 - b \cot b = 1 - h^{[-1]}(x) \cot h^{[-1]}(x).$$

Then (1.19) follows easily. □

5. ZERO DISTRIBUTION

Recall the notation (1.10) and (1.11) as well as the definition (1.8) of the inverse conformal map ϕ. Our proof of Theorem 1.3 begins with the following lemma:

Lemma 5.1 We have

(5.1) $$\lim_{k\to\infty} |\hat{D}_k(z)|^{1/k} = |ze^{\phi(z)-1}|,$$

uniformly in compact subsets of $\mathbb{C}\backslash[0,1]$. Moreover, there exists a non–negative measure μ on $[0,1]$ with

(5.2) $$\int_0^1 d\mu = 1,$$

and satisfying

(5.3) $$\int_0^1 \log|z-t|\, d\mu(t) = \log|z| + \mathrm{Re}\, \phi(z) - 1,\ z \in \mathbb{C}\backslash[0,1].$$

If we define for $x \in (0,1)$

$$\mathrm{Re}\, \phi(x) := \lim_{y\to 0+} \mathrm{Re}\, \phi(x+iy),$$

whenever the limit exists, and

(5.4) $$F(x) := \log|x| + \mathrm{Re}\, \phi(x) - 1,$$

then

(5.5) $$\int_0^1 \log|x-t|\, d\mu(t) = F(x),\ x \in (0,1).$$

Proof First note that

$$|\hat{D}_k(z)|^{1/k} = \exp(\int_0^1 \log |z-t| \, d\mu_k(t)), \; k \geq 1.$$

Now by Helly's Theorem [6], [15] (or if the reader prefers, the Banach–Alaoglu Theorem on weak–*–compactness), we can choose an infinite subsequence $\mathscr{S}$ of positive integers and a measure μ such that

$$\mu_k \xrightarrow{*} \mu, \; k \to \infty, \; k \in \mathscr{S}.$$

Necessarily μ is non–negative, has support in [0,1] and satisfies (5.2) as each μ_k does. Moreover, weak convergence gives

(5.6) $$\lim_{\substack{k\to\infty \\ k\in\mathscr{S}}} |\hat{D}_k(z)|^{1/k} = \exp(\int_0^1 \log |z-t| \, d\mu(t)),$$

$z \in \mathbb{C}\backslash[0,1]$. The uniform convergence in compact subsets follows from equicontinuity and the Arzela–Ascoli Theorem (cf. [6], [15]). But also, for such z, Theorem 1.2 and Stirling's formula gives

(5.7) $$\lim_{\substack{k\to\infty \\ k\in\mathscr{S}}} |\hat{D}_k(z)|^{1/k} = |ze^{\phi(z)-1}|.$$

Comparing with (5.6) gives (5.3). It remains to prove (5.5). Now by Lebesgue's Monotone Convergence Theorem, for all $x \in (0,1)$,

$$\lim_{y\to 0+} \int_0^1 \log|(x+iy)-t| \, d\mu(t) = \int_0^1 \log|x-t| \, d\mu(t).$$

Consequently, we deduce from (5.3) that (5.5) holds at all $x \in (0,1)$ where there exists

(5.8) $$\lim_{y\to 0+} \operatorname{Re} \phi(x+iy) = \operatorname{Re} \phi(x).$$

But from (1.7) and Theorem 1.1(b) Ψ is continuous and one–one on $\overline{\mathscr{A}} \cap \{z: \operatorname{Im} z > 0\}$, so the inverse ϕ of Ψ satisfies (5.8) for each $x \in (0,1)$. □

We next turn to solving (5.5):

<u>**Lemma 5.2**</u> The equation (5.5) subject to μ being a non–negative measure with support in (0,1) and satisfying (5.2) has a unique solution, given by

(5.9) $$d\mu(x) = \mu'(x)\,dx = -\frac{dx}{\pi h'(h^{[-1]}(x))}, \; x \in (0,1).$$

Proof We already know from the previous lemma that there is a solution $d\mu$ and must just show that it is unique and given by (5.9). Let us write

$$\phi(z) = U(x,y) + i\,V(x,y) \text{ for } z = x + iy.$$

Then (5.3) becomes

$$\frac{1}{2}\int_0^1 \log((x-t)^2 + y^2)\, d\mu(t) = \frac{1}{2}\log(x^2 + y^2) + U(x,y) - 1.$$

Differentiating partially w.r.t. y gives

$$\int_0^1 \frac{y}{(x-t)^2 + y^2}\, d\mu(t) = \frac{y}{x^2 + y^2} + \frac{\partial}{\partial y} U(x,y)$$

$$= \frac{y}{x^2 + y^2} - \frac{\partial}{\partial x} V(x,y),$$

by the Cauchy–Riemann equations. Now the left–hand side of the last expression is the Poisson integral for the upper–half plane of the measure μ, apart from a multiplicative constant. It is known that for a.e. $x \in (0,1)$, the left–hand side approaches $\pi\,\mu'(x)$ as $y \to 0+$. See for example [10,p.244, Thm. 11.24]. Thus for a.e. $x \in (0,1)$,

$$\pi\,\mu'(x) = -\lim_{y\to 0+} \frac{\partial}{\partial x} V(x,y).$$

But we see that if $w = a + ib$, where a, b satisfy (2.2), (2.3), and in particular $x = h(b)$, then

$$\lim_{y\to 0+} (U + iV)(x,y) = \lim_{y\to 0+} \phi(x + iy) = w,$$

so

$$\lim_{y\to 0+} V(x,y) = \operatorname{Im} w = b = h^{[-1]}(x).$$

Because $h^{[-1]}(x)$ is continuously differentiable in (0,1) and because V(x,y) is harmonic in the upper–half plane, we deduce that also for a.e. $x \in (0,1)$,

$$\lim_{y\to 0+} \frac{\partial}{\partial x} V(x,y) = \frac{db}{dx} = \frac{d}{dx} h^{[-1]}(x) = \frac{1}{h'(h^{[-1]}(x))}.$$

So we have for a.e. $x \in (0,1)$,

$$\mu'(x) = -\frac{1}{\pi h'(h^{[-1]}(x))}.$$

Finally, write

$$d\mu(x) = \mu'(x)\,dx + d\nu(x),$$

where ν is the singular part of μ, and in particular is non–negative and has support in [0,1]. Then

$$0 \le \int_0^1 d\nu = \int_0^1 [d\mu(x) - \mu'(x)\,dx]$$

$$= 1 - \int_0^1 \frac{-1}{\pi h'(h^{[-1]}(x))}\,dx \qquad \text{(by (5.2) and (5.9))}$$

$$= 1 - \int_0^\pi \frac{h'(b)}{\pi h'(b)}\,db = 0.$$

Thus, ν has total mass 0, so is the zero measure and we have (5.9). □

Proof of Theorem 1.3 We have from Lemma 5.1 that (1.18) holds where μ satisfies (5.3) and (5.5). Then by Lemma 5.2, μ has the form (1.13) and satisfies (1.14). The weak convergence (1.15) follows from (1.18) and the uniqueness of the measure μ in Lemma 5.2. It remains to prove (1.16) and (1.17). To this end, recall from (2.6) that

$$h'(b) = \frac{e^{b\cot b - 1}}{b}\left[2\cos b - \frac{\sin b}{b} - \frac{b}{\sin b}\right],$$

so we obtain using Maclaurin series for sin and cos, that

$$h'(b) = -b + O(b^2),\ b \to 0+, \tag{5.10}$$

and similarly from the definition (1.12) of h,

$$h(b) = 1 - \frac{b^2}{2} + O(b^4),\ b \to 0+. \tag{5.11}$$

We deduce that for $x = h(b)$, $b \to 0+$,

$$b = h^{[-1]}(x) = \sqrt{2(1-x)}\,\{1 + o(1)\},$$

so from (5.10), (5.11) and this last relation,

$$\mu'(x) = -\frac{1}{\pi h'(h^{[-1]}(x))} = -\frac{1}{\pi h'(b)} = \frac{1}{\pi b}(1 + o(1)) \tag{5.12}$$

$$= \frac{1}{\pi\sqrt{2(1-x)}}(1 + o(1)),\ x \to 1-.$$

Also as $b \to \pi-$, we see that

$$x = h(b) = \frac{\pi - b}{\pi} e^{-\frac{\pi}{\pi - b}}(1 + O(\pi - b)),$$

which yields

(5.13) $$\pi - b = \frac{\pi}{|\log\ x|}(1 + o(1)).$$

Then from (2.6),

$$h'(b) = \frac{x}{\sin\ b}\left[2\cos b - \frac{\sin\ b}{b} - \frac{b}{\sin\ b}\right]$$

$$= -\frac{x\pi}{(\pi - b)^2}[1 + o(1)]$$

$$= -\frac{x\ |\log\ x|^2}{\pi}[1 + o(1)].$$

Thus the density function $\mu'(x)$ satisfies

$$\mu'(x) = \frac{1}{x\ |\log\ x|^2}[1 + o(1)],\ x \to 0+. \qquad \square$$

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