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FURTHER RESULTS ON CONVERGENCE AND STABILITY OF A GENERALIZATION OF THE RICHARDSON EXTRAPOLATION PROCESS *

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Abstract.

In an earlier paper by the author a detailed convergence and stability analysis of a generalization of the Richardson extrapolation process was given under certain conditions. In the present work these conditions are modified and relaxed considerably, and results are obtained on convergence and stability under the new conditions. As the previous ones, these new results are asymptotic in nature, and contain the former. The conditions of the present paper are naturally satisfied, e.g., by the trapezoidal rule approximations of finite range integrals of functions having algebraic and logarithmic end point singularities.

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1 Introduction and review of earlier results.

Let $A(y)$ be a scalar function of a discrete or continuous variable y, defined for $0 < y \leq b < \infty$. Let there exist constants A and $\alpha_k, \quad k = 1, 2, ...,$ and functions $\phi_k(y), \quad k = 1, 2, ...,$ which form an asymptotic sequence in the sense that

$$\phi_{k+1}(y) = o(\phi_k(y)) \quad \text{as } y \to 0+, \tag{1.1}$$

and assume that $A(y)$ has the asymptotic expansion

$$A(y) \sim A + \sum_{k=1}^{\infty} \alpha_k \phi_k(y) \quad \text{as } y \to 0+ \,. \tag{1.2}$$

The functions $A(y)$ and $\phi_k(y), \quad k = 1, 2, ...,$ are assumed to be known for $0 < y \leq b$, but A and $\alpha_k, \quad k = 1, 2, ...,$ are unknown. The problem is to obtain (or approximate) A, which in many cases is $\lim_{y \to 0+} A(y)$ when the latter exists. (When $\lim_{y \to 0+} A(y)$ does not exist, A is said to be the antilimit of $A(y)$ as $y \to 0+$.) An effective means for achieving this goal is the generalized Richardson extrapolation that we define in the next paragraph.

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Pick a decreasing sequence $\{y_l\}_{l=0}^{\infty}$, such that $y_l \in (0,b]$, $l = 0,1,\ldots$, and $\lim_{l\to\infty} y_l = 0$. Then, for each pair (j,p) of nonnegative integers, the solution for A_p^j of the system of linear equations

$$A(y_l) = A_p^j + \sum_{k=1}^{p} \bar{\alpha}_k \phi_k(y_l), \quad j \le l \le j+p, \tag{1.3}$$

is taken as an approximation to A. (Note that $\bar{\alpha}_1,\ldots,\bar{\alpha}_p$ are the additional unknowns in (1.3), so that the total number of unknowns there is the same as the number of equations, namely, $p+1$.)

The approximations A_p^j to A can be arranged in a two-dimensional table in the form

$$\begin{array}{llllll} p=0 & p=1 & p=2 & p=3 & p=4 & \cdots \\ A_0^0 & & & & & \\ A_0^1 & A_1^0 & & & & \\ A_0^2 & A_1^1 & A_2^0 & & & \\ A_0^3 & A_1^2 & A_2^1 & A_3^0 & & \\ A_0^4 & A_1^3 & A_2^2 & A_3^1 & A_4^0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \qquad A_0^j = A(y_j), \quad j = 0,1,\ldots\,. \tag{1.4}$$

Let us set for simplicity of notation

$$\begin{aligned} a(l) &= A(y_l), \quad l = 0,1,\ldots, \\ g_k(l) &= \phi_k(y_l), \quad l = 0,1,\ldots; \quad k = 1,2,\ldots, \\ I(l) &= 1, \quad l = 0,1,\ldots\,. \end{aligned} \tag{1.5}$$

Then the following results are true:

THEOREM 1.1. *For any sequence $b(l)$, $l = 0,1,\ldots$, let $f_p^j(b)$ be defined by*

$$f_p^j(b) = \begin{vmatrix} g_1(j) & g_2(j) & \cdots & g_p(j) & b(j) \\ g_1(j+1) & g_2(j+1) & \cdots & g_p(j+1) & b(j+1) \\ \vdots & \vdots & & \vdots & \vdots \\ g_1(j+p) & g_2(j+p) & \cdots & g_p(j+p) & b(j+p) \end{vmatrix}. \tag{1.6}$$

Then A_p^j can be expressed as the quotient of two determinants in the form

$$A_p^j = \frac{f_p^j(a)}{f_p^j(I)}. \tag{1.7}$$

THEOREM 1.2. *Define the polynomial $H_p^j(\lambda)$ by*

$$H_p^j(\lambda) = \begin{vmatrix} g_1(j) & \cdots & g_p(j) & 1 \\ g_1(j+1) & \cdots & g_p(j+1) & \lambda \\ \vdots & & \vdots & \vdots \\ g_1(j+p) & \cdots & g_p(j+p) & \lambda^p \end{vmatrix}. \tag{1.8}$$

Then A_p^j can also be expressed in the form

$$A_p^j = \sum_{i=0}^{p} \gamma_{p,i}^j \; a(j+i), \tag{1.9}$$

where $\gamma_{p,i}^j$ are uniquely defined by

$$\sum_{i=0}^{p} \gamma_{p,i}^j \; \lambda^i = \frac{H_p^j(\lambda)}{H_p^j(1)}, \tag{1.10}$$

from which we also have, by letting $\lambda = 1$,

$$\sum_{i=0}^{p} \gamma_{p,i}^j = 1. \tag{1.11}$$

The general setting of this generalization of the Richardson extrapolation process that is given in (1.1)–(1.3) can be found in Hart et al. [3]. Levin [5] seems to be the first to give the determinant representation in Theorem 1.1. This result can be obtained from (1.3) by applying Cramer's rule. While the results in (1.9) and (1.11) are simple consequences of (1.7), that of (1.10) is somewhat complicated to obtain, and it was given first in Sidi [10]. For the case in which $\phi_k(y_l)$ have no particular structure, Schneider [7] gave the first recursive algorithm for the A_p^j, which has been denoted the E-algorithm. By using different techniques the E-algorithm was later rederived by Håvie [4] and by Brezinski [1]. Recently, Ford and Sidi [2] derived another recursive algorithm for the A_p^j, which is substantially more economical than the E-algorithm. (This new algorithm actually forms an essential part of the $W^{(m)}$-algorithm of Ford and Sidi (1987) that is used in implementing the generalized Richardson extrapolation process of Sidi [8], which further generalizes the one we discuss in the present work.)

The important problems concerning the extrapolation procedure above are those of convergence and stability. For the limiting process in which $j \to \infty$ and p is held fixed, the following results are known:

THEOREM 1.3. *Assume that*

$$\lim_{l\to\infty} \frac{\phi_k(y_{l+1})}{\phi_k(y_l)} = \lim_{l\to\infty} \frac{g_k(l+1)}{g_k(l)} = b_k \neq 1, \quad k = 1, 2, \ldots, \tag{1.12}$$

and that

$$b_j \neq b_k \text{ for } j \neq k. \tag{1.13}$$

(i) The $\gamma_{p,i}^j$ satisfy

$$\lim_{j\to\infty} \sum_{i=0}^{p} \gamma_{p,i}^j \; \lambda^i = \sum_{i=0}^{p} \tilde{\gamma}_{p,i} \; \lambda^i \equiv \prod_{i=1}^{p} \frac{\lambda - b_i}{1 - b_i}. \tag{1.14}$$

(ii) Let μ be the smallest positive integer for which $\alpha_{p+\mu} \neq 0$ in the asymptotic expansion of $A(y)$ given in (1.2). Then, whether $\lim_{y\to 0+} A(y)$ exists or not, A_p^j satisfies

$$A_p^j - A \sim \alpha_{p+\mu}\left[\prod_{i=1}^{p}\left(\frac{b_{p+\mu}-b_i}{1-b_i}\right)\right] g_{p+\mu}(j) \quad \text{as} \quad j\to\infty. \tag{1.15}$$

(iii) Set $\bar{\alpha}_k = \alpha_{p,k}^j$ in (1.3). Then, with μ as above,

$$\alpha_{p,k}^j - \alpha_k \sim \alpha_{p+\mu}\left(\frac{b_{p+\mu}-1}{b_k-1}\right)\left[\prod_{\substack{i=1\\ i\neq k}}^{p}\left(\frac{b_{p+\mu}-b_i}{b_k-b_i}\right)\right]\frac{g_{p+\mu}(j)}{g_k(j)} \tag{1.16}$$

as $j\to\infty$.

The results in (1.14), (1.15), and (1.16) are those given as, respectively, Theorem 2.4, Theorem 2.2, and Theorem 2.3 in Sidi [10]. Actually, we have generalized Theorems 2.2 and 2.3 in Sidi [10] by accounting for the possibility that α_{p+1} may become zero in (1.2). (For additional results of a different nature, see Sidi [9, Section 4].)

REMARKS:

1. The condition in (1.12) implies $\limsup_{n\to\infty} |g_k(n)|^{1/n} = b_k$, which, in turn, implies that $g_k(n)$ behaves, roughly speaking, like b_k^n as $n\to\infty$. The conditions in (1.1) and (1.12) together imply that $|b_{k+1}| \le |b_k|$ for all k, and this shows that the condition in (1.13) is, in fact, an independent one.

2. If $|b_{p+\mu}| < 1$ in (1.15), then $A_p^j \to A$ as $j\to\infty$, whether $\lim_{j\to\infty} A_k^j$, $k = 0,1,\ldots,p-1$, exist or not. The error $A_p^j - A$ tends to zero, roughly speaking, like $b_{p+\mu}^j$ for $j\to\infty$.

3. If $\mu = 1$, i.e., $\alpha_{p+1}\neq 0$, then the sequences $\{A_p^j\}_{j=0}^\infty$ and $\{A_{p+1}^j\}_{j=0}^\infty$ satisfy

$$\lim_{j\to\infty}\frac{A_{p+1}^j - A}{A_p^j - A} = 0, \tag{1.17}$$

whether they converge or not. In case they both converge, (1.17) is said to imply that $\{A_{p+1}^j\}_{j=0}^\infty$ converges more quickly than $\{A_p^j\}_{j=0}^\infty$. If $\mu > 1$, i.e., $\alpha_{p+1} = 0$, however, the μ sequences $\{A_{p+i}^j\}_{j=0}^\infty$, $i = 0,1,\ldots,\mu-1$, all have the same behavior; they all converge or diverge at exactly the same rate, namely, like $g_{p+\mu}(j)$ for $j\to\infty$. In summary, each column of the extrapolation table in (1.4) is at least as good as the one preceding it; it may be better or may behave in exactly the same way.

4. A weaker version of (1.15) has been proved also in Wimp [10, pp. 189–190). There it is assumed, in particular, that (1.1) holds uniformly in k

and $|\alpha_k| < \lambda^k$, $k = 1, 2, ...$, for some λ, in which case $a(n) = A(y_n)$ has the *convergent* expansion $a(n) = A + \sum_{k=1}^{\infty} \alpha_k g_k(n)$ and that this expansion converges *absolutely*, and *uniformly* in n. The weak result in (1.17) that is already contained in Theorem 1.3 has been proved in Brezinski [1] under the additional assumption that $\lim_{y\to 0+} A(y) = A$. A careful reading of the relevant proof in Brezinski [1] also reveals that, as in Wimp [11], at least a convergent expansion $a(n) = A + \sum_{k=1}^{\infty} \alpha_k g_k(n)$ is assumed. Note that in most problems of interest the asymptotic expansion in (1.2) is divergent. Furthermore, $\alpha_{p+1} \neq 0$ both in Wimp [11] and Brezinski [1].

The purpose of the present work is to provide further results on the convergence and stability of the A_p^j for $j \to \infty$ under conditions much weaker than those given in (1.1) and (1.13). The new conditions are described in detail in Section 2, in which the new results are stated as Theorems 2.1 - 2.3 for stability and convergence. The proofs of Theorems 2.1 and 2.2 are provided in Sections 3 and 4. An important result concerning $H_p^j(\lambda)$ is given as Theorem 3.1 in Section 3, and this result forms the basis of those in Theorems 2.1 - 2.3. Since the proof of Theorem 2.3 is almost identical to those of Theorems 2.1 and 2.2, it is omitted.

2 Generalized and extended theory.

2.1 Modified assumptions.

In this section we present new convergence and stability results for A_p^j as $j \to \infty$ under much weaker conditions than those that were used in Section 1. In particular, we will modify and relax the conditions in (1.1) and (1.13) considerably.

Let $\psi_k(y)$, $k = 1, 2, ...$, be functions defined for $y \in (0, b]$, where y is a discrete or continuous variable, and assume that

$$\psi_{k+1}(y) = O(\psi_k(y)) \text{ as } y \to 0+, \quad k = 1, 2, ... \tag{2.1}$$

What is implied by (2.1) is that the sequence $\{\psi_k(y)\}_{k=1}^{\infty}$, unlike $\{\phi_k(y)\}_{k=1}^{\infty}$ in Section 1, is not necessarily an asymptotic sequence in the strict sense of the concept.

Pick a decreasing sequence $\{y_l\}_{l=0}^{\infty}$ such that $y_l \in (0, b]$, $l = 0, 1, ...$, and $\lim_{l\to\infty}(y_{l+1}/y_l) = \omega$, for some $\omega \in (0, 1)$. Obviously, $y_l \to 0$ as $l \to \infty$ at least exponentially in l. Assume that

$$\lim_{n\to\infty} \frac{\psi_k(y_{n+1})}{\psi_k(y_n)} = c_k \neq 1, \quad k = 1, 2, ..., \tag{2.2}$$

and that the c_k are distinct, i.e.,

$$c_j \neq c_k \text{ if } j \neq k. \tag{2.3}$$

Note that (2.2) implies that for any $\varepsilon > 0$ there exist two positive constants L_1 and L_2 such that $L_1(|c_k| - \varepsilon)^n \leq |\psi_k(y_n)| \leq L_2(|c_k| + \varepsilon)^n$ for all large n. As a result of this we can show that

$$|c_1| \geq |c_2| \geq |c_3| \geq \cdots, \tag{2.4}$$

and that

$$|c_k| > |c_{k+1}| \text{ implies } \psi_{k+1}(y) = o(\psi_k(y)) \text{ as } y \to 0+. \tag{2.5}$$

We must emphasize that the converse of (2.5) is not necessarily true, and it need not be assumed to hold in our work.

Despite (2.3) and (2.4) we do not restrict the $|c_j|$ to be distinct. All we demand is that there be at most a finite number of the c_j having the same modulus. The implication of this demand is that $|c_k| > |c_{k+1}|$ holds for infinitely many values of k.

An immediate example of functions $\psi_k(y)$ satisfying all of the conditions in (2.1)–(2.3) is $\psi_k(y) = y^{\sigma_k}$, $\text{Re}\,\sigma_1 \geq \text{Re}\,\sigma_2 \geq \cdots$. For these $\psi_k(y)$ we have $c_k = \omega^{\sigma_k}$ and $|c_k| = \omega^{\text{Re}\sigma_k}$, $k = 1, 2, \ldots$. Also $|\psi_k(y)| = |\psi_s(y)|$ when $\text{Re}\,\sigma_k = \text{Re}\,\sigma_s$.

With the $\psi_k(y)$ and $\{y_l\}_{l=0}^{\infty}$ as described above, we now assume that the function $A(y)$ has the asymptotic expansion

$$A(y) \sim A + \sum_{k=1}^{\infty} \left[\sum_{i=0}^{q_k} \alpha_{ki} (\log y)^i \right] \psi_k(y) \quad \text{as } y \to 0+. \tag{2.6}$$

As before, A is $\lim_{y \to 0+} A(y)$ when this limit exists. Otherwise, A is the antilimit of $A(y)$. The q_k are some known nonnegative integers. The constants α_{ki} are unknown.

Functions $A(y)$ that satisfy (2.6) arise very naturally as Euler-Maclaurin expansions in the trapezoidal rule approximations of integrals of the form

$$\int_0^1 x^{\sigma} (\log x)^q g(x) dx,$$

where $\sigma > -1, q$ is a positive integer, and $g(x)$ is infinitely differentiable over $[0, 1]$. See Navot [6] for $q = 1$. For a brief survey see also Sidi [9].

For the sake of completeness, we mention that what is meant by (2.6) is that for any positive integer N

$$A(y) = A + \sum_{k=1}^{N-1} \left[\sum_{i=0}^{q_k} \alpha_{ki} (\log y)^i \right] \psi_k(y) + O((\log y)^{\hat{q}} \psi_N(y)) \quad \text{as } y \to 0+, \tag{2.7}$$

where $\hat{q}$ is the maximum of q_k, $k = N, N+1, \ldots$, for which the corresponding c_k have the same modulus.

Finally, the condition that $\lim_{l \to \infty} (y_{l+1}/y_l) = \omega$, for some $\omega \in (0, 1)$ implies that

$$\frac{y_{n+1}}{y_n} = \omega + \varepsilon_n, \quad \varepsilon_n = o(1) \quad \text{as } n \to \infty. \tag{2.8}$$

We supplement (2.8) by the extra condition that

$$(\log y_n)^{\nu} \varepsilon_n = o(1) \quad \text{as } n \to \infty, \quad \text{all } \nu \geq 0. \tag{2.9}$$

This is satisfied, e.g., when $\varepsilon_n = O(y_n^\tau)$ as $n \to \infty$, for some $\tau > 0$. Obviously, (2.9) holds trivially when $\varepsilon_n = 0$ for all n, i.e., when $y_n = \omega^n y_0$, $n = 0, 1, \ldots$.

We mentioned in the beginning of this section that we would modify and relax the conditions in (1.1) and (1.13). We now describe how this comes about in the present setting. First, let us rename the functions $\psi_k(y)(\log y)^s$, $s = 0, 1, \ldots, q_k$, $k = 1, 2, \ldots$, by $\phi_i(y)$, $i = 1, 2, \ldots$, for short, and order them such that $\phi_{i+1}(y) = O(\phi_i(y))$ as $y \to 0+$. If, for some i and k, $\phi_{i+1}(y) = \psi_{k+1}(y)$, and $\lim_{y\to 0+} |\psi_{k+1}(y)/\psi_k(y)| = C$ for some $C > 0$, so that $|c_k| = |c_{k+1}|$ in addition, everything being consistent with (2.1), then we have $\lim_{y\to 0+} |\phi_{i+1}(y)/\phi_i(y)| = C$ and *not* $\phi_{i+1}(y) = o(\phi_i(y))$ as $y \to 0+$. (The latter holds by (2.5) if $|c_k| > |c_{k+1}|$ above, which occurs for infinitely many values of k.) This shows that (1.1) is not necessarily satisfied by all members of the sequence $\{\phi_i(y)\}_{i=1}^{\infty}$. Next, for all $s = 0, 1, \ldots$, we have

$$\lim_{n\to\infty} \frac{\psi_k(y_{n+1})(\log y_{n+1})^s}{\psi_k(y_n)(\log y_n)^s} = c_k,$$

independently of s, and this shows that (1.13) is not satisfied when at least one of the integers q_k is nonzero.

2.2 Statement of Main Results

We now state Theorems 2.1 and 2.2, which are two of the main results of this work.

THEOREM 2.1. *Let the integer p be given by*

$$p = \sum_{k=1}^{t}(q_k + 1) = \sum_{k=1}^{t} \nu_k, \quad \nu_k \equiv q_k + 1, \quad k = 1, 2, \ldots , \tag{2.10}$$

and let $\phi_1(y), \ldots, \phi_p(y)$ in (1.3) stand for the p functions $\psi_k(y)(\log y)^i$, $i = 0, 1, \ldots, q_k$, $k = 1, \ldots, t$. Then the $\gamma_{p,i}^j$ in (1.9) satisfy

$$\lim_{j\to\infty} \sum_{i=0}^{p} \gamma_{p,i}^j \lambda^i = \sum_{i=0}^{p} \tilde{\gamma}_{p,i} \lambda^i \equiv \prod_{i=1}^{t} \left(\frac{\lambda - c_i}{1 - c_i}\right)^{\nu_i}, \tag{2.11}$$

as a result of which we also have

$$\lim_{j\to\infty} \sum_{i=0}^{p} |\gamma_{p,i}^j| = \sum_{i=0}^{p} |\tilde{\gamma}_{p,i}| < \infty. \tag{2.12}$$

Note that the major implication of the result in Theorem 2.1 is that the extrapolation procedure involving the A_p^j with p as in (2.10) is stable for all j as $\sum_{i=0}^{p} |\gamma_{p,i}^j|$ is bounded in j. Note also that Theorem 2.1 does not depend on $A(y)$ and its asymptotic expansion is (2.6), but only on the properties of the sequences $\{\psi_k(y_n)(\log y_n)^i\}_{n=0}^{\infty}$, $i = 0, 1, \ldots, q_k$, $k = 1, 2, \ldots$, namely, on (2.2), (2.3), (2.8), and (2.9).

THEOREM 2.2. *Let p be as in Theorem 2.1, and write (2.7) in the form*

$$A(y) = A + \sum_{k=1}^{t} \left[\sum_{i=0}^{q_k} \alpha_{ki} (\log y)^i \right] \psi_k(y) + R_t(y). \tag{2.13}$$

Then

$$A_p^j - A = O(R_t(y_j)) \quad \text{as } j \to \infty. \tag{2.14}$$

A much more refined and quantitative version of (2.14) can be given as follows: Let μ be the integer for which

$$|c_{t+1}| = \cdots = |c_{t+\mu}| > |c_{t+\mu+1}|, \tag{2.15}$$

and assume without loss of generality that not all of the coefficients α_{kq_k}, $k = t+1, ..., t+\mu$, are zero. (In case all of the α_{kq_k}, $k = t+1, ..., t+\mu$, are zero, the $\psi_k(y)$ and q_k with $k > t$, i.e., those $\psi_k(y)$ and q_k that are present in the asymptotic expansion of $R_t(y)$ for $y \to 0+$, can be renamed so that this assumption is realized.) Then

$$A_p^j - A = \sum_{k=t+1}^{t+\mu} \left\{ \alpha_{kq_k} \left[\prod_{i=1}^{t} \left(\frac{c_k - c_i}{1 - c_i} \right)^{\nu_i} \right] \psi_k(y_j)(\log y_j)^{q_k} + \eta_{j,k} \right\}, \tag{2.16}$$

where

$$\eta_{j,k} = o(\psi_k(y_j)(\log y_j)^{q_k}) \quad \text{as } j \to \infty, \quad k = t+1, ..., t+\mu. \tag{2.17}$$

COROLLARY 1. *If $\mu = 1$, i.e., $|c_{t+1}| > |c_{t+2}|$, and if $\alpha_{t+1,q_{t+1}} \neq 0$, then precisely*

$$A_p^j - A \sim \alpha_{t+1,q_{t+1}} \left[\prod_{i=1}^{t} \left(\frac{c_{t+1} - c_i}{1 - c_i} \right)^{\nu_i} \right] \psi_{t+1}(y_j)(\log y_j)^{q_{t+1}} \quad \text{as } j \to \infty. \tag{2.18}$$

COROLLARY 2. *If*

$$|c_{t+1}| > |c_{t+2}| > |c_{t+3}| \tag{2.19}$$

and $\alpha_{kq_k} \neq 0$, $k = t+1, t+2$, and we set

$$p_s = \sum_{k=1}^{s} (q_k + 1) = \sum_{k=1}^{s} \nu_k, \quad s = 1, 2, ..., \tag{2.20}$$

then

$$\lim_{j \to \infty} \frac{A_{p_{t+1}}^j - A}{A_{p_t}^j - A} = 0. \tag{2.21}$$

Comparing these results with the earlier ones reviewed in the previous section, we see that the latter are actually contained in the former.

Now Theorems 2.1 and 2.2 do not cover all values of p, but only those values given in (2.10). Similar results hold for all the remaining values of p. These results are given as Theorem 2.3 below.

THEOREM 2.3. *Let the integer p be given as*

$$p = \sum_{k=1}^{t} \nu_k + s, \quad 1 \leq s \leq \nu_{t+1} - 1, \quad \text{for } \nu_{t+1} > 1, \tag{2.22}$$

and let $\phi_1(y), ..., \phi_p(y)$ in (1.3) stand for the p functions $\psi_k(y)(\log y)^i$, $i = 0, 1, ..., q_k$, $k = 1, ..., t$, and $k = t + 1$, $i = 0, 1, ..., s - 1$. Here $t = 0$ is also possible. Then the following are analogous to Theorems 2.1 and 2.2.

(i) The $\gamma_{p,i}^{j}$ satisfy

$$\lim_{j\to\infty} \sum_{i=0}^{p} \gamma_{p,i}^{j}\ \lambda^i = \sum_{i=0}^{p} \tilde{\gamma}_{p,i}\ \lambda^i \equiv \left[\prod_{i=1}^{t} \left(\frac{\lambda - c_i}{1 - c_i}\right)^{\nu_i}\right] \left(\frac{\lambda - c_{t+1}}{1 - c_{t+1}}\right)^{s}. \tag{2.23}$$

Hence (2.12) is satisfied as well.

(ii) Write (2.7) in the form

$$\begin{aligned} A(y) = A \quad &+ \quad \sum_{k=1}^{t} \left[\sum_{i=0}^{q_k} \alpha_{ki} (\log y)^i\right] \psi_k(y) \\ &+ \quad \left[\sum_{i=0}^{s-1} \alpha_{t+1,i} (\log y)^i\right] \psi_{t+1}(y) + R_{t,s}(y). \end{aligned} \tag{2.24}$$

Then

$$A_p^j - A = O(R_{t,s}(y_j)) \quad \text{as } j \to \infty. \tag{2.25}$$

3 Proof of Theorem 2.1.

We start with the determinantal representation given in (1.10). Obviously, it is enough to treat only $H_p^j(\lambda)$ for $j \to \infty$. First, $H_p^j(\lambda)$ is as given in (1.8) with

$$\begin{aligned} g_i(j) &= \psi_1(y_j)(\log y_j)^{i-1}, \quad 1 \leq i \leq \nu_1 \equiv q_1 + 1, \\ g_{\nu_1+i}(j) &= \psi_2(y_j)(\log y_j)^{i-1}, \quad 1 \leq i \leq \nu_2 \equiv q_2 + 1, \\ g_{\nu_1+\nu_2+i}(j) &= \psi_3(y_j)(\log y_j)^{i-1}, \quad 1 \leq i \leq \nu_3 \equiv q_3 + 1, \text{ and so on.} \end{aligned} \tag{3.1}$$

Next, we perform only *column* transformations on the determinant $H_p^j(\lambda)$. We can actually perform these transformations on the first ν_1 columns, then on the next ν_2 columns, then on the next ν_3 columns, etc., independently. To demonstrate the technique we shall treat the first ν_1 columns.

Dividing each of these ν_1 columns by $g_1(j) = \psi_1(y_j)$, we obtain

(3.2) $$\frac{H_p^j(\lambda)}{[\psi_1(y_j)]^{\nu_1}} =$$

$$\begin{vmatrix} 1 & (\log y_j) & \cdots & (\log y_j)^{q_1} & g_{\nu_1+1}(j) & \cdots & g_p(j) & 1 \\ \tilde{\psi}_{1,1}^j & \tilde{\psi}_{1,1}^j(\log y_{j+1}) & \cdots & \tilde{\psi}_{1,1}^j(\log y_{j+1})^{q_1} & g_{\nu_1+1}(j+1) & \cdots & g_p(j+1) & \lambda \\ \tilde{\psi}_{1,2}^j & \tilde{\psi}_{1,2}^j(\log y_{j+2}) & \cdots & \tilde{\psi}_{1,2}^j(\log y_{j+2})^{q_1} & g_{\nu_1+1}(j+2) & \cdots & g_p(j+2) & \lambda^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \tilde{\psi}_{1,p}^j & \tilde{\psi}_{1,p}^j(\log y_{j+p}) & \cdots & \tilde{\psi}_{1,p}^j(\log y_{j+p})^{q_1} & g_{\nu_1+1}(j+p) & \cdots & g_p(j+p) & \lambda^p \end{vmatrix},$$

where

(3.3) $$\tilde{\psi}_{1,s}^j = \frac{\psi_1(y_{j+s})}{\psi_1(y_j)}, \quad s = 0, 1, \ldots .$$

Now from (2.8) we have $y_{j+s} = y_j \prod_{i=0}^{s-1}(\omega + \varepsilon_{j+i})$, $s = 1, 2, \ldots$, from which we obtain

(3.4) $$\log y_{j+s} = \log y_j + s \log \omega + O(\tilde{\varepsilon}_j) \quad \text{as } j \to \infty,$$

with

(3.5) $$\tilde{\varepsilon}_j = \max(|\varepsilon_j|, |\varepsilon_{j+1}|, \ldots, |\varepsilon_{j+p-1}|) = o(1) \quad \text{as } j \to \infty.$$

Consequently,

(3.6) $$(\log y_{j+s})^i = (\log y_j + s \log \omega)^i + O((\log y_j)^{i-1} \tilde{\varepsilon}_j) \quad \text{as } j \to \infty,$$

which, upon invoking the supplementary condition in (2.9), becomes

(3.7) $$(\log y_{j+s})^i = \sum_{k=0}^{i} \binom{i}{k} (\log y_j)^k (s \log \omega)^{i-k} + \varepsilon_{j,s,i},$$

with $\varepsilon_{j,s,i} = o(1)$ as $j \to \infty$. Substituting (3.7) in (3.2), and performing elementary column transformations on the 2nd, 3rd,..., ν_1th columns in this order, we eliminate all of the terms involving $\log y_j$, and obtain

(3.8) $$\frac{H_p^j(\lambda)}{[\psi_1(y_j)]^{\nu_1}} =$$

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & g_{\nu_1+1}(j) & \cdots & 1 \\ \tilde{\psi}_{1,1}^j & \tilde{\psi}_{1,1}^j[\log \omega + \varepsilon'_{j,1,1}] & \cdots & \tilde{\psi}_{1,1}^j[(\log \omega)^{q_1} + \varepsilon'_{j,1,q_1}] & g_{\nu_1+1}(j+1) & \cdots & \lambda \\ \tilde{\psi}_{1,2}^j & \tilde{\psi}_{1,2}^j[2\log \omega + \varepsilon'_{j,2,1}] & \cdots & \tilde{\psi}_{1,2}^j[(2\log \omega)^{q_1} + \varepsilon'_{j,2,q_1}] & g_{\nu_1+1}(j+2) & \cdots & \lambda^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \tilde{\psi}_{1,p}^j & \tilde{\psi}_{1,p}^j[p\log \omega + \varepsilon'_{j,p,1}] & \cdots & \tilde{\psi}_{1,p}^j[(p\log \omega)^{q_1} + \varepsilon'_{j,p,q_1}] & g_{\nu_1+1}(j+p) & \cdots & \lambda^p \end{vmatrix}$$

with $\varepsilon'_{j,s,i} = o(1)$ as $j \to \infty$. From (3.3) and (2.2) we have for $s = 1, 2, \ldots, p$

(3.9) $$\tilde{\psi}_{1,s}^j = \frac{\psi_1(y_{j+s})}{\psi_1(y_{j+s-1})} \frac{\psi_1(y_{j+s-1})}{\psi_1(y_{j+s-2})} \cdots \frac{\psi_1(y_{j+1})}{\psi_1(y_j)} = c_1^s + o(1) \quad \text{as } j \to \infty.$$

Thus, if we let $j \to \infty$ in the first ν_1 columns of the determinant on the right hand side of (3.8), these become

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ c_1 & c_1(\log\omega) & c_1(\log\omega)^2 & \cdots & c_1(\log\omega)^{q_1} \\ c_1^2 & c_1^2(2\log\omega) & c_1^2(2\log\omega)^2 & \cdots & c_1^2(2\log\omega)^{q_1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_1^p & c_1^p(p\log\omega) & c_1^p(p\log\omega)^2 & \cdots & c_1^p(p\log\omega)^{q_1} \end{bmatrix}. \tag{3.10}$$

In addition, we can divide the 1st, 2nd,..., ν_1th columns by $1, (\log\omega), ..., (\log\omega)^{q_1}$, respectively, as a result of which, (3.10) becomes

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ c_1 & c_1 1 & c_1 1^2 & \cdots & c_1 1^{q_1} \\ c_1^2 & c_1^2 2 & c_1^2 2^2 & \cdots & c_1^2 2^{q_1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_1^p & c_1^p p & c_1^p p^2 & \cdots & c_1^p p^{q_1} \end{bmatrix}. \tag{3.11}$$

We must note, however, that we should not let $j \to \infty$ in (3.8) without performing analogous transformations on the next $\nu_2, \nu_3, ..., \nu_t$ columns of $H_p^j(\lambda)$. The reason for this is that, as $j \to \infty$, columns of $H_p^j(\lambda)$ either tend to zero or are unbounded or are bounded but have no limit, and in any case the result is useless and/or meaningless.

Let us, therefore, assume that we have performed all the necessary column transformations on all columns of $H_p^j(\lambda)$. If we now let $j \to \infty$, we obtain

$$\lim_{j\to\infty} \frac{H_p^j(\lambda)}{\prod_{i=1}^t [\psi_i(y_j)]^{\nu_i}} = (\log\omega)^{\frac{1}{2}\sum_{i=1}^t q_i\nu_i} \det[\tilde{H}_1|\tilde{H}_2|\cdots|\tilde{H}_t|\boldsymbol{\lambda}], \tag{3.12}$$

where

$$\tilde{H}_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ c_i & c_i 1^1 & c_i 1^2 & \cdots & c_i 1^{q_i} \\ c_i^2 & c_i^2 2^1 & c_i^2 2^2 & \cdots & c_i^2 2^{q_i} \\ \vdots & \vdots & \vdots & & \vdots \\ c_i^p & c_i^p p^1 & c_i^p p^2 & \cdots & c_i^p p^{q_i} \end{bmatrix}, \quad i = 1, 2, ..., \tag{3.13}$$

and

$$\boldsymbol{\lambda} = (1, \lambda, ..., \lambda^p)^T. \tag{3.14}$$

We show in the appendix to this work that

$$\det[\tilde{H}_1|\tilde{H}_2|\cdots|\tilde{H}_t|\boldsymbol{\lambda}] = \left[\prod_{i=1}^t \left(\prod_{s=0}^{q_i} s!\right) c_i^{q_i\nu_i/2}\right] \left[\prod_{1\le k<s\le t} (c_s - c_k)^{\nu_k\nu_s}\right] \left[\prod_{i=1}^t (\lambda - c_i)^{\nu_i}\right]. \tag{3.15}$$

Combining (3.12) and (3.15), we have the following key result:

THEOREM 3.1. *The determinant $H_p^j(\lambda)$ satisfies*

$$\lim_{j\to\infty} \frac{H_p^j(\lambda)}{\prod_{i=1}^{t}[\psi_i(y_j)]^{\nu_i}} = K\prod_{i=1}^{t}(\lambda - c_i)^{\nu_i}, \tag{3.16}$$

where K is a constant that depends only on $\omega, c_1, ..., c_t, \nu_1, ..., \nu_t$, given by

$$K = \left[\prod_{i=1}^{t}\left(\prod_{s=0}^{q_i} s!\right)(c_i \log\omega)^{q_i\nu_i/2}\right]\left[\prod_{1\leq k<s\leq t}(c_s - c_k)^{\nu_k\nu_s}\right]. \tag{3.17}$$

The proof of Theorem 2.1 now follows from Theorem 3.1 if we also note that the c_i are distinct and different than 1 and $\omega \neq 1$ so that the right hand side of (3.16) is finite and nonzero as long as $\lambda \notin \{c_1, c_2, ..., c_t\}$.

4 Proof of Theorem 2.2.

As in Sidi [10], letting

$$r(l) = A(y_l) - A, \quad l = 0, 1, ..., \tag{4.1}$$

we have

$$A_p^j - A = \frac{f_p^j(r)}{f_p^j(I)}. \tag{4.2}$$

Now from (4.1), (2.13), and (3.1), we have

$$r(l) = \sum_{i=1}^{p} \delta_i g_i(l) + R_t(y_l), \quad l = 0, 1, ..., \tag{4.3}$$

where δ_i are the appropriate α_{ks}. Substituting (4.3) in the determinant representation of $f_p^j(r)$, c.f. (1.6), we have

$$f_p^j(r) = \sum_{i=1}^{p} \delta_i f_p^j(g_i) + f_p^j(\rho_t), \tag{4.4}$$

where g_i and ρ_t stand for the sequences $\{g_i(n)\}_{n=0}^{\infty}$ and $\{\rho_t(n) = R_t(y_n)\}_{n=0}^{\infty}$ respectively. For $i = 1, 2, ..., p$, we have $f_p^j(g_i) = 0$ as its determinant representation has two identical columns, namely, the ith and $(p+1)$st columns. Consequently, (4.4) becomes

$$f_p^j(r) = f_p^j(\rho_t). \tag{4.5}$$

Substituting (4.5) in (4.2), we have, in a manner analogous to (1.7) and (1.9), the result

$$A_p^j - A = \sum_{i=0}^{p} \gamma_{p,i}^j \rho_t(j+i) = \sum_{i=0}^{p} \gamma_{p,i}^j R_t(y_{j+i}), \tag{4.6}$$

from which we have

$$\text{(4.7)} \qquad \begin{aligned} |A_p^j - A| &\leq \sum_{i=0}^{p} |\gamma_{p,i}^j|\,|R_t(y_{j+i})| \\ &\leq \left(\sum_{i=0}^{p} |\gamma_{p,i}^j|\right)\left(\max_{0\leq i\leq p} |R_t(y_{j+i})|\right). \end{aligned}$$

From Theorem 2.1 we have that the sum $\sum_{i=0}^{p} |\gamma_{p,i}^j|$ is bounded for all large j. Also, (2.7) and (2.1) imply that $R_t(y_{j+i}) = O(R_t(y_j))$ as $j \to \infty$, $i = 1, 2, \ldots$, even when some of the coefficients α_{ks} with $k > t$ vanish. Combining these facts in (4.7), we finally obtain (2.14).

To prove the quantitative result in (2.15)–(2.17) we proceed from (4.6) and the expansion

$$\text{(4.8)} \qquad \rho_t(n) = R_t(y_n) = \sum_{k=t+1}^{t+\mu} \left[\alpha_{kq_k} h_k(n) + \Theta_k(n)\right],$$

where

$$\text{(4.9)} \qquad h_k(n) = \psi_k(y_n)(\log y_n)^{q_k}$$

$$\text{(4.10)} \qquad \Theta_k(n) = o(h_k(n)) \quad \text{as } n \to \infty.$$

That (4.8)–(4.10) is true can be shown by employing (2.7) and (2.5). (Also, more explicit expressions for the $\Theta_k(n)$ can be obtained directly from (2.7), although (4.10) is sufficient for our purposes.) Substituting (4.8) in (4.6), and changing the order of summation, we obtain

$$\text{(4.11)} \quad A_p^j - A = \sum_{k=t+1}^{t+\mu} \left[\alpha_{kq_k}\left(\sum_{i=0}^{p} \gamma_{p,i}^j h_k(j+i)\right) + \sum_{i=0}^{p} \gamma_{p,i}^j \Theta_k(j+i)\right],$$

which we rewrite in the form

$$\text{(4.12)} \; A_p^j - A = \sum_{k=t+1}^{t+\mu} \left[\alpha_{kq_k}\left(\sum_{i=0}^{p} \gamma_{p,i}^j \frac{h_k(j+i)}{h_k(j)}\right) + \sum_{i=0}^{p} \gamma_{p,i}^j \frac{\Theta_k(j+i)}{h_k(j)}\right] h_k(j).$$

But

$$\text{(4.13)} \qquad \frac{h_k(j+i)}{h_k(j)} = \frac{\psi_k(y_{j+i})}{\psi_k(y_j)}\left(\frac{\log y_{j+i}}{\log y_j}\right)^{q_k} = c_k^i + o(1), \quad \text{as } j \to \infty, \quad i = 0, 1, \ldots, p,$$

which can be shown by using (3.7) and (3.9). Similarly,

$$\text{(4.14)} \qquad \frac{\Theta_k(j+i)}{h_k(j)} = o\left(\frac{h_k(j+i)}{h_k(j)}\right) = o(1) \quad \text{as } j \to \infty,$$

by (4.13). Consequently,

$$\text{(4.15)} \qquad \lim_{j\to\infty} \sum_{i=0}^{p} \gamma_{p,i}^j \frac{h_k(j+i)}{h_k(j)} = \sum_{i=0}^{p} \tilde{\gamma}_{p,i} c_k^i = \prod_{i=1}^{t} \left(\frac{c_k - c_i}{1 - c_i}\right)^{\nu_i},$$

where we have employed (2.11) (from Theorem 2.1) and (4.13). Similarly,

$$\lim_{j\to\infty}\sum_{i=0}^{p}\gamma_{p,i}^{j}\frac{\Theta_k(j+i)}{h_k(j)}=0. \tag{4.16}$$

Combining (4.15) and (4.16) in (4.12), the result in (2.15)–(2.17) now follows.

Corollary 1 follows from (2.16) in a straightforward manner. Corollary 2 follows directly from Corollary 1.

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Appendix. Proof of (3.15).

The proof of (3.15) can be achieved by performing column transformations on each of the matrices $\tilde{H}_i$, $i = 1, 2, ..., t$, independently, as follows: First notice that for $m > 0$

$$m^s=\sum_{k=0}^{s}\tau_{sk}\binom{m}{k},\quad \tau_{s0}=0,\quad \tau_{ss}=s!, \tag{A.1}$$

where τ_{sk} are constants independent of m. That (A.1) holds follows from the fact that the binomial coefficients $\binom{m}{k}$, $k = 0, 1, ..., s$, form a basis for polynomials in m of degree $\leq s$. Substitute now (A.1) in the matrix $\tilde{H}_i$ given in (3.13). Next, leaving the 1st and 2nd columns of $\tilde{H}_i$ unchanged, perform the following transformations on the 3rd, 4th,...,$(\nu_i = q_i + 1)$st columns:

for $n = 3, 4, ..., \nu_i$ do
  for $l = 2, ..., n-1$ do
    multiply the lth column by $\tau_{n-1,l-1}/\tau_{l-1,l-1}$
    and subtract from the nth column, overwriting the latter.
  end do
end do

Let us denote this transformed $\tilde{H}_i$ by $\hat{H}_i$. We have

$$(A.2) \qquad \hat{H}_i = \begin{bmatrix} \binom{0}{0}c_i^0 & 0 & 0 & \cdots & 0 \\ \binom{1}{0}c_i^1 & \binom{1}{1}1!c_i^1 & 0 & \cdots & 0 \\ \binom{2}{0}c_i^2 & \binom{2}{1}1!c_i^2 & \binom{2}{2}2!c_i^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{p}{0}c_i^p & \binom{p}{1}1!c_i^p & \binom{p}{2}2!c_i^p & \cdots & \binom{p}{q_i}q_i!c_i^p \end{bmatrix}.$$

Taking the common factor $s!c_i^s$ out of the $(s+1)$st column of $\hat{H}_i$, $s = 1, 2, ..., q_i$, we obtain the matrix

$$(A.3) \qquad \bar{H}_i = \begin{bmatrix} \binom{0}{0}c_i^0 & 0 & 0 & \cdots & 0 \\ \binom{1}{0}c_i^1 & \binom{1}{1}c_i^0 & 0 & \cdots & 0 \\ \binom{2}{0}c_i^2 & \binom{2}{1}c_i^1 & \binom{2}{2}c_i^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{p}{0}c_i^p & \binom{p}{1}c_i^{p-1} & \binom{p}{2}c_i^{p-2} & \cdots & \binom{p}{q_i}c_i^{p-q_i} \end{bmatrix}.$$

As a result of all the above

$$(A.4) \qquad \det[\tilde{H}_1|\tilde{H}_2|\cdots|\tilde{H}_t|\boldsymbol{\lambda}] = \left[\prod_{i=1}^{t}\left(\prod_{s=1}^{q_i} s!c_i^s\right)\right]\det\bar{H},$$

where

$$\bar{H} = [\bar{H}_1|\bar{H}_2|\cdots|\bar{H}_t|\boldsymbol{\lambda}].$$

But $\bar{H}$ is the generalized Vandermonde matrix whose determinant is given by

$$(A.6) \qquad \det\bar{H} = \left[\prod_{1\leq i<j\leq t}(c_j - c_i)^{\nu_i\nu_j}\right]\prod_{i=1}^{t}(\lambda - c_i)^{\nu_i}.$$

From (A.4) and (A.6) the result in (3.15) now follows.