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Acceleration of convergence of (generalized) Fourier series by the *d*-transformation

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Dedicated to Prof. Luigi Gatteschi on the occasion of his 70th birthday

In this paper we consider the efficient acceleration of the convergence of classical trigonometric Fourier series and their generalizations, such as Fourier-Legendre and Fourier-Bessel series, and many others as well. A common approach to this problem has been through the application of nonlinear sequence transformations to the series in question. Most well known sequence transformations, however, are either ineffective in general, or lose their effectiveness near points of singularity of the corresponding limit functions. The recent d-transformation of Levin and Sidi, on the other hand, has been observed to be very effective on a large family of infinite series that includes the generalized Fourier series above when applied in the appropriate manner. In the present work we propose a new approach involving the d-transformation for the efficient summation of generalized Fourier series, by which we produce very accurate and stable approximations in an economical way. In this approach one first extends the given series in a suitable fashion by including the corresponding functions of the second kind, and then applies the d-transformation to the extended series with the few parameters of the transformation properly adjusted. This new approach is economical in the sense that, for a given required level of accuracy, the number of terms of the series that it uses in the acceleration procedure is about half the number used when applying the d-transformation directly to the given series. In addition, it achieves high accuracy in a stable way near points of singularity of the limit function. Finally, it is also very effective for summing divergent series. We give a detailed convergence and stability theory pertaining to the new approach in an important special case. We also demonstrate the effectiveness of this approach with numerical examples.

Keywords: Acceleration of convergence, d-transformation, $W^{(m)}$-algorithm, Fourier series, generalized Fourier series, Fourier-Legendre series, Fourier-Bessel series, series of special functions, asymptotic expansions.

AMS (MOS) subject classification: 40A05, 40A25, 42A20, 42C15, 65B05, 65B10.

1. Introduction

In this paper we shall be concerned with the problem of efficient acceleration of the convergence of trigonometric Fourier series and their generalizations. In particular, we will be treating convergent and divergent infinite series of the form

$$F(x) := \sum_{n=0}^{\infty} [a_n \phi_n(x) + b_n \psi_n(x)], \tag{1.1}$$

where the functions $\phi_n(x)$ and $\psi_n(x)$ are assumed to satisfy

$$\rho_n^{\pm}(x) := \phi_n(x) \pm \mathrm{i}\psi_n(x) = e^{\pm \mathrm{i} n\omega x} g_n^{\pm}(x), \tag{1.2}$$

ω being some fixed real parameter, and

$$g_n^{\pm}(x) \sim n^{\gamma} \sum_{j=0}^{\infty} \delta_j^{\pm}(x) n^{-j} \quad \text{as } n \to \infty, \tag{1.3}$$

for some fixed γ, which, in general, can be complex.

The simplest and most widely treated members of the class above are the classical Fourier series

$$F(x) := \sum_{n=0}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x),$$

for which $\phi_n(x) = \cos n\omega x$ and $\psi_n(x) = \sin n\omega x$, so that $\rho_n^{\pm}(x) = e^{\pm \mathrm{i} n\omega x}$ and hence $g_n^{\pm}(x) \equiv 1$.

Additional examples involving orthogonal polynomial expansions, "non-classical" Fourier series, and Fourier-Bessel series will be provided in the next section. Of course, further examples involving different special functions exist, and can be dealt with by employing the approach that we take in the present paper.

In general, $\phi_n(x)$ and $\psi_n(x)$ may be some linear combinations of the nth eigenfunction of a Sturm-Liouville problem and of the corresponding second linearly independent solution of the relevant O.D.E.

Proposed method of acceleration

Step 1. Define the infinite series $A^{\pm}(x)$ and $B^{\pm}(x)$ by

$$\begin{aligned} A^{\pm}(x) &:= \sum_{n=0}^{\infty} a_n \rho_n^{\pm}(x), \\ B^{\pm}(x) &:= \sum_{n=0}^{\infty} b_n \rho_n^{\pm}(x), \end{aligned} \tag{1.4}$$

and observe that

$$\begin{aligned} F_{\phi}(x) &:= \sum_{n=0}^{\infty} a_n \phi_n(x) = \tfrac{1}{2}[A^{+}(x) + A^{-}(x)], \\ F_{\psi}(x) &:= \sum_{n=0}^{\infty} b_n \psi_n(x) = \frac{1}{2\mathrm{i}}[B^{+}(x) - B^{-}(x)], \end{aligned} \tag{1.5}$$

and that

$$F(x) = F_\phi(x) + F_\psi(x). \tag{1.6}$$

Step 2. Apply the d-transformation of Levin and Sidi [8], which is a nonlinear sequence transformation, to the series $A^\pm(x)$ and $B^\pm(x)$ to obtain good approximations to their limits. (Details are given in section 3.)

Step 3. Use these approximations in conjunction with (1.5) and (1.6) to obtain good approximations to the limit of $F(x)$.

In connection with steps 2 and 3 we note that in case the functions $\phi_n(x)$ and $\psi_n(x)$ and the coefficients a_n and b_n are real, it is enough to treat the two series $A^+(x)$ and $B^+(x)$, as $A^-(x) = \overline{A^+(x)}$ and $B^-(x) = \overline{B^+(x)}$, so that $F_\phi(x) = \operatorname{Re} A^+(x)$ and $F_\psi(x) = \operatorname{Im} B^+(x)$.

Besides the d-transformation there are other nonlinear sequence transformations that one can use to accelerate the convergence of the series $A^\pm(x)$ and $B^\pm(x)$, but these are not as powerful as the d-transformation, in general. The best known of these transformations are the transformation of Shanks [11] (or the equivalent ϵ-algorithm of Wynn [19]) and the u-transformation of Levin [7]. Numerical experiments with these transformations on the series $A^\pm(x)$ and $B^\pm(x)$ indicate the following:

1. Both methods are effective when the coefficients a_n and b_n have a "smooth" behavior for $n \to \infty$, such as that described by (3.5) in section 3, and x is not close to a point of singularity of the limit function.
2. The u-transformation fails when a_n and b_n have a "nonsmooth" behavior for $n \to \infty$; for instance, a_n and b_n may be oscillating like $\alpha_n \sin n\sigma + \beta_n \cos n\sigma$ for some constants α_n and β_n that behave "smoothly" themselves for $n \to \infty$. The ϵ-algorithm may be effective in this case provided x is not close to a point of singularity of the limit function.
3. Both methods fail when x is close to a point of singularity of the limit function. Due to round-off errors, they at most achieve limited accuracy which diminishes quickly by the addition of more terms in the acceleration process.

Numerical experiments with the d-transformation, on the other hand, show that this transformation is very effective in all the cases mentioned above. In fact, it has been observed numerically in many examples of varying nature and complexity that the d-transformation can be applied successfully to series for which the ϵ-algorithm and/or the u-transformation produce good results, and, in addition, it can accelerate the convergence of additional classes of series for which the other methods fail.

Our proposed method of acceleration assumes that the coefficients a_n and b_n have been provided individually. If, instead, $c_n := a_n \phi_n(x) + b_n \psi_n(x)$ are given as one piece (e.g., c_n may be given numerically), and there is no easy way of determining a_n and b_n separately, the method is, of course, not applicable. In this case the ϵ-algorithm and the d-transformation can be applied to the series $\sum_{n=0}^{\infty} c_n$, while the u-transformation fails. Again, if x is close to a point of singularity of the limit function, the ϵ-algorithm fails, but the d-transformation remains effective.

Note that although $F(x)$ is the sum of the two series $F_\phi(x) := \sum_{n=0}^{\infty} a_n \phi_n(x)$ and $F_\psi(x) := \sum_{n=0}^{\infty} b_n \psi_n(x)$ only, in the method we propose in this work we require the series $\sum_{n=0}^{\infty} a_n \psi_n(x)$ and $\sum_{n=0}^{\infty} b_n \phi_n(x)$, in addition. On the grounds that this involves twice as much computational work, one may object to the proposed approach at first, and may prefer the approach in which one directly accelerates the convergence of the series $F_\phi(x)$ and $F_\psi(x)$ or of the series $\sum_{n=0}^{\infty} c_n$ with $c_n := a_n \phi_n(x) + b_n \psi_n(x)$. A close look at the "tail" $\sum_{n=R}^{\infty} c_n$ of $F(x)$ for $R \to \infty$ and a consideration of the philosophy behind the d-transformation reveal, however, that the approach of the present work is about *twice* as economical in comparison with the other approach as far as the *number of terms* of the series used to obtain a given level of accuracy is concerned. (These points will be discussed in detail in subsection 3.2 of the present work.) This becomes an important factor when (i) there is an upper limit to the number of terms available, and (ii) the terms of the series are growing rapidly, which may result in instabilities in acceleration. Furthermore, we have observed in many cases that the new approach produces considerably higher accuracy by the addition of more terms of the series than the other direct approach.

The efficient summation of classical Fourier series by nonlinear convergence acceleration methods has been considered in different papers. Wynn [20] proposed that a real cosine series $\sum a_n \cos n\omega x$ be written as $\mathrm{Re}\left(\sum a_n z^n\right)$ with $z = e^{\mathrm{i}\omega x}$, and then the ϵ-algorithm be used to accelerate the convergence of the complex power series $\sum a_n z^n$. Obviously, this approach can be applied to real sine series too. Later Sidi [14] proposed that the u-transformation be used to accelerate the convergence of Fourier series in their complex power series form $\sum a_n z^n$ when the coefficients a_n have a "smooth" behavior for $n \to \infty$, providing a convergence acceleration theory for this approach at the same time. The ϵ-algorithm was used by Crump [3] in accelerating the convergence of Fourier series occurring in the numerical inversion of the Laplace transform.

The use of the ϵ-algorithm, a generalization of summation by parts, and the u-transformation on Fourier series in their real form was considered by Kiefer and Weiss [6], who concluded that the ϵ-algorithm and summation by parts are useful, while the u-transformation fails. Similarly, in their comparative survey, Smith and Ford [18] concluded that the ϵ-algorithm is useful, while the u-transformation and the θ-algorithm of Brezinski [2] both fail when applied to Fourier series in their real form. The conclusions of both [6] and [18] were based on extensive numerical testing and not on theoretical results.

The summation of classical Fourier series in their real form and also of Fourier-Legendre and Fourier-Bessel series by the d-transformation was considered in [8, section 7]. The numerical results there clearly show that the d-transformation is very effective in all cases, as we mentioned before.

In a recent paper Longman [9] has proposed to write Fourier series, Chebyshev series, and Legendre series as integrals involving some functions $f(u)$ when the coefficients of the series are moments of $f(u)$. A clever manipulation of these integrals results in some linear methods that turn out to be efficient especially when $f(u)$ has [0, 1] as its support. A nice feature of the method is that $f(u)$ need not be known.

Finally, we would like to mention the recent method of Homeier [5] for classical Fourier series, which is actually based on the Generalized Richardson Extrapolation Process (GREP) of Sidi [12] and is a variant of the d-transformation with $m = 2$ (to be explained in subsection 3.1). The performance of this method is very similar to that of the d-transformation with $m = 2$ in its simplest form when the coefficients of the Fourier series have a "smooth" behavior and provided we stay away from points of singularity of the limit function. Close to points of singularity, however, this method becomes very prone to roundoff errors and loses its effectiveness just as other methods. It achieves limited accuracy at best, and addition of more terms of the series in the acceleration procedure results in much loss of accuracy. This is also borne out by the numerical results of [5]. In addition, for Fourier series with "nonsmooth" coefficients the method of Homeier does not produce any acceleration at all.

In the next section we give examples of generalized Fourier series for which the approach of the present work is effective. A nice feature of these examples is that they are very common and that the functions $\phi_n(x)$ and $\psi_n(x)$ are available simultaneously and require no undue effort to obtain.

In section 3 we describe the d-transformation briefly, and explain how to adjust its few parameters to make it effective for Fourier series and their generalizations. This adjustment is justified in a rigorous manner by the convergence and stability theory that we provide in section 4 for an important special case. The main results of section 4 are theorems 4.1–4.4. Of these, theorems 4.1 and 4.2 deal with the derivation of asymptotic expansions for $n \to \infty$ of the partial sums $A_n = \sum_{i=0}^{n} u_i$ of convergent or divergent series $\sum_{i=0}^{\infty} u_i$ under certain conditions. Both of these theorems apply to the partial sums of $A^{\pm}(x)$ and $B^{\pm}(x)$ when a_n and b_n have a "smooth" behavior for $n \to \infty$. Theorems 4.3 and 4.4 provide optimal convergence and stability results for the d-transformation under the conditions of theorems 4.1 and 4.2 for a limiting procedure called Process I. The results of this section are a major extension and improvement of some corresponding results in [14].

Finally, in section 5 we illustrate the effectiveness of the approach proposed above with three examples involving classical Fourier and Fourier-Legendre series.

2. Examples of generalized Fourier series

In the previous section we mentioned the classical Fourier series as the simplest example of the class of series that is characterized through (1.1)–(1.3). We now give further examples. Some of the mathematical results cited below have been taken from Abramowitz and Stegun [1] and Olver [10].

2.1. *Chebyshev series*

$$F_T(x) := \sum_{n=0}^{\infty} d_n T_n(x) \text{ or } F_U(x) := \sum_{n=0}^{\infty} e_n U_n(x), \quad -1 \le x \le 1, \tag{2.1}$$

where $T_n(x)$ and $U_n(x)$ are the Chebyshev polynomials of degree n, of the first and second kinds, respectively. Defining $x = \cos\theta$, $0 \le \theta \le \pi$, we have $T_n(x) = \cos n\theta$ and $U_n(x) = \sin(n+1)\theta / \sin\theta$. Therefore, $F_T(x) := \sum_{n=0}^{\infty} d_n \cos n\theta$ and $F_U(x) := \left[\sum_{n=0}^{\infty} e_n \sin(n+1)\theta\right] / \sin\theta$, and can thus be treated as ordinary Fourier series. Here $\sin\theta = \sqrt{1-x^2}$ in terms of $x \in [-1, 1]$.

2.2. *"Nonclassical" Fourier series*

$$F(x) := \sum_{n=0}^{\infty} (a_n \cos\lambda_n x + b_n \sin\lambda_n x), \tag{2.2}$$

where

$$\lambda_n \sim n \sum_{j=0}^{\infty} \alpha_j n^{-j} \quad \text{as } n \to \infty, \ \alpha_0 > 0, \tag{2.3}$$

so that $\lambda_n \sim \alpha_0 n$ as $n \to \infty$.

Here $\phi_n(x) = \cos\lambda_n x$ and $\psi_n(x) = \sin\lambda_n x$, so that $\rho_n^{\pm}(x) = \exp(\pm i \lambda_n x)$ and (1.2) is seen to hold with $\omega = \alpha_0$ and $g_n^{\pm}(x) = \exp[\pm i(\lambda_n - \alpha_0 n)x]$. By invoking the asymptotic expansion in (2.3), it is easy to verify that $g_n^{\pm}(x)$ also satisfies (1.3) with $\gamma = 0$.

Functions $\phi_n(x)$ and $\psi_n(x)$ such as the ones here arise, e.g., when one solves an eigenvalue problem associated with a boundary value problem involving the O.D.E. $u'' + \lambda^2 u = 0$ on the interval $[0, l]$, which, in turn, may arise when one uses separation of variables in the solution of some appropriate heat equation, wave equation, or Laplace's equation. For instance, the eigenfunctions of the problem

$$u'' + \lambda^2 u = 0, \ 0 < x < l; \ u(0) = 0, \ u'(l) = -hu(l), \quad h > 0,$$

are $\sin\lambda_n x$, $n = 1, 2, \ldots$, where λ_n is the nth positive solution of the nonlinear equation $\lambda\cos\lambda l = -h\sin\lambda l$. By straightforward asymptotic techniques it can be shown that

$$\lambda_n \sim \left(n - \frac{1}{2}\right)\frac{\pi}{l} + \epsilon_1 n^{-1} + \epsilon_2 n^{-2} + \cdots \quad \text{as } n \to \infty.$$

Consequently, we also have that $\omega = \alpha_0 = \pi/l$.

2.3. *Fourier-Legendre series*

$$F_P(x) := \sum_{n=0}^{\infty} d_n P_n(x) \text{ or } F_Q(x) := \sum_{n=0}^{\infty} e_n Q_n(x), \quad -1 < x < 1, \tag{2.4}$$

where $P_n(x)$ is the Legendre polynomial of degree n and $Q_n(x)$ is the associated Legendre function of the second kind of order 0 of degree n. They are both generated by the recursion relation

$$M_{n+1}(x) = \frac{2n+1}{n+1} x M_n(x) - \frac{n}{n+1} M_{n-1}(x), \quad n = 1, 2, \ldots, \tag{2.5}$$

(here $M_n(x)$ is either $P_n(x)$ or $Q_n(x)$), with the initial conditions

$$P_0(x) = 1, \ P_1(x) = x \text{ and } Q_0(x) = \tfrac{1}{2} \log \frac{1+x}{1-x}, \ Q_1(x) = xQ_0(x) - 1, \\ \text{when } -1 < x < 1. \tag{2.6}$$

We now show that, with $\theta = \cos^{-1} x$, $\phi_n(\theta) = P_n(x)$ and $\psi_n(\theta) = -(2/\pi)Q_n(x)$, so that $\rho_n^{\pm}(\theta) = P_n(x) \mp \mathrm{i}(2/\pi)Q_n(x)$ and $\omega = 1$ in (1.2), and $\gamma = -1/2$ in (1.3).

First, we recall that $P_n(x) = P_n^0(x)$ and $Q_n(x) = Q_n^0(x)$, where $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are the associated Legendre functions of degree ν and of order μ. Next, letting $x = \cos\theta$, $0 < \theta < \pi$, and $u = n + 1/2$ in [10, p. 473, Ex. 13.3], it follows that, for any fixed real m, there exist two symptotic expansions $A_m(\theta; u)$ and $B_m(\theta; u)$,

$$\begin{aligned} A_m(\theta; u) &:= \sum_{s=0}^{\infty} \frac{A_s^{-m}(\theta^2)}{u^{2s}} \quad \text{as } n \to \infty, \\ B_m(\theta; u) &:= \sum_{s=0}^{\infty} \frac{B_s^{-m}(\theta^2)}{u^{2s}} \quad \text{as } n \to \infty, \end{aligned} \tag{2.7}$$

such that

$$P_n^{-m}(\cos\theta) \mp \mathrm{i}\frac{2}{\pi} Q_n^{-m}(\cos\theta) \sim \frac{1}{u^m}\left(\frac{\theta}{\sin\theta}\right)^{1/2} \\ \times \left\{ H_m^{(\pm)}(u\theta) A_m(\theta; u) + \frac{\theta}{u} H_{m-1}^{(\pm)}(u\theta) B_m(u; \theta) \right\}. \tag{2.8}$$

Here $H_m^{(+)}(z)$ and $H_m^{(-)}(z)$ stand for the Hankel functions $H_m^{(1)}(z)$ and $H_m^{(2)}(z)$, respectively. Finally, we also have

$$H_m^{(\pm)}(z) \sim e^{\pm \mathrm{i}z} \sum_{j=0}^{\infty} \frac{C_{mj}^{\pm}}{z^{j+1/2}} \quad \text{as } z \to +\infty. \tag{2.9}$$

Substituting (2.7) and (2.9) in (2.8), letting $m = 0$ there, and invoking $u = n + 1/2$, the result now follows. We leave the details to the interested reader.

It is now also clear that in the series $F_P(x)$ and $F_Q(x)$ above, $P_n(x)$ and $Q_n(x)$ can be replaced by $P_n^\mu(x)$ and $Q_n^\mu(x)$, μ being an arbitrary real number.

2.4. *Fourier-Bessel series*

$$F(x) := \sum_{n=1}^{\infty} [d_n J_\nu(\lambda_n x) + e_n Y_\nu(\lambda_n x)], \quad 0 < x \le r, \text{ some } r, \tag{2.10}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions of order $\nu \geq 0$ of the first and second kinds respectively, and λ_n are scalars satisfying (2.3). Normally, such λ_n result from boundary value problems involving the Bessel equation

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \left(\lambda^2 x - \frac{\nu^2}{x}\right)u = 0,$$

and r and α_0 are related through $\alpha_0 r = \pi$. For example, λ_n can be the nth positive zero of $J_\nu(z)$ or of $J_\nu'(z)$ or of some linear combination of them. In these cases, for all ν, $\alpha_0 = \pi$ in (2.3).

We now show that $\phi_n(x) = J_\nu(\lambda_n x)$ and $\psi_n(x) = Y_\nu(\lambda_n x)$ so that $\rho_n^\pm(x) = J_\nu(\lambda_n x) \pm \mathrm{i}\,Y_\nu(\lambda_n x)$ and $\omega = \alpha_0$ in (1.2), and $\gamma = -1/2$ in (1.3). From (2.9) and the fact that $\lambda_n \to +\infty$ as $n \to \infty$, it follows that

$$\rho_n^\pm(x) = H_\nu^{(\pm)}(\lambda_n x) \sim e^{\pm \mathrm{i}\lambda_n x} \sum_{j=0}^{\infty} \frac{C_{\nu j}^\pm}{(\lambda_n x)^{j+1/2}} \quad \text{as } n \to \infty, \tag{2.11}$$

which, when combined with the asymptotic expansion in (2.3), leads to (1.3).

3. The d-transformation

Let A_r, $r = 0, 1, 2, \ldots$, be a sequence whose limit or antilimit we denote by A. The sequence can be complex, in general. Let us define the differences of the A_r by $u_0 = A_0$ and $u_r = A_r - A_{r-1}$, $r = 1, 2, \ldots$. Thus $A_r = \sum_{j=0}^{r} u_j$, $r = 0, 1, \ldots$. (The definition of antilimit pertinent to the series we deal with in this work is given following (3.4) below.)

3.1. *Description of the d-transformation*

Pick a positive integer m and integers R_l, $l = 0, 1, \ldots$, such that $0 \leq R_0 < R_1 < R_2 < \cdots$. Denote $n = (n_1, \ldots, n_m)$, where n_k are nonnegative integers. Then the d-transformation produces a table of approximations $A_n^{(m,j)}$ to A that are defined through the linear systems

$$A_{R_l} = A_n^{(m,j)} + \sum_{k=1}^{m} (R_l + \alpha)^k (\Delta^{k-1} u_{R_l}) \sum_{i=0}^{n_k} \frac{\bar{\beta}_{ki}}{(R_l + \alpha)^i}, \quad j \leq l \leq j + N, \tag{3.1}$$

where $N = \sum_{k=1}^{m} (n_k + 1)$, and $A_n^{(m,j)}$ and the $\bar{\beta}_{ki}$ are the $N + 1$ unknowns. Here $\alpha > 0$ is arbitrary but fixed, and the $\Delta^k u_j$ are defined by $\Delta^0 u_j = u_j$, $\Delta u_j = u_{j+1} - u_j$, and $\Delta^k u_j = \Delta(\Delta^{k-1} u_j)$, $k \geq 2$.

This form of the d-transformation is a slightly modified version of the one given in [8] originally, and was given in [4] with $\alpha = 1$. It is also the most user-friendly in that it requires less input from the reader. Only the integers m and R_l are to be supplied by the user.

The d-transformation was obtained by analyzing rigorously the behavior of the "remainder" $A_r - A = \sum_{j=0}^{r} u_j - A$ as $r \to \infty$ for a very large family of sequences $\{u_r\}_{r=0}^{\infty}$ that was denoted $\tilde{B}^{(m)}$ in [8]. Roughly speaking, if $\{u_r\}_{r=0}^{\infty}$ is in $\tilde{B}^{(m)}$, then the u_r satisfy a linear homogeneous mth order difference equation of the form

$$u_r = \sum_{k=1}^{m} p_k(r) \Delta^k u_r, \tag{3.2}$$

such that $p_k(r)$ have asymptotic expansions given as

$$p_k(r) \sim r^{i_k} \sum_{i=0}^{\infty} \frac{p_{ki}}{r^i} \quad \text{as } r \to \infty, \tag{3.3}$$

i_k being integers that satisfy $i_k \leq k$, $k = 1, \ldots, m$. Theorem 2 in [8] states that for $\{u_r\}_{r=0}^{\infty}$ in $\tilde{B}^{(m)}$, under mild additional assumptions, there holds

$$A_r \sim A + \sum_{k=1}^{m} r^{j_k} (\Delta^{k-1} u_r) \sum_{i=0}^{\infty} \frac{\beta_{ki}}{r^i} \quad \text{as } r \to \infty, \tag{3.4}$$

j_k being integers that satisfy $j_k \leq k$, $k = 1, \ldots, m$. The special case of this theorem with $m = 1$ is stated and proved in detail as theorem 6.1 in [13]. This is the simplest case, and it arises, e.g., when the coefficients a_n and b_n in (1.1) have a "smooth" behavior described by (3.5) below. We shall elaborate on this in subsection 3.2 and subsection 4.1. In case the asymptotic expansion remains valid also when $\{A_r\}_{r=0}^{\infty}$ diverges, A is said to be the antilimit of $\{A_r\}_{r=0}^{\infty}$.

Let us now pick a sequence of integers R_l, $l = 0, 1, \ldots$, such that $0 \leq R_0 < R_1 < R_2 \cdots$. The equations defining the d-transformation are now obtained by truncating the infinite expansions in (3.4) at the terms $i = n_k$, collocating at $r = R_l$, $j \leq l \leq j + N$, replacing A and β_{ki} by $A_n^{(m,j)}$ and $\bar{\beta}_{ki}$, respectively. In addition, we insert α in the appropriate places. Finally, we also replace each integer j_k by its upper bound k, $k = 1, \ldots, m$, thus relieving the user of the burden to supply exact values for the j_k. As a result, the user is required to supply only the values of the integers m and R_l, $l = 0, 1, \ldots$, and a (positive) value for α. The determination of these parameters for the series treated in this work will be discussed in subsection 3.2.

We would like to note that, on account of the option that is given the user to pick the R_l, the d-transformation can be made to enjoy very favorable accuracy and stability properties compared to most other methods.

Two limiting processes for the table of $A_n^{(m,j)}$ have been discussed in [12] and subsequent work: (i) Process I, in which $n = (n_1, \ldots, n_m)$ is held fixed and $j \to \infty$, and (ii) Process II, in which j is held fixed and $n_k \to \infty$, $k = 1, \ldots, m$, simultaneously. Both theoretical results and numerical experience suggest that Process II has much better convergence properties. Therefore, we normally compute the sequence of approximations $A_n^{(m,0)}$, where $n = (\nu, \nu, \ldots, \nu)$, $\nu = 0, 1, 2, \ldots$.

When $m = 1$, $\alpha = 1$, and $R_l = l$, $l = 0, 1, \ldots$, the d-transformation reduces to the u-transformation of Levin.

A very efficient recursive algorithm that can be used for implementing the d-transformation when $m = 1$ and R_l are arbitrary has been given in [16] and has been designated the W-algorithm. For arbitrary m and R_l, a very sophisticated and efficient recursive algorithm has been proposed in [4] and denoted the $W^{(m)}$-algorithm. This algorithm reduces to the W-algorithm of [16] for $m = 1$. A user-friendly FORTRAN 77 program that implements the $W^{(m)}$-algorithm can be found in [4, appendix B]. (The use of this program for generalized Fourier series is explained in very simple terms in section 5 of the present work.) The sequence of approximants $A_n^{(m,0)}$, $n = (\nu, \nu, \dots, \nu)$, $\nu = 0, 1, 2, \dots$, that was mentioned above and that has the best convergence properties, is actually a subsequence of the one generated by this computer program. We finally note that both the W- and $W^{(m)}$-algorithms can be used in the implementation of other extrapolation procedures besides the d-transformation.

3.2. *Remarks on the correct and efficient use of the d-transformation*

As mentioned earlier, the user is to supply the values of m, R_l, $l = 0, 1, \dots$, and α. We now discuss the suitable choice of these parameters.

1. *Choice of* α: One can give α any positive value. The value of $\alpha = 1$ is the one used in the computer program of [4] and seems to be sufficient. Other values can be incorporated in this program in a very simple manner, however.
2. *Choice of the* R_l: For the (generalized) Fourier series treated in this paper the values $R_l = sl$, $l = 0, 1, \dots$, for some positive integer s, produce excellent results. When x is away from a point of singularity of the limit function, $s = 1$ is sufficient. As x is made to approach a point of singularity, we need to increase s to $2, 3, \dots$. The closer x gets to the singular point, the larger s needs to become. We would like to emphasize that this choice of the R_l is *not* ad hoc as may appear in the beginning. In fact, it has a very sound mathematical justification that is supplied following theorem 4.4 of section 4 of the present work.

   This strategy (with $\alpha = 1$) was suggested in [4] and incorporated in the computer implementation of the $W^{(m)}$-algorithm there. An almost identical strategy, in which $\alpha = 0$ and $R_l = \xi + sl$, $l = 0, 1, \dots$, where ξ is a positive integer, was originally suggested in [8]. In either case no theoretical justification was provided, however.
3. *Determination of* m: Upper bounds on the appropriate value of m can be obtained by invoking lemma 3 and corollaries 1–3 in [8], and these upper bounds coincide with the true values in most cases. (Even when they do not, the d-transformation remains effective.) We do not repeat these results here; instead, we suggest the following practical approach for the generalized Fourier series of this work:

   (a) For the computation of the series $A^{\pm}(x)$ and $B^{\pm}(x)$ in (1.4), normally pick $m = 1$. This choice is the correct one if a_n and b_n have asymptotic expansions of the form

$$a_n \text{ or } b_n \sim q^n \sum_{j=0}^{\infty} \xi_j n^{\epsilon-j} \quad \text{as } n \to \infty,\ \xi_0 \neq 0, \tag{3.5}$$

with arbitrary q and ϵ, which, in general, can be complex. (That is to say, the sequences $\{a_n \rho_n^{\pm}(x)\}_{n=0}^{\infty}$ and $\{b_n \rho_n^{\pm}(x)\}_{n=0}^{\infty}$ are in $\tilde{B}^{(1)}$ precisely, i.e., (3.2) and (3.3) hold with $m = 1$, consequently, (3.4) holds with $m = 1$.) Note that when $|q| < 1$ the series $A^{\pm}(x)$ and $B^{\pm}(x)$ converge absolutely and uniformly to functions that are analytic in the parameter q for $|q| < 1$. Let us denote those functions generically by $G(x;q)$. The functions $G(x;q)$ are singular at $|q| = 1$ and $\tilde{x} = -(\arg q)/\omega$ for $A^{+}(x)$, $B^{+}(x)$ and at $|q| = 1$ and $\tilde{x} = +(\arg q)/\omega$ for $A^{-}(x)$, $B^{-}(x)$. We assume in the sequel that, apart from a possible branch cut in the q-plane, the functions $G(x;q)$ can be continued analytically for all $|q| \geq 1$. (This assumption holds in many cases. See theorems 4.1 and 4.2 in the next section.) When $|q| > 1$ the series diverge. When $|q| = 1$ the series converge absolutely for $\operatorname{Re}(\gamma + \epsilon) < -1$, converge for $-1 \leq \operatorname{Re}(\gamma + \epsilon) < 0$ provided also that $x \neq \tilde{x}$, and diverge for $\operatorname{Re}(\gamma + \epsilon) \geq 0$ but represent generalized functions in x. Here γ is the parameter that appears in (1.3).

Now that we have discussed the singular points of $G(x;q)$, we can conclude that when $|q|$ is close to 1 and x gets close to $\tilde{x}$, it is appropriate to take $R_l = sl$, with $s = 2, 3, \ldots$, depending on the degree of proximity to the singular point.

In case a_n and b_n have a behavior more complicated than the one in (3.5), we suggest that $m \geq 2$ be used. If the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are in $\tilde{B}^{(m')}$ for some m', then $m = m'$ is true. (Note that if a_n and b_n are as in (3.5), then $m' = 1$.)

(b) In case the a_n and b_n in (1.1) are not known individually, we can apply the d-transformation directly to the series $F(x)$ with $m = 2$ at least. The choice $m = 2$ is the correct one when a_n and b_n are as in (3.5). If $m = 2$ is not sufficient, then we need to take $m = 4, 6, \ldots$, depending on how complicated (or "nonsmooth") the a_n and b_n are. If, for example, $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are in $\tilde{B}^{(m')}$ for some m', then $m = 2m'$ is true, in general.

In the introduction we mentioned that the proposed method of summing $A^{\pm}(x)$ and $B^{\pm}(x)$ separately and then forming the sum of $F(x)$ is about twice as economical in comparison with summing $F(x)$ directly. We would now like to explain this point briefly. Loosely speaking, the d-transformation "knocks out" consecutive terms from each of the asymptotic expansions in the remainder of A_r given in (3.4). For example, if $j_k = k$ in (3.4), then in $A_n^{(m,j)}$ with $n = (\nu, \ldots, \nu)$, these asymptotic expansions will hopefully start with the term $r^{-\nu-1}$. Now the number of terms needed for $A_n^{(m,0)}$ is $ms(\nu + 1) + m$. If the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are in $\tilde{B}^{(m')}$ for some m', then $m = m'$ for the series $A^{\pm}(x)$ and $B^{\pm}(x)$, while $m = 2m'$ for the series $F(x)$, the respective number of terms for the same order of accuracy being $m's(\nu+1)+m'$ and $2m'(\nu+1)+2m'$. This has been observed in many applications and will be demonstrated through the examples of section 5.

4. Convergence and stability analysis of the d-transformation for $m=1$ and $R_l = sl$

In this section we develop the convergence and stability theory of the d-transformation as it is applied to the series $A^{\pm}(x)$ and $B^{\pm}(x)$ in (1.4), with a_n and b_n there as in (3.5). As we mentioned in the previous section, for this case $m=1$. Specifically, we examine the convergence and stability of $A_n^{(1,j)}$ under Process I, i.e., when $j\to\infty$, n being held fixed.

Throughout this section we will be considering a sequence $\{A_n\}_{n=0}^{\infty}$, $A_n = \sum_{i=0}^{n} u_i$, $n=0,1,\ldots$, for which u_n has the asymptotic expansion

$$u_n \sim \zeta^n n^{\sigma} \sum_{i=0}^{\infty} e_i n^{-i} \quad \text{as } n\to\infty,\ e_0\neq 0, \zeta \text{ and } \sigma \text{ complex.} \tag{4.1}$$

The motivation for this is that the terms $a_n\rho_n^{\pm}(x)$ and $b_n\rho_n^{\pm}(x)$ of $A^{\pm}(x)$ and $B^{\pm}(x)$, respectively, have asymptotic expansions of the form given in (4.1) when a_n and b_n are as in (3.5). For example, when $A_n = \sum_{i=0}^{n} a_i\rho_i^{\pm}(x)$, i.e., A_n is the nth partial sum of $A^{\pm}(x)$, and a_n is as in (3.5), $u_n = a_n\rho_n^{\pm}(x)$ satisfies (4.1) with $\zeta = qe^{\pm i\omega x}$ and $\sigma = \gamma+\epsilon$, and $e_0 = \xi_0\delta_0^{\pm}(x)$. This follows from (1.2), (1.3), and (3.5). Note also that if $u_n = a_n\lambda^n$ with a_n independent of λ, then $\sum_{n=0}^{\infty} u_n$ becomes a power series. If a_n is as in (3.5), in addition, then u_n is as in (4.1) with $\zeta = q\lambda$ and $\sigma=\epsilon$. Hence the results of this section apply to such power series as well.

Now $\lim_{n\to\infty} A_n$ exists when (i) $|\zeta|<1$, or (ii) $|\zeta|=1$ but $\zeta\neq 1$ and $\operatorname{Re}\sigma<0$, or (iii) $\zeta=1$ and $\operatorname{Re}\sigma<-1$. Otherwise, $\lim_{n\to\infty} A_n$ does not exist. Note that in cases (i) and (ii) the sequence $\{A_n\}_{n=0}^{\infty}$ is linearly convergent, whereas in case (iii) it is logarithmically convergent. In many cases of interest, when $\lim_{n\to\infty} A_n$ does not exist, the antilimit turns out to be the analytic continuation of the function $f(\zeta)$ that is defined by $f(\zeta) = \sum_{i=0}^{\infty} u_i$ for $|\zeta|<1$, whenever this is possible. For $|\zeta|=1$ but $\zeta\neq 1$ the antilimit is also defined to be $\lim_{\tau\to 1-}\sum_{i=0}^{\infty} u_i\tau^i$.

4.1. *Asymptotic expansion of A_n for $n\to\infty$*

In order to be able to analyze the convergence of $A_n^{(1,j)}$ we need to know the nature of A_n for $n\to\infty$. This is accomplished below in theorems 4.1 and 4.2, where the antilimits are explicitly constructed in case $\lim_{n\to\infty} A_n$ does not exist.

Theorem 4.1
Let $u_n = \zeta^n n^p w(n)$, where p is a nonnegative integer and $w(n)$ is a Laplace transform given by

$$w(n) = \int_0^{\infty} e^{-nt}\varphi(t)\,dt, \tag{4.2}$$

such that $\varphi(t)$ is continuous in a neighborhood of 0 except possibly at 0, and satisfies

$$\int_0^\infty e^{-t}|\varphi(t)|\,dt < \infty; \quad \varphi(t) \sim \sum_{i=0}^\infty \mu_i t^{\eta+i-1} \quad \text{as } t \to 0+, \ \mu_0 \neq 0, \ \operatorname{Re}\eta > 0. \tag{4.3}$$

Then, for $\zeta \notin [1, \infty)$ in the complex plane, we have

$$A_n \sim A + u_n \sum_{i=0}^\infty \beta_i n^{-i} \quad \text{as } n \to \infty, \ \beta_0 \neq 0, \tag{4.4}$$

where

$$A = u_0 + \int_0^\infty \varphi(t) \left[\left(z \frac{d}{dz} \right)^p \frac{z}{1-z} \right] \bigg|_{z=\zeta e^{-t}} dt, \tag{4.5}$$

whether $\lim_{n\to\infty} A_n$ exists or not. Here A is the limit or antilimit of $\{A_n\}_{n=0}^\infty$, and is analytic in the ζ-plane cut along $[1, \infty)$.

The proof of theorem 4.1 is provided in the last paragraph of section 2 of [17]. Although the form of u_n assumed in the present theorem looks strange, u_n really satisfies (4.1) with $\sigma = p - \eta$. This can be verified by applying Watson's lemma to (4.2). The proof of $\beta_0 \neq 0$ in (4.4) is not given in [17]; it follows from lemma 2.1 in [17] and from the assumption that $\zeta \neq 1$. We leave the details to the reader.

The extra conditions on u_n that were imposed in theorem 4.1 are needed for (4.4) to hold both for $|\zeta| \leq 1$ but $\zeta \neq 1$ *and* for $|\zeta| > 1$. In order for (4.4) to hold only for $|\zeta| \leq 1$ but $\zeta \neq 1$ these conditions are not needed and (4.1) is enough as we show in theorem 4.2. As we shall see, theorem 4.1 plays an important role in the proof of theorem 4.2.

Theorem 4.2

Let u_n be as in (4.1) only. Then, for $|\zeta| \leq 1$ but $\zeta \neq 1$, A_n satisfies (4.4) whether $\lim_{n\to\infty} A_n$ exists or not. The precise antilimit of $\{A_n\}_{n=0}^\infty$ when $\lim_{n\to\infty} A_n$ does not exist will be derived in the proof below. This antilimit is also $\lim_{\tau\to 1-} \sum_{i=0}^\infty u_i \tau^i$.

Proof

When $\lim_{n\to\infty} A_n$ exists, the result is simply theorem 2.2 of [14]. We now turn to the remaining case of divergence, which prevails when $|\zeta| = 1$ but $\zeta \neq 1$ and $\operatorname{Re}\sigma \geq 0$.

Let $N = \lfloor \operatorname{Re}\sigma + 1 \rfloor$. Then N is the unique positive integer that satisfies $\operatorname{Re}\sigma < N \leq \operatorname{Re}\sigma + 1$. Define

$$v_n = \zeta^n n^\sigma \sum_{i=0}^{N-1} e_i n^{-i} \quad \text{and} \quad \hat{u}_n = u_n - v_n, \quad n = 0, 1, \ldots. \tag{4.6}$$

Obviously, $\hat{u}_n$ has the asymptotic expansion

$$\hat{u}_n \sim \zeta^n n^{\sigma-N} \sum_{i=0}^{\infty} \hat{e}_i n^{-i} \quad \text{as } n \to \infty, \tag{4.7}$$

with $\hat{e}_i = e_{N+i}$, $i = 0, 1, \ldots$. Thus, since $\operatorname{Re}(\sigma - N) < 0$, the sequence $\{\hat{A}_n\}_{n=0}^{\infty}$, where $\hat{A}_n = \sum_{i=0}^{n} \hat{u}_i$, $n = 0, 1, \ldots,$ converges. If we let $\hat{A} = \lim_{n\to\infty} \hat{A}_n$, then we have

$$\hat{A}_n \sim \hat{A} + \hat{u}_n \sum_{i=0}^{\infty} \hat{\beta}_i n^{-i} \quad \text{as } n \to \infty,\ \hat{\beta}_0 \neq 0. \tag{4.8}$$

Let us now consider $\{B_n\}_{n=0}^{\infty}$, where $B_n = A_n - \hat{A}_n = \sum_{i=0}^{n} v_i$, $n = 0, 1, \ldots$. Obviously, $\lim_{n\to\infty} B_n$ does not exist. However, theorem 4.1 applies to $\{B_n\}_{n=0}^{\infty}$ as $v_n = \zeta^n n^p w(n)$, with $p = N$, $w(n)$ as in (4.2) such that $\varphi(t) = \sum_{i=0}^{N-1} [e_i/\Gamma(\eta + i)] t^{\eta+i-1}$ and $\eta = N - \sigma$ in (4.3). Hence

$$B_n \sim B + v_n \sum_{i=0}^{\infty} \gamma_i n^{-i} \quad \text{as } n \to \infty,\ \gamma_0 \neq 0. \tag{4.9}$$

Here

$$B = v_0 + \int_0^{\infty} \varphi(t) \left[\left(z \frac{d}{dz} \right)^N \frac{z}{1-z} \right] \bigg|_{z=\zeta e^{-t}} dt. \tag{4.10}$$

Combining (4.8) and (4.9), we have

$$A_n = \hat{A}_n + B_n \sim \hat{A} + B + u_n \left(\frac{\hat{u}_n}{u_n} \sum_{i=0}^{\infty} \hat{\beta}_i n^{-i} + \frac{v_n}{u_n} \sum_{i=0}^{\infty} \gamma_i n^{-i} \right) \quad \text{as } n \to \infty. \tag{4.11}$$

A careful analysis of the term inside the parentheses reveals that this term is exactly of the form $\sum_{i=0}^{\infty} \beta_i n^{-i}$ as $n \to \infty$, with $\beta_0 = \gamma_0 \neq 0$. As a result, (4.4) holds, with $A = \hat{A} + B$ as the antilimit. The rest of the proof is left to the reader. □

4.2. *Convergence and stability results*

In the following lemma we give a closed form expression for $A_n^{(m,j)}$ when $m = 1$, $n_1 = \nu$, and $R_l = sl$, $l = 0, 1, \ldots$. Throughout the remainder of this section we denote this $A_n^{(m,j)}$ by $W_\nu^{(j)}$ for short.

Lemma 4.1
$W_\nu^{(j)}$ can be expressed in the form

$$W_\nu^{(j)} = \frac{\Delta^{\nu+1} \left[\dfrac{(sj+\alpha)^{\nu-1}}{u_{sj}} A_{sj} \right]}{\Delta^{\nu+1} \left[\dfrac{(sj+\alpha)^{\nu-1}}{u_{sj}} \right]}, \tag{4.12}$$

where the forward difference operator Δ operates on j.

Proof
The defining equations in (3.1) with $m = 1$ and $n_1 = \nu$ take on the simple form

$$A_{R_l} = W_\nu^{(j)} + (R_l + \alpha) u_{R_l} \sum_{i=0}^{\nu} \bar{\beta}_i (R_l + \alpha)^{-i}, \quad j \le l \le j + \nu + 1. \tag{4.13}$$

These equations can be expressed also as

$$\frac{(R_l + \alpha)^{\nu-1} A_{R_l}}{u_{R_l}} = W_\nu^{(j)} \frac{(R_l + \alpha)^{\nu-1}}{u_{R_l}} + Q(R_l), \quad j \le l \le j + \nu + 1, \tag{4.14}$$

where $Q(r) = \sum_{i=0}^{\nu} \bar{\beta}_i (r + \alpha)^{\nu - i}$ is a polynomial in r of degree at most ν. Substituting now $R_l = sl$ in (4.14), we obtain

$$\frac{(sl + \alpha)^{\nu-1} A_{sl}}{u_{sl}} = W_\nu^{(j)} \frac{(sl + \alpha)^{\nu-1}}{u_{sl}} + Q(sl), \quad j \le l \le j + \nu + 1. \tag{4.15}$$

Observe that $Q(sl)$ is a polynomial in l of degree at most ν. Hence, $\Delta^{\nu+1} Q(sj) = 0$, and the result in (4.12) follows from theorem 2 in [15]. □

We now state a result concerning the convergence of Process I for $m = 1$ in the d-transformation.

Theorem 4.3
Let $A_n = \sum_{i=0}^{n} u_i$, $n = 0, 1, \ldots$, where u_n satisfies (4.1) with $\zeta \ne 1$. In case $|\zeta| > 1$ in (4.1), assume that u_n satisfies the additional conditions of theorem 4.1. Thus, A_n satisfies (4.4), A there being the limit or the appropriate antilimit of $\{A_n\}_{n=0}^{\infty}$. Reexpress (4.4) in the form

$$A_n \sim A + u_n \sum_{i=0}^{\infty} \tilde{\beta}_i (n + \alpha)^{-i} \quad \text{as } n \to \infty. \tag{4.16}$$

Obviously, $\tilde{\beta}_0 = \beta_0 \ne 0$. Let μ be the smallest nonnegative integer for which $\tilde{\beta}_{\nu+\mu} \ne 0$. Then, whether $\lim_{n\to\infty} A_n$ exists or not, we have

$$W_\nu^{(j)} - A = z^j j^{\sigma - 2\nu - \mu - 1} [D_\nu + O(j^{-1})] \quad \text{as } j \to \infty, \tag{4.17}$$

where

$$z = \zeta^s \quad \text{and} \quad D_\nu = e_0 \tilde{\beta}_{\nu+\mu} (\mu + 1)_{\nu+1} s^{\sigma - \nu - \mu} \left(\frac{z}{z-1}\right)^{\nu+1}, \tag{4.18}$$

provided also that $z \ne 1$. Consequently,

$$\frac{W_\nu^{(j)} - A}{A_{s(j+\nu+1)} - A} \sim K j^{-2\nu - \mu - 1} \quad \text{as } j \to \infty, \text{ for some } K \ne 0. \tag{4.19}$$

Proof

Subtracting A from both sides of (4.12), we have

$$W_\nu^{(j)} - A = \frac{\Delta^{\nu+1}\left[\dfrac{(sj+\alpha)^{\nu-1}}{u_{sj}}(A_{sj}-A)\right]}{\Delta^{\nu+1}\left[\dfrac{(sj+\alpha)^{\nu-1}}{u_{sj}}\right]} \equiv \frac{\text{Num}}{\text{Den}}. \tag{4.20}$$

Invoking (4.16), we obtain for the numerator

$$\text{Num} \sim \Delta^{\nu+1}\left[\sum_{i=0}^{\infty} \tilde{\beta}_i (sj+\alpha)^{\nu-i-1}\right] \quad \text{as } j \to \infty. \tag{4.21}$$

By $\Delta^{\nu+1}\left[\sum_{i=0}^{\nu-1} \tilde{\beta}_i (sj+\alpha)^{\nu-i-1}\right] = 0$, (4.21) becomes

$$\text{Num} \sim \Delta^{\nu+1}\left[\sum_{i=\nu}^{\infty} \tilde{\beta}_i (sj+\alpha)^{\nu-i-1}\right] \quad \text{as } j \to \infty. \tag{4.22}$$

Since $\tilde{\beta}_{\nu+\mu}$ is the first nonzero $\tilde{\beta}_i$ for $i \geq \nu$, then

$$\begin{aligned}\text{Num} &\sim \tilde{\beta}_{\nu+\mu}\Delta^{\nu+1}(sj+\alpha)^{-\mu-1}\\ &\sim \{(-1)^{\nu+1}(\mu+1)_{\nu+1}s^{-\mu-1}\tilde{\beta}_{\nu+\mu}\} j^{-\nu-\mu-2} \quad \text{as } j \to \infty.\end{aligned} \tag{4.23}$$

As for the denominator, we have

$$\text{Den} = \sum_{i=0}^{\nu+1} (-1)^{\nu+1-i}\binom{\nu+1}{i}\frac{(sj+si+\alpha)^{\nu-1}}{u_{s(j+i)}}, \tag{4.24}$$

which, upon invoking (4.1), and taking into account only the most dominant terms, becomes

$$\begin{aligned}\text{Den} &\sim \sum_{i=0}^{\nu+1} (-1)^{\nu+1-i}\binom{\nu+1}{i}\frac{(sj)^{\nu-1}}{\zeta^{sj+si}(sj)^{\sigma}e_0} \quad \text{as } j \to \infty,\\ &\sim \frac{1}{e_0}(sj)^{\nu-1-\sigma}\zeta^{-sj}(\zeta^{-s}-1)^{\nu+1} \quad \text{as } j \to \infty.\end{aligned} \tag{4.25}$$

The result in (4.17) and (4.18) now follows by combining (4.23) and (4.25) in (4.20). The rest of the proof is simple and is left to the reader. □

It is clear from (4.17) that $W_\nu^{(j)} \to A$ as $j \to \infty$ provided either (i) $|\zeta| < 1$ or (ii) $|\zeta| = 1$ but $\zeta \neq 1$ and $2\nu > \operatorname{Re}(\sigma - \mu - 1)$ even when $\lim_{n\to\infty} A_n$ does not exist. For $|\zeta| > 1$, however, $W_\nu^{(j)}$ is unbounded as $j \to \infty$. Also, (4.19) implies that when $\{A_n\}_{n=0}^{\infty}$ converges $W_\nu^{(j)}$ converges even faster, and when $\{A_n\}_{n=0}^{\infty}$ diverges $W_\nu^{(j)}$ either converges or diverges but more slowly than $\{A_n\}_{n=0}^{\infty}$. Note that $A_{s(j+\nu+1)}$ in (4.19) is the A_r with the largest index that is being used in the computation of $W_\nu^{(j)}$.

With the convergence question settled, we next turn to that of stability.

As explained in [12], the $W_\nu^{(j)}$ can be expressed in the general form

$$W_\nu^{(j)} = \sum_{i=0}^{\nu+1} \gamma_{\nu,i}^{(j)} A_{s(j+i)}, \quad \sum_{i=0}^{\nu+1} \gamma_{\nu,i}^{(j)} = 1. \tag{4.26}$$

Furthermore, the stability of $W_\nu^{(j)}$ is determined by the quantity $\Gamma_\nu^{(j)}$, where

$$\Gamma_\nu^{(j)} = \sum_{i=0}^{\nu+1} |\gamma_{\nu,i}^{(j)}| \geq 1. \tag{4.27}$$

The larger $\Gamma_\nu^{(j)}$, the less stable $W_\nu^{(j)}$ is numerically. For a more detailed discussion of this point and the significance of $\Gamma_\nu^{(j)}$ see [12, section 6]. Specifically, we have

$$\gamma_{\nu,i}^{(j)} = \frac{(-1)^{\nu+1-i}\binom{\nu+1}{i}\dfrac{(sj+si+\alpha)^{\nu-1}}{u_{s(j+i)}}}{\Delta^{\nu+1}\left[\dfrac{(sj+\alpha)^{\nu-1}}{u_{sj}}\right]}, \quad 0 \leq i \leq \nu+1, \tag{4.28}$$

and, therefore,

$$\Gamma_\nu^{(j)} = \frac{\displaystyle\sum_{i=0}^{\nu+1} \binom{\nu+1}{i}\frac{(sj+si+\alpha)^{\nu-1}}{|u_{s(j+i)}|}}{\left|\displaystyle\sum_{i=0}^{\nu+1} (-1)^{\nu+1-i}\binom{\nu+1}{i}\frac{(sj+si+\alpha)^{\nu-1}}{u_{s(j+i)}}\right|}. \tag{4.29}$$

Theorem 4.4

Under the conditions of theorem 4.3, and with the notation therein, Process I is stable in the sense that

$$\Gamma_\nu^{(j)} \sim \left|\frac{|z|+1}{z-1}\right|^{\nu+1} < \infty \quad \text{as } j \to \infty. \tag{4.30}$$

Proof

As follows from (4.25), the denominator of (4.29) is asymptotically equivalent to

$$|e_0|^{-1}|(sj)^{\nu-1-\sigma}||z|^{-j}|z^{-1}-1|^{\nu+1} \quad \text{as } j \to \infty.$$

Similarly, the numerator of (4.29) is asymptotically equivalent to

$$|e_0|^{-1}|(sj)^{\nu-1-\sigma}||z|^{-j}(|z|^{-1}+1)^{\nu+1} \quad \text{as } j \to \infty,$$

and this result is obtained precisely as that in (4.25). The rest follows easily. □

Remarks

(1) We have carried out the convergence and stability analysis above for the "user-friendly" version of the d-transformation, in which the j_k in (3.4) have been

replaced by their upper bounds k, $k = 1, \ldots, m$. In our case $j_1 = 0$ by (4.4), so that we can also have u_{R_l} instead of $(R_l + \alpha)u_{R_l}$ in (4.13). Then $(sj + \alpha)^{\nu-1}$ in (4.12) of lemma 4.1 is to be replaced by $(sj + \alpha)^{\nu}$. Despite this change, theorem 4.3 remains essentially the same, with minor modifications in (4.17) and (4.18), which now become, respectively,

$$W_\nu^{(j)} - A = z^j j^{\sigma-2\nu-\mu-2}[D_\nu + O(j^{-1})] \quad \text{as } j \to \infty \tag{4.17$'$}$$

and

$$z = \zeta^s \quad \text{and} \quad D_\nu = e_0 \tilde{\beta}_{\nu+\mu+1}(\mu+1)_{\nu+1} s^{\sigma-\nu-\mu-1}\left(\frac{z}{z-1}\right)^{\nu+1}, \tag{4.18$'$}$$

where μ is the smallest nonnegative integer for which $\tilde{\beta}_{\nu+\mu+1} \neq 0$ this time. Consequently, (4.19) becomes

$$\frac{W_\nu^{(j)} - A}{A_{s(j+\nu+1)} - A} \sim Kj^{-2\nu-\mu-2} \quad \text{as } j \to \infty, \text{ for some } K \neq 0. \tag{4.19$'$}$$

Thus, a small improvement may take place, e.g., if $\tilde{\beta}_\nu \neq 0$ and $\tilde{\beta}_{\nu+1} \neq 0$. Theorem 4.4 remains the same under the change mentioned above. Numerically, both versions of the d-transformation behave very similarly.

(2) Let $u_0', u_1', \ldots,$ be such that u_n', for $n \to \infty$, has an asymptotic expansion of the form (4.1), with the same ζ and σ, but possibly different e_i's there, and replace u_{sj} in (4.12) of lemma 4.1 by u_{sj}' while keeping $A_{sj} = \sum_{i=0}^{sj} u_i$ as before. Then theorem 4.3 remains the same, this time the $\tilde{\beta}_i$ being the coefficients in the expansion

$$A_n \sim A + u_n' \sum_{i=0}^{\infty} \tilde{\beta}_i (n+\alpha)^{-i} \quad \text{as } n \to \infty, \tag{4.16$'$}$$

that replaces (4.16). Similarly, theorem 4.4 remains the same. In addition, the previous remark remains valid.

We now turn to the theoretical justification of the choice $R_l = sl$ in the acceleration procedure. First, as we have already mentioned, the stability of $W_\nu^{(j)}$ is determined by $\Gamma_\nu^{(j)}$. As $\Gamma_\nu^{(j)}$ becomes large, $W_\nu^{(j)}$ becomes less stable. Therefore, we should aim at keeping $\Gamma_\nu^{(j)}$ close to 1, its lowest possible value. Now, from (4.30), $\lim_{j\to\infty} \Gamma_\nu^{(j)}$ is proportional to $1/|z-1|^{\nu+1}$, which, from $z = \zeta^s$, is unbounded for $\zeta \to 1$. Thus, if we keep s fixed, $s = 1$ say, and let ζ get too close to 1, there will be numerical instabilities in acceleration. (Recall that for $m = 1$ and $s = 1$ in $R_l = sl$, the d-transformation is simply the u-transformation.) On the other hand, if we increase s, we cause $z = \zeta^s$ to separate from 1 in modulus and/or in phase, thus causing $\Gamma_\nu^{(j)}$ to stay bounded. This provides the theoretical justification for the introduction of the integer s in the choice of the R_l. We can even see that, as we approach the point of singularity, if we keep $z = \zeta^s$ approximately fixed by increasing s gradually, we can maintain an almost uniform level of accuracy.

In addition to $\Gamma_\nu^{(j)}$, also D_ν, the coefficient of the leading term in the error $W_\nu^{(j)} - A$ in (4.17), gives a good indication as to whether the acceleration is effective or not. Surprisingly, D_ν, just like $\lim_{j\to\infty}\Gamma_\nu^{(j)}$, is proportional to $1/|z-1|^{\nu+1}$ as can be seen from (4.18). Again, when ζ is close to 1, we can cause D_ν to stay bounded by increasing s. It is thus very interesting that by forcing the acceleration process to become stable numerically, we are also preserving the quality of the theoretical error $W_\nu^{(j)} - A$.

Note

The convergence and stability results of theorems 4.3 and 4.4 concern only Process I, in which ν is held fixed and $j\to\infty$. Results for Process II, in which j is held fixed and $\nu\to\infty$, can be obtained as is done in [14, section 4] for the Levin transformations. The techniques for doing this are quite involved, however, and we shall not pursue this matter here. In many numerical experiments we have observed that near points of singularity $\lim_{\nu\to\infty}\Gamma_\nu^{(j)}$ seems to remain bounded and small for increasing values of s. For small values of s, however, $\Gamma_\nu^{(j)}$ assumes extremely large values as ν increases.

5. Numerical examples

We now demonstrate the effectiveness of the present approach with three numerical examples involving real Fourier cosine and Fourier-Legendre series, i.e., $F(x) := \sum_{n=0}^{\infty} a_n\phi_n(x)$, with $\phi_n(x) = \cos n\omega x$ and $\phi_n(x) = P_n(\cos x)$, respectively. As we mentioned earlier, in these cases $F(x) = \operatorname{Re} A^+(x)$, where $A^+(x) := \sum_{n=0}^{\infty} a_n\rho_n^+(x)$.

We have applied the d-transformation to the complex series $A^+(x)$ precisely along the lines described in section 3. If we let $A_n^{(m,j)}$ be the approximations obtained by applying the d-transformation to $A^+(x)$, then $\operatorname{Re} A_n^{(m,j)}$ are approximations to the limit or antilimit of $F(x)$, which we shall denote $\hat{F}(x)$. In our examples we have computed the sequences $L_\nu^{(m)}(x;s) := \operatorname{Re} A_{(\nu,\nu,\ldots,\nu)}^{(m,0)}$, $\nu = 0,1,\ldots,$ by letting $R_l = sl$, for some positive integer $s\geq 1$, in the d-transformation. In all the tables

$$\Delta_\nu(m,s) := \left|\frac{L_\nu^{(m)}(x;s) - \hat{F}(x)}{\hat{F}(x)}\right|, \quad \text{when } \hat{F}(x)\neq 0. \tag{5.1}$$

We have similarly applied the d-transformation to the real series $F(x)$. We have computed the approximations $L_\nu^{(m)}(x;s) := A_{(\nu,\nu,\ldots,\nu)}^{(m,0)}$, $\nu = 0,1,\ldots,$ by letting $R_l = sl$, for some positive integer $s\geq 1$, in the d-transformation. The relative errors $\Delta_\nu(m,s)$ for the real series are also defined as in (5.1) with the appropriate $L_\nu^{(m)}(x;s)$.

Finally, we have also applied Homeier's method to the real Fourier series. In this case $\Delta_\nu^{\text{Homeier}}$ denotes the relative error in the approximation $s_0^{(\nu+1)}$ of [5] obtained from the first $2(\nu+1)+1$ terms of the series, and is analogous to $\Delta_\nu(2,1)$ for the d-transformation as applied to real Fourier series with $m=2$ and $s=1$, using the first $2(\nu+1)+2$ terms of the series.

We have used the computer program of [4, appendix B] to implement the d-transformation for all values of m. For the new approach that involves complex arithmetic this program needs to be modified slightly by the addition of suitable type declaration statements. Note that in the second PARAMETER statement of the listing of the program the integer constant INCR is to be set equal to our integer s, while the real constant SIGMA must be kept equal to $1D0$. The integer constant M in the same statement is simply our m, and the integer constant LMAX there is the index l of the largest R_l that is being used in the extrapolation. The relevant approximations are stored in the two-dimensional array APPROX, the sequence APPROX (0,I), $\mathrm{I}=1,2,\ldots$, containing $\{A^{(m,0)}_{(\nu,\nu,\ldots,\nu)}\}_{\nu=0}^{\infty}$ as a subsequence. In fact, APPROX $(0, \mathrm{M}*(\nu+1)) = A^{(m,0)}_{(\nu,\nu,\ldots,\nu)}$, $\nu=0,1,\ldots$.

We must also warn the reader that the $W^{(m)}$-algorithm, just like the ϵ- and θ-algorithms, the u-transformation and further non-linear algorithms, cannot handle zero terms in the series $\sum c_n$. This is due to the fact that the recursions in the $W^{(m)}$-algorithm involve division by the c_n. Occasional zero terms do not, however, affect the numerical solution by Gaussian elimination of the linear equations (3.1) that define the d-transformation, in general.

All computations reported here were done in extended precision arithmetic (approximately 33 decimal digits) on an IBM-370 computer at the Computer Center of the Technion.

Example 1

$$F(x) := -2\sum_{j=1}^{\infty}\frac{q^j}{j}\cos jx = \log(1-2q\cos x+q^2), \quad |q|\le 1,\ q\ne e^{\pm\mathrm{i}x}. \tag{5.2}$$

The limit function is singular for $q=e^{\pm\mathrm{i}x}$.

In the notation of section 1, we have $a_n=-2q^{n+1}/(n+1)$, $\phi_n(x)=\cos(n+1)x$, $\psi_n(x)=\sin(n+1)x$, hence $\rho_n^{\pm}(x)=e^{\pm\mathrm{i}(n+1)x}$. This example has been used in [5] in the comparison of the ϵ-algorithm with the method proposed there.

Obviously, the sequence $\{a_n\}_{n=0}^{\infty}$ satisfies (3.5), so that the complex sequence $\{a_n\rho_n^{\pm}(x)\}_{n=0}^{\infty}$ is in $\tilde{B}^{(1)}$, while the real sequence $\{a_n\phi_n(x)\}_{n=0}^{\infty}$ is in $\tilde{B}^{(2)}$. Thus the d-transformation is applicable to the former with $m=1$, and to the latter with $m=2$. In our numerical experiments below we have fixed the parameter q so that $q=1$.

In table 1 we compare the performances of the d-transformation (with $m=2$ and $R_l=l$) with the method of Homeier. Both methods are being applied to the *real* series $F(x)$ for $x=\pi/3.1$. It is clear that the results produced by the two methods are very similar as we mentioned earlier. We also include the results obtained by applying the d-transformation (with $m=1$ and $R_l=l$, i.e., the u-transformation) to the *complex* series $A^{+}(x)$ and taking the real parts of the

Table 1
Relative errors in approximations for example 1 from Homeier's method, the d-transformation with $m = 2$ and $R_l = l$ on real series, and the d-transformation with $m = 1$ and $R_l = l$ on complex series (new approach) for $x = \pi/3.1$. Number of terms used for each entry is $2(\nu+1)+1$ for the first column, $2(\nu+2)$ for the second column, and $\nu+2$ for the third column. (We have taken $q = 1$.)

ν	$\Delta_\nu^{\text{Homeier}}$ on real series	$\Delta_\nu(2,1)$ on real series	$\Delta_\nu(1,1)$ on complex series
3	3.4D − 02	2.7D − 01	1.6D − 02
7	6.7D − 06	3.8D − 06	1.2D − 05
11	6.1D − 10	4.0D − 09	3.4D − 09
15	7.7D − 13	2.3D − 13	9.6D − 13
19	2.1D − 16	5.6D − 17	1.4D − 15
23	7.5D − 21	1.7D − 20	6.2D − 19
27	1.2D − 21	2.0D − 22	1.5D − 23
31	5.4D − 20	2.2D − 17	1.5D − 25
35	1.2D − 18	1.4D − 18	2.2D − 27
39	6.7D − 17	9.6D − 17	4.1D − 27

results. It is seen that this way of applying the d-transformation requires about half of the coefficients a_n of the series $F(x)$ as compared to the former approaches.

A similar comparison is brought in table 2 for $x = \pi/6.1$. This time the d-transformation is being applied with $R_l = 2l$. The superiority of the d-transformation is clearly demonstrated. (This x is relatively close to 0, the point of singularity of the limit function.) Again the new approach is the most economical and accurate. Note that the method of Homeier achieves a limited accuracy which eventually is destroyed by the addition of more terms of the series.

Yet another comparison is provided in table 3 for $x = \pi/50.1$, which is very close to 0, the point of singularity of the limit function. Now the d-transformation is

Table 2
Relative errors in approximations for example 1 from Homeier's method, the d-transformation with $m = 2$ and $R_l = 2l$ on real series, and the d-transformation with $m = 1$ and $R_l = 2l$ on complex series (new approach) for $x = \pi/6.1$. Number of terms used for each entry is $2(\nu+1)+1$ for the first column, $2[2(\nu+1)+1]$ for the second column, and $2(\nu+1)+1$ for the third column. (We have taken $q = 1$.)

ν	$\Delta_\nu^{\text{Homeier}}$ on real series	$\Delta_\nu(2,2)$ on real series	$\Delta_\nu(1,2)$ on complex series
3	5.2D − 02	8.8D − 03	2.6D − 03
7	5.8D − 05	1.1D − 06	7.4D − 07
11	4.4D − 06	1.6D − 10	1.1D − 10
15	8.8D − 08	1.2D − 14	2.4D − 13
19	1.2D − 09	3.4D − 17	1.2D − 16
23	1.1D − 11	1.5D − 21	2.3D − 20
27	3.5D − 09	1.5D − 23	1.3D − 23
31	3.0D − 05	8.9D − 21	1.4D − 26
35	2.3D − 01	2.2D − 19	1.5D − 28

Table 3
Relative errors in approximations for example 1 from Homeier's method, the d-transformation with $m = 2$ and $R_l = 10l$ on real series, and the d-transformation with $m = 1$ and $R_l = 10l$ on complex series (new approach) for $x = \pi/50.1$. Number of terms used for each entry is $2(\nu + 1) + 1$ for the first column, $2[10(\nu + 1) + 1]$ for the second column, and $10(\nu + 1) + 1$ for the third column. (We have taken $q = 1$.)

ν	$\Delta_\nu^{\text{Homeier}}$ on real series	$\Delta_\nu(2, 10)$ on real series	$\Delta_\nu(1, 10)$ on complex series
3	3.8D + 03	2.2D − 02	6.3D − 03
7	2.0D + 05	9.8D − 06	1.2D − 05
11	1.4D + 07	6.9D − 08	2.1D − 08
15	1.4D + 09	1.2D − 10	3.4D − 11
19	2.5D + 03	3.0D − 10	4.1D − 14
23	1.7D − 01	1.2D − 14	1.0D − 17
27	1.8D + 00	2.7D − 13	1.6D − 19
31	3.9D + 00		7.3D − 22

being applied with $R_l = 10l$. In this case Homeier's method does not produce any accuracy, while the d-transformation on the real series with $m = 2$ gives 14 correct significant figures. The d-transformation on the complex series with $m = 1$ produces 21 correct significant figures.

Finally, the d-transformation remains effective also for $|q| < 1$ and $|q| > 1$, the conclusions above being the same.

Example 2

$$F_P(x) := \sum_{n=0}^{\infty} \frac{P_n(x)}{(1 - 2n)(2n + 3)} = \frac{1}{2}\sqrt{\frac{1 - x}{2}}, \quad -1 \le x \le 1. \tag{5.3}$$

The limit function is singular at $x = 1$.

With $\theta = \cos^{-1} x$, we can write $F_P(x) := \sum_{n=0}^{\infty} a_n \phi_n(\theta)$, where $a_n = 1/[(1 - 2n)(2n + 3)]$ and $\phi_n(\theta) = P_n(x)$. As was shown in section 2, $\psi_n(\theta) = -(2/\pi)Q_n(x)$ in this case, hence $\rho_n^{\pm}(\theta) = P_n(x) \mp \mathrm{i}(2/\pi)Q_n(x)$. This example has been used in [8] to demonstrate the use of the d-transformation.

Since the coefficients a_n are smooth in the sense of (3.5), the sequence $\{a_n \rho_n^{\pm}(\theta)\}_{n=0}^{\infty}$ is in $\tilde{B}^{(1)}$, while $\{a_n P_n(x)\}_{n=0}^{\infty}$ is in $\tilde{B}^{(2)}$. Thus the d-transformation is applicable to the former with $m = 1$, and to the latter with $m = 2$.

Tables 4 and 5 compare the performances of the d-transformation (with $m = 2$) on the real series $F_P(x) := \sum_{n=0}^{\infty} a_n P_n(x)$ and the d-transformation (with $m = 1$) on the complex series $A^+(\theta) := \sum_{n=0}^{\infty} a_n[P_n(x) - \mathrm{i}(2/\pi)Q_n(x)]$ for, respectively, $x = 0.5$ and $x = 0.95$. We have taken $R_l = l$ for $x = 0.5$ and $R_l = 5l$ for $x = 0.95$. Note that $x = 0.95$ is very close to the point of singularity. As before, the conclusion is that the new approach achieves a given level of accuracy with about half the number of coefficients a_n of the series $F(x)$ as compared to the other approach.

Table 4
Relative errors in approximations for example 2 from the d-transformation with $m = 2$ and $R_l = l$ on real series and the d-transformation with $m = 1$ and $R_l = l$ on complex series (new approach) for $x = 0.5$. Number of terms used for each entry is $2(\nu + 2)$ for the first column and $\nu + 2$ for the second column.

ν	$\Delta_\nu(2,1)$ on real series	$\Delta_\nu(1,1)$ on complex series
3	2.0D − 04	5.1D − 05
7	7.4D − 10	4.4D − 08
11	4.2D − 12	3.2D − 11
15	1.1D − 15	1.3D − 14
19	1.5D − 18	2.7D − 18
23	3.9D − 23	8.5D − 22
27	1.7D − 23	1.1D − 24
31		5.6D − 28
35		2.3D − 29

Example 3

$$G(\theta,\beta) := \sum_{n=0}^{\infty} \cos\left((n+\tfrac{1}{2})\beta\right)P_n(\cos\theta)$$

$$= \begin{cases} [2(\cos\beta - \cos\theta)]^{-1/2}, & 0 \le \beta < \theta < \pi, \\ 0, & 0 < \theta < \beta \le \pi. \end{cases} \tag{5.4}$$

The limit function is singular for $\beta = \theta$.

We treat $G(\theta,\beta)$ as a Fourier-Legendre series $F_P(x) := \sum_{n=0}^{\infty} a_n P_n(x)$, with $x = \cos\theta$, and $a_n = \cos(n + 1/2)\beta$. The sequences $\{a_n \rho_n^{\pm}(\theta)\}_{n=0}^{\infty}$ and $\{a_n P_n(x)\}_{n=0}^{\infty}$ are in $\tilde{B}^{(2)}$ and $\tilde{B}^{(4)}$, respectively. Thus the d-transformation is applicable to the former with $m = 2$ and to the latter with $m = 4$. This example too has been used in [8].

Table 5
Relative errors in approximations for example 2 from the d-transformation with $m = 2$ and $R_l = 5l$ on real series and the d-transformation with $m = 1$ and $R_l = 5l$ on complex series (new approach) for $x = 0.95$. Number of terms used for each entry is $2[5(\nu + 1) + 1]$ for the first column and $5(\nu + 1) + 1$ for the second column.

ν	$\Delta_\nu(2,5)$ on real series	$\Delta_\nu(1,5)$ on complex series
3	2.0D − 05	1.1D − 04
7	1.0D − 10	4.5D − 09
11	2.1D − 16	1.2D − 12
15	8.7D − 21	5.3D − 17
19	7.7D − 26	1.3D − 20
23	3.4D − 32	6.7D − 25
27	2.1D − 31	1.4D − 28
31		2.1D − 31

Table 6
Relative errors in approximations for example 3 from the d-transformation with $m=4$ and $R_l = l$ on real series and the d-transformation with $m=2$ and $R_l = l$ on complex series (new approach) for $\beta = \pi/6$ and $\phi = 2\pi/3$. Number of terms used for each entry is $4(\nu+2)$ for the first column and $2(\nu+2)$ for the second column.

ν	$\Delta_\nu(4,1)$ on real series	$\Delta_\nu(2,1)$ on complex series
3	3.8D − 03	3.4D − 03
7	1.2D − 05	1.7D − 06
11	8.4D − 11	1.7D − 09
15	2.6D − 16	2.4D − 14
19	1.3D − 20	2.9D − 17
23	4.8D − 25	3.4D − 20
27	4.8D − 29	1.4D − 23
31	1.2D − 32	2.0D − 27
35	6.4D − 34	1.8D − 30

Tables 6 and 7 compare the performances of the d-transformation (with $m=4$) on the real series $F_P(x)$ and the d-transformation (with $m=2$) on the complex series $A^+(\theta)$ for, respectively, (i) $\beta = \pi/6$ and $\theta = 2\pi/3$ and (ii) $\beta = 0.6\pi$ and $\theta = 2\pi/3$. In the first case we have taken $R_l = l$ while in the second $R_l = 10l$. Note that for β and θ as in (ii), the limit function is very close to being singular. Again, the conclusion is that the new approach achieves a given accuracy level with about half the number of coefficients a_n of the series $F_P(x)$ as compared to the other approach.

We can also treat $G(\theta, \beta)$ as a Fourier series of the form $\sum_{n=0}^{\infty}(a_n \cos n\beta + b_n \sin n\beta)$, where $a_n = P_n(\cos\theta)\cos(\beta/2)$ and $b_n = -P_n(\cos\theta)\sin(\beta/2)$. Again, the approach of the present paper can be successfully applied to this series; i.e.,

Table 7
Relative errors in approximations for example 3 from the d-transformation with $m=4$ and $R_l = 10l$ on real series and the d-transformation with $m=2$ and $R_l = 10l$ on complex series (new approach) for $\beta = 0.6\pi$ and $\phi = 2\pi/3$. Number of terms used for each entry is $4[10(\nu+1)+1]$ for the first column and $2[10(\nu+1)+1]$ for the second column.

ν	$\Delta_\nu(4,10)$ on real series	$\Delta_\nu(2,10)$ on complex series
3	2.2D − 05	8.0D − 04
7	5.8D − 08	2.3D − 08
11	2.0D − 10	9.7D − 13
15	3.1D − 12	3.9D − 17
19	6.7D − 15	5.8D − 19
23	2.1D − 17	1.0D − 21
27	9.5D − 18	1.6D − 24
31	3.7D − 18	1.1D − 27
35		1.0D − 30
39		5.7D − 33

the d-transformation can be applied to the appropriate complex Fourier series with $m = 2$. The method of Homeier, however, fails completely for this case as the coefficients a_n and b_n are not smooth.

6. Concluding remarks

We have given a novel approach to the important problem of accelerating the convergence of classical Fourier series and their common generalizations. In this approach the d-transformation, which is a very effective nonlinear sequence transformation, is applied not to the given series, but to suitable extensions of it. The resulting method possesses three significant properties: (i) For a given level of accuracy there are great savings in the number of coefficients of the series actually used in the acceleration procedure compared with application of the d-transformation directly to the original series, (ii) very accurate and stable approximations are obtained also near points of singularity of the limit function, and (iii) in many cases the new method produces considerably higher accuracy by the addition of more terms of the series than does the direct approach. All of these properties have been verified numerically for several problems.

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