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Extension and Completion of Wynn's Theory on Convergence of Columns of the Epsilon Table*

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Let $\{S_n\}_{n=0}^{\infty}$ be such that $S_n \sim S + \sum_{j=1}^{\infty} a_j \lambda_j^n$ as $n \to \infty$, with $1 > |\lambda_1| > |\lambda_2| > \cdots$, such that $\lim_{j\to\infty} \lambda_j = 0$. A well-known result by Wynn states that when the Shanks transformation or its equivalent ε-algorithm is applied to $\{S_n\}_{n=0}^{\infty}$, then $\varepsilon_{2k}^{(n)} - S \sim a_{k+1} [\prod_{i=1}^{k} (\lambda_{k+1} - \lambda_i)/(1-\lambda_i)]^2 \lambda_{k+1}^n$ as $n \to \infty$. In the present work we extend this result (i) by allowing some of the λ_j to have the same modulus and (ii) by replacing the constants a_j by some polynomials $P_j(n)$ in n. Sequences $\{S_n\}_{n=0}^{\infty}$ with these characteristics arise frequently, e.g., in fixed point iterative solution of linear systems and in trapezoidal rule approximation of finite range integrals with logarithmic endpoint singularities and their multidimensional analogues. The results of this work are obtained by exploiting the connection between the Shanks transformation and Padé approximants and by using some recent results of the author on Padé approximants for meromorphic functions.

1. INTRODUCTION

Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of complex numbers whose limit or antilimit we denote by S. An effective means for computing approximations to S, in certain cases of importance, whether it is the limit or antilimit, is the well known transformation of Shanks [Sh]. This transformation generates an array of approximations denoted $e_k(S_n)$ that are defined by

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$$e_k(S_n) = \frac{\begin{vmatrix} S_n & S_{n+1} & \cdots & S_{n+k} \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \vdots & \vdots & & \vdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+k} \\ \vdots & \vdots & & \vdots \\ \Delta S_{n+k-1} & \Delta S_{n+k} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}}, \tag{1.1}$$

where $\Delta S_i = S_{i+1} - S_i$, $i = 0, 1, \ldots$. Normally, the $e_k(S_n)$ are computed with the help of the ε-algorithm of Wynn [W1] that is defined through the recursions

$$\begin{aligned} &\varepsilon_{-1}^{(n)} = 0, \qquad \varepsilon_0^{(n)} = S_n, \qquad n = 0, 1, ..., \\ &\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + \frac{1}{\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)}}, \qquad n, k = 0, 1, ..., \end{aligned} \tag{1.2}$$

and there holds

$$e_k(S_n) = \varepsilon_{2k}^{(n)}, \qquad n, k = 0, 1, 2, \ldots. \tag{1.3}$$

Commonly, the $\varepsilon_k^{(n)}$ are arranged in a two-dimensional table in the form

$$\begin{array}{cccccc} \varepsilon_{-1}^{(0)} = 0 & & & & & \\ & \varepsilon_0^{(0)} = S_0 & & & & \\ \varepsilon_{-1}^{(1)} = 0 & & \varepsilon_1^{(0)} & & & \\ & \varepsilon_0^{(1)} = S_1 & & \varepsilon_2^{(0)} & & \\ \varepsilon_{-1}^{(2)} = 0 & & \varepsilon_1^{(1)} & & \varepsilon_3^{(0)} & \\ & \varepsilon_0^{(2)} = S_2 & & \varepsilon_2^{(1)} & & \ddots \\ \varepsilon_{-1}^{(3)} = 0 & & \varepsilon_1^{(2)} & & \varepsilon_3^{(1)} & \\ \vdots & \varepsilon_0^{(3)} = S_3 & & \varepsilon_2^{(2)} & & \ddots \\ \vdots & \vdots & \varepsilon_1^{(3)} & & \varepsilon_3^{(2)} & \\ \vdots & \vdots & \vdots & \varepsilon_2^{(3)} & & \ddots \\ \vdots & \vdots & \vdots & \vdots & \varepsilon_3^{(3)} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{1.4}$$

The first theoretical results concerning the convergence of the columns of the ε-table in (1.4) were given by Wynn [W2]. An improved version of one of these famous results is given below as Theorem 1.1.

THEOREM 1.1. *Let $\lambda_1, \lambda_2, \ldots,$ be distinct nonzero complex numbers that satisfy*

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \cdots; \qquad \lambda_j \neq 1, \quad j = 1, 2, \ldots; \qquad \lim_{j \to \infty} \lambda_j = 0. \quad (1.5)$$

Let the sequence $\{S_n\}_{n=0}^{\infty}$ be such that

$$S_n \sim S + \sum_{j=1}^{\infty} a_j \lambda_j^n \qquad as \quad n \to \infty, \qquad (1.6)$$

for some complex numbers $a_j \neq 0$, $j = 1, 2, \ldots$. Then

$$\varepsilon_{2k}^{(n)} - S \sim a_{k+1} \left(\prod_{i=1}^{k} \frac{\lambda_{k+1} - \lambda_i}{1 - \lambda_i} \right)^2 \lambda_{k+1}^n \qquad as \quad n \to \infty. \qquad (1.7)$$

In connection with the sequence $\{S_n\}_{n=0}^{\infty}$ in Theorem 1.1, we note that if $|\lambda_1| < 1$, $\lim_{n \to \infty} S_n$ exists and we have $S = \lim_{n \to \infty} S_n$. Otherwise, $\lim_{n \to \infty} S_n$ does not exist, and S is said to be the antilimit of $\{S_n\}_{n=0}^{\infty}$. The same applies to all the sequences $\{S_n\}_{n=0}^{\infty}$ that we will be considering throughout this paper.

Theorem 1.1 differs from the corresponding result in [W2] with respect to the assumptions made on the λ_j. In [W2] it is assumed that the λ_j are either all positive or all negative and that $|\lambda_1| < 1$. A close look at the proof of [W2] reveals that it remains valid under the more general conditions given in (1.5).

Note that Theorem 1.1 does not cover the more general case in which some of the λ_j, either for $1 \leqslant j \leqslant k$ or for $j \geqslant k+1$, may have the same modulus, a situation that arises frequently when these λ_j can be present in the form of complex conjugate pairs. The relevant results for this case are given in Theorems 2.1 and 2.2 in the next section. Of these, Theorem 2.1 provides a complete expansion for $\varepsilon_{2k}^{(n)}$ and Theorem 2.2 is obtained by analyzing the dominant terms of this expansion. Theorem 2.2 shows that a result very similar to (1.7) of Theorem 1.1 holds for $\varepsilon_{2k}^{(n)}$ also in this case provided $|\lambda_k| > |\lambda_{k+1}|$; namely, there holds $\varepsilon_{2k}^{(n)} - S = O(|\lambda_{k+1}|^n)$ as $n \to \infty$. (Actually, Theorem 2.2 gives an optimally refined version of this result.) When $|\lambda_k| = |\lambda_{k+1}|$, however, it follows from Theorem 2.1 that the best we can say in general is that, under certain conditions, there may exist a subsequence $\{\varepsilon_{2k}^{(n_i)}\}_{i=1}^{\infty}$ for which $\varepsilon_{2k}^{(n_i)} - S = O(|\lambda_{k+1}|^{n_i})$ as $i \to \infty$. Hence our theory covers *all* of the even numbered columns of the ε-table.

Following the generalization of Section 2, in Section 3 we give a further generalization and completion of Wynn's result. In particular, the results of Section 3 cover those sequences $\{S_n\}_{n=0}^{\infty}$ that satisfy (1.6), where now

(i) some of the λ_j may have equal moduli, and

(ii) the *constants* a_j are replaced by some *polynomials* $P_j(n)$ in n.

This situation seems to be the ultimate generalization of (1.6) that occurs naturally. It arises, e.g., as a result of iterative solution of linear systems in the presence of defective (nondiagonalizable) iteration matrices, see [SiBr]. It also arises in numerical integration of functions that have logarithmic singularities (at the endpoints in one-dimensional integration, at corners or boundaries in multidimensional integration) through trapezoidal rule approximations. For results on one-dimensional singular integrals, see [Na1, Na2, LyNi], and for those on multidimensional singular integrals see [Ly, MLy, LyM, Si].

In Section 3 we present a complete convergence theory for the columns of the ε-algorithm as the latter is applied to the sequences mentioned in the previous paragraph. This theory too covers *all* of the even numbered columns of the ε-table. The main results of Section 3 are Theorem 3.1 and Theorem 3.2 that generalize Theorem 2.2, and a corollary to Theorems 3.1 and 3.2.

The proofs of the results of Sections 2 and 3 are provided in Section 4. The results of the recent paper [Si2] concerning the row convergence of the Padé table for meromorphic functions turn out to be crucial in the proofs. In Section 5 we will review some problems in which sequences $\{S_n\}_{n=0}^{\infty}$ covered in Section 3 occur.

2. Main Results: First Generalization

In this section we assume that the sequence $\{S_n\}_{n=0}^{\infty}$ is such that

$$S_n = S + \sum_{j=1}^{v} a_j \lambda_j^n + U_n, \qquad n = 0, 1, ..., \tag{2.1}$$

where $a_j \neq 0$ for all j and λ_j are, in general, complex distinct nonzero scalars that satisfy

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_v| > R^{-1}; \qquad \lambda_j \neq 1, \quad j = 1, ..., v; \qquad \text{some } R > 0, \tag{2.2}$$

and

$$U_n = O(\xi^n) \qquad \text{as} \quad n \to \infty, \tag{2.3}$$

with arbitrary ξ in the open interval $(R^{-1}, |\lambda_v|)$.

Theorem 2.1 below gives a complete expansion of $\varepsilon_{2k}^{(n)} - S$ under the conditions above. A nice feature of this result is that it is expressed in a simple manner in terms of Vandermonde determinants only.

THEOREM 2.1. *Let $\{S_n\}_{n=0}^{\infty}$ be precisely as in the first paragraph of this section. Then*

$$\varepsilon_{2k}^{(n)} - S = \frac{N_k^{(n)}}{D_k^{(n)}}, \tag{2.4}$$

with

$$N_k^{(n)} = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{k+1} \leqslant \nu} \left(\prod_{s=1}^{k+1} a_{j_s} \right) [V(\lambda_{j_1}, ..., \lambda_{j_k}, \lambda_{j_{k+1}})]^2 \left(\prod_{s=1}^{k+1} \lambda_{j_s} \right)^n + G_{n,k}, \tag{2.5}$$

and

$$D_k^{(n)} = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant \nu} \left(\prod_{s=1}^{k} a_{j_s} \right) [V(\lambda_{j_1}, ..., \lambda_{j_k}, 1)]^2 \left(\prod_{s=1}^{k} \lambda_{j_s} \right)^n + E_{n,k}, \tag{2.6}$$

where $V(\xi_1, ..., \xi_s)$ is the Vandermonde determinant defined by

$$V(\xi_1, ..., \xi_s) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_s \\ \vdots & \vdots & & \vdots \\ \xi_1^{s-1} & \xi_2^{s-1} & \cdots & \xi_s^{s-1} \end{vmatrix} = \prod_{1 \leqslant i < j \leqslant s} (\xi_j - \xi_i), \tag{2.7}$$

and

$$G_{n,k} = O\left(\left| \prod_{j=1}^{k} \lambda_j \right|^n \xi^n \right), \qquad E_{n,k} = O\left(\left| \prod_{j=1}^{k-1} \lambda_j \right|^n \xi^n \right) \qquad \text{as} \quad n \to \infty. \tag{2.8}$$

If $U_n = 0$ for all $n \geqslant N$, then $G_{n,k} = 0$ and $E_{n,k} = 0$ for all $n \geqslant N$ as well.

Note that $D_k^{(n)}$ is simply the denominator of the quotient on the right-hand side of (1.1).

Starting now with Theorem 2.1, we obtain the following generalization of Theorem 1.1.

THEOREM 2.2. *Let $\{S_n\}_{n=0}^{\infty}$ be as in the first paragraph of this section.*

(i) *Let $k = \nu$. Then*

$$\varepsilon_{2k}^{(n)} - S = O(\xi^n) \qquad \text{as} \quad n \to \infty. \tag{2.9}$$

If $U_n = 0$ for $n \geqslant N$, we have $\varepsilon_{2k}^{(n)} = S$ for $n \geqslant N$.

(ii) *Assume that* $|\lambda_t| > |\lambda_{t+1}|$ *for some* $t < \nu$, *and let* $k = t$. *Let* r *be that integer for which*

$$|\lambda_1| \geqslant \cdots \geqslant |\lambda_t| > |\lambda_{t+1}| = \cdots = |\lambda_{t+r}| > |\lambda_{t+r+1}| \geqslant \cdots, \qquad (2.10)$$

where $|\lambda_{t+r+1}|$ *is meant to stand for* ξ *when* $t + r = \nu$. *Then, whether* $|\lambda_{t+1}| < 1$ *or not,*

$$\begin{aligned} \varepsilon_{2k}^{(n)} - S &= \sum_{j=t+1}^{t+r} a_j \left(\prod_{i=1}^{t} \frac{\lambda_j - \lambda_i}{1 - \lambda_i} \right)^2 \lambda_j^n + o(\lambda_{t+1}^n) \qquad \text{as} \quad n \to \infty, \\ &= O(|\lambda_{k+1}|^n) \qquad \text{as} \quad n \to \infty. \end{aligned} \qquad (2.11)$$

Note. Theorem 2.2 covers the cases in which $k = t$ with $|\lambda_t| > |\lambda_{t+1}|$. It does not, however, cover the remaining values of k, i.e., those k's for which $t + 1 \leqslant k \leqslant t + r - 1$, with t and r as in (2.10) and $r > 1$. The best that we can say for these values of k is that, under certain conditions, there may exist a subsequence of $\{\varepsilon_{2k}^{(n)}\}_{n=0}^{\infty}$ that satisfies $\varepsilon_{2k}^{(n)} - S = O(|\lambda_{k+1}|^n)$ as $n \to \infty$. It can be shown that such a subsequence exists when $k = t + 1$.

In all our results above we have assumed that S_n satisfies (2.1), the right-hand side of which is not a genuine asymptotic expansion. We now state a theorem that says that these results hold also when we replace the right-hand side of (2.1) by a genuine asymptotic expansion, under an additional mild assumption on the λ_j.

THEOREM 2.3. *Let the sequence* $\{S_n\}_{n=0}^{\infty}$ *be such that*

$$S_n \sim S + \sum_{j=1}^{\infty} a_j \lambda_j^n \qquad \text{as} \quad n \to \infty, \qquad (2.12)$$

where $a_j \neq 0$ *for all* j, *and* λ_j *are in general complex distinct nonzero scalars that satisfy*

$$|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots; \qquad \lambda_j \neq 1, \quad j = 1, 2, ...; \qquad \lim_{j \to \infty} \lambda_j = 0. \qquad (2.13)$$

This implies that there is an infinite number of integer pairs (t, r) *for which* (2.10) *holds. Then, with* $\nu = \infty$, *part* (ii) *of Theorem 2.2 applies to the* ε*-table of* $\{S_n\}_{n=0}^{\infty}$ *without any changes. Similarly, the contents of the note following Theorem 2.2 remain true.*

Proof. By (2.12) and (2.13), the sequence $\{S_n\}_{n=0}^{\infty}$ automatically satisfies the conditions of Theorems 2.1 and 2.2 with arbitrary ν. The rest is now obvious. ∎

3. Main Results: Further Generalization and Completion

In Theorems 3.1 and 3.2 and their corollary below we assume that the sequence $\{S_n\}_{n=0}^{\infty}$ is such that

$$S_n = S + \sum_{j=1}^{v} P_j(n)\,\lambda_j^n + U_n, \qquad n = 0, 1, ..., \tag{3.1}$$

where, for each j, $P_j(n)$ is a polynomial in n of degree exactly p_j for some p_j, which we choose to express in the form

$$P_j(n) = \sum_{i=0}^{p_j} a_{ji} \binom{n}{i}, \qquad a_{jp_j} \neq 0, \tag{3.2}$$

the λ_j are, in general, complex distinct nonzero scalars that satisfy

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_v| > R^{-1}; \qquad \lambda_j \neq 1, \quad j = 1, 2, ..., v;\ \text{some } R > 0, \tag{3.3}$$

and

$$U_n = O(\xi^n) \qquad \text{as} \quad n \to \infty, \tag{3.4}$$

with arbitrary ξ in the open interval $(R^{-1}, |\lambda_v|)$. For convenience, we shall also denote

$$\omega_j = p_j + 1, \qquad j = 1, ..., v. \tag{3.5}$$

Theorem 3.1. *Let $\{S_n\}_{n=0}^{\infty}$ be as in the first paragraph of this section*:

(i) *Let*

$$k = \sum_{j=1}^{v} \omega_j. \tag{3.6}$$

Then

$$\varepsilon_{2k}^{(n)} - S = O(\xi^n) \qquad \textit{as} \quad n \to \infty. \tag{3.7}$$

If $U_n = 0$ for $n \geqslant N$, we have $\varepsilon_{2k}^{(n)} = S$ for $n \geqslant N$.

(ii) *Assume that $|\lambda_t| > |\lambda_{t+1}|$ for some $t < v$, and let*

$$k = \sum_{j=1}^{t} \omega_j. \tag{3.8}$$

Let r be that integer for which

$$|\lambda_1| \geqslant \cdots \geqslant |\lambda_t| > |\lambda_{t+1}| = \cdots = |\lambda_{t+r}| > |\lambda_{t+r+1}|, \tag{3.9}$$

where $|\lambda_{t+r+1}|$ *is meant to stand for* ξ *when* $t+r=v$. *Assume that* $\lambda_{t+1}, ..., \lambda_{t+r}$ *are ordered such that* $p_{t+1} \geqslant p_{t+2} \geqslant \cdots \geqslant p_{t+r}$, *and let* μ *be that integer*, $1 \leqslant \mu \leqslant r$, *for which*

$$\bar{p} \equiv p_{t+1} = \cdots = p_{t+\mu} > p_{t+\mu+1} \geqslant \cdots \geqslant p_{t+r}. \tag{3.10}$$

Then, whether $|\lambda_{t+1}| < 1$ *or not*, $\varepsilon_{2k}^{(n)}$ *satisfies*

$$\begin{aligned} \varepsilon_{2k}^{(n)} - S &= \frac{n^{\bar{p}}}{\bar{p}!} \sum_{j=t+1}^{t+\mu} a_{jp_j} \left[\prod_{i=1}^{t} \left(\frac{\lambda_j - \lambda_i}{1-\lambda_i} \right)^{2\omega_i} \right] \lambda_j^n + o(n^{\bar{p}} |\lambda_{t+1}|^n) \qquad \text{as} \quad n \to \infty, \\ &= O(n^{\bar{p}} |\lambda_{t+1}|^n) \qquad \text{as} \quad n \to \infty. \end{aligned} \tag{3.11}$$

We would like to note here that the qualitative result, $\varepsilon_{2k}^{(n)} - S = O(n^{\bar{p}} |\lambda_{t+1}|^n)$ as $n \to \infty$, in part (ii) of Theorem 3.1 was first mentioned in [SiBr, p. 42, Note] and it was obtained as a by-product in the analysis of the topological epsilon algorithm for vector sequences.

Theorem 3.1 gives the solution to the convergence problem associated with $\{\varepsilon_{2k}^{(n)}\}_{n=0}^{\infty}$ for which k is either $\sum_{j=1}^{v} \omega_j$ or $\sum_{j=1}^{t} \omega_j$ for some $t < v$ such that $|\lambda_t| > |\lambda_{t+1}|$. We now turn to the *remaining* values of k that are less than $\sum_{j=1}^{v} \omega_j$.

THEOREM 3.2. *Let the integers* t *and* r *be as in part* (ii) *of Theorem* 3.1, *and pick* k *such that*

$$\sum_{j=1}^{t} \omega_j < k < \sum_{j=1}^{t+r} \omega_j, \tag{3.12}$$

and let

$$\tau = k - \sum_{j=1}^{t} \omega_j. \tag{3.13}$$

This time, however, we also allow $t=0$ *and define* $\sum_{j=1}^{0} \omega_j = 0$. *Denote by* IP($\tau$) *the nonlinear integer programming problem*:

$$\begin{aligned} &\text{maximize} \quad g(\boldsymbol{\sigma}); \; g(\boldsymbol{\sigma}) = \sum_{j=t+1}^{t+r} (\omega_j \sigma_j - \sigma_j^2) \\ &\text{subject to} \quad \sum_{j=t+1}^{t+r} \sigma_j = \tau, \qquad 0 \leqslant \sigma_j \leqslant \omega_j, \quad t+1 \leqslant j \leqslant t+r, \end{aligned} \tag{3.14}$$

and denote by $G(\tau)$ *the* (*optimal*) *value of* $g(\boldsymbol{\sigma})$ *at the solution to* IP(τ).

Provided IP(τ) *has a unique solution for* σ_j, $j = t+1, ..., t+r$, $\varepsilon_{2k}^{(n)}$ *satisfies*

$$\varepsilon_{2k}^{(n)} - S = O(n^{G(\tau+1) - G(\tau)} |\lambda_{t+1}|^n) \qquad \text{as} \quad n \to \infty, \tag{3.15}$$

whether $|\lambda_{t+1}|<1$ *or not.* (*Here* $\mathrm{IP}(\tau+1)$ *is not required to have a unique solution.*)

Note. When k and τ are as in (3.12) and (3.13) of Theorem 3.2, but $\mathrm{IP}(\tau)$ does not have a unique solution for σ_j, $j=t+1, ..., t+r$, the best that we can say is that, under certain conditions, there may exist a subsequence of $\{\varepsilon_{2k}^{(n)}\}_{n=0}^{\infty}$ that satisfies (3.15), again whether $|\lambda_{t+1}|<1$ or not. It can be shown that such a subsequence exists when k is such that $\tau=1$. Note that the values of k that we are dealing with here are all of the remaining values not covered by Theorem 3.1 and Theorem 3.2. We thus have a complete convergence theory for all of the even numbered columns of the ε-table for which $k \leqslant \sum_{j=1}^{\nu} \omega_j$.

In connection with $\mathrm{IP}(\tau)$, we would like to mention that algorithms for its solution have been given in [P, KamSi] and recently in [LiSa]. The algorithms of [KamSi, LiSa] also enable one to decide whether or not the solution is unique in a simple manner. Some properties of the solutions to $\mathrm{IP}(\tau)$ have been given in [Si2], and we mention them here for convenience and further reference.

Let σ_j, $t+1 \leqslant j \leqslant t+r$, be a solution of $\mathrm{IP}(\tau)$:

1. $\sigma_j'=\omega_j-\sigma_j$, $t+1 \leqslant j \leqslant t+r$, is a solution of $\mathrm{IP}(\tau')$ with $\tau'=\sum_{j=t+1}^{t+r} \omega_j-\tau$.

2. If $\omega_{j'}=\omega_{j''}$ for some j', j'', $t+1 \leqslant j', j'' \leqslant t+r$, and if $\sigma_{j'}=\delta_1$, $\sigma_{j''}=\delta_2$ in a solution to $\mathrm{IP}(\tau)$, $\delta_1 \neq \delta_2$, then there is another solution to $\mathrm{IP}(\tau)$ with $\sigma_{j'}=\delta_2$, $\sigma_{j''}=\delta_1$. Consequently, a solution to $\mathrm{IP}(\tau)$ cannot be unique unless $\sigma_{j'}=\sigma_{j''}$. One implication of this is that for $\omega_{t+1}=\cdots=\omega_{t+r}=\bar{\omega}>1$, $\mathrm{IP}(\tau)$ has a unique solution only for $\tau=qr$, $q=1, ..., \bar{\omega}-1$, and in this solution $\sigma_j=q$, $t+1 \leqslant j \leqslant t+r$. For $\omega_{t+1}=\cdots=\omega_{t+r}=1$ no unique solution to $\mathrm{IP}(\tau)$ exists with $1 \leqslant \tau \leqslant r-1$. Another implication is that for $\omega_{t+1}=\cdots=\omega_{t+\mu}>\omega_{t+\mu+1} \geqslant \cdots \geqslant \omega_{t+r}$, $\mu<r$, no unique solution to $\mathrm{IP}(\tau)$ exists for $\tau=1, ..., \mu-1$, and a unique solution exists for $\tau=\mu$, this solution being $\sigma_{t+1}=\cdots=\sigma_{t+\mu}=1$, $\sigma_j=0$, $t+\mu+1 \leqslant j \leqslant t+r$.

3. A unique solution to $\mathrm{IP}(\tau)$ exists when ω_j, $t+1 \leqslant j \leqslant t+r$, are all even or all odd, and $\tau=qr+\frac{1}{2}\sum_{j=t+1}^{t+r}(\omega_j-\omega_{t+r})$, $0 \leqslant q \leqslant \omega_{t+r}$. This solution is given by $\sigma_j=q+\frac{1}{2}(\omega_j-\omega_{t+r})$, $t+1 \leqslant j \leqslant t+r$.

4. Obviously, when $r=1$ a unique solution to $\mathrm{IP}(\tau)$ exists for all possible τ and is given as $\sigma_{t+1}=\tau$. When $r=2$ and $\omega_1+\omega_2$ is odd, a unique solution to $\mathrm{IP}(\tau)$ exists for all possible τ, as shown in [KamSi].

From what we know about the case in which $r=1$ we can now state the following immediate corollary to Theorems 3.1 and 3.2, which will be of use in Section 5.

COROLLARY. *Let* $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_\nu|$. *Also define* $\sum_{j=1}^{0} c_j = 0$ *and* $\prod_{j=1}^{0} c_j = 1$:

(i) *If* $k = \sum_{j=1}^{\nu} \omega_j$, *then*

$$\varepsilon_{2k}^{(n)} - S = O(\xi^n) \qquad as \quad n \to \infty. \tag{3.16}$$

If $U_n = 0$ *for* $n \geqslant N$, *we have* $\varepsilon_{2k}^{(n)} = S$ *for* $n \geqslant N$.

(ii) *If* $k = \sum_{j=1}^{t} \omega_j$ *for any* $t = 0, 1, ..., \nu - 1$, *then*

$$\varepsilon_{2k}^{(n)} - S \sim a_{t+1,\, p_{t+1}} \left[\prod_{i=1}^{t} \left(\frac{\lambda_{t+1} - \lambda_i}{1 - \lambda_i} \right)^{2\omega_i} \right] \frac{n^{p_{t+1}}}{p_{t+1}!} \lambda_{t+1}^n \qquad as \quad n \to \infty. \tag{3.17}$$

(iii) *If* $\sum_{j=1}^{t} \omega_j < k < \sum_{j=1}^{t+1} \omega_j$ *for any* $t = 0, 1, ..., \nu - 1$, *then*

$$\varepsilon_{2k}^{(n)} - S \sim C n^{p_{t+1} - 2\tau} \lambda_{t+1}^n \qquad as \quad n \to \infty, \tag{3.18}$$

where $\tau = k - \sum_{j=1}^{t} \omega_j$ *hence* $0 < \tau < \omega_{t+1}$ *and*

$$C = (-1)^\tau \frac{\tau!}{(p_{t+1} - \tau)!} a_{t+1,\, p_{t+1}} \left(\frac{\lambda_{t+1}}{1 - \lambda_{t+1}} \right)^{2\tau} \left[\prod_{i=1}^{t} \left(\frac{\lambda_{t+1} - \lambda_i}{1 - \lambda_i} \right)^{2\omega_i} \right]. \tag{3.19}$$

That is, the sequence $\{\varepsilon_{2k}^{(n)}\}_{n=0}^{\infty}$ *is better than* $\{\varepsilon_{2k-2}^{(n)}\}_{n=0}^{\infty}$. *Thus, if* $|\lambda_1| < 1$, *then all the sequences* $\{\varepsilon_{2k}^{(n)}\}_{n=0}^{\infty}$, $k = 0, 1, ..., \nu$, *converge, each one converging more quickly than the ones preceding it.*

In all our results above we have assumed that S_n satisfies (3.1), the right-hand side of which is not a genuine asymptotic expansion. Theorem 3.3 below, that is analogous to Theorem 2.3, summarizes the results for the case in which the right-hand side of (3.1) is replaced by a genuine asymptotic expansion. The proof of Theorem 3.3 is identical to that of Theorem 2.3.

THEOREM 3.3. *Let the sequence* $\{S_n\}_{n=0}^{\infty}$ *be such that*

$$S_n \sim S + \sum_{j=1}^{\infty} P_j(n)\, \lambda_j^n \qquad as \quad n \to \infty, \tag{3.20}$$

where the polynomials $P_j(n)$ *are precisely as in* (3.2) *and the* λ_j *are, in general, complex distinct nonzero scalars that satisfy*

$$|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \cdots; \qquad \lambda_j \neq 1, \quad j = 1, 2, ...; \quad \lim_{j \to \infty} \lambda_j = 0. \tag{3.21}$$

This implies that there is an infinite number of integer pairs (t, r) *for which* (3.9) *holds. Then, with* $\nu = \infty$, *part* (ii) *of Theorem* 3.1, *all of Theorem* 3.2, *and their corollary apply to the ε-table of* $\{S_n\}_{n=0}^{\infty}$ *without any changes. Similarly, the contents of the note following Theorem* 3.2 *remain true.*

Obviously, Theorem 3.3 covers *all* of the even numbered columns of the ε-table. In particular, it says that when $r=1$ for all $t=1, 2, \ldots$ and $|\lambda_1|<1$, the even numbered columns of the ε-table converge, each converging more quickly than the ones preceding it.

4. Proofs of Main Results

We begin by recalling the connection between the Shanks transformation and Padé approximants. Specifically, let us consider the formal power series $f(z) := \sum_{i=0}^{\infty} c_i z^i$, whose partial sums we denote by $f_n(z)$, i.e., $f_n(z) = \sum_{i=0}^{n} c_i z^i$, $n=0, 1, \ldots$. If we apply the Shanks transformation to the sequence of the partial sums $\{f_n(z)\}_{n=0}^{\infty}$, then, as is shown in [Sh],

$$e_k(f_n(z)) = f_{n+k,\,k}(z), \tag{4.1}$$

where $f_{p,\,q}(z)$ stands for the (p/q) Padé approximant associated with the series $f(z)$. (For Padé approximants see, e.g., [Ba].) Thus, if $f(z) := \sum_{i=0}^{\infty} c_i z^i$, with $c_0 = S_0$ and $c_i = S_i - S_{i-1}$, $i=1, 2, \ldots$, then $S_n = f_n(1)$, $n=0, 1, \ldots$, and hence

$$e_k(S_n) = f_{n+k,\,k}(1), \tag{4.2}$$

It turns out that when $R>1$ the proofs of the main results of Sections 2 and 3 can be based *directly* on the results of [Si2] pertaining to convergence of rows of the Padé table for the function represented by $\sum_{i=0}^{\infty} c_i z^i$ essentially by substituting $z=1$ in the latter. This function turns out to be analytic at $z=0$ and meromorphic for $|z|<R$. This case is treated in the following two subsections. We turn to the case $R \leqslant 1$ briefly in the third subsection.

4.1. *Proofs of Theorems* 2.1 *and* 2.2 *When* $R>1$

Invoking (2.1), we have

$$c_n = S_n - S_{n-1} = \sum_{j=1}^{\nu} A_j \lambda_j^n + V_n, \qquad n=1, 2, \ldots, \tag{4.3}$$

where

$$A_j = (1-\lambda_j^{-1})\, a_j, \qquad j=1, \ldots, \nu, \tag{4.4}$$

and

$$V_0 = S_0 - \sum_{j=1}^{\nu} A_j, \qquad V_i = U_i - U_{i-1}, \qquad i = 1, 2, \ldots. \tag{4.5}$$

By (2.3), $V_n = O(\xi^n)$ as $n \to \infty$. Consequently, the power series $g(z) := \sum_{i=0}^{\infty} V_i z^i$ converges for $|\xi z| < 1$. Since ξ^{-1} is in $(|\lambda_\nu|^{-1}, R)$ but is arbitrary otherwise, the series $g(z)$ represents a function, also denoted $g(z)$, that is analytic in the open disk $K = \{z : |z| < R\}$. With this, we conclude that the series $f(z) := \sum_{i=0}^{\infty} c_i z^i$ is the Maclaurin expansion of a function also denoted $f(z)$, that is given by

$$f(z) = \sum_{j=1}^{\nu} \frac{A_j}{1 - \lambda_j z} + g(z). \tag{4.6}$$

Obviously, $f(z)$ is analytic at $z = 0$ and meromorphic in K with simple poles at $\lambda_1^{-1}, \ldots, \lambda_\nu^{-1}$.

Now $z = 1 \in K$ by $R > 1$, and $\lambda_j \neq 1$, $j = 1, \ldots, \nu$, by (2.2). Therefore, $z = 1$ is a point of analyticity of $f(z)$ in K. In addition, $f(1) = S$, whether $\lim_{n \to \infty} S_n$ exists or not, as can also be verified by setting $z = 1$ in (4.6). Consequently, $e_k(S_n) - S = f_{n+k,k}(1) - f(1)$.

When $U_n = 0$, $n = 0, 1, \ldots$, we have $g(z) \equiv V_0$ so that $f(z)$ is a rational function with simple poles. By dividing Eq. (5.10′) on page 278 of [Si2] by Eq. (4.12′) on page 271 of [Si2], and replacing the ζ's, there by λ's we obtain the error $f(z) - f_{n+k-1,k}(z)$ in the $(n+k-1, k)$ Padé approximant of $f(z)$. Theorem 2.1 now follows by noting the relation between a_j and A_j given in (4.4), and by setting $z = 1$ and replacing n by $n+1$, with $G_{n,k} = 0$ and $E_{n,k} = 0$ in (2.5) and (2.6).

When U_n, $n = 0, 1, \ldots$, are as in (2.3), then, as follows from the proofs of Theorems 4.2 and 5.2 in [Si2], $G_{n,k}$ and $E_{n,k}$ do have to be present in (2.5) and (2.6), and they satisfy (2.8). This completes the proof of Theorem 2.1.

The results given in Theorem 2.2 are obtained from Theorem 2.1 by extracting the most dominant parts of $N_k^{(n)}$ and $D_k^{(n)}$ in (2.5) and (2.6) for $n \to \infty$. For both part (i) and part (ii) of Theorem 2.2, $D_k^{(n)}$ has only one most dominant term, namely, that with the indices $j_1 = 1$, $j_2 = 2$, ..., $j_k = k$, when $k = \nu$, we have $N_k^{(n)} = G_{n,k}$. When k, t, and r are as in part (ii), $N_k^{(n)}$ has r most dominant terms, namely, those with the indices $j_1 = 1$, $j_2 = 2$, ..., $j_k = k = t$, $j_{k+1} = t + i$, $i = 1, 2, \ldots, r$.

The contents of the note following Theorem 2.2 can be seen to be true by observing that the most dominant part of $D_k^{(n)}$ for $n \to \infty$ is $|\prod_{j=1}^{k} \lambda_j|^n$ multiplied by a trigonometric sum, which, under certain conditions, is not identically zero, this being the case, e.g., when $k = t + 1$. Also $N_k^{(n)} = O(|\prod_{j=1}^{k+1} \lambda_j|^n)$ as $n \to \infty$.

4.2. *Proofs of Theorems* 3.1 *and* 3.2 *and Their Corollary When* $R > 1$.

Invoking (3.1)–(3.3), we have

$$c_n = S_n - S_{n-1} = \sum_{j=1}^{v} \left[\sum_{l=0}^{p_j} \tilde{A}_{jl} \binom{n}{l} \right] \lambda_j^n + V_n, \qquad n = 1, 2, ..., \tag{4.7}$$

with some constants $\tilde{A}_{jl}$, $l = 0, 1, ..., p_j$, $j = 1, ..., v$,

$$\tilde{A}_{jp_j} = (1 - \lambda_j^{-1})\, a_{jp_j} \neq 0, \qquad j = 1, ..., v, \tag{4.8}$$

V_n, for $n \geqslant 1$, being as in (4.5). Again it can be proved that $f(z) := \sum_{i=0}^{\infty} c_i z^i$ is the Maclaurin series of a function, denoted also by $f(z)$, that is now given as

$$f(z) = \sum_{j=1}^{v} \sum_{i=0}^{p_j} \frac{A_{ji}}{(1 - \lambda_j z)^{i+1}} + g(z), \qquad A_{jp_j} \neq 0, \quad 1 \leqslant j \leqslant v, \tag{4.9}$$

where the A_{ji} are uniquely determined by the $\tilde{A}_{jl}$ through

$$\tilde{A}_{jl} = \sum_{i=l}^{p_j} A_{ji} \binom{i}{i-l}, \qquad l = 0, 1, ..., p_j, \quad j = 1, ..., v, \tag{4.10}$$

and $g(z)$ is analytic in the open disk $K = \{z : |z| < R\}$. In fact, $g(z) = \sum_{i=0}^{\infty} V_i z^i$, $|z| < R$, as before, with an appropriate V_0. Thus $f(z)$ is analytic at $z = 0$ and meromorphic in K with poles $\lambda_1^{-1}, ..., \lambda_v^{-1}$, whose respective multiplicities are $\omega_1, ..., \omega_v$. All this follows from Lemma 4.1 of [Si2].

Another useful expression for $f(z)$ can be obtained directly in terms of (3.1)–(3.4) and is given as

$$f(z) = S + U_0 + (1 - z) \sum_{j=1}^{v} R_j(z) + \sum_{n=1}^{\infty} (U_n - U_{n-1})\, z^n, \qquad z \in K, \tag{4.11}$$

where, for each j, $j = 1, 2, ..., v$, $R_j(z)$ is a rational function with a single pole of multiplicity ω_j at λ_j^{-1}, whose Maclaurin series is

$$R_j(z) = \sum_{n=0}^{\infty} P_j(n)(\lambda_j z)^n. \tag{4.12}$$

Again, $z = 1$ is a point of analyticity of $f(z)$ in K, and $f(1) = S$, whether $\lim_{n \to \infty} S_n$ exists of not, as can be seen by setting $z = 1$ in (4.11). Consequently, $e_k(S_n) - S = f_{n+k,k}(1) - f(1)$.

Next, by Theorem 3.3 of [Si2] we have the following results for the function $f(z)$ described above and its Padé approximants $f_{p,q}(z)$:

1. If $k=\sum_{j=1}^{\nu}\omega_j$, then

$$f(z)-f_{m,k}(z)=O(|\xi z|^m) \qquad \text{as} \quad m\to\infty. \tag{4.13}$$

uniformly in any compact subset of $K\backslash\{\lambda_1^{-1},...,\lambda_\nu^{-1}\}$.

2. If $k=\sum_{j=1}^{t}\omega_j$, and t, r, and μ are as in (3.9) and (3.10), then, with $m=n+k-1$,

$$f(z)-f_{m,k}(z)=\frac{n^{\bar{p}}}{\bar{p}!}\sum_{j=t+1}^{t+\mu}\phi_j(z)(\lambda_j z)^{n+2k}+o(n^{\bar{p}}|\lambda_{t+1}z|^n) \qquad \text{as} \quad n\to\infty, \tag{4.14}$$

uniformly in any compact subset of $K\backslash\{\lambda_1^{-1},...,\lambda_\nu^{-1}\}$, where

$$\phi_j(z)=\frac{A_{jp_j}}{1-\lambda_j z}\left[\frac{Q(\lambda_j^{-1})}{Q(z)}\right]^2, \qquad Q(z)=\prod_{j=1}^{t}(1-\lambda_j z)^{\omega_j}. \tag{4.15}$$

Part (i) of Theorem 3.1 follows by setting $z=1$ in (4.13). Part (ii) of Theorem 3.1 follows by setting $z=1$ in (4.14) and (4.15) and replacing n by $n+1$ and by observing that $A_{jp_j}=(1-\lambda_j^{-1})\,a_{jp_j}$, $j=1,...,\nu$, that follows from (4.8) and (4.10).

The proof of Theorem 3.2 can be achieved by invoking Theorem 6.1 of [Si2]. For a better understanding of the contents of the note following Theorem 3.2, see [Si2, p. 284, Note 6.3].

We now come to the proof of the corollary to Theorems 3.1 and 3.2. First, we realize that parts (i) and (ii) there follow directly from parts (i) and (ii), respectively, of Theorem 3.1. As far as part (iii) of the corollary is concerned, we note that Theorem 3.2, with $r=1$ and, hence, with $G(m)=\omega_{t+1}m-m^2$, $m=\tau,\tau+1$, already produces the qualitative result $\varepsilon_{2k}^{(n)}-S=O(n^{p_{t+1}-2\tau}\lambda_{t+1}^n)$ as $n\to\infty$. The proof of the asymptotic equivalence in (3.18) with (3.19) is much more complicated, however, and can be achieved by going into the fine details of the proof of Theorem 6.1 of [Si2]. First, an expression like (2.4) exists, $D_k^{(n)}$ being the denominator of the quotient on the right-hand side of (1.1). Since IP(τ) and IP$(\tau+1)$ both have unique solutions, $N_k^{(n)}$ has a unique nonzero most dominant term that is asymptotically $C_1 n^{G(\tau+1)}[(\prod_{j=1}^{t}\lambda_j^{\omega_j})\,\lambda_{t+1}^{\tau+1}]^n$ while $D_k^{(n)}$ has a unique nonzero most dominant term that is asymptotically $C_2 n^{G(\tau)}[(\prod_{j=1}^{t}\lambda_j^{\omega_j})\,\lambda_{t+1}^{\tau}]^n$. Both C_1 and C_2 can be obtained with the help of Appendix A in [Si2] after long and tedious manipulations. The result in (3.18) and (3.19) follows by taking the quotient of these two terms.

4.3. *Brief Description of Proofs with* $R \leqslant 1$

As is clear from the above, the assumption that $R > 1$ enables us to construct a function $f(z)$ with Maclaurin series $\sum_{i=0}^{\infty} c_i z^i$, $c_0 = S_0$, $c_i = S_i - S_{i-1}$, $i = 1, 2, ...,$ that is meromorphic in the open disk $K = \{z : |z| < R\}$ and analytic at the point $z = 1 \in K$. As a result, we have $f(1) = S$, and, hence, $e_k(S_n) - S = f_{n+k,k}(1) - f(1)$. When $R \leqslant 1$, however, we are not able to reach such a conclusion, and we have to analyze $e_k(S_n) - S$ for $n \to \infty$ almost from first principles.

First, Theorem 2.1 can be proved directly by substituting (2.1) in (1.1) and then by employing Lemma 2.1 of [Si2] both in the numerator and in the denominator and by making use of (A.11)–(A.13) of [Si2]. (Lemma 2.1 in [Si2] originally appeared as Lemma A.1 in [SiFSm].) For the details of this technique see also [SiFSm]. Theorem 2.2 follows from Theorem 2.1, as was mentioned before.

As for Theorems 3.1 and 3.2, we proceed as follows: With $c_n = S_n - S_{n-1}$ and $V_n = U_n - U_{n-1}$, $n = 1, 2, ...,$ (4.7) and (4.8) are valid with

$$\tilde{A}_{jl} = a_{jl} - \lambda_j^{-1} \sum_{i=l}^{p_j} (-1)^{i-l} a_{ji}, \qquad l = 0, 1, ..., p_j. \tag{4.16}$$

Next, defining

$$\begin{aligned} \hat{B}_{jl} &= \tilde{A}_{jl}\delta_j + \sum_{q=l+1}^{p_j} \tilde{A}_{jq}(\delta_j + 1)\,\delta_j^{q-1}, \qquad 0 \leqslant l \leqslant p_j - 1, \\ \hat{B}_{jp_j} &= \tilde{A}_{jp_j}\delta_j; \qquad \delta_j = \lambda_j/(1-\lambda_j), \end{aligned} \tag{4.17}$$

it can be shown after tedious manipulations that

$$\hat{B}_{jl} = -a_{jl} \qquad \text{for all } j \text{ and } l. \tag{4.18}$$

Consequently, (3.1) can be expressed also as

$$S - S_n = \sum_{j=1}^{\nu} \left[\sum_{l=0}^{p_j} \hat{B}_{jl} \binom{n}{l} \right] \lambda_j^n - U_n, \qquad n = 0, 1, 2, \ldots. \tag{4.19}$$

Comparing (4.17) and (4.19) with Eq. (5.4)–(5.6) in Lemma 5.1 of [Si2], we see that the former and the latter are similar in form, with $f(z) - S_n(z)$, z, ζ_j, and $\hat{B}_{jl}(z)$ in the latter replaced by $S - S_n$, 1, λ_j, and $\hat{B}_{jl}$, respectively. Consequently, the numerator and denominator determinants in (1.1) of the present work have expansions identical to those given in Theorems 5.2 and 4.2, respectively, of [Si2]. Theorems 3.1 and 3.2 of the present work now follow from these just as Theorems 3.1, 3.3, and 6.1 of [Si2] do.

5. Examples

5.1. *Iterative Solution of Linear Systems*

Denote by s the unique solution of the nonsingular linear system $x = Ax + b$, and consider the fixed-point iterative solution of this system as in

$$x_{j+1} = Ax_j + b, \qquad j = 0, 1, \ldots; \qquad x_0 \text{ given.} \tag{5.1}$$

The matrix A may be defective, in general. It is shown in [SiBr, Section 2] that the vector x_n has an expansion of the form

$$x_n = s + \sum_{j=1}^{M} \left[\sum_{i=0}^{p_j} y_{ji} \binom{n}{i} \right] \lambda_j^n, \qquad \text{all large } n. \tag{5.2}$$

Here $\lambda_1, \ldots, \lambda_M$ are some or all of the distinct nonzero eigenvalues of A, and, since the matrix $(I - A)$ is nonsingular, none of the λ_j can be unity. For each j, the vectors y_{ji}, $i = 0, 1, \ldots, p_j$, are in the invariant subspaces of A corresponding to the eigenvalue λ_j. In particular, y_{jp_j} is an eigenvector corresponding to λ_j; i.e., $Ay_{jp_j} = \lambda_j y_{jp_j}$. The invariant subspaces are given as

$$Y_i = \text{span}\{ y_{j,p_j - r}, r = 0, 1, \ldots, i\}, \qquad i = 0, 1, \ldots, p_j,$$

and they obviously satisfy $Y_0 \subset Y_1 \subset \cdots \subset Y_{p_j}$.

The ε-algorithm can now be applied to each component of the vectors x_n separately, recalling the column convergence theorems of Section 3. Indeed, this was one of the first approaches used in accelerating the convergence of fixed point iterative methods for linear and nonlinear systems of equations.

5.2. *Euler–Maclaurin Expansions for Integrands with Logarithmic End Point Singularities*

Consider the integral

$$I_p = \int_0^1 (\log x)^p \, x^\alpha g(x) \, dx, \qquad p = 0, 1, \ldots, \quad \alpha > -1, \quad g(x) \in C^\infty[0, 1], \tag{5.3}$$

and the trapezoidal rule approximation to it,

$$\begin{aligned} T_p(h) &= h \sum_{i=1}^{m-1} G_p(ih) + \frac{h}{2} G_p(1), \\ G_p(x) &\equiv (\log x)^p \, x^\alpha g(x), \qquad h = 1/m, \quad m = 1, 2, \ldots . \end{aligned} \tag{5.4}$$

Theorem 5.1 below gives the Euler–Maclaurin expansion for the error, $T_p(h) - I_p$ as $h \to 0$.

THEOREM 5.1. *The approximation $T_p(h)$ satisfies*

$$T_p(h)-I_p \sim \sum_{j=1}^{\infty} a_j^{(p)} h^{2j} + \sum_{j=0}^{\infty} \left[\sum_{i=0}^{p} b_{ji}^{(p)} (\log h)^i \right] h^{\alpha+j+1} \qquad \textit{as} \quad h \to 0, \tag{5.5}$$

for some constants $a_j^{(p)}$ and $b_{ji}^{(p)}$ that are independent of h. Actually,

$$\begin{aligned} a_j^{(p)} &= \frac{B_{2j}}{(2j)!} G_p^{(2j-1)}(1), \qquad j=1, 2, ..., \\ b_{ji}^{(p)} &= \binom{p}{i} \left[\frac{d^{p-i}}{d\alpha^{p-i}} \zeta(-\alpha-j) \right] \frac{g^{(j)}(0)}{j!}, \qquad 0 \leqslant i \leqslant p, \quad j=0, 1, ..., \end{aligned} \tag{5.6}$$

where B_i are the Bernoulli numbers and $\zeta(z)$ is the Riemann zeta function.

Proof. The result in (5.5) and (5.6) when $p=0$ is a special case of that given in [Na1]. The result for $p=1$ is similarly a special case of that given in [Na2], and it is obtained by differentiating both sides of (5.5) (with $p=0$ there) once with respect to α. Applying this technique of differentiation with respect to α p times on both sides of (5.5) (with $p=0$ there), we obtain the required result. ▮

Note that the results of [Na1, Na2] were rederived in [LyNi] by using generalized function techniques.

Letting now $h=h_n=2^{-n}$ in (5.5), and denoting $S_n=T_p(h_n)$, $n=0, 1, \ldots,$ and $S=I_p$, after some manipulation (5.5) becomes

$$S_n \sim S + \sum_{j=1}^{\infty} \tilde{a}_j \rho_j^n + \sum_{j=0}^{\infty} \left(\sum_{i=0}^{p} \tilde{b}_{ji} n^i \right) \mu_j^n \qquad \text{as} \quad n \to \infty, \tag{5.7}$$

where $\rho_j = 4^{-j}$, $\mu_j = 2^{-\alpha-j-1}$, $\tilde{a}_j = a_j^{(p)}$, $\tilde{b}_{ji} = b_{ji}^{(p)}(-\log 2)^i$.

It is important to note that for all values of $\alpha > -1$, whether integral or not, and for all integers $p \geqslant 0$, S_n in (5.7) is precisely of the form treated in the corollary to Theorems 3.1 and 3.2, with $1 > |\lambda_1| > |\lambda_2| > \cdots$. Also $p_j \leqslant p$ for all j. Consequently, by Theorem 3.3, the corollary applies with $v=\infty$ there, and *all* of the even numbered columns of the ε-stable converge, the rates of convergence being as described in parts (ii) and (iii) of the corollary. Thus, each of these columns converges more quickly than the ones preceding it.

Needless to say, the algorithm is suitable when one does not have complete knowledge of the singularities of the integrand. When one knows the precise nature of the singularities, however, the generalized Richardson extrapolation process turns out to be more economical.

As an example, let us consider the case in which $\alpha = 0$. We also assume that $g^{(j)}(0) \neq 0$, $j = 0, 1, \ldots$. It is known that (see [O, p. 63])

$$\zeta(0) = -\frac{1}{2} \neq 0, \qquad \zeta(-2m) = 0, \qquad \zeta(1-2m) = -\frac{B_{2m}}{2m} \neq 0, \qquad m = 1, 2, \ldots. \tag{5.8}$$

Also, the reflection formula for the zeta function (see again [O, p. 63]) on differentiation yields

$$\zeta'(-2m) = (-1)^m \, 2^{-2m-1} \pi^{-2m} \Gamma(2m+1)\, \zeta(2m+1) \neq 0, \qquad m = 1, 2, \ldots. \tag{5.9}$$

Therefore, $\tilde{b}_{0p} \neq 0$, $\tilde{b}_{jp} \neq 0$, $j = 1, 3, 5, \ldots$, and $\tilde{b}_{jp} = 0$, $\tilde{b}_{j,p-1} \neq 0$ for $j = 2, 4, 6, \ldots$. This implies that

$$\begin{aligned} &\lambda_j = 2^{-j}, \qquad j = 1, 2, \ldots; \\ &p_1 = p, \qquad p_{2j} = p, \qquad p_{2j+1} = p - 1, \qquad j = 1, 2, \ldots. \end{aligned} \tag{5.10}$$

Hence from the corollary we have

$$\begin{aligned} &\text{if} \quad 0 \leqslant k \leqslant p, && \varepsilon_{2k}^{(n)} - S = O(n^{p-2k} 2^{-n}) && \text{as} \quad n \to \infty, \\ &\text{if} \quad p+1 \leqslant k \leqslant 2p+1, && \varepsilon_{2k}^{(n)} - S = O(n^{3p-2k+2} 2^{-2n}) && \text{as} \quad n \to \infty, \\ &\text{if} \quad 2p+2 \leqslant k \leqslant 3p+1, && \varepsilon_{2k}^{(n)} - S = O(n^{5p-2k+3} 2^{-3n}) && \text{as} \quad n \to \infty, \\ & && \text{and so on.} && \end{aligned} \tag{5.11}$$

In connection with this problem it is interesting to observe that if α, p, and $g(0)$ are known, then the $j = 0$ term in the second summation on the right-hand side of (5.5) can be subtracted from $T_p(h)$ to obtain the "corrected" trapezoidal rule

$$\hat{T}_p(h) = T_p(h) - \sum_{i=0}^{p} b_{0i}^{(p)} (\log h)^i \, h^{\alpha+1}, \tag{5.12}$$

where $b_{0i}^{(p)}$, given in (5.6), involve only $g(0)$. Consequently, if we now replace $T_p(h)$ by $\hat{T}_p(h)$ and let $S_n = \hat{T}_p(h_n)$, then (5.7) holds, except that the second summation on its right-hand side starts with the $j = 1$ term. Obviously, this is also more favorable as far as applying the ε-algorithm is concerned. This kind of a correction for the trapezoidal rule has been proposed in [SiI] in case the singularity of the integrand is in the interior of the interval of integration.

In (5.3) we have assumed that $g(x) \in C^{\infty}[0, 1]$. In case $g(x)$ has a finite number of continuous derivatives on $[0, 1]$, the expansions in (5.5) and (5.7) are finite and there is also a remainder term of some well-known order for $h \to 0$. For instance, if $g(x) \in C^{2q}[0, 1]$, then in the first summations of (5.5) and (5.7), $1 \leqslant j \leqslant q$, while in the second summations, $0 \leqslant j \leqslant 2q-1$, and the remainder is $O(h^{2q})$ as $h \to 0$. The truth of this is already known for $p=0$ and $p=1$ (see [Na1, Na2]) and can be shown in a similar way for arbitrary p. Thus $\{S_n\}_{n=0}^{\infty}$ in (5.7) is precisely of the form treated in the corollary to Theorems 3.1 and 3.2 with a finite value of v that depends on p and q. We leave the details to the interested reader.

Before closing, we mention that the use of the ε-algorithm for accelerating the convergence of sequences of trapezoidal rule approximations for finite range singular integrals such as those in (5.3) with $p=0$ and $p=1$ was originally proposed in [CGR, Kah]. The only convergence result relevant to these problems that was known at that time was Theorem 1.1, that is valid only for $p=0$ in (5.3), and this was mentioned later in [G]. We have shown that the ε-algorithm produces convergence acceleration for all values of $p = 0, 1, 2, \ldots$. At the same time we have provided the precise rates of acceleration.

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