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CONVERGENCE ANALYSIS FOR A GENERALIZED RICHARDSON EXTRAPOLATION PROCESS WITH AN APPLICATION TO THE $d^{(1)}$-TRANSFORMATION ON CONVERGENT AND DIVERGENT LOGARITHMIC SEQUENCES

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ABSTRACT. In an earlier work by the author the Generalized Richardson Extrapolation Process (GREP) was introduced and some of its convergence and stability properties were discussed. In a more recent work by the author a special case of GREP, which we now call $\mathrm{GREP}^{(1)}$, was considered and its properties were reviewed with emphasis on oscillatory sequences. In the first part of the present work we give a detailed convergence and stability analysis of $\mathrm{GREP}^{(1)}$ as it applies to a large class of logarithmic sequences, both convergent and divergent. In particular, we prove several theorems concerning the columns and the diagonals of the corresponding extrapolation table. These theorems are very realistic in the sense that they explain the remarkable efficiency of $\mathrm{GREP}^{(1)}$ in a very precise manner. In the second part we apply this analysis to the Levin-Sidi $d^{(1)}$-transformation, as the latter is used with a new strategy to accelerate the convergence of infinite series that converge logarithmically, or to sum the divergent extensions of such series. This is made possible by the observation that, when the proper analogy is drawn, the $d^{(1)}$-transformation is, in fact, a $\mathrm{GREP}^{(1)}$. We append numerical examples that demonstrate the theory.

1. INTRODUCTION

In [13] the author introduced a generalization of the well-known Richardson extrapolation process and discussed some of its convergence and stability properties. This generalization—called GREP for short—has proved to be very useful in accelerating the convergence of a large class of infinite sequences with varying degrees of complexity in their behavior. Such sequences arise naturally in the computation of infinite series and infinite integrals that may be oscillatory or monotonic, or that may behave in a more complicated manner. They also arise from trapezoidal rule approximations of finite-range simple or multiple integrals of regular or singular functions, etc. In addition, these sequences may be convergent or divergent. For a brief survey and areas of application, see also [17].

The sequences for which GREP is useful arise from, and are identified with, functions $A(y)$ that belong to some general sets that were defined in [13] and denoted there by $F^{(m)}$, m being a positive integer.

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The simplest case of GREP that is applicable to sequences identified with functions in $F^{(1)}$ was considered by the author in [16], and will be called GREP $^{(1)}$ in the present work. In [16] an efficient recursive technique—the W-algorithm—for the implementation of GREP $^{(1)}$ was proposed, and some of the convergence and stability properties of GREP$^{(1)}$ were summarized with more emphasis on *oscillatory* sequences. Such sequences arise, e.g., in the computation of convergent or divergent (very) oscillatory infinite integrals (see, e.g., [18] and [20]). Recently, a very economical recursive implementation for GREP, as it applies to sequences that arise from functions in $F^{(m)}$, with arbitrary m, was proposed in [4], and denoted the $W^{(m)}$-algorithm. For $m = 1$, the $W^{(m)}$-algorithm reduces exactly to the W-algorithm. A FORTRAN 77 program that implements the $W^{(m)}$-algorithm is included in the appendix of [4].

In the present work we would like to continue our study of GREP $^{(1)}$ in the context of *logarithmically convergent* sequences and their divergent extensions that are associated with functions in $F^{(1)}$. In this connection we note that several results pertaining to the T-transformation of Levin [7] have already been published by the author in [14] and [15]. Some of these have been reviewed recently in [19], see also [2, p. 116]. (We recall that the T-transformation is a GREP $^{(1)}$, and that the t-, u-, and v-transformations are particular cases of it.) The results of the present work, however, are totally different from those given in [14] and [15], and so are the analytical techniques leading to them.

We start by giving the descriptions of the set $F^{(1)}$ and of the accompanying extrapolation method GREP $^{(1)}$. This is done in Definitions 1.1 and 1.2, respectively, which also establish some of the notation that we use throughout this paper.

Definition 1.1. We shall say that a function $A(y)$, defined for $0 < y \le b$, for some $b > 0$, where y can be a discrete or continuous variable, belongs to the set $F^{(1)}$, if there exist functions $\phi(y)$ and $\beta(y)$ and a constant A such that

$$A = A(y) + \phi(y)\beta(y), \tag{1.1}$$

where $\beta(\xi)$, as a function of the continuous variable ξ, is continuous for $0 \le \xi \le b$, and, for some constant $r > 0$, has a Poincaré-type asymptotic expansion of the form

$$\beta(\xi) \sim \sum_{i=0}^{\infty} \beta_i \xi^{ir} \quad \text{as } \xi \to 0+. \tag{1.2}$$

If, in addition, the function $B(t) \equiv \beta(t^{1/r})$, as a function of the continuous variable t, is infinitely differentiable for $0 \le t \le b^r$, we shall say that $A(y)$ belongs to the set $F_{\infty}^{(1)}$. Note that $F_{\infty}^{(1)} \subset F^{(1)}$.

Remark. We have $A = \lim_{y\to 0+} A(y)$ whenever this limit exists, in which case $\lim_{y\to 0+} \phi(y) = 0$. If $\lim_{y\to 0+} A(y)$ does not exist, then A is said to be the antilimit of $A(y)$. In this case, $\lim_{y\to 0+} \phi(y)$ does not exist, as is obvious from (1.1) and (1.2).

It is assumed that the functions $A(y)$ and $\phi(y)$ are computable for $0 < y \le b$ (keeping in mind that y may be discrete or continuous depending on the situation) and that the constant r is known. The constants A and β_i are not assumed to be known. In attempting to accelerate the convergence of a sequence that can be identified with $A(y)$, the idea, thus the problem, is to find (or approximate) A whether it is the limit or the antilimit of $A(y)$ as $y \to 0+$, and GREP $^{(1)}$, the extrapolation procedure that corresponds to $F^{(1)}$, is designed to tackle precisely this problem. The β_i are not required in most cases of interest, although GREP $^{(1)}$ produces approximations (usually not very good ones) to them as well.

Definition 1.2. Let $A(y) \in F^{(1)}$, with $\phi(y)$, $\beta(y)$, A, and r being as in Definition 1.1. Pick $y_l \in (0, b]$, $l = 0, 1, 2, \ldots$, such that $y_0 > y_1 > y_2 > \cdots$, and $\lim_{l\to\infty} y_l = 0$. Then A_n^j, the approximation to A, and the parameters $\overline{\beta}_i$, $i = 0, 1, \ldots, n-1$, are defined to be the solution of the system of $n+1$ linear equations

$$A_n^j = A(y_l) + \phi(y_l)\sum_{i=0}^{n-1} \overline{\beta}_i y_l^{ir}, \quad j \le l \le j+n, \tag{1.3}$$

provided the matrix of this system is nonsingular. It is this process that generates the approximations A_n^j that we call GREP $^{(1)}$.

As is seen, GREP $^{(1)}$ produces a two-dimensional table of approximations of the form

$$\begin{array}{ccccc} n=0 & n=1 & n=2 & n=3 & \cdots \\ A_0^0 & & & & \\ A_0^1 & A_1^0 & & & \\ A_0^2 & A_1^1 & A_2^0 & & \\ A_0^3 & A_1^2 & A_2^1 & A_3^0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}, \quad A_0^j \equiv A(y_j), \ j = 0, 1, \ldots. \tag{1.4}$$

Numerical experiments and the theory that exists for some cases suggest that when $\lim_{y\to 0+} A(y)$ exists, the columns of this table converge, each column converging at least as quickly as those preceding it, while the diagonals converge more quickly than the columns.

Going down a column corresponds to letting $j \to \infty$ while n is being held fixed in A_n^j, and this limiting process was called Process I in [13]. Going along a diagonal corresponds to letting $n \to \infty$ while j is being held fixed in A_n^j, and this limiting process was called Process II in [13].

Before going on, we shall let $t = y^r$ and $t_l = y_l^r$, $l = 0, 1, \ldots$, and define $a(t) \equiv A(y)$ and $\varphi(t) \equiv \phi(y)$. Then the equations in (1.3) take on the more convenient form

$$A_n^j = a(t_l) + \varphi(t_l)\sum_{i=0}^{n-1} \overline{\beta}_i t_l^i, \quad j \le l \le j+n. \tag{1.3'}$$

A closed-form expression for A_n^j is given by the following theorem.

Theorem 1.1. *Let D_k^s denote the divided difference operator of order k over the set of points $t_s, t_{s+1}, \ldots, t_{s+k}$, where, for any function $g(t)$ defined at these* *points,*

(1.5)

$$D_k^s\{g(t)\} = g[t_s, t_{s+1}, \dots, t_{s+k}] = \sum_{l=s}^{s+k} \left(\prod_{\substack{i=s \\ i\neq l}}^{s+k} \frac{1}{t_l - t_i} \right) g(t_l) \equiv \sum_{i=0}^{k} c_{k,i}^s g(t_{s+i}).$$

Then A_n^j *is given by*

$$A_n^j = \frac{D_n^j\{a(t)/\varphi(t)\}}{D_n^j\{1/\varphi(t)\}}. \tag{1.6}$$

The result in (1.6) is used in obtaining the W-algorithm for the efficient recursive computation of the A_n^j. This algorithm is summarized in Theorem 1.2 below. In the present work we make use of (1.6) also in the analysis of A_n^j, where it proves to be a rather powerful tool.

Theorem 1.2 (The W-algorithm). *Let*

$$M_0^s = a(t_s)/\varphi(t_s), \quad N_0^s = 1/\varphi(t_s), \quad s = 0, 1, 2, \dots, \tag{1.7}$$

and define recursively

(1.8)

$$M_k^s = \frac{M_{k-1}^{s+1} - M_{k-1}^s}{t_{s+k} - t_s}, \quad N_k^s = \frac{N_{k-1}^{s+1} - N_{k-1}^s}{t_{s+k} - t_s}, \quad s = 0, 1, \dots, \quad k = 1, 2, \dots.$$

Then

$$A_k^s = \frac{M_k^s}{N_k^s}, \quad s = 0, 1, \dots, \quad k = 0, 1, \dots. \tag{1.9}$$

For all these developments and the proofs of Theorems 1.1 and 1.2 we refer the reader to [16]. We only mention that the notation of the present work is slightly different from that used in [16]. For instance, the A_n^j of the present work are related to the $A_n^{(j)}$ of [16] through $A_n^j = A_{n-1}^{(j)}$.

When $\varphi(t) = t$, GREP $^{(1)}$ reduces to the classical Richardson extrapolation process that has been analyzed thoroughly in [6] and [3]. As follows from this analysis, and as is observed numerically, this process is quite unstable when the t_l approach 0 slowly, e.g., $t_l = O(l^{-1})$ as $l \to \infty$, but is very stable and accurate when $t_{l+1}/t_l \le \omega$ for some fixed $\omega \in (0, 1)$, i.e., when the t_l approach 0 at least exponentially. This suggests that, whenever feasible computationally, we should prefer the choice $t_{l+1}/t_l \le \omega$, $\omega \in (0, 1)$.

The purpose of the present work is to carry out a detailed convergence and stability analysis for GREP $^{(1)}$ in the presence of functions $\varphi(t)$ that are more complicated than $\varphi(t) = t$, again with the choice $t_{l+1}/t_l \le \omega$, $\omega \in (0, 1)$ (or another similar one). The $\varphi(t)$ that we will concern ourselves with behave essentially like t^δ as $t \to 0+$ for some $\delta \ne 0, -1, -2, \dots,$ and they arise naturally in a large class of logarithmically convergent sequences and their divergent extensions. It seems that these divergent extensions have not been treated elsewhere before.

The plan of this paper is as follows.

In §2 we provide a complete convergence and stability analysis for GREP$^{(1)}$ under Process I with the condition $\lim_{l\to\infty}(t_{l+1}/t_l) = \omega, \ \omega \in (0, 1)$. The main results of this section are Theorem 2.1 on convergence and Theorem 2.2 on stability.

In §3 we derive upper bounds for the error in GREP$^{(1)}$ under Process II with the condition $t_{l+1}/t_l \leq \omega, \ \omega \in (0, 1)$. From these bounds we obtain a powerful convergence result very similar to those of [6] and [3]. In addition, we provide theoretical and numerical stability analyses. The latter can be carried out simultaneously with the computation of the A_n^j, also by the W-algorithm, and at no extra cost. The main results of this section are Theorem 3.1 and its corollary on convergence, and Theorems 3.2 and 3.3 on stability.

Section 4 is devoted to the acceleration of convergence by the Levin-Sidi $d^{(1)}$-transformation of some infinite series $\sum_{n=1}^{\infty} a_n$, whose terms a_n behave essentially like $n^{-\delta-1}$ for $n \to \infty$, where $\delta \neq 0, -1, -2, \ldots$, but δ is arbitrary otherwise. These series converge for $\operatorname{Re}\delta > 0$, and diverge otherwise. If we denote $S_n = \sum_{i=1}^{n} a_i, \ n = 1, 2, \ldots$, and $S = \lim_{n\to\infty} S_n$ in case of convergence, then (see, e.g., [15])

$$S_n \sim S + na_n(\beta_0' + \beta_1' n^{-1} + \beta_2' n^{-2} + \cdots) \quad \text{as } n \to \infty. \tag{1.10}$$

The first main result of §4 is Theorem 4.1, which says that (1.10) holds for some well-defined antilimit S also when $\lim_{n\to\infty} S_n$ does not exist. The theorem actually gives S exactly. In many cases, S turns out to be a function that is analytic in the parameter δ, and thus the $d^{(1)}$-transformation proves to be an effective tool for analytic continuation of S in δ to regions in the δ-plane where $\sum_{n=1}^{\infty} a_n$ diverges, within the limits of finite precision arithmetic. It seems that extrapolation methods have not been employed for such applications before. The reason for this may be that the existence of an antilimit and its meaning for divergent series of logarithmic type was not understood properly. By letting $y = n^{-1}$, $A(y) \equiv S_{y^{-1}} = S_n$, and $\phi(y) \equiv y^{-1}a_{y^{-1}} = na_n$, we see that $A(y) \in F^{(1)}$, y being a discrete variable. Similarly, the $d^{(1)}$-transformation is shown to be a GREP$^{(1)}$. Finally, it is shown that all the results of §§2 and 3 apply directly to the $d^{(1)}$-transformation when this is implemented using a strategy that was first proposed in [4] for use with logarithmic sequences. This strategy has been observed to be extremely stable and accurate, and has proved to be the best in all examples done by the author. In many cases, where the sequence $\{S_n\}_{n=1}^{\infty}$ converges or diverges mildly, the $d^{(1)}$-transformation in conjunction with this strategy seems to be capable of producing approximations to S that are correct almost to machine accuracy.

In §5 we give some numerical examples that support the results of §§2, 3, and 4. These include both convergent and divergent series of the type discussed in §4, and their convergence is accelerated by the $d^{(1)}$-transformation.

Finally, we note that the $d^{(1)}$-transformation is the simplest form of the $d^{(m)}$-transformation of Levin and Sidi that was developed in [8]. The $d^{(m)}$-transformation, by way of its construction, is capable of accelerating the convergence of a very large class of sequences with great success, and has a larger scope than most other acceleration methods. Being a GREP itself, it can be implemented very efficiently by the $W^{(m)}$-algorithm of [4]. In the recent paper [22] the $d^{(m)}$-transformation was compared with various other convergence acceleration methods as these are applied to some class of logarithmically convergent sequences. For all cases treated in [22] the $d^{(m)}$-transformation was observed to give very stable and accurate results. See also [5] , where an extension is proposed.

2. THEORY FOR PROCESS I : n FIXED, $j \to \infty$

Even though $\varphi(t)$ may be a complicated-looking function in general, for many logarithmically convergent sequences that arise in practical problems its most dominant behavior for $t \to 0+$ is quite simple. A commonly occurring behavior is t^δ for some δ. For this and even for some more complicated behavior of $\varphi(t)$ we are able to give a precise *quantitative* analysis of Process I when the t_l are suitably chosen. This analysis is based on some of the results of the recent paper [19] by the author. See also [2, p. 68].

2.1. Convergence analysis of Process I.

Theorem 2.1. *Pick the t_l in GREP$^{(1)}$ to satisfy*

$$\lim_{l\to\infty} \frac{t_{l+1}}{t_l} = \omega \text{ for some } \omega \in (0, 1). \tag{2.1}$$

Assume that $\varphi(t)$ is such that

$$\lim_{l\to\infty} \frac{\varphi(t_{l+1})}{\varphi(t_l)} = \omega^\delta \text{ for some (complex) } \delta \neq 0, -1, -2, \dots, \tag{2.2}$$

and define

$$b_k = \omega^{\delta+k-1}, \quad k = 1, 2, \dots . \tag{2.3}$$

Then, whether $\lim_{l\to\infty} a(t_l)$ exists or not, we have

$$A - A_n^j \sim \beta_{n+\mu} \left[\prod_{i=1}^{n} \left(\frac{b_{n+\mu+1} - b_i}{1 - b_i}\right)\right] \varphi(t_j) t_j^{n+\mu} \text{ as } j \to \infty, \tag{2.4}$$

where $\beta_{n+\mu}$ is the first nonzero β_i for $i \geq n$.

Proof. Defining $\varphi_k(t) = \varphi(t)t^{k-1}$ and $\alpha_k = -\beta_{k-1}$, $k = 1, 2, \dots,$ we can rewrite (1.1) in the form

$$a(t) \sim A + \sum_{k=1}^{\infty} \alpha_k \varphi_k(t) \text{ as } t \to 0+ . \tag{2.5}$$

From (2.1) and (2.2) we also have

$$\lim_{l\to\infty} \frac{\varphi_k(t_{l+1})}{\varphi_k(t_l)} = \omega^{\delta+k-1} = b_k, \quad k = 1, 2, \dots . \tag{2.6}$$

By the assumption on δ, we have $b_k \neq 1$ for all k. Also, $\lim_{k\to\infty} b_k = 0$, and $|b_1| > |b_2| > \cdots$, so that the b_k are distinct. Consequently, a slightly generalized form of Theorem 2.2 in [19] applies, and we obtain (2.4). We leave the details to the reader. □

Remarks. (1) Combining (2.2) with (2.4), we see that

$$\limsup_{j\to\infty} \left|A - A_n^j\right|^{1/j} \le \left|b_{n+\mu+1}\right| = \omega^{\mathrm{Re}\,\delta+n+\mu}, \tag{2.7}$$

from which we also have

$$A - A_n^j = O((\omega+\epsilon)^{(\mathrm{Re}\,\delta+n+\mu)j}) \quad \text{as } j\to\infty, \tag{2.8}$$

where $\epsilon > 0$ is arbitrarily close to 0.

(2) Now $\lim_{t\to 0+} a(t)$ exists if $\mathrm{Re}\,\delta > 0$. If $\mathrm{Re}\,\delta \le 0$, however, $\lim_{t\to 0+} a(t)$ does not exist when $\beta_0 \ne 0$. In case the limit exists, all columns of the table in (1.4) converge, each column converging at least as quickly as the ones preceding it. When $\mathrm{Re}\,\delta \le 0$ and $\delta \ne 0, -1, -2, \ldots$, all the columns in (1.4) with $n = n_0, n_0+1, n_0+2, \ldots$, where $n_0 = \lfloor -\mathrm{Re}\,\delta + 1\rfloor$, converge, each one converging more quickly than the ones preceding it. The columns with $0 \le n \le n_0 - 1$ may diverge. If a column diverges, it diverges at most as quickly as the column preceding it. If $\beta_m \ne 0$, but $\beta_{m+1} = \cdots = \beta_{s-1} = 0$, and $\beta_s \ne 0$, then we have

$$\begin{aligned} &A - A_p^j = o(A - A_m^j) \quad \text{as } j\to\infty, \quad m+1 \le p \le s, \\ &A - A_p^j \sim \theta_p(A - A_s^j) \quad \text{as } j\to\infty, \quad m+1\le p\le s-1, \quad \text{some } \theta_p, \\ &A - A_{s+1}^j = o(A - A_s^j) \quad \text{as } j\to\infty. \end{aligned} \tag{2.9}$$

(3) Concerning the condition in (2.2), the important point to realize is that $\lim_{l\to\infty} \varphi(t_{l+1})/\varphi(t_l) = K$ is assumed to exist. With K defined, we now determine $\delta = \log K/\log\omega$. Finally, the condition in (2.2) is satisfied, e.g., when

$$\varphi(t) \sim \rho|\log t|^\nu t^\delta \quad \text{as } t\to 0+, \quad \rho, \nu \text{ and } \delta \text{ complex}, \quad \delta \ne 0, -1, -2, \ldots. \tag{2.10}$$

Note. The proof of Theorem 2.1 of this work was achieved by employing Theorem 2.2 of [19]. This result concerns the acceleration of convergence under Process I of the generalized Richardson extrapolation process for a function $a(t)$ that satisfies (2.5) with $\lim_{l\to\infty} \varphi_k(t_{l+1})/\varphi_k(t_l) = b_k$, and $b_k \ne 1$, and $b_k \ne b_j$ if $k \ne j$. With these conditions, this result is asymptotically best possible for $j\to\infty$. Recently another result for Process I, with different assumptions on the $\varphi_k(t)$ has been given in [9, Theorem 3]. It is interesting to note that this result too applies to the case treated in [19], see [9, Example 1], but produces a much weaker theorem than [19, Theorem 2.2].

2.2. **Stability analysis of Process I.** With the problem of convergence resolved, we now go on to tackle that of stability. We recall that A_n^j can be expressed in the form

$$A_n^j = \sum_{i=0}^n \gamma_{n,i}^j a(t_{j+i}), \tag{2.11}$$

with

$$\sum_{i=0}^n \gamma_{n,i}^j = 1. \tag{2.12}$$

The exact expression for $\gamma_{n,i}^j$ is not very crucial at this point. What is important to realize is that under the conditions of Theorem 2.1 we can employ Theorem 2.4 of [19] to conclude that Process I is stable in the sense that

$$\sup_j \sum_{i=0}^{n} |\gamma_{n,i}^j| < \infty. \tag{2.13}$$

Actually, we can state a much more precise result as follows:

Theorem 2.2. *Under the conditions of Theorem* 2.1 *and with the notation therein, we have*

$$\lim_{j\to\infty} \gamma_{n,i}^j = \tilde{\gamma}_{n,i}, \quad i = 0, 1, \dots, n, \tag{2.14}$$

where the $\tilde{\gamma}_{n,i}$ *are defined by*

$$\prod_{i=1}^{n} \left(\frac{\lambda - b_i}{1 - b_i}\right) = \sum_{i=0}^{n} \tilde{\gamma}_{n,i}\lambda^i. \tag{2.15}$$

Consequently, (2.13) *holds. Furthermore, if* δ *is real, then*

$$\lim_{j\to\infty} \sum_{i=0}^{n} |\gamma_{n,i}^j| = \prod_{i=1}^{n} \frac{1 + b_i}{|1 - b_i|} = \prod_{i=1}^{n} \frac{1 + \omega^{\delta+i-1}}{|1 - \omega^{\delta+i-1}|}, \tag{2.16}$$

and if δ *is complex, then*

$$\lim_{j\to\infty} \sum_{i=0}^{n} |\gamma_{n,i}^j| \le \prod_{i=1}^{n} \frac{1 + |b_i|}{|1 - b_i|} = \prod_{i=1}^{n} \frac{1 + \omega^{\mathrm{Re}\,\delta+i-1}}{|1 - \omega^{\delta+i-1}|}. \tag{2.17}$$

Proof. The relations (2.15) and (2.16) are direct consequences of Theorem 2.4 and its corollary in [19]. The proof of (2.17) is similar to that of (2.16). □

As is well known, when computations are done in finite precision arithmetic, the accuracy and stability of the computed A_n^j (call them $\bar{A}_n^j$), as opposed to the exact A_n^j, is dictated by $\Gamma_n^j \equiv \sum_{i=0}^{n} |\gamma_{n,i}^j|$, in the sense that

$$|A_n^j - \bar{A}_n^j| \le \Gamma_n^j \left(\max_{j\le i\le j+n} |\epsilon_i|\right), \tag{2.18}$$

where ϵ_i is the error in $A(y_i)$. Therefore, for an extrapolation procedure to be reliable, the associated Γ_n^j should stay bounded, or at most should increase mildly, with increasing j in Process I and with increasing n in Process II.

3. Theory for Process II: j fixed, $n \to \infty$

We noted in §1 that Process II has a much better convergence behavior than Process I. Yet Process II has always proved to be much more difficult to analyze. Normally, in order to obtain results that can truly explain the numerically observed behavior of Process II, we have to assume more about the function $\varphi(t)$ than we do for Process I. For example, an asymptotic condition such as (2.10) (or, more generally, (2.2)) that is *local* in nature will not be very helpful. The reason for this is that Process II is based on information coming from the interval $(0, t_j]$ (see the defining equations in (1.3) and (1.3′)), and this interval is *fixed* as j is held fixed. This implies that we need to specify a *global* condition on $\varphi(t)$, valid in $(0, t_j]$. Simple, yet realistic, global conditions satisfied by $\varphi(t)$ in many cases of interest will be given in Lemmas 3.4 and 3.5 below. (For Process I, on the other hand, the information comes from the points $t_j, t_{j+1}, \ldots, t_{j+n}$, and since $j \to \infty$, hence $t_j \to 0+$, the information comes from a shrinking (right) neighborhood of $t = 0$. This explains why (2.2) is sufficient for obtaining the optimal result of (2.4).)

3.1. **Convergence analysis of Process II.** We start by deriving an error expression for A_n^j.

Lemma 3.1. *The error in* A_n^j *is given by*

$$A - A_n^j = \frac{D_n^j\{B(t)\}}{D_n^j\{1/\varphi(t)\}}, \tag{3.1}$$

where $B(t) \equiv \beta(t^{1/r})$.

Proof. The result follows from $A - A(y) = A - a(t) = \varphi(t)B(t)$, cf. (1.1), and from the linearity of the divided difference operator D_n^j. □

We now go on to investigate the numerator and denominator of (3.1) separately. We begin with the numerator.

3.1.1. *Upper bounds for the numerator of* (3.1).

Lemma 3.2. *Pick the* t_l *in GREP*$^{(1)}$ *to satisfy*

$$\frac{t_{l+1}}{t_l} \le \omega \text{ for some } \omega \in (0, 1). \tag{3.2}$$

Define the positive constants $M_n^{(j)}$ *by*

$$M_n^{(j)} = \max_{0 \le t \le t_j} \left(\left| B(t) - \sum_{i=0}^{n-1} \beta_i t^i \right| / t^n \right). \tag{3.3}$$

Then

$$|D_n^j\{B(t)\}| \le C_n M_n^{(j)} < C_\infty M_n^{(j)}, \tag{3.4}$$

where C_n *and* C_∞ *are defined by*

$$C_n = \prod_{i=1}^{n} \frac{1 + \omega^i}{1 - \omega^i}, \quad n = 1, 2, \ldots; \quad C_\infty = \lim_{n \to \infty} C_n. \tag{3.5}$$

Proof. The proof of (3.4) and (3.5) is quite involved, but can be done by extending and refining the analyses of [6] and [3]. We leave the details to the interested reader. □

We now give a result that is similar to (3.4) but does not impose any conditions on the t_l, such as (3.2). As we will see, the proof of this result is much simpler than that of (3.4).

Lemma 3.3. *Let* $A(y) \in F_\infty^{(1)}$, *cf. Definition* 1.1. *This implies that the function* $B(t)$ *is infinitely differentiable for* $0 \le t \le b^r$. *Define the positive constants* $R_n^{(j)}$

by

$$R_n^{(j)} = \frac{1}{n!} \max_{0\le t\le t_j} |B^{(n)}(t)|, \tag{3.6}$$

where $B^{(n)}(t)$ *denotes the* nth *derivative of* $B(t)$. *Then*

$$\left|D_n^j\{B(t)\}\right| \le R_n^{(j)}. \tag{3.7}$$

Proof. The result follows from the fact that

$$|D_n^j\{g(t)\}| = |g[t_j, t_{j+1}, \dots, t_{j+n}]| \le \frac{1}{n!} \max_{t_{j+n}\le t\le t_j} |g^{(n)}(t)|, \tag{3.8}$$

whenever $g(t)$ is in general complex and at least n times continuously differentiable on $[t_{j+n}, t_j]$. The inequality in (3.8) is a consequence of the Hermite-Gennochi formula stated as Lemma A.1 in the appendix to this work. □

Note that when $B(t)$ is infinitely differentiable on $[0, b^r]$, the constants $M_n^{(j)}$ and $R_n^{(j)}$, defined in (3.3) and (3.6), respectively, seem to be approximately of the same order of magnitude. They have the common lower bound $|\beta_n| = |B^{(n)}(0)|/n!$, and satisfy $M_n^{(j)} \le R_n^{(j)}$ as well.

3.1.2. *Lower bounds for the denominator of* (3.1). We now turn to the analysis of the denominator of (3.1), namely, $D_n^j\{1/\varphi(t)\}$.

First of all, we would like to note the exact result

$$D_n^j\{t^{-1}\} = (-1)^n/(t_j t_{j+1} \cdots t_{j+n}) \tag{3.9}$$

which can be proved by induction. (Actually, (3.9) holds with no restrictions on the t_l.) Thus, when $\varphi(t) = t$, combining Lemma 3.2 and (3.9), we have

$$|A - A_n^j| \le C_n M_n^{(j)} (t_j t_{j+1} \cdots t_{j+n}), \tag{3.10}$$

which is the well-known result of [6] and [3] for the classical Richardson extrapolation. This result is especially powerful when we invoke the condition (3.2) in the product $\prod_{l=j}^{j+n} t_l$, which therefore satisfies

$$\prod_{l=j}^{j+n} t_l \le t_j^{n+1} \omega^{n(n+1)/2}, \tag{3.11}$$

and hence tends to 0 extremely quickly (practically like $\omega^{n^2/2}$) as $n \to \infty$. As a result, the combination of (3.10) and (3.11) gives an excellent explanation of the quick convergence of A_n^j when (3.2) is satisfied and $\varphi(t) = t$.

It is observed numerically in many cases in which $\varphi(t) \sim t^\delta$ as $t \to 0+$ for some (complex) $\delta \ne 0, -1, -2, \dots$, that the convergence behavior of A_n^j to A, under the condition (3.2), depends on δ and is practically independent of what exactly $\varphi(t)$ is, and is very similar to that implied by (3.10) for $\varphi(t) = t$. A theoretical result similar to (3.10) for the general $\varphi(t)$ mentioned above, under the condition (3.2), does not seem to be known, however. The only result known to the author in this connection is one given in [3] for $\varphi(t) = t^\delta$, $\delta > 0$, when *equality* holds in (3.2), and this result is very similar to (3.10). The analysis of A_n^j as $n \to \infty$ for general $\varphi(t)$ and/or under the condition in (3.2) seems to have posed a serious problem in the past. In the context and developments of the present work, the source of this problem seems to be the difficulty in analyzing $D_n^j\{1/\varphi(t)\}$ for general $\varphi(t)$ and under the condition in (3.2). As we shall see below, the knowledge that D_n^j is a divided difference operator helps in tackling this problem effectively in many cases.

Going back to $D_n^j\{1/\varphi(t)\}$, we see that a simple closed-form expression for it that is similar to (3.9) is practically impossible to obtain. We therefore aim at obtaining either its dominant asymptotic behavior for $n \to \infty$ or a good lower bound for it, both of which will, in essence, behave like the product $t_j t_{j+1} \cdots t_{j+n}$ for $n \to \infty$. It turns out that this is possible when suitable conditions are imposed on $\varphi(t)$. In Lemmas 3.4 and 3.5 below we present this approach with realistic conditions on $\varphi(t)$ which are indeed met in many common applications involving logarithmically convergent sequences and their divergent extensions. These lemmas are based on the various developments in the appendix to this work, and turn out to be crucial in Theorems 3.1 and 3.2 on convergence and stability. We believe that the contents of the appendix are of importance and interest in themselves and may form the basis for further developments.

Lemma 3.4. *Let* $\varphi(t) = t^\delta h(t)$, *where* δ *and* $h(t)$ *are in general complex,* $\delta \neq 0, -1, -2, \ldots$, *and* $h(t)$ *is infinitely differentiable and nonzero on* $[0, t_j]$ *and satisfies* $\max_{0\le t\le t_j} |h^{(k)}(t)| \le K(pk)!\rho^k k^\theta$, $k = 0, 1, 2, \ldots$, *for some* K, p, ρ, *and* θ. *Pick* t_l, $l = 0, 1, \ldots$, *to satisfy the condition in* (3.2). *Then, provided that either*

(i) δ *is a positive integer, or*

(ii) δ *is real but not an integer, and* $g(t) = 1/h(t)$ *is a polynomial, or*

(iii) δ *is real but not an integer, and* $g(t) = 1/h(t)$ *is a completely monotonic function on* $[0, t_j]$, *or*

(iv) δ *is complex,* $g(t) = 1/h(t)$ *is a polynomial, and equality holds in* (3.2), *we have*

$$D_n^j\{1/\varphi(t)\} = Q_n^{(j)} D_n^j\{t^{-\delta}\}, \tag{3.12}$$

with $Q_n^{(j)} \sim g(0) = 1/h(0)$ *as* $n \to \infty$, *independently of* j, *for* (i), (ii), *and* (iv), *and* $|Q_n^{(j)}| \ge L_n^{(j)} \sim |g(0)| = 1/\,|h(0)|$ *as* $n \to \infty$ *for* (iii). *In all cases,*

$$|D_n^j\{1/\varphi(t)\}| \ge \frac{|Q_n^{(j)}|}{|\omega^{\delta n + n(n-1)/2}\, \hat{t}_j^{\delta+n}|} \left| \prod_{i=1}^{n} \frac{1-\omega^{\delta+i-1}}{1-\omega^i} \right|, \tag{3.13}$$

where $\hat{t}_l = \omega^l t_0$, $l = 0, 1, \ldots$, *and equality holds in* (3.13) *when* $t_l = \hat{t}_l$, $l = 0, 1, \ldots$. *If* $\varphi(t) \equiv t^\delta$, *then* $Q_n^{(j)} = 1$ *in all cases.*

The results in Lemma 3.4 follow from Lemmas A.6–A.8. An important point to note is that the constants $|Q_n^{(j)}|$ are bounded below by a *positive* constant *independent* of n. This implies that $|D_n^j\{1/\varphi(t)\}|$ tends to infinity as $n \to \infty$ practically at the rate $\omega^{-n^2/2}$, which is what we, in fact, wanted to establish.

Note that for the cases (ii), (iii), and (iv) of Lemma 3.4, in which δ is not a positive integer, we need to impose extra conditions on the function $h(t)$. By imposing different conditions on $h(t)$ we are able to obtain a result of a more general nature but weaker than those given in Lemma 3.4. This is done in Lemma 3.5.

Lemma 3.5. *Let $\varphi(t)$ be complex in general, and infinitely differentiable and nonzero on $(0, t_j]$. Define $\psi(t) = 1/\varphi(t)$, and assume that $\psi^{(n)}(t)$ is nonzero on $(0, t_j]$ for all large n, and let*

$$L_n^{(j)} = \left[\min_{t_{j+n}\le t\le t_j} |\mathrm{Re}\, G_n(t)|^2 + \min_{t_{j+n}\le t\le t_j} |\mathrm{Im}\, G_n(t)|^2\right]^{\frac{1}{2}}, \tag{3.14}$$

where

$$G_n(t) = \psi^{(n)}(t)/\Delta^{(n)}(t); \quad \Delta(t) = t^{-\alpha}, \ \alpha \text{ real}, \ \alpha \ne 0, -1, -2, \ldots . \tag{3.15}$$

Then, for all large n,

$$|D_n^j\{1/\varphi(t)\}| \ge L_n^{(j)} |D_n^j\{t^{-\alpha}\}|. \tag{3.16}$$

If t_l, $l = 0, 1, \ldots$, also satisfy (3.2), *then*

$$|D_n^j\{1/\varphi(t)\}| \ge \frac{L_n^{(j)}}{\omega^{\alpha n + n(n-1)/2}\, \hat{t}_j^{\alpha+n}} \left|\prod_{i=1}^n \frac{1-\omega^{\alpha+i-1}}{1-\omega^i}\right|. \tag{3.17}$$

The results of Lemma 3.5 follow from Lemma A.9 and Lemma A.4. Obviously, Lemma 3.5 may be useful when $\varphi(t) = t^{-\delta} h(t)$, with $\mathrm{Re}\,\delta = \alpha$ and $h(t)$ infinitely differentiable on $[0, t_j]$, provided we have a way of bounding $L_n^{(j)}$ in (3.14) from below. This lower bound on $L_n^{(j)}$ does not have to be a constant. For the ultimate convergence theory it is enough if we can establish that at worst it goes to zero like $\rho^{n^{1+\epsilon}}$ for some $\rho \in (0, 1)$ and $\epsilon < 1$.

3.1.3. *Convergence theorem for Process* II. Combining the upper bounds for $D_n^j\{B(t)\}$ with the lower bounds for $D_n^j\{1/\varphi(t)\}$, we finally have the main result of this section concerning the convergence of Process II.

Theorem 3.1. *Pick the t_l in GREP$^{(1)}$ to satisfy the inequalities in* (3.2). *Let*

$$U_n^{(j)} = \begin{cases} R_n^{(j)} & \text{when } B(t) \in C^\infty[0, t_j], \\ C_n M_n^{(j)} & \text{otherwise}, \end{cases} \tag{3.18}$$

with $M_n^{(j)}$, C_n, and $R_n^{(j)}$ as defined in (3.3), (3.5), *and* (3.6), *respectively. Let also $\hat{t}_l = \omega^l t_0$, $l = 0, 1, \ldots$.*

(i) *Provided that $\varphi(t) = t^\delta h(t)$, with δ, $h(t)$, and t_l, $l = 0, 1, \ldots$, as in any one of the four parts of Lemma* 3.4, *we have*

(3.19)
$$|A - A_n^j| \le \frac{U_n^{(j)}}{|Q_n^{(j)}|} |D_n^j\{t^{-\delta}\}|^{-1} \le \frac{U_n^{(j)}}{|Q_n^{(j)}|} \left[\left|\prod_{i=1}^n \frac{1-\omega^i}{1-\omega^{\delta+i-1}}\right| |\hat{t}_j^{\delta+n}|\, |\omega^{\delta n}| \omega^{n(n-1)/2}\right],$$

with $Q_n^{(j)}$ as in the different parts of Lemma 3.4.

(ii) *If* $\varphi(t)$ *is as in Lemma* 3.5, *then we have*

(3.20)

$$|A - A_n^j| \le \frac{U_n^{(j)}}{L_n^{(j)}} |D_n^j\{t^{-\alpha}\}|^{-1} \le \frac{U_n^{(j)}}{L_n^{(j)}} \left[\left| \prod_{i=1}^{n} \frac{1-\omega^i}{1-\omega^{\alpha+i-1}} \right| \hat{t}_j^{\alpha+n} \omega^{\alpha n} \omega^{n(n-1)/2} \right],$$

with α *and* $L_n^{(j)}$ *as in Lemma* 3.5.

We now discuss the bounds in (3.19) and (3.20). We recall that the products $\prod_{i=1}^{n}(1-\omega^i)/(1-\omega^{\nu+i-1})$, with $\nu = \delta$ in (3.19) and with $\nu = \alpha$ in (3.20), are bounded in n since their limits for $n \to \infty$ exist. The factors $\hat{t}_j^{\delta+n}\omega^{\delta n}$ and $\hat{t}_j^{\alpha+n}\omega^{\alpha n}$ are dominated by $\omega^{n(n-1)/2}$ for $n \to \infty$. Therefore, the square brackets in (3.19) and (3.20) tend to zero practically at the rate $\omega^{n^2/2}$ as $n \to \infty$, for all δ and α. Now $|A - A_n^j|$ will tend to zero also at the rate $\omega^{n^2/2}$ provided $U_n^{(j)}/|Q_n^{(j)}|$ and $U_n^{(j)}/L_n^{(j)}$ grow with n at most like $e^{\gamma n^{1+\tau}}$ for some γ and $\tau < 1$, which may even dominate $(pn)!\rho^n n^\theta$ for arbitrary p, ρ, and θ. We already know that the $|Q_n^{(j)}|$ are bounded below by constants independent of n. We similarly expect the constant $L_n^{(j)}$ either to be bounded below by a constant independent of n or to go to zero in a mild fashion (e.g., like $e^{-\nu n}$, $\nu > 0$) as $n \to \infty$. As for the $U_n^{(j)}$, different types of behavior may occur depending on the nature of the function $B(t)$. If $B(t)$ is analytic on $[0, t_j]$, then $U_n^{(j)} = R_n^{(j)} = O(\rho^n)$ as $n \to \infty$ for some $\rho > 0$, at worst. If $B(t)$ is not analytic on $[0, t_j]$ (normally, $B(t)$ fails to be analytic at $t = 0$) but is infinitely differentiable there, then usually $U_n^{(j)} = R_n^{(j)} = O((pn)!)$ as $n \to \infty$ for some $p > 0$. Under these circumstances, $U_n^{(j)}/|Q_n^{(j)}|$ and $U_n^{(j)}/L_n^{(j)}$ may grow with n at most like $(pn)!$ for some p, and hence $|A - A_n^j|$ tends to zero as $n \to \infty$ practically at the rate $\omega^{n^2/2}$. We summarize this discussion in the following corollary to Theorem 3.1.

Corollary. *Assume that* $U_n^{(j)}/|Q_n^{(j)}|$ *or* $U_n^{(j)}/L_n^{(j)}$ *are* $O(e^{\gamma n^{1+\tau}})$ *as* $n \to \infty$ *for some* γ *and* $\tau < 1$. *Pick* $\epsilon > 0$ *such that* $\omega + \epsilon < 1$. *Then there exists a positive integer* N *for which*

$$|A - A_n^j| \le (\omega + \epsilon)^{n^2/2} \quad \textit{when } n \ge N. \tag{3.21}$$

Remarks. (1) We believe that the discussion above shows clearly that the approach that we have taken to the convergence theory of Process II is a valid one, as the accompanying results give a realistic explanation of the observed behavior of A_n^j for $n \to \infty$.

(2) Theorem 3.1 contains the known results for the cases (a) $\varphi(t) = t$, $t_{l+1}/t_l \le \omega$, and (b) $\varphi(t) = t^\delta$, $\delta > 0$, and $t_{l+1}/t_l = \omega$. The rest of the results in Theorem 3.1 seem to be entirely new.

Note. In the recent paper [11] some new results concerning Process II are provided, primarily under the conditions of [9, Theorem 3] and other additional ones. For example, Theorem 3 in [11] treats the special case of our problem, namely, that with $\varphi(t) = t$, that has already been treated in [6] and [3], under the growth condition $\beta_k = O(r^k)$ as $k \to \infty$. Clearly, this growth condition is very stringent compared to the one discussed following the statement of Theorem 3.1 in the present work. In particular, it implies that $\sum_{i=0}^{\infty}\beta_i t^i$ converges for t sufficiently close to zero, and this is not required in the present work. Theorem 4 in [11] produces an upper bound on $S - S_p^j$ under the additional condition $S_p^j \le S$. In the present work the functions $\varphi(t)$ are quite general and we do not expect $S_p^j \le S$ to be satisfied in general. In particular, when $a(t)$ is a complex function, $S_p^j \le S$ may not necessarily have a meaning.

3.2. Stability analysis of Process II.

3.2.1. *Theoretical stability analysis.* A thorough stability analysis of Process II for the case $\varphi(t) = t$ under the condition (3.2) has been provided in [6] (see also [3]). By refining their analyses, we are able to show (see notation of (2.11)) that

$$\sum_{i=0}^{n} |\gamma_{n,i}^{j}| \le C_n < C_\infty \quad \text{for all } j \text{ and } n, \tag{3.22}$$

with C_n and C_∞ precisely as in (3.5). Furthermore, when equality holds in (3.2), the first inequality in (3.22) becomes an equality. (The constants that are provided by [6] and [3] and that are analogous to C_n in (3.22) are quite complicated compared to C_n.)

Again, a thorough analysis for the case $\varphi(t) = t^\delta$, δ real and $\delta \ne 0, -1, -2, \dots,$ when equality holds in (3.2), follows from that given in [3], and it reads

$$\sum_{i=0}^{n} |\gamma_{n,i}^{j}| = \left| \prod_{i=1}^{n} \frac{1+\omega^{\delta+i-1}}{1-\omega^{\delta+i-1}} \right|. \tag{3.23}$$

We note that the case $\delta < 0$ is not considered in [3], even though their analysis can easily be extended to *all* real $\delta \ne 0, -1, -2, \dots,$ and this is what we have done to obtain (3.23).

As it turns out, we can use the technique of [3] to treat the case in which $\varphi(t) = t^\delta$ when δ is complex and equality holds in (3.2). First, we have

$$p(z) = \sum_{i=0}^{n} \gamma_{n,i}^{j} z^i = \prod_{i=1}^{n} \frac{z-\omega^{\delta+i-1}}{1-\omega^{\delta+i-1}}, \quad \text{independently of } j, \tag{3.24}$$

for *all* δ. By using the known relations between the coefficients $\gamma_{n,i}^{j}$ of $p(z)$ and its zeros $\omega^{\delta+i-1}$, after some manipulation we obtain $|\gamma_{n,i}^{j}| \le \hat{\gamma}_{n,i}$, $0 \le i \le n$, where $\sum_{i=0}^{n} \hat{\gamma}_{n,i} z^i = \prod_{i=1}^{n} \frac{z+|\omega^{\delta+i-1}|}{|1-\omega^{\delta+i-1}|}$. Letting now $z = 1$, we have

$$\sum_{i=0}^{n} |\gamma_{n,i}^{j}| \le \prod_{i=1}^{n} \frac{1+\omega^{\mathrm{Re}\,\delta+i-1}}{|1-\omega^{\delta+i-1}|} \equiv \hat{\Gamma}_n(\delta), \tag{3.25}$$

and this result seems to be new.

Since the products on the right-hand sides of (3.23) and (3.25) have finite limits for $n \to \infty$, the absolute stability of GREP$^{(1)}$ with $\varphi(t) = t^\delta$, δ in general complex and $\delta \ne 0, -1, -2, \dots,$ and $t_l = \omega^l t_0$, $l = 0, 1, \dots,$ is now established.

We now go on to derive upper bounds for $\sum_{i=0}^{n}|\gamma_{n,i}^{j}|$ when $\varphi(t)=t^{\delta}h(t)$, from which we can also obtain stability results in some cases.

Theorem 3.2. *Under the conditions of Lemma* 3.4 *and with the notation therein, we have*

$$\sum_{i=0}^{n}|\gamma_{n,i}^{j}|\leq V_{n}^{(j)}\Gamma_{n}^{j}(\delta) \tag{3.26}$$

and also

$$\sum_{i=0}^{n}|\gamma_{n,i}^{j}|\leq \tilde{V}_{n}^{(j)}\Gamma_{n}^{j}(1)\leq C_{n}\tilde{V}_{n}^{(j)}, \tag{3.27}$$

where C_n *is as defined in* (3.5),

$$\begin{aligned} V_{n}^{(j)}&=\frac{\max_{t_{j+n}\leq t\leq t_j}|1/h(t)|}{|Q_{n}^{(j)}|},\\ \tilde{V}_{n}^{(j)}&=\frac{1}{|Q_{n}^{(j)}|}\frac{|D_{n}^{j}\{t^{-1}\}|}{|D_{n}^{j}\{t^{-\delta}\}|}\max_{t_{j+n}\leq t\leq t_j}\left|\frac{t^{1-\delta}}{h(t)}\right|, \end{aligned} \tag{3.28}$$

and $\Gamma_{n}^{j}(\delta)$ *is the sum of the moduli of the* $\gamma_{n,i}^{j}$ *corresponding to the special case* $\varphi(t)=t^{\delta}$.

Proof. From (1.5) and (1.6) we first have

$$\gamma_{n,i}^{j}=\frac{1}{D_{n}^{j}\{1/\varphi(t)\}}\frac{c_{n,i}^{j}}{\varphi(t_{j+i})},\quad 0\leq i\leq n. \tag{3.29}$$

Therefore,

$$\sum_{i=0}^{n}|\gamma_{n,i}^{j}|=\frac{1}{|D_{n}^{j}\{1/\varphi(t)\}|}\sum_{i=0}^{n}\frac{|c_{n,i}^{j}|}{|\varphi(t_{j+i})|}. \tag{3.30}$$

Rewriting (3.30) in the form

$$\sum_{i=0}^{n}|\gamma_{n,i}^{j}|=\frac{|D_{n}^{j}\{t^{-\delta}\}|}{|D_{n}^{j}\{1/\varphi(t)\}|}\left[\frac{1}{|D_{n}^{j}\{t^{-\delta}\}|}\sum_{i=0}^{n}\frac{|c_{n,i}^{j}|}{|t_{j+i}^{\delta}|}\frac{1}{|h(t_{j+i})|}\right], \tag{3.31}$$

and invoking (3.12) of Lemma 3.4, we have

$$\sum_{i=0}^{n}|\gamma_{n,i}^{j}|\leq V_{n}^{(j)}\left[\frac{1}{|D_{n}^{j}\{t^{-\delta}\}|}\sum_{i=0}^{n}\frac{|c_{n,i}^{j}|}{|t_{j+i}^{\delta}|}\right], \tag{3.32}$$

the expression inside the square brackets being nothing but $\Gamma_{n}^{j}(\delta)$. From this, (3.26) follows. The proof of (3.27) can be done in a similar fashion. □

Corollary. *GREP*$^{(1)}$ *for Process* II *is stable*

(i) *when* $\delta=1$ *and the* t_l *satisfy* (3.2), *or*

(ii) *when* $\delta\neq 1$ *and is in general complex and the* t_l *satisfy* (3.2) *with equality there.*

When δ is real and the t_l satisfy (3.2), *we have, in general,*

$$\sum_{i=0}^{n} |\gamma_{n,i}^{j}| \le K n^{1-\delta} t_{j+n}^{-|1-\delta|} \quad \text{for some } K > 0 \text{ and all large } n. \tag{3.33}$$

Proof. Case (i) of the first part follows by letting $\delta = 1$ in (3.26), recalling that $\Gamma_n^j(1) \le C_n < C_\infty$, and observing that $V_n^{(j)} = O(1)$ as $n \to \infty$.

Case (ii) of the first part follows again from (3.26) by recalling that $\Gamma_n^j(\delta)$ is bounded for all n when the t_l satisfy (3.2) with equality there, both for real and complex δ.

The proof of (3.33) in the second part is achieved from (3.27) by showing that $\tilde{V}_n^{(j)} = O(n^{1-\delta} t_{j+n}^{-|1-\delta|})$ as $n \to \infty$. This, in turn, can be achieved by recalling that $|Q_n^{(j)}|$ is bounded below by a positive constant independent of n, by invoking Lemma A.10, and by a proper analysis of $|t^{1-\delta}/h(t)|$ in $[t_{j+n}, t_j]$ both for $\delta > 1$ and for $\delta < 1$. □

Remark. Although the upper bound for $\sum_{i=0}^{n} |\gamma_{n,i}^{j}|$ given in (3.33) for arbitrary real δ goes to infinity as $n \to \infty$ like $\omega^{-|1-\delta| n}$, it is not necessarily true that $\sum_{i=0}^{n} |\gamma_{n,i}^{j}| \to \infty$ as $n \to \infty$. In fact, we believe that $\sum_{i=0}^{n} |\gamma_{n,i}^{j}|$ is bounded above by a finite constant, although we do not have a proof of this at this time. Judging from (3.26), one way of proving this would be by showing that $\Gamma_n^j(\delta)$ is bounded for all n. Even this seems to be a difficult problem.

3.2.2. *Numerical assessment of stability by the W-algorithm.* Before closing this section we would like to show how the W-algorithm itself can be used to actually compute $\Gamma_n^j \equiv \sum_{i=0}^{n} |\gamma_{n,i}^{j}|$ for each j and n, at no additional cost. As will become clear soon, the computation of Γ_n^j can be done simultaneously with that of A_n^j. All of this follows from Theorem 3.3 below.

Theorem 3.3. *Define the function $P(t)$ arbitrarily for all t, except for $t_0, t_1, t_2, \ldots$, where it is defined by*

$$P(t_j) = (-1)^j / |\varphi(t_j)|, \quad j = 0, 1, 2, \ldots. \tag{3.34}$$

Then

$$\Gamma_n^j \equiv \sum_{i=0}^{n} |\gamma_{n,i}^{j}| = \frac{|D_n^j\{P(t)\}|}{|D_n^j\{1/\varphi(t)\}|}. \tag{3.35}$$

Proof. From (1.5) we first observe that $c_{n,i}^j c_{n,i+1}^j < 0$, $i = 0, 1, \ldots, n-1$. Consequently,

$$|D_n^j\{P(t)\}| = \sum_{i=0}^{n} |c_{n,i}^j| / |\varphi(t_{j+i})|. \tag{3.36}$$

The result now follows from (3.30). □

Comparing (3.35) with (1.6), we see that the computation of Γ_n^j can be done simultaneously with that of A_n^j by simply augmenting the W-algorithm of Theorem 1.2 as follows:

(1) Add to (1.7)

$$H_0^s = P(t_s) = (-1)^s/|\varphi(t_s)|.$$

(2) Add to (1.8)

$$H_k^s = \frac{H_{k-1}^{s+1} - H_{k-1}^s}{t_{s+k} - t_s}.$$

(3) Add to (1.9)

$$\Gamma_k^s = \frac{|H_k^s|}{|N_k^s|}.$$

4. An application: acceleration of convergence of some convergent and divergent logarithmic sequences by the $d^{(1)}$-transformation

4.1. Existence of asymptotic expansions. Consider the infinite sequence $\{S_n\}_{n=1}^{\infty}$, where

$$S_n = \sum_{i=1}^{n} a_i, \quad n = 1, 2, \ldots. \tag{4.1}$$

Let $w(n) = a_n$, and assume that $w(x)$, as a function of the continuous variable x, has an asymptotic expansion of the form

$$w(x) \sim x^{-\delta-1} \sum_{i=0}^{\infty} \nu_i x^{-i} \text{ as } x \to \infty; \quad \nu_0 \neq 0, \quad \delta \neq 0, -1, -2, \ldots. \tag{4.2}$$

As is known, $S = \lim_{n\to\infty} S_n$ exists and is finite, i.e., the infinite series $\sum_{i=1}^{\infty} a_i$ converges, if and only if $\operatorname{Re}\delta > 0$. In this case, Theorems 2.1 and 2.2 in [15] apply, and we have

$$S = S_n + n a_n f(n), \tag{4.3}$$

where

$$f(n) \sim \sum_{i=0}^{\infty} \beta_i n^{-i} \text{ as } n \to \infty, \quad \beta_0 \neq 0. \tag{4.4}$$

Hence, the sequence $\{S_n\}_{n=1}^{\infty}$ belongs to the set LOGSF of sequences, which in turn is a subset of LOG, the set of logarithmically convergent sequences. For appropriate definitions we refer the reader to [2, p. 41]. We mention that Theorem 2.1 of [15] is a special case of a more general result given in [8], and a detailed proof of it can be found in [14].

The result that we give in Theorem 4.1 below is new, however, and is a nontrivial extension of Theorem 2.2 of [15] for $\operatorname{Re}\delta \leq 0$.

Theorem 4.1. *Let S_n, a_n, and $w(x)$ be as described in the first paragraph of this subsection. Consider* $\operatorname{Re}\delta \leq 0$ *in* (4.2), *so that* $\lim_{n\to\infty} S_n$ *does not exist. Then there exists a constant S that serves as the antilimit of $\{S_n\}_{n=1}^{\infty}$ and a function $f(n)$ such that* (4.3) *and* (4.4) *continue to hold. The antilimit S is given in the proof below.*

Proof. Let N be some positive integer for which $\operatorname{Re}\delta + N > 0$, and define

$$\hat{w}(n) = \hat{a}_n = a_n - \sum_{i=0}^{N-1} \nu_i n^{-\delta-i-1}. \tag{4.5}$$

Obviously, as a function of the continuous variable x, $\hat{w}(x)$ has the asymptotic expansion

$$\hat{w}(x) \sim x^{-\delta-N-1} \sum_{i=0}^{\infty} \hat{\nu}_i x^{-i} \quad \text{as } x \to \infty, \tag{4.6}$$

with $\hat{\nu}_i = \nu_{N+i}$, $i = 0, 1, \ldots$. Thus, since $\operatorname{Re}\delta + N > 0$, the sequence $\{\hat{S}_n\}_{n=1}^{\infty}$, where $\hat{S}_n = \sum_{i=1}^{n} \hat{a}_i$, $n = 1, 2, \ldots$, converges. If we let $\hat{S} = \lim_{n\to\infty} \hat{S}_n$, then (4.3) and (4.4) become

$$\hat{S} = \hat{S}_n + n\hat{a}_n \hat{f}(n), \tag{4.7}$$

and

$$\hat{f}(n) \sim \sum_{i=0}^{\infty} \hat{\beta}_i n^{-i} \quad \text{as } n \to \infty, \tag{4.8}$$

respectively, for some $\hat{f}(n)$. Consider now $\{U_n\}_{n=1}^{\infty}$, where $U_n = S_n - \hat{S}_n = \sum_{i=1}^{n} u_i$, and $u_n = \sum_{i=0}^{N-1} \nu_i n^{-\delta-i-1}$, $n = 1, 2, \ldots$. Since the function $\sum_{i=0}^{N-1} \nu_i x^{-\delta-i-1}$ is infinitely differentiable for all $x > 0$, we can apply the Euler-Maclaurin summation formula to $\sum_{i=1}^{n} u_i = U_n$, and obtain

$$U = U_n + n^{-\delta} g(n), \tag{4.9}$$

where

$$g(n) \sim \sum_{i=0}^{\infty} \gamma_i n^{-i} \quad \text{as } n \to \infty, \quad \gamma_0 \neq 0. \tag{4.10}$$

Actually, $U = \sum_{i=0}^{N-1} \nu_i \zeta(\delta + i + 1)$, $\zeta(z)$ being the Riemann zeta function. (For real δ this result follows immediately from [10, p. 292, Ex. 3.2]. The case of complex δ can be treated in a similar fashion. See also Example 5.1 in §5 of this work.) Combining (4.7) and (4.9) in $S_n = \hat{S}_n + U_n$, we have

$$\hat{S} + U = S_n + n a_n \left[\frac{\hat{a}_n}{a_n} \hat{f}(n) + \frac{n^{-\delta-1}}{a_n} g(n) \right]. \tag{4.11}$$

Now let $S = \hat{S} + U$, and denote the term in the square brackets by $f(n)$. Invoking the asymptotic expansions of a_n, $\hat{a}_n$, $\hat{f}(n)$, and $g(n)$, we can easily show that $f(n)$ satisfies (4.4) with $\beta_0 = \gamma_0/\nu_0 \neq 0$. This completes the proof. □

As far as we know, divergent sequences of the logarithmic type considered here have not been treated in the literature of extrapolation methods before.

4.2. The Levin-Sidi $d^{(1)}$-transformation.

Definition 4.1. Let the sequence $\{S_n\}_{n=1}^{\infty}$, where $S_n = \sum_{i=1}^{n} a_i$, $n = 1, 2, \ldots$, be given, and denote its limit or antilimit by S. Pick a sequence of integers $\{R_l\}_{l=0}^{\infty}$ such that $0 < R_0 < R_1 < R_2 < \cdots$. Then S_n^j, the approximation to S, and the parameters $\bar{\beta}_i$, $i = 0, 1, \ldots, n-1$, are defined to be the solution of the system of $n+1$ linear equations

$$S_n^j = S_{R_l} + R_l a_{R_l} \sum_{i=0}^{n-1} \bar{\beta}_i / R_l^i, \quad j \le l \le j+n. \tag{4.12}$$

This procedure thus generates a nonlinear sequence transformation, which we call the $d^{(1)}$-transformation.

We mention that for $R_l = l+1$, $l = 0, 1, 2, \ldots$, the $d^{(1)}$-transformation reduces precisely to the u-transformation of Levin [7].

By drawing the proper analogy, we can now show that the sequence $\{S_n\}_{n=1}^{\infty}$ considered in the previous subsection is actually identified with a function $A(y)$ in $F^{(1)}$, and that the $d^{(1)}$-transformation is a GREP$^{(1)}$. This analogy runs as follows:

(1) $A(y) = a(t) \leftrightarrow S_n$, thus $y \leftrightarrow n^{-1}$. Therefore, y is a *discrete* variable that takes on the values $1, 1/2, 1/3, \ldots$. Also $r = 1$ in (1.2) so that $t = y$ for this case.

(2) $\phi(y) = \varphi(t) \leftrightarrow n a_n$, $n = 1, 2, \ldots$. Furthermore, by $a_n = w(n)$ and by (4.2), $\varphi(t) = t^{-1} w(t^{-1})$ is exactly of the form $\varphi(t) = t^{\delta} h(t)$, with $h(t) \sim \sum_{i=0}^{\infty} \nu_i t^i$ as $t \to 0+$, that was considered in §3.

(3) $y_l = t_l = 1/R_l$, $l = 0, 1, 2, \ldots$, and $A_n^j \leftrightarrow S_n^j$. Consequently, the W-algorithm of Theorem 1.2 can be used to implement the $d^{(1)}$-transformation in an efficient manner by making the appropriate substitutions. In addition, it can also be augmented as shown at the end of the previous section to obtain the Γ_n^j exactly. We thus have

$$\text{(a) } M_0^j = S_{R_j}/(R_j a_{R_j}),\ N_0^j = 1/(R_j a_{R_j}),\ H_0^j = (-1)^j |N_0^j|, \quad j = 0, 1, 2, \ldots,$$

$$\text{(b) } M_k^j = \frac{M_{k-1}^{j+1} - M_{k-1}^j}{1/R_{j+k} - 1/R_j},\quad N_k^j = \frac{N_{k-1}^{j+1} - N_{k-1}^j}{1/R_{j+k} - 1/R_j},$$
$$H_k^j = \frac{H_{k-1}^{j+1} - H_{k-1}^j}{1/R_{j+k} - 1/R_j},\quad j = 0, 1, \ldots,\ k = 1, 2, \ldots, \tag{4.13}$$

$$\text{(c) } S_k^j = \frac{M_k^j}{N_k^j},\quad \Gamma_k^j = \frac{|H_k^j|}{|N_k^j|},\quad j = 0, 1, \ldots,\ k = 0, 1, \ldots.$$

It is important to note that we do not need to know δ in (4.2) in order to be able to apply the $d^{(1)}$-transformation. In this sense the $d^{(1)}$-transformation is a user-friendly procedure.

4.3. **Choice of the** R_l, $l = 0, 1, \ldots$. We recall that the R_l in (4.12) are at our disposal. This provides the $d^{(1)}$-transformation with a large amount of flexibility that most other methods of acceleration do not possess.

The simplest choice of the R_l is given by $R_l = l+1$, $l = 0, 1, \ldots$. As mentioned already, for this choice the $d^{(1)}$-transformation reduces to the Levin u-transformation. A detailed analysis of the u-transformation for both Process I and Process II, in the context of linearly and logarithmically convergent sequences, has been given by the author in [14] and [15]. As has been established in the survey [21], among most of the known nonlinear sequence transformations, the u-transformation produces the best results when applied to *convergent* sequences of the form described in this section with real δ. It is also known, however, that when applied to such sequences, the u-transformation is prone to roundoff error propagation. This does not enable one to increase the accuracy by adding more terms of the sequence $\{S_n\}_{n=1}^{\infty}$ in the extrapolation procedure. On the contrary, addition of more terms ultimately results in total loss of accuracy. It must be mentioned, though, that the u-transformation is not the only extrapolation procedure that suffers from numerical instabilities; almost all other well-known sequence transformations as well suffer from the same problem.

By a judicious choice of the R_l we can cause the $d^{(1)}$-transformation to become extremely stable. The following was first suggested in [4, Appendix B] and incorporated in the FORTRAN 77 code that implements GREP and the $d^{(m)}$-transformation that was included there:

$$R_0 = 1, \quad R_{l+1} = \lfloor \sigma R_l \rfloor + 1, \quad l = 0, 1, \ldots, \text{ for some } \sigma > 1. \tag{4.14}$$

(Actually, the R_l proposed here are slightly different than those in [4], but the difference is insignificant.)

The important point to note is that

$$\sigma R_l < R_{l+1} \leq \sigma R_l + 1, \tag{4.15}$$

which implies

$$\sigma^l < R_l \leq \sum_{i=0}^{l} \sigma^i = \frac{\sigma^{l+1}-1}{\sigma - 1}, \quad l \geq 1. \tag{4.16}$$

Thus, R_l increases exponentially in l like σ^l. From the equations in (4.12) we realize that S_n^j is determined from the sequence elements S_i, $1 \leq i \leq R_{j+n}$. Obviously, the number R_{j+n} of these S_i is greater than $\sigma^{R_{j+n}}$ by (4.16). This shows that if we pick σ too large, e.g., $\sigma \geq 2$, then the number of the sequence elements S_i used in the extrapolation procedure increases at a prohibitive rate for the sequence S_n^j, $n = 0, 1, 2, \ldots$, i.e., for Process II. This means that σ should take on moderate values for practical purposes. We have found that, depending on the finite-precision arithmetic being used, σ in the range [1.1, 1.5] produces excellent results, with the R_l increasing relatively midly.

Finally, we would like to emphasize that any other strategy for which the R_l increase exponentially in l will also do. (For example, we can pick $R_l = l+1$ for $l \leq (\sigma-1)^{-1}$ and $R_{l+1} = \lfloor \sigma R_l \rfloor$ for $l > (\sigma-1)^{-1}$.) Note also that if we let $\sigma = 1$ in (4.14), what we have is precisely the u-transformation.

4.4. **Application of the theory.** As can be deduced from (4.15), the choice of the R_l given in (4.14) results in $t_l = 1/R_l$, $l = 0, 1, \ldots$, which satisfy

$$\frac{\omega t_l}{1+\omega t_l} \le t_{l+1} < \omega t_l, \quad l = 0, 1, \ldots; \quad \omega \equiv \sigma^{-1} \in (0, 1). \tag{4.17}$$

Consequently, the t_l satisfy *both* (2.1) and (3.2).

As mentioned in §4.2, $\varphi(t) = t^{-1}w(t^{-1}) = t^{\delta}h(t) \sim t^{\delta}\sum_{i=0}^{\infty}\nu_i t^i$ as $t \to 0+$. Consequently, Theorems 2.1 and 3.1 apply directly to the approximations S_n^j, whether $\{S_n\}_{n=1}^{\infty}$ converges or not. The excellent results obtained by applying the $d^{(1)}$-transformation with R_l as in (4.14) are thus explained in a very accurate manner by Theorem 2.1 and Theorem 3.1 and its corollary.

5. Numerical examples

We have applied the $d^{(1)}$-transformation with the strategy described by (4.14) to many infinite series of the logarithmic type discussed in the previous section. In particular, we have applied it to all the (real) logarithmically convergent test series in Table 6.1 of [21]. For all of these series the limits were obtained almost to machine accuracy. We do not bring the relevant numerical results. Instead, we concentrate on the series that define the Riemann zeta function $\zeta(z)$ and the Gauss hypergeometric function ${}_2F_1(b, c; d; 1)$, and use the $d^{(1)}$-transformation to analytically continue these functions in their parameters. We also use the zeta function series to demonstrate and verify numerically several features of our convergence theory.

Example 5.1. Consider the series $\sum_{n=1}^{\infty} a_n$ with $a_n = n^{-\delta-1}$, $\delta \ne 0, -1, -2, \ldots$, which converges for $\operatorname{Re}\delta > 0$ and diverges otherwise. Let $S_n = \sum_{i=1}^{n} a_i$, $n = 1, 2, \ldots$. We have

$$S_{n-1} \sim \zeta(\delta+1) - \frac{n^{-\delta}}{\delta}\sum_{i=0}^{\infty}\binom{-\delta}{i}B_i n^{-i} \quad \text{as } n \to \infty, \tag{5.1}$$

provided $\delta \ne 0$. (See [10, p. 292, Ex. 3.2] for real δ.) Here, B_i are the Bernoulli numbers. For our purposes it is enough to note that $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_{2i+1} = 0$, $i = 1, 2, \ldots$, while B_{2i}, $i = 1, 2, \ldots$, are all nonzero. Adding a_n to both sides of (5.1), we see that S_n satisfies (4.3) and (4.4), with $S = \zeta(\delta+1)$ and $\beta_i = \delta^{-1}\binom{-\delta}{i}B_i$ for $i = 0$ and $i \ge 2$ and $\beta_1 = -\frac{1}{2}$. Thus $\beta_3 = \beta_5 = \beta_7 = \cdots = 0$, the remaining β_i being nonzero.

We have applied to this series the $d^{(1)}$-transformation with the R_l as in (4.14) and $\sigma = 1.2$. We have considered both $\operatorname{Re}\delta > 0$ and $\operatorname{Re}\delta \le 0$.

Since $\varphi(t) = t^{\delta}$ precisely for this case, *all* of the results of §2 pertaining to Process I apply with the same notation. In particular, Theorem 2.1 implies that, whether $\lim_{n\to\infty} S_n$ exists or not, $S - S_n^j$ is roughly speaking, $O(b_1^j)$ for $n = 0$, $O(b_2^j)$ for $n = 1$, $O(b_3^j)$ for $n = 2$, $O(b_{2i+1}^j)$ for $n = 2i-1, 2i$, and $i = 2, 3, \ldots$. We also have that $\lim_{j\to\infty}(S - S_n^{j+1})/(S - S_n^j)$ is exactly equal to b_1 for $n = 0$, b_2 for $n = 1$, b_3 for $n = 2$, b_{2i+1} for $n = 2i-1, 2i$, and $i = 2, 3, \ldots$. Note that, with $\omega = \sigma^{-1}$, we have $b_k = \omega^{\delta+k-1}$, $k = 1, 2, \ldots$, in this example.

TABLE 5.1.1. The ratios $|S - S_n^{j+1}|/|S - S_n^j|$, $j = 0, 1, 2, \ldots$ for the series of $\zeta(\delta + 1)$ with $\delta = -1.1 + 10i$ in Example 5.1. The $d^{(1)}$-transformation is implemented with $\sigma = 1.2$ in (4.14)

j \ n	0	1	2	3	4	5	6	7
0	0.68198							
1	1.01959	0.17519						
2	1.13949	0.47730	0.09301					
3	1.15994	0.74946	0.28310	0.08471				
4	1.15351	0.87191	0.51851	0.20117	0.08967			
5	1.28769	1.20705	0.80118	0.40371	0.21609	0.10765		
6	1.23637	0.84141	0.82108	0.50811	0.35545	0.18820	0.10717	
7	1.19784	0.92683	0.60510	0.46617	0.41161	0.25852	0.15936	0.10255
8	1.25511	1.08213	0.79574	0.41558	0.42521	0.29770	0.21918	0.14155
9	1.20856	0.93818	0.82359	0.51676	0.40189	0.29706	0.25260	0.17717
10	1.23536	1.04450	0.78542	0.55520	0.50133	0.28675	0.27139	0.19975
11	1.24360	1.02312	0.86440	0.53707	0.54167	0.35177	0.27757	0.20814
12	1.24023	1.00958	0.82832	0.57892	0.53204	0.37367	0.33687	0.20400
13	1.23067	1.00301	0.81363	0.54771	0.56062	0.36111	0.35798	0.23893
14	1.21838	1.00070	0.81273	0.54197	0.53690	0.37741	0.35113	0.24874
15	1.22833	1.02929	0.83760	0.55862	0.54381	0.36694	0.37024	0.24329
16	1.22736	1.01629	0.85277	0.57713	0.56089	0.37385	0.36641	0.25646
17	1.22018	1.00871	0.83506	0.58292	0.57395	0.38531	0.37400	0.25277
18	1.22319	1.02139	0.84064	0.57603	0.58058	0.39609	0.38574	0.25851
19	1.22965	1.02628	0.85533	0.58433	0.57913	0.40240	0.39661	0.26738
20	1.22655	1.01456	0.84845	0.58921	0.58421	0.40000	0.40157	0.27440
21	1.22580	1.01748	0.84033	0.58281	0.58673	0.40276	0.39971	0.27751
22	1.22519	1.01766	0.84413	0.57836	0.58153	0.40411	0.40181	0.27575
23	1.22362	1.01647	0.84405	0.58171	0.57837	0.40052	0.40274	0.27707
24	1.22511	1.02027	0.84644	0.58368	0.58226	0.39935	0.40056	0.27812
25	1.22315	1.01604	0.84671	0.58467	0.58344	0.40211	0.39977	0.27655
26	1.22177	1.01666	0.84406	0.58514	0.58408	0.40319	0.40212	0.27612
27	1.22232	1.01901	0.84724	0.58518	0.58523	0.40442	0.40346	0.27823
28	1.22270	1.01884	0.84927	0.58816	0.58594	0.40579	0.40485	0.27951
29	1.22190	1.01740	0.84791	0.58901	0.58808	0.40637	0.40605	0.28062

In Table 5.1.1 we give the numbers $|(S - S_n^{j+1})/(S - S_n^j)|$ obtained by taking $\delta = -1.1 + 5i$. The agreement of these numbers with the theory is simply remarkable. For this value of δ the $n = 0$ and $n = 1$ columns in the extrapolation table of (1.4) diverge, while the remaining ones converge.

Similarly, all the results of §3 pertaining to Process II apply, again with the same notation. For example, if we let δ be real, then (3.19) in Theorem 3.1 holds with $\omega = \sigma^{-1}$, and $Q_n^{(j)} = 1$, and $\hat{t}_l = \omega^l t_0$, $l = 0, 1, \ldots$. In addition, for this case, $M_n^{(j)} = O(n!(2\pi)^{-n})$ as $n \to \infty$, as a result of which Theorem 3.1 predicts that $|S - S_n^j| \to 0$ as $n \to \infty$ practically at the rate of $\omega^{n^2/2}$, and (3.21) holds. (When δ is complex, Theorem 3.1 makes the same prediction provided we pick σ to be a positive integer ≥ 2 and $R_l = \sigma^l R_0$ so that $t_l = \omega^l t_0$ with $\omega = \sigma^{-1}$. Note that numerical results indicate very clearly that $S_n^j \to S$ as $n \to \infty$ very quickly even when R_l are picked to satisfy (4.14).)

In Table 5.1.2 we give the relative errors $|(S - S_n^j)/S|$ and the corresponding Γ_n^j, for $j = 0$ and $n = 0, 1, 2, \ldots$. In addition, we give the corresponding results obtained from the u-transformation.

TABLE 5.1.2. Relative errors in S_n^0 and Γ_n^0, $n = 0, 1, \dots,$ for the series of $\zeta(\delta + 1)$ with $\delta = -1.1 + 10i$ in Example 5.1. The $d^{(1)}$-transformation is implemented once with $\sigma = 1.2$ and once with $\sigma = 1$ in (4.14). ($\sigma = 1$ in (4.14) gives rise precisely to the u-transformation.)

	$\sigma = 1.2$ in (4.14)		$\sigma = 1$ in (4.14)	
n	$\lvert(S_n^0 - S)/S\rvert$	Γ_n^0	$\lvert(S_n^0 - S)/S\rvert$	Γ_n^0
0	4.49D −01	1.00000D +00	4.49D −01	1.00000D +00
1	8.20D −01	2.13032D +00	8.20D −01	2.13032D +00
2	3.43D −01	1.80641D +00	3.43D −01	1.80641D +00
3	5.32D −02	1.17365D +00	5.32D −02	1.17365D +00
4	7.25D −03	1.10671D +00	7.25D −03	1.10671D +00
5	9.59D −04	1.33628D +00	9.59D −04	1.33628D +00
6	1.38D −04	1.55307D +00	1.23D −04	1.98440D +00
7	1.89D −05	1.67944D +00	1.59D −05	3.47167D +00
8	2.34D −06	1.81933D +00	2.06D −06	6.90773D +00
9	2.60D −07	2.06963D +00	2.64D −07	1.52175D +01
10	2.60D −08	2.51579D +00	3.43D −08	3.63622D +01
11	2.32D −09	3.09382D +00	4.43D −09	9.27666D +01
12	1.87D −10	3.69876D +00	5.71D −10	2.49591D +02
13	1.36D −11	4.23788D +00	7.43D −11	7.01407D +02
14	8.61D −13	4.68989D +00	9.56D −12	2.04316D +03
15	4.95D −14	5.16084D +00	1.24D −12	6.13177D +03
16	2.44D −15	5.76105D +00	1.61D −13	1.88668D +04
17	1.08D −16	6.53878D +00	2.06D −14	5.92820D +04
18	4.19D −18	7.45320D +00	2.69D −15	1.89611D +05
19	1.41D −19	8.39573D +00	3.45D −16	6.15708D +05
20	4.31D −21	9.25756D +00	4.45D −17	2.02540D +06
21	1.10D −22	9.99091D +00	5.77D −18	6.73731D +06
22	2.60D −24	1.05601D +01	7.39D −19	2.26280D +07
23	5.08D −26	1.09772D +01	9.56D −20	7.66368D +07
24	8.94D −28	1.13184D +01	1.23D −20	2.61455D +08
25	1.19D −29	1.16504D +01	1.58D −21	8.97694D +08
26	9.39D −30	1.20158D +01	2.09D −22	3.09953D +09
27	1.06D −29	1.24180D +01	6.03D −23	1.07550D +10
28	1.12D −29	1.28386D +01	3.39D −22	3.74821D +10
29	7.73D −30	1.32600D +01	1.09D −21	1.31136D +11

Example 5.2. Let $a_{n+1} = [(b)_n(c)_n]/[(d)_n n!]$, $n = 0, 1, \dots$. Provided $\operatorname{Re} d > \operatorname{Re}(b + c)$, we have

$$(5.2) \qquad \sum_{n=1}^{\infty} a_n = {}_2F_1(b, c; d; 1) = \frac{\Gamma(d-b-c)\Gamma(d)}{\Gamma(d-b)\Gamma(d-c)} = S$$

which is a well-known result concerning Gauss' hypergeometric function.

By the fact that $(e)_n = \Gamma(e+n)/\Gamma(e)$, $n = 0, 1, \dots,$ and by Stirling's formula for the gamma function, we have that $a_n = w(n)$ is precisely as in (4.2) with $\delta = d - (b + c)$. Furthermore, (5.2) can be continued analytically in b, c, and d, and this is a well-known fact.

We have applied the $d^{(1)}$- transformation to the series above with the R_l as in (4.14) and $\sigma = 1.2$. In Table 5.2 we give the relative errors $\lvert(S - S_n^j)/S\rvert$ and the corresponding Γ_n^j for $j = 0$ and $n = 0, 1, 2, \dots$. We have done the computations with (i) $b = 0.5$, $c = 0.5$, and $d = 1.5$ (convergent series) and (ii) $b = 0.6$, $c = 0.4$, and $d = 1 + 10i$ (divergent series).

TABLE 5.2. Relative errors in S_n^0 and Γ_n^0, $n = 0, 1, \ldots$, for the series of ${}_2F_1(b, c; d; 1)$ in Example 5.2. The $d^{(1)}$-transformation is implemented with $\sigma = 1.2$ in (4.14)

	$b=0.5,\ c=0.5,\ d=1.5$		$b=0.6,\ c=0.4,\ d=1+10i$	
n	$\lvert(S-S_n^0)/S\rvert$	Γ_n^0	$\lvert(S-S_n^0)/S\rvert$	Γ_n^0
0	3.63D −01	1.00000D +00	2.40D −02	1.00000D +00
1	2.04D −01	2.00000D +00	1.53D −03	1.05157D +00
2	2.99D −02	1.12857D +01	2.15D −05	1.24945D +00
3	3.83D −03	5.36087D +01	3.76D −08	1.57834D +00
4	2.58D −05	2.15573D +02	1.23D −09	2.12335D +00
5	2.74D −05	8.28418D +02	4.18D −11	3.03909D +00
6	1.42D −06	1.77857D +03	1.37D −12	2.43071D +00
7	1.78D −07	3.03959D +03	5.10D −14	2.13733D +00
8	2.02D −08	5.50246D +03	2.33D −15	2.46923D +00
9	4.06D −10	9.53210D +03	1.19D −16	2.57753D +00
10	1.44D −10	1.65700D +04	6.46D −18	2.78215D +00
11	3.29D −12	2.66015D +04	3.80D −19	2.97763D +00
12	5.09D −13	3.84489D +04	2.38D −20	3.07529D +00
13	2.62D −14	5.06996D +04	1.52D −21	3.11742D +00
14	7.24D −16	6.37515D +04	9.73D −23	3.22193D +00
15	7.37D −17	8.10522D +04	6.07D −24	3.51280D +00
16	1.72D −19	1.03698D +05	3.65D −25	3.95987D +00
17	9.75D −20	1.31024D +05	2.08D −26	4.47974D +00
18	1.59D −21	1.62253D +05	1.10D −27	5.03799D +00
19	6.11D −23	1.95171D +05	5.38D −29	5.61874D +00
20	1.66D −24	2.25862D +05	2.41D −30	6.17390D +00
21	1.54D −26	2.52777D +05	1.11D −31	6.66761D +00
22	1.21D −27	2.75075D +05	3.37D −32	7.08243D +00
23	3.11D −28	2.94182D +05	1.96D −32	7.44890D +00
24	5.73D −28	3.13027D +05	2.50D −32	7.82183D +00
25	4.18D −28	3.31981D +05	2.88D −32	8.21410D +00
26	3.13D −28	3.51567D +05	2.46D −32	8.62300D +00
27	5.79D −28	3.71911D +05	2.02D −32	9.04079D +00
28	3.05D −28	3.92375D +05	2.75D −32	9.45763D +00
29	6.74D −28	4.12079D +05	2.00D −32	9.86025D +00

APPENDIX. DIVIDED DIFFERENCES OF POWERS WITH APPLICATIONS

Lemma A.1 (Hermite-Gennochi). *Let $f(x)$ be in $C^n[a, b]$, and let $x_0, x_1, \ldots, x_n$ be all in $[a, b]$. Then*

$$f[x_0, x_1, \ldots, x_n] = \int_{T_n} f^{(n)}\Big(\sum_{i=0}^{n} \xi_i x_i\Big) d\xi_1 \cdots d\xi_n, \tag{A.1}$$

where

$$T_n = \{(\xi_1, \ldots, \xi_n) : 0 \le \xi_i \le 1,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} \xi_i \le 1\}; \quad \xi_0 = 1 - \sum_{i=1}^{n} \xi_i. \tag{A.2}$$

For a proof of this lemma see, e.g., [1, p. 120]. Note that the argument $z = \sum_{i=0}^{n} \xi_i x_i$ of $f^{(n)}$ in (A.1) is actually a convex combination of $x_0, x_1, \ldots, x_n$ as $0 \le \xi_i \le 1$, $i = 0, 1, \ldots, n$, and $\sum_{i=0}^{n} \xi_i = 1$. If we order the x_i such that $x_0 < x_1 < \cdots < x_n$, then $z \in [x_0, x_n] \subseteq [a, b]$.

As a consequence of the Hermite-Gennochi formula we obtain the following result, which says that if $f^{(n)}(x)$ is monotonic on $[a, b]$, then so is the nth-order divided difference of $f(x)$, in a sense to be made clear below.

Lemma A.2. *Let* $f^{(n)}(x)$ *be nondecreasing on* $[a, b]$. *Let* $x_i \le \hat{x}_i$, $a \le x_i, \hat{x}_i \le b$, $i = 0, 1, \dots, n$, *and assume* $x_i < \hat{x}_i$ *at least for one value of* i. *Then*

$$f[x_0, x_1, \dots, x_n] \le f[\hat{x}_0, \hat{x}_1, \dots, \hat{x}_n]. \tag{A.3}$$

If $f^{(n)}(x)$ *is strictly increasing on* $[a, b]$, *then strict inequality holds in* (A.3).

Proof. Since $\xi_i \ge 0$, $i = 0, 1, \dots, n$, we have $z = \sum_{i=0}^{n} \xi_i x_i < \sum_{i=0}^{n} \xi_i \hat{x}_i = \hat{z}$. Therefore, since both z and $\hat{z}$ are in $[a, b]$, $f^{(n)}(z) \le f^{(n)}(\hat{z})$. The result in (A.3) now follows by employing (A.1). The rest is simple. □

We now apply Lemma A.1 to powers. Throughout the remainder of this appendix, $t_0 > t_1 > t_2 > \cdots$, and D_n^j are exactly as in §1.

We shall also be making use of the following result.

Lemma A.3. *Let* $\hat{t}_i = \omega^i \hat{t}_0$, $i = 0, 1, \dots$, *and define* $\hat{D}_n^j$ *to be the divided difference operator of order* n *over the set of points* $\hat{t}_j, \hat{t}_{j+1}, \dots, \hat{t}_{j+n}$. *Define* $\Delta(t) = t^{-\delta}$, δ *being a complex number in general. Then*

$$\hat{D}_n^j\{\Delta(t)\} = \Delta[\hat{t}_j, \hat{t}_{j+1}, \dots, \hat{t}_{j+n}] = \frac{(-1)^n}{\omega^{\delta n + n(n-1)/2}\, \hat{t}_j^{\delta+n}} \prod_{i=1}^{n} \frac{1 - \omega^{\delta+i-1}}{1 - \omega^i}. \tag{A.4}$$

Proof. The assertion (A.4) can be proved by induction on n. A direct proof is possible by proper manipulation of the determinant representation of divided differences, see [12, p. 45]. □

It is important to analyze the behavior of $\hat{D}_n^j\{\Delta(t)\}$ for $n \to \infty$. Note that the product $\prod_{i=1}^{n}[(1 - \omega^{\delta+i-1})/(1 - \omega^i)]$ has a finite and nonzero limit as $n \to \infty$. Consequently, $|\hat{D}_n^j\{\Delta(t)\}| \sim C_j \mu_j^{-n} \omega^{-n^2/2}$ for some $C_j > 0$ and $\mu_j = \hat{t}_j \omega^{\delta - 1/2}$, which means that $|\hat{D}_n^j\{\Delta(t)\}| \to \infty$ as $n \to \infty$ practically like $\omega^{-n^2/2}$. This implies that, as $n \to \infty$, $\hat{D}_n^j\{\Delta(t)\}$ dominates $(pn)!\rho^n n^\theta$ for any p, ρ, and θ. Also, $\hat{D}_n^j\{t^{-\delta_2}\}/\hat{D}_n^j\{t^{-\delta_1}\} = O(\omega^{\mathrm{Re}(\delta_1 - \delta_2)n}) = o(1)$ as $n \to \infty$, when $\mathrm{Re}\,\delta_1 > \mathrm{Re}\,\delta_2$, and $\delta_1 \ne 0, -1, -2, \dots$.

Lemma A.4. *Let* $t_0, t_1, \dots$, *satisfy* $t_{i+1}/t_i \le \omega$ *for some* $\omega \in (0, 1)$, *and define* $\hat{t}_i = \omega^i t_0$, $i = 0, 1, \dots$. *Define also* $\Delta(t) = t^{-\delta}$, *where* δ *is real. Then, for* $n > -\delta$,

$$|D_n^j\{\Delta(t)\}| \ge |\hat{D}_n^j\{\Delta(t)\}| = \frac{1}{\omega^{\delta n + n(n-1)/2}\, \hat{t}_j^{\delta+n}} \left| \prod_{i=1}^{n} \frac{1 - \omega^{\delta+i-1}}{1 - \omega^i} \right|. \tag{A.5}$$

Proof. First, $\Delta^{(n)}(t) = (-1)^n (\delta)_n t^{-\delta-n}$, where $(\delta)_n$ is the Pochhammer symbol, is monotonic and of one sign for $t > 0$. Obviously, $|\Delta^{(n)}(t)|$ is strictly decreasing for $t > 0$ when $n > -\delta$. Next, $t_i \le \hat{t}_i$, $i = 0, 1, 2, \dots$, so that, if we define $z = \sum_{i=0}^{n} \xi_i t_{j+i}$ and $\hat{z} = \sum_{i=0}^{n} \xi_i \hat{t}_{j+i}$, with $(\xi_1, \dots, \xi_n) \in T_n$ and $\xi_0 = 1 - \sum_{i=1}^{n} \xi_i$, then we have $z \le \hat{z}$. Consequently, $|\Delta^{(n)}(z)| \ge |\Delta^{(n)}(\hat{z})|$.

Applying now Lemma A.1, we obtain

(A.6)

$$|D_n^j\{\Delta(t)\}| = \int_{T_n} |\Delta^{(n)}(z)| d\xi_1 \cdots d\xi_n \geq \int_{T_n} |\Delta^{(n)}(\hat{z})| d\xi_1 \cdots d\xi_n = |\hat{D}_n^j\{\Delta(t)\}|.$$

The rest follows from Lemma A.3. □

Lemma A.5. *Let* δ_1 *and* δ_2 *be two real numbers and* $\delta_1 \neq \delta_2$. *Define* $\Delta_i(t) = t^{-\delta_i}$, $i = 1, 2$. *Let* $t_0 > t_1 > t_2 > \cdots > 0$. *Then, provided* $\delta_1 \neq 0, -1, -2, \ldots$,

$$D_n^j\{\Delta_2(t)\} = \frac{(\delta_2)_n}{(\delta_1)_n} \tilde{t}^{\delta_1 - \delta_2} D_n^j\{\Delta_1(t)\} \quad \textit{for some } \tilde{t} \in (t_{j+n}, t_j). \tag{A.7}$$

Proof. From Lemma A.1,

$$D_n^j\{\Delta_2(t)\} = \int_{T_n} \Delta_2^{(n)}(z) d\xi_1 \cdots d\xi_n = \int_{T_n} \left[\frac{\Delta_2^{(n)}(z)}{\Delta_1^{(n)}(z)}\right] \Delta_1^{(n)}(z) d\xi_1 \cdots d\xi_n. \tag{A.8}$$

Since $\Delta_1^{(n)}(z)$ is of one sign on T_n, we can apply the mean value theorem to the second integral to obtain

(A.9)

$$D_n^j\{\Delta_2(t)\} = \frac{\Delta_2^{(n)}(\tilde{t})}{\Delta_1^{(n)}(\tilde{t})} \int_{T_n} \Delta_1^{(n)}(z) d\xi_1 \cdots d\xi_n = \frac{\Delta_2^{(n)}(\tilde{t})}{\Delta_1^{(n)}(\tilde{t})} D_n^j\{\Delta_1(t)\}, \quad \tilde{t} \in (t_{j+n}, t_j).$$

This proves (A.7). □

Corollary. *When* $\delta_1 > \delta_2$ *in Lemma* A.5, *then*

$$\left|\frac{(\delta_2)_n}{(\delta_1)_n}\right| t_{j+n}^{\delta_1 - \delta_2} \leq \frac{|D_n^j\{\Delta_2(t)\}|}{|D_n^j\{\Delta_1(t)\}|} \leq \left|\frac{(\delta_2)_n}{(\delta_1)_n}\right| t_j^{\delta_1 - \delta_2}, \tag{A.10}$$

from which we also have, for some constant $K > 0$,

$$\frac{D_n^j\{\Delta_2(t)\}}{D_n^j\{\Delta_1(t)\}} \leq K n^{\delta_2 - \delta_1} = o(1) \quad \textit{as } n \to \infty. \tag{A.11}$$

Proof. That (A.10) is true is obvious from (A.7). The result in (A.11) follows by substituting in the right inequality of (A.10) the identity

$$\frac{(\delta_2)_n}{(\delta_1)_n} = \frac{\Gamma(\delta_1)}{\Gamma(\delta_2)} \frac{\Gamma(n + \delta_2)}{\Gamma(n + \delta_1)},$$

and by invoking Stirling's formula. □

We now go on to investigate $D_n^j\{\psi(t)\}$ for $n \to \infty$, where $\psi(t) = t^{-\delta} g(t)$, $g(t)$ being infinitely differentiable in $[0, t_j]$. This is a problem of crucial importance in the analysis of Process II considered in §3 of this work.

Lemma A.6. *Pick* $t_0 > t_1 > t_2 > \cdots > 0$ *such that* $t_{i+1}/t_i \leq \omega$ *for some* $\omega \in (0, 1)$, *and let* $\hat{t}_i = \omega^i t_0$, $i = 0, 1, \ldots$. *Consider the function* $\psi(t) = t^{-\delta} g(t)$, *where* δ *is a positive integer and* $g(t)$ *is in* $C^\infty[0, t_j]$ *such that* $g(0) \neq 0$ *and* $\max_{0 \leq t \leq t_j} |g^{(n)}(t)| = O((pn)! \rho^n)$ *as* $n \to \infty$, *for arbitrary* $p \geq 0$ *and* $\rho \geq 0$. *(If*

$g(t)$ is analytic, then $p \le 1$.) Then

$$D_n^j\{\psi(t)\} = Q_n^{(j)} D_n^j\{t^{-\delta}\}; \quad Q_n^{(j)} \sim g(0) \text{ as } n \to \infty. \tag{A.12}$$

In addition,

$$|D_n^j\{\psi(t)\}| \ge \frac{|Q_n^{(j)}|}{\omega^{\delta n + n(n-1)/2}\,\hat{t}_j^{\delta+n}} \prod_{i=1}^{n} \frac{1-\omega^{\delta+i-1}}{1-\omega^i}. \tag{A.13}$$

Equality holds in (A.13) *when* $t_i = \hat{t}_i$, $i = 0, 1, \ldots$.

Proof. We start by expressing $\psi(t)$ in the form

$$\psi(t) = \sum_{i=0}^{\delta-1} \varepsilon_i t^{-\delta+i} + \tilde{g}(t), \quad \varepsilon_0 = g(0), \tag{A.14}$$

where $\tilde{g}(t)$ is in $C^\infty[0, t_j]$. By the linearity of D_n^j, we have

$$D_n^j\{\psi(t)\} = \sum_{i=0}^{\delta-1} \varepsilon_i D_n^j\{t^{-\delta+i}\} + D_n^j\{\tilde{g}(t)\}. \tag{A.15}$$

Thus, $Q_n^{(j)}$ in (A.12) is given by

$$Q_n^{(j)} = \varepsilon_0 + \sum_{i=1}^{\delta-1} \varepsilon_i \frac{D_n^j\{t^{-\delta+i}\}}{D_n^j\{t^{-\delta}\}} + \frac{D_n^j\{\tilde{g}(t)\}}{D_n^j\{t^{-\delta}\}}. \tag{A.16}$$

From the corollary of Lemma A.5, the summation on the right-hand side of (A.16) is $o(1)$ as $n \to \infty$. Furthermore, from (3.8) and by our assumption on $g(t)$, we have

$$|D_n^j\{\tilde{g}(t)\}| \le \frac{1}{n!} \max_{t_{j+n} \le t \le t_j} |\tilde{g}^{(n)}(t)| = O((p'n)!) \text{ as } n \to \infty, \quad \text{some } p'. \tag{A.17}$$

By (A.17), (A.5), and the discussion following Lemma A.3, $D_n^j\{\tilde{g}(t)\}/D_n^j\{\Delta(t)\} \to 0$ as $n \to \infty$. This completes the proof of (A.12). The rest follows from Lemma A.4. □

We do not know whether Lemma A.6 remains valid for δ not a positive integer. Imposing additional conditions on $g(t)$ and/or the t_i, however, we are able to obtain results of the form similar to (A.13). This is done in Lemmas A.7 and A.8. These lemmas suggest that Lemma A.6 might hold also when δ is not a positive integer, but this is an open problem.

Lemma A.7. *Let t_i and $\hat{t}_i$, $i = 0, 1, \ldots$, be as in Lemma* A.6. *Consider the function $\psi(t) = t^{-\delta} g(t)$, where δ is not an integer and can be complex, and $g(t) = \sum_{k=0}^{q} \varepsilon_k t^k$, $\varepsilon_0 = g(0) \ne 0$, where q is an integer ≥ 0.*

(i) *If δ is real, then $D_n^j\{\psi(t)\}$ satisfies* (A.12) *and* (A.13).

(ii) *If δ is complex, in general, with $\alpha = \operatorname{Re}\delta$, then $\hat{D}_n^j\{\psi(t)\}$ satisfies*

(A.18) $$\hat{D}_n^j\{\psi(t)\} = Q_n^{(j)}\hat{D}_n^j\{t^{-\delta}\}; \quad Q_n^{(j)} \sim g(0) \text{ as } n \to \infty,$$

and hence

(A.19) $$|\hat{D}_n^j\{\psi(t)\}| = \frac{|Q_n^{(j)}|}{\omega^{\alpha n + n(n-1)/2}\,\hat{t}_j^{\alpha+n}}\left|\prod_{i=1}^{n}\frac{1-\omega^{\delta+i-1}}{1-\omega^i}\right|.$$

Proof. The proof of part (i) is almost identical to that of Lemma A.6. The proof of part (ii) can be achieved in a similar manner by recalling the last remark following Lemma A.3. We leave the details to the reader. □

Lemma A.8. *Let* t_i *and* $\hat{t}_i$, $i = 0, 1, \ldots$, *and* $g(t)$ *be as in Lemma* A.6, *and consider the function* $\psi(t) = t^{-\delta}g(t)$, δ *real and not an integer. Assume also that* $g(t)$ *is nonzero on* $[0, t_j]$ *and that* $(-1)^k g^{(k)}(t) \geq 0$, $k = 0, 1, 2, \ldots$, *for* $t \in [0, t_j]$. *Then*

(A.20) $$D_n^j\{\psi(t)\} = Q_n^{(j)}D_n^j\{t^{-\delta}\}; \quad |Q_n^{(j)}| \geq L_n^{(j)} \sim |g(0)| \text{ as } n \to \infty.$$

Hence, $D_n^j\{\psi(t)\}$ *satisfies* (A.13) *too.*

Proof. From Leibniz's formula for divided differences (see, e.g., [12, p.50]), we have

(A.21) $$D_n^j\{\psi(t)\} = \sum_{i=0}^{n} D_i^j\{t^{-\delta}\}D_{n-i}^{j+i}\{g(t)\}.$$

Now since $D_k^s\{h(t)\} = h^{(k)}(\xi)/k!$, $\xi \in (t_{s+k}, t_s)$, we have

(A.22) $$\begin{aligned} C_i &\equiv D_i^j\{t^{-\delta}\}D_{n-i}^{j+i}\{g(t)\} \\ &= (-1)^n\frac{(\delta)_i}{i!}\xi_i^{-\delta-i}\frac{|g^{(n-i)}(\eta_i)|}{(n-i)!}, \quad \xi_i \in (t_{j+i}, t_j), \ \eta_i \in (t_{j+n}, t_{j+i}), \end{aligned}$$

where we have also used the assumption on the sign of $g^{(n-i)}(t)$. From (A.22) it is obvious that C_i, for $i \geq i_0$, where $i_0 = 0$ if $\delta > 0$ and $i_0 = \lfloor 1-\delta \rfloor$ if $\delta < 0$, all have the same sign, so that

(A.23) $$\left|\sum_{i=i_0}^{n} C_i\right| \geq |C_n| = |g(t_{j+n})||D_n^j\{t^{-\delta}\}| \sim |g(0)||D_n^j\{t^{-\delta}\}| \text{ as } n \to \infty.$$

This implies that $|\sum_{i=i_0}^n C_i|$ grows at least like $|D_n^j\{t^{-\delta}\}|$ for $n \to \infty$. The summation $\sum_{i=0}^{i_0} C_i$, on the other hand, is either empty or has a fixed number of terms, and, by our assumption on $g(t)$, has a rate of growth bounded by $(p'n)!$ as $n \to \infty$, for some $p' \geq 0$. Since $|D_n^j\{t^{-\delta}\}|$ grows with n, roughly speaking, like $\omega^{-n^2/2}$, we see that $(\sum_{i=0}^{i_0} C_i / \sum_{i=i_0}^n C_i) \to 0$ as $n \to \infty$. Consequently, $\sum_{i=0}^n C_i \sim \sum_{i=i_0}^n C_i$ as $n \to \infty$. The result in (A.20) follows from this and from (A.23). □

Note that the conditions $(-1)^k g^{(k)}(t) \geq 0$ on $[0, T]$, $k = 0, 1, \ldots$, imply that $g(t)$ is completely monotonic on $[0, T]$. For completely monotonic functions, see e.g., [23, Chapter IV].

Finally, we have the following more general, but weaker, result, which holds for arbitrary $\psi(t)$, but may be useful for $\psi(t) = t^{-\delta} g(t)$, with $\operatorname{Re}\delta = \alpha$ and $g(t)$ infinitely differentiable in $[0, t_j]$.

Lemma A.9. *Let $t_0 > t_1 > \cdots > 0$ be arbitrary, and let $\psi(t)$ be in general complex, infinitely differentiable on $(0, t_j]$, such that $\psi^{(n)}(t)$ is nonzero there for all large n. Let also*

$$L_n^{(j)} = \left[\min_{t_{j+n}\le t\le t_j} |\operatorname{Re} G_n(t)|^2 + \min_{t_{j+n}\le t\le t_j} |\operatorname{Im} G_n(t)|^2\right]^{\frac{1}{2}}, \tag{A.24}$$

where

$$G_n(t) = \psi^{(n)}(t)/\Delta^{(n)}(t); \quad \Delta(t) = t^{-\alpha}, \quad \alpha \text{ real.} \tag{A.25}$$

Then, for all large n,

$$|D_n^j\{\psi(t)\}| \ge L_n^{(j)} |D_n^j\{t^{-\delta}\}|. \tag{A.26}$$

Proof. Manipulating the Hermite-Gennochi formula for $D_n^j\{\psi(t)\}$, we have

$$D_n^j\{\psi(t)\} = \int_{T_n} G_n(z)\Delta^{(n)}(z)d\xi_1\cdots d\xi_n, \tag{A.27}$$

in the notation of Lemma A.4. Since $\Delta^{(n)}(z)$ is real and of one sign on T_n, we can apply the mean value theorem to the real and imaginary parts of (A.27) to obtain

$$D_n^j\{\psi(t)\} = [\operatorname{Re} G_n(\theta_r) + i \operatorname{Im} G_n(\theta_i)] \int_{T_n} \Delta^{(n)}(z)d\xi_1\cdots d\xi_n. \tag{A.28}$$

The result in (A.26) follows by taking the modulus of both sides and invoking the Hermite-Gennochi formula once more. The details are left to the reader. □

By adding the condition $t_{i+1}/t_i \le \omega \in (0, 1)$ we can, by using Lemma A.4, replace the right-hand side of (A.26) by $L_n^{(j)} \hat{D}_n^j\{t^{-\alpha}\}$.

Before ending this appendix, we give lower bounds on $|D_n^j\{t^{-\delta}\}|$ for δ real and $\delta \ne 0, -1, -2, \dots$, which are expressible explicitly in terms of the t_i, where $t_0 > t_1 > t_2 > \cdots > 0$, with no other restrictions on the t_i.

Lemma A.10. *Let δ be real and $\delta \ne 0, -1, -2, \dots$.*

(i) *When $\delta > 1$, there exists a constant $K_1 > 0$ such that*

$$|D_n^j\{t^{-\delta}\}| \ge K_1 n^{\delta-1}(t_j t_{j+1}\cdots t_{j+n})^{-1}. \tag{A.29}$$

(ii) *When $\delta < 1$, there exists a constant $K_2 > 0$ such that*

$$|D_n^j\{t^{-\delta}\}| \ge K_2 n^{\delta-1} t_{j+n}^{1-\delta}(t_j t_{j+1}\cdots t_{j+n})^{-1}. \tag{A.30}$$

Proof. The inequalities (A.29) and (A.30) follow by letting $(\delta_1, \delta_2) = (\delta, 1)$ and $(\delta_1, \delta_2) = (1, \delta)$ in (A.10), and by invoking

$$D_n^j\{t^{-1}\} = (-1)^n(t_j t_{j+1}\cdots t_{j+n})^{-1}. \quad \square$$

These results can be used in Lemma A.6, part (i) of Lemma A.7, and in Lemmas A.8 and A.9.

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