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Optimal Error Bounds for Convergents of a Family of Continued Fractions

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Let $\mathscr{F}$ be the family of continued fractions $K(a_p/1)$, where $a_1 = -g_1$, $a_p = (1 - g_{p-1})g_p x_p$, $p = 2, 3, \dots$, with $0 \le g_p \le 1$, g_p fixed, and $|x_p| \le 1$, $p = 2, 3, \dots$. In this work, we derive upper bounds on the errors in the convergents of $K(a_p/1)$ that are uniform for $\mathscr{F}$, and optimal in the sense that they are attained by some continued fraction in $\mathscr{F}$. For the special case $g_i = g < 1/2$, $i = 1, 2, \dots,$ this bound turns out to be especially simple, and for $g_i = g = 1/2$, $i = 1, 2, \dots,$ the known best form of the theorem of Worpitzki is obtained as an immediate corollary.

1. INTRODUCTION

Let $\mathscr{F}$ be the family of continued fractions $K(a_p/1)$, where

$$K(a_p/1) = \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \cdots}}}, \tag{1.1}$$

with

$$a_1 = -g_1, \qquad a_p = (1 - g_{p-1})g_p x_p, \qquad p = 2, 3, \dots;$$
$$0 \le g_p \le 1, \qquad g_p \text{ fixed}, \qquad p = 1, 2, \dots; \qquad |x_p| \le 1, \qquad p = 2, 3, \dots . \tag{1.2}$$

Denote the special continued fraction in $\mathscr{F}$ for which $x_p = -1$, $p =$

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$2, 3, \ldots,$ by $K(b_p/1)$, i.e.,

$$K(b_p/1) = \cfrac{b_1}{1 + \cfrac{b_2}{1 + \cfrac{b_3}{1 + \cdots}}}, \tag{1.3}$$

with

$$b_1 = -g_1, \qquad b_p = -(1 - g_{p-1})g_p, \qquad p = 2, 3 \ldots . \tag{1.4}$$

For any integer $m \geq 1$, let us define

$$f_n^{(m)} = \cfrac{a_m}{1 + \cfrac{a_{m+1}}{1 + \ddots + \cfrac{a_{m+n-1}}{1}}}, \qquad n = 1, 2, \ldots,$$

$$h_n^{(m)} = \cfrac{b_m}{1 + \cfrac{b_{m+1}}{1 + \ddots + \cfrac{b_{m+n-1}}{1}}}, \qquad n = 1, 2, \ldots,$$

$$f^{(m)} = \lim_{n \to \infty} f_n^{(m)},$$

$$h^{(m)} = \lim_{n \to \infty} h_n^{(m)},$$

$$S_n = 1 + \sum_{p=1}^{n} \prod_{k=1}^{p} \frac{g_k}{1 - g_k} \qquad \text{when } 0 \leq g_p < 1, \qquad p = 1, \ldots, n,$$

$$S_n = +\infty \qquad \text{when } 0 \leq g_p < 1, \qquad p = 1, \ldots, N-1,$$

$$g_N = 1, \qquad \text{and } n \geq N,$$

$$S = \lim_{n \to \infty} S_n \qquad (\text{possibly } +\infty). \tag{1.5}$$

Note that as soon as $g_{N-1} = 1$ or $g_N = 0$ for some N, we have $a_N = 0$, hence $f_n^{(1)} = f_{N-1}^{(1)} = f^{(1)}$, $h_n^{(1)} = h_{N-1}^{(1)} = h^{(1)}$, and $S_n = S_{N-1} = S$ for $n = N, N+1, \ldots$.

In Theorem 1.1 below, we state a fundamental result that is proved in [7, pp. 45–46, Theorem 11.1].

THEOREM 1.1. (i) *The* $K(a_p/1)$ *in* $\mathscr{F}$ *converge uniformly (in the* x_p, $p = 2, 3, \ldots$).

(ii) $f^{(1)}$ *and* $f_n^{(1)}$, $n = 1, 2, \ldots$, *are all in the disk* $\{z\colon |z| \le 1 - 1/S\}$, *and* $h^{(1)} = 1/S - 1$.

(iii) $f^{(1)}$ *and* $f_n^{(1)}$, $n = 1, 2, \ldots$, *are also in the disk* $\{z\colon |z + 1/(2 - g_1)| \le (1 - g_1)/(2 - g_1)\}$.

The purpose of the present work is to give an upper bound on $|f^{(1)} - f_n^{(1)}|$ that is (i) independent of the x_p, hence is uniform for $\mathscr{F}$, and (ii) is optimal in the sense that it is attained by a member of $\mathscr{F}$, namely, by $K(b_p/1)$.

Theorem 1.2 below is the main result of the present work.

THEOREM 1.2. (i) *For any two integers* l *and* n, $l > n \ge 1$, *we have*

$$|f_l^{(1)} - f_n^{(1)}| \le |h_l^{(1)} - h_n^{(1)}| = \frac{1}{S_n} - \frac{1}{S_l}. \tag{1.6}$$

(ii) *For any integer* $n \ge 1$ *we have*

$$|f^{(1)} - f_n^{(1)}| \le |h^{(1)} - h_n^{(1)}| = \frac{1}{S_n} - \frac{1}{S}. \tag{1.7}$$

Both bounds in (1.6) *and* (1.7) *are independent of the* x_p *and are optimal for* $\mathscr{F}$.

The following corollaries are simple consequences of Theorem 1.2, and their proofs are omitted.

COROLLARY 1. *In the case* $S = +\infty$, *we have*

$$|f^{(1)} - f_n^{(1)}| \le |h^{(1)} - h_n^{(1)}| = \frac{1}{S_n}, \qquad n = 1, 2, \ldots. \tag{1.8}$$

COROLLARY 2. (i) *In the case* $g_i = g$, $i = 1, 2, \ldots$, *with* $0 < g < 1/2$, *we have*

$$|f^{(1)} - f_n^{(1)}| \le |h^{(1)} - h_n^{(1)}| = \left(\frac{1 - \alpha}{1 - \alpha^{n+1}}\right)\alpha^{n+1}, \qquad n = 1, 2, \ldots, \tag{1.9}$$

where

$$\alpha = \frac{g}{1-g} \in (0,1). \tag{1.10}$$

(ii) *In the case* $g_i = g = 1/2$, $i = 1, 2, \dots,$ *we have*

$$|f^{(1)} - f_n^{(1)}| \le |h^{(1)} - h_n^{(1)}| = \frac{1}{n+1}, \qquad n = 1, 2, \dots . \tag{1.11}$$

First, $\mathscr{F}$ in part (i) of Corollary 2 is the family of continued fractions $K(a_p/1)$ for which $|a_p| \le (1-g)g < 1/4$, $p = 2, 3, \dots$. The convergence problem for this family is also considered in [6, p. 118], where a bound of the form $|f^{(1)} - f_n^{(1)}| \le C\alpha^n$, $n = 1, 2, \dots,$ is given, $C > 0$ being a constant independent of n. Clearly, this bound is not optimal. From (1.9), we can also obtain the bound $|f^{(1)} - f_n^{(1)}| \le [\alpha/(1+\alpha)]\alpha^n$, $n = 1, 2, \dots,$ and our constant $\alpha/(1+\alpha)$ is smaller than C of [6].

Next, $\mathscr{F}$ in part (ii) of Corollary 2 is the family of continued fractions for which $|a_p| \le 1/4$, $p = 2, 3, \dots$. The result in (1.11) then is the optimal form of the theorem of Worpitzki. For different proofs of this classical theorem, see [2, 3, 6, 7]. This optimal form of Worpitzki's theorem is not new, however, and is given in [2, p. 513, Problem 2]. Note that (1.11) follows from Corollary 1 by noting that $S_n = n + 1$ when $g_i = 1/2$, $i = 1, 2, \dots$. It also follows from (1.9) by letting $\alpha \to 1$ there. (The bound of [6] mentioned in the previous paragraph does not produce any information on rate of convergence or convergence as we let $\alpha \to 1$ there.)

The simplicity of the result in part (i) of Corollary 2 is due to the fact that S_n in (1.5) is a partial sum of a geometric series, hence is known analytically. This observation enables us to obtain simple bounds also for cases more complicated than that treated in Corollary 2. For example, when $g_{i+kq} = g_i \in (0,1)$, $k = 1, 2, \dots,$ $i = 1, 2, \dots, q$, for some positive integer q, i.e., when $K(b_p/1)$ is a periodic continued fraction, S turns out to be the sum of q geometric series, and S_n can again be expressed in a simple manner. If we let $\delta = \prod_{k=1}^{q} g_k/(1-g_k)$ in this case, the following can be shown to hold:

$$\text{when } \delta < 1, \quad |f^{(1)} - f_n^{(1)}| = O(\delta^{n/q}) \quad \text{as } n \to \infty,$$

$$\text{when } \delta > 1, \quad |f^{(1)} - f_n^{(1)}| = O(\delta^{-n/q}) \quad \text{as } n \to \infty, \quad \text{and}$$

$$\text{when } \delta = 1, \quad |f^{(1)} - f_n^{(1)}| = O(n^{-1}) \quad \text{as } n \to \infty.$$

Examples in which S involves series other than geometric can also easily be constructed. For instance, when $g_i = 1/(i+1)$, $i = 1, 2, \dots,$ we have $S_n = \sum_{p=0}^{n} 1/p!$, $S = e$, hence (1.7) gives $|f^{(1)} - f_n^{(1)}| =$

$O(1/(n+1)!)$ as $n \to \infty$. When $g_i = i/(2i+1)$, $i = 1, 2, \ldots,$ we have $S_n = \sum_{p=0}^{n} 1/(p+1) = H_{n+1}$, the $(n+1)$st harmonic number, $S = +\infty$, hence (1.7) gives $|f^{(1)} - f_n^{(1)}| \le 1/H_{n+1} = O(1/\log n)$ as $n \to \infty$.

Finally, the authors have been informed by the referee that the techniques of the present work bear some relation to those used in [1, 4, 5].

2. PROOF OF MAIN RESULT

LEMMA 2.1. *The convergents* $h_n^{(1)}$ *are given by*

$$h_n^{(1)} = \frac{1}{S_n} - 1. \tag{2.1}$$

Proof. The proof of (2.1) can be achieved by induction on n by noting that

$$h_{n+1}^{(1)} = \frac{-g_1}{1 + (1 - g_1)\tilde{h}_n}, \tag{2.2}$$

where

$$\tilde{h}_n = \cfrac{-g_2}{1 + \cfrac{a_3}{1 + \cfrac{a_4}{1 + \ddots + \cfrac{a_{n+1}}{1}}}}. \tag{2.3}$$

We leave the details to the reader. ∎

By letting $n \to \infty$ in (2.1), we obtain

$$h^{(1)} = \frac{1}{S} - 1 \tag{2.4}$$

that is part of Theorem 1.1.

LEMMA 2.2. *Assume* $0 \le g_i < 1$, $i = 1, 2, \ldots$. *Then for all* $m \ge 1$ *and* $n \ge 1$,

$$|f_n^{(m)}| \le -h_n^{(m)} \in [0, 1). \tag{2.5}$$

Proof. First, $h_n^{(m)} \in (-1, 0]$ follows from (2.1) and from $1 \le S_n < \infty$. Obviously, (2.5) holds for $n = 1$ and all $m \ge 1$. Suppose it holds for some

$n \geq 1$ and all m. Then

$$|f_{n+1}^{(m)}| = \left| \frac{a_m}{1 + f_n^{(m+1)}} \right| \leq \frac{|a_m|}{|1 - |f_n^{(m+1)}||} \leq \frac{-b_m}{1 + h_n^{(m+1)}} = -h_{n+1}^{(m)}. \quad (2.6)$$

The second inequality in (2.6) follows from the fact that $|a_m| \leq -b_m$ and from the induction hypothesis $|f_n^{(m+1)}| \leq -h_n^{(m+1)} \in [0, 1)$. ▮

LEMMA 2.3. *For any integers l and n, $l > n \geq 1$, and $m \geq 1$, there holds*

$$f_l^{(m)} - f_n^{(m)} = (-1)^n f_l^{(m)} \prod_{k=1}^{n} \frac{f_{l-k}^{(m+k)}}{1 + f_{n-k}^{(m+k)}}, \quad (2.7)$$

where we define $f_0^{(m)} = 0$ for $m \geq 1$.

Proof. We have

$$f_l^{(m)} - f_n^{(m)} = \frac{a_m}{1 + f_{l-1}^{(m+1)}} - \frac{a_m}{1 + f_{n-1}^{(m+1)}}$$

$$= -\frac{f_l^{(m)}}{1 + f_{n-1}^{(m+1)}} \left(f_{l-1}^{(m+1)} - f_{n-1}^{(m+1)} \right). \quad (2.8)$$

Repeating (2.8) $n - 1$ times, we obtain (2.7). ▮

We can now prove part (i) of Theorem 1.2. Under the assumption $0 \leq g_i < 1$, $i = 1, 2, \ldots,$ from Lemmas 2.2 and 2.3 we have

$$|f_l^{(m)} - f_n^{(m)}| \leq |h_l^{(m)}| \prod_{k=1}^{n} \frac{|h_{l-k}^{(m+k)}|}{1 + h_{n-k}^{(m+k)}} = |h_l^{(m)} - h_n^{(m)}|. \quad (2.9)$$

The first part of (1.6) now follows by letting $m = 1$ in (2.9). The second part follows by invoking Lemma 2.1.

Part (ii) of Theorem 1.2 is obtained by letting $l \to \infty$ in part (i).

To complete the proof of Theorem 1.2, let us now consider $g_i < 1$, $i = 1, \ldots, N - 1$, and $g_N = 1$. For this case, $f^{(1)} = f_N^{(1)}$ and $h^{(1)} = h_N^{(1)} = -1$, and Lemmas 2.1, 2.2, and 2.3 apply to all $f_n^{(m)}$ and $h_n^{(m)}$ that are determined by $a_1, a_2, \ldots, a_N$. We leave the details to the reader.

Before closing, we also mention that the proof technique of the present paper also provides an independent proof of convergence for the sequence $\{f_n^{(1)}\}_{n=1}^{\infty}$. This is seen as follows: By the fact that $h^{(1)} = \lim_{n \to \infty} h_n^{(1)}$ exists, $\{h_n^{(1)}\}_{n=1}^{\infty}$ is a Cauchy sequence. From this and from (1.6), we therefore have that $\{f_n^{(1)}\}_{n=1}^{\infty}$ is a Cauchy sequence as well. Consequently, $f^{(1)} = \lim_{n \to \infty} f_n^{(1)}$ exists.

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