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Computation of infinite integrals involving Bessel functions of arbitrary order by the $\bar{D}$-Transformation

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Abstract

The $\bar{D}$-transformation due to the author is an effective extrapolation method for computing infinite oscillatory integrals of various kinds. In this work two new variants of this transformation are designed for computing integrals of the form $\int_a^\infty g(t)\mathscr{C}_\nu(t)\,dt$, where $g(x)$ is a nonoscillatory function and $\mathscr{C}_\nu(x)$ may be an arbitrary linear combination of the Bessel functions of the first and second kinds $J_\nu(x)$ and $Y_\nu(x)$, of arbitrary real order ν. When applied to such integrals, the $\bar{D}$-transformation and its new variants are observed to produce very accurate results. It is also seen that their performance is very similar to that of the modified W-transformation due to the author, as extended in a recent work by Lucas and Stone with $\mathscr{C}_\nu(x) = J_\nu(x)$. The present paper is concluded by stating the relevant convergence and stability results and by appending a numerical example.

Keywords: Numerical integration; Infinite oscillatory integrals; Generalized Richardson extrapolation; Bessel functions; Hankel transforms

AMS classification: 41A25, 41A60, 65B05, 65D30, 65R10

1. Introduction

The computation of infinite oscillatory integrals of the form

$$I = \int_a^\infty g(t)K(t)\,dt, \quad a \geqslant 0, \tag{1.1}$$

where $g(x)$ is a smooth monotonic function for $x \to \infty$ and $K(x)$ has an infinite number of oscillations for $x \to \infty$, is an important practical problem that has been considered in a number of publications. Examples of commonly occurring functions $K(x)$ are the trigonometric functions $\cos x$ and $\sin x$ and the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ of the first and second kinds, respectively, of

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real order ν. One very efficient way of tackling this problem is through the generalized Richardson-extrapolation process (GREP), see [9], that may take on different forms: (1) If not much is known about $K(x)$, then the D-transformation of Levin and Sidi [6] may be applied successfully. (2) If the asymptotic behavior as $x \to \infty$ or the zero structure of $K(x)$ is known, however, then two modifications of the D-transformation, namely, the $\bar{D}$-transformation of Sidi [10] and the modified W-transformation (mW-transformation) of Sidi [14] are much more efficient.

Extensive numerical tests have shown that both the $\bar{D}$-transformation and the modified W-transformation are the most effective procedures for the cases $K(x) = \cos x$ and $K(x) = \sin x$; see [10, Section 3, Example 1;5]. (In [5] the W-transformation of Sidi [11] was used, but both the W- and the mW-transformations produce practically the same numerical results.)

The use of the $\bar{D}$-, W-, and mW-transformations for computing (1.1) with $K(x) = J_\nu(x)$ and $K(x) = Y_\nu(x)$ was considered in [10, Section 3, Example 2; 11,14]. It was observed numerically in these papers that all three transformations produce excellent results when ν is moderate.

Finally, the mW-transformation has been used very effectively also in the inversion of the Kontorovich–Lebedev transform numerically; see [2].

More recently, Lucas and Stone [7] have gone back to the problem of numerically computing (1.1) with $K(x) = J_\nu(x)$ only, where ν may take on arbitrarily large values. They have compared three different extrapolation approaches that involve the Euler transformation, the ε-algorithm, and the mW-transformation and that employ the exact zeros or extrema of $J_\nu(x)$, and concluded that the approach involving the mW-transformation is the most effective. Unfortunately, these authors have overlooked the $\bar{D}$-transformation in their numerical study, even though this transformation was the first GREP to utilize the exact zeros of $K(x)$, and produce results practically as good as those obtained in [7] from the mW-transformation approach. In the present work we wish to close this gap in the comparative study of the methods. We also bring to the attention of the reader the relevant convergence and stability theories that exist for the $\bar{D}$- and mW-transformations and that were completely left out of [7].

In the next section we review the development of the $\bar{D}$-transformation and also derive two new variants of it. We also suggest an additional approach involving the mW-transformation that is as efficient as the others. Finally, we discuss the convergence and stability properties of both the $\bar{D}$-transformation and the mW-transformation in all of their forms.

In Section 3 we show the effectiveness of all methods with a numerical example.

2. The $\bar{D}$-transformation

The following definition of the $\bar{D}$-transformation and two additional variations of it for the integral (1.1), with $K(x) = J_\nu(x)$ or $K(x) = Y_\nu(x)$ or any linear combination of the two, that we now turn to is actually taken from Sections 2 and 3 in [10]. (It is important to emphasize that the $\bar{D}$-transformation is a general method that is applicable to a large class of oscillatory infinite integrals that includes the ones treated in this work and in [7].)

Definition 2.1. The approximations $\bar{D}_n^{(j)}$ to I in (1.1) with $K(x) = \mathscr{C}_\nu(x)$, where $\mathscr{C}_\nu(x)$ stands for either $J_\nu(x)$ or $Y_\nu(x)$ or any linear combination of them and where $g(x)$ is monotonic at infinity, is defined through the solution of the linear system of equations

$$F(x_l) = \bar{D}_n^{(j)} + \psi(x_l)\sum_{i=0}^{n} \bar{\beta}_i x_l^{-i}, \quad j \leqslant l \leqslant j+n+1, \tag{2.1}$$

where

$$F(x) = \int_a^x g(t)K(t)\,dt, \tag{2.2}$$

and $\bar{\beta}_i$ are the remaining unknowns. Here the x_l and the function $\psi(x)$ can be chosen in three different ways:

1. The x_l are consecutive zeros of $K(x) = \mathscr{C}_\nu(x)$ in (a,∞) and $\psi(x) = g(x)K'(x)$, thus $\psi(x_l) = g(x_l)\mathscr{C}_{\nu+1}(x_l)$.
2. The x_l are consecutive zeros of $K'(x) = \mathscr{C}_\nu'(x)$ in (a,∞), and $\psi(x) = g(x)K(x)$, thus $\psi(x_l) = g(x_l)\mathscr{C}_\nu(x_l)$.
3. The x_l are consecutive zeros of $\mathscr{C}_{\nu+1}(x)$ in (a,∞), and $\psi(x)=g(x)K(x)$, thus $\psi(x_l)=g(x_l)\mathscr{C}_\nu(x_l)$.

The solution of (2.2) for $\bar{D}_n^{(j)}$ can be obtained recursively with the help of the W-algorithm of Sidi [12] as follows: Set

$$M_{-1}^{(s)} = F(x_s)/\psi(x_s) \quad \text{and} \quad N_{-1}^{(s)} = 1/\psi(x_s), \quad s = 0,1,\ldots, \tag{2.3}$$

and compute, for $s = 0,1,\ldots,\ p = 0,1,\ldots,$

$$\begin{aligned} M_p^{(s)} &= (M_{p-1}^{(s)} - M_{p-1}^{(s+1)})/(x_s^{-1} - x_{s+p+1}^{-1}),\\ N_p^{(s)} &= (N_{p-1}^{(s)} - N_{p-1}^{(s+1)})/(x_s^{-1} - x_{s+p+1}^{-1}),\\ \bar{D}_p^{(s)} &= M_p^{(s)}/N_p^{(s)}. \end{aligned} \tag{2.4}$$

The finite integrals $F(x_s)$ can best be determined from $F(x_s) = \sum_{l=0}^{s} u_l$, where the integrals $u_l = \int_{x_{l-1}}^{x_l} f(t)\,dt$ with $x_{-1} = a$ can easily be computed using a numerical quadrature formula, such as a low-order Gaussian rule.

We next present a summary of the theoretical developments that lead to Definition 2.1.

Definition 2.2. We shall say that a function $\alpha(x)$, defined for $x > a \geqslant 0$, belongs to the set $A^{(\gamma)}$ if it is infinitely differentiable for all $x > a$ and has a Poincaré-type asymptotic expansion of the form

$$\alpha(x) \sim x^{\gamma}\sum_{i=0}^{\infty} \alpha_i x^{-i} \quad \text{as} \quad x \to \infty, \tag{2.5}$$

and all its derivatives have Poincaré-type asymptotic expansions for $x \to \infty$ that are obtained by differentiating the right-hand side of (2.5) term by term.

Theorem 2.3. *Let $f(x)=g(x)K(x)$, where $K(x)=\mathscr{C}_\nu(x)$ and $g(x)=h(x)\exp[\phi(x)]$ such that $\phi \in A^{(m)}$ for some nonnegative integer m and $\lim_{x\to\infty}\phi(x)=-\infty$ when $m > 0$ and $h \in A^{(\gamma)}$ for some γ. Then we have*

$$F(x) = I + x^{\rho_0} f(x)\beta_0(x) + x^{\rho_1} f'(x)\beta_1(x), \tag{2.6}$$

where ρ_0 and ρ_1 are nonpositive integers given by

$$\rho_0 = \begin{cases} -m+1 & \text{if } m>0, \\ -1 & \text{if } m=0, \end{cases} \quad \text{and} \quad \rho_1 = \begin{cases} -2m+2 & \text{if } m>0, \\ 0 & \text{if } m=0, \end{cases} \tag{2.7}$$

and $\beta_0(x)$ and $\beta_1(x)$ are functions in $A^{(0)}$. Here I is $\int_a^\infty f(t)\,dt$ when this integral converges. In case of divergence, I is the Abel sum $\lim_{\varepsilon\to 0+}\int_a^\infty e^{-\varepsilon t} f(t)\,dt$ of the divergent integral $\int_a^\infty f(t)\,dt$. (This occurs when $m=0$ and $\gamma\geqslant 1/2$.)

For a discussion of Abel-summable oscillatory infinite integrals; see [13].

The first form of the $\bar{D}$-transformation in Definition 2.1 is obtained from (2.6) as follows: First, let $x=x_l$, where $K(x_l)=\mathscr{C}_\nu(x_l)=0$, thus eliminating the term $x^{\rho_0}f(x)\beta_0(x)$. Next, use the fact that $f'(x_l)=g(x_l)\mathscr{C}'_\nu(x_l)=-g(x_l)\mathscr{C}_{\nu+1}(x_l)$ that in turn follows from

$$\mathscr{C}'_\nu(x)=\frac{\nu}{x}\mathscr{C}_\nu(x)-\mathscr{C}_{\nu+1}(x); \tag{2.8}$$

see [1, p. 361, Formula 9.1.27]. Following that truncate the asymptotic expansion of $\beta_1(x)$ at the power x^{-n}, replace I by $\bar{D}_n^{(j)}$ and β_i by $\bar{\beta}_i$, $i=0,1,\ldots,n$, and replace ρ_1 by its known upper bound 0. Finally, collocate at the points x_l, $l=j,j+1,\ldots,j+n+1$.

For the second form of the $\bar{D}$-transformation we begin by expressing (2.6) in the form

$$F(x)=I+x^{\rho_0}f(x)\tilde{\beta}_0(x)+x^{\rho_1}g(x)K'(x)\beta_1(x), \tag{2.9}$$

where $\tilde{\beta}_0(x)=\beta_0(x)+x^{\rho_1-\rho_0}[g'(x)/g(x)]\beta_1(x)$ and is in $A^{(0)}$. Next, we let $x=x_l$, where $K'(x_l)=0$, thus eliminating the term $x^{\rho_1}g(x)K'(x)\beta_1(x)$. Now we continue as before.

For the third form of the $\bar{D}$-transformation we begin by expressing (2.6) in the form

$$F(x)=I+x^{\rho_0}f(x)\hat{\beta}_0(x)-x^{\rho_1}g(x)\mathscr{C}_{\nu+1}(x)\beta_1(x), \tag{2.10}$$

where $\hat{\beta}_0(x)=\beta_0(x)+x^{\rho_1-\rho_0}[g'(x)/g(x)]\beta_1(x)+\nu x^{\rho_1-\rho_0-1}\beta_1(x)$ and is in $A^{(0)}$. Next, we let $x=x_l$, where $\mathscr{C}_{\nu+1}(x_l)=0$, thus eliminating the term $x^{\rho_1}g(x)\mathscr{C}_{\nu+1}(x)\beta_1(x)$. Now we continue as before.

Note that while the first form of the $\bar{D}$-transformation is already in [10], the second and third forms are new and have been obtained by employing the philosophy of the $\tilde{D}$-transformation that is also in [10].

Here we recall that the mW-transformation for (1.1) with $K(x)=J_\nu(x)$ or $K(x)=Y_\nu(x)$ is also defined through a linear system of the form (2.1), where now x_l are the zeros of $\cos x$ (or $\sin x$), and thus $x_l=(q+l)\pi$, $l=0,1,\ldots$, for some integer (or half integer) q depending on a, and $\psi(x_l)=F(x_{l+1})-F(x_l)$, $l=0,1,\ldots$, and $x_{-1}=a$. The idea of picking the x_l to be equidistant with a distance of π between two consecutive x_l's was originally published in the framework of the $\tilde{D}$-transformation in [10] and later in the framework of the W-transformation in [11]. This idea was later also used in [8] in the framework of the Euler-transformation. When ν is large, better results are obtained from the mW-transformation if the x_l are chosen to be the zeros of $K(x)$ or $K'(x)$, as was suggested in [7]. We now suggest that the mW-transformation is as effective with the x_l chosen as the consecutive zeros of $\mathscr{C}_{\nu+1}(x)$ as well. With all three choices of the x_l the mW-transformation becomes completely analogous to the $\bar{D}$-transformation.

We end this section by stating the following convergence and stability theorems for the $\bar{D}$-transformation and its two variants and the mW-transformation with the x_l being the zeros of $\mathscr{C}_\nu(x)$ or $\mathscr{C}'_\nu(x)$, or $\mathscr{C}_{\nu+1}(x)$.

Theorem 2.4. *Let the function $f(x)$ be as in Theorem 2.3, and let $\bar{D}_n^{(j)}$ be the approximations to I obtained from the $\bar{D}$-transformation. Then, whether $\int_a^\infty f(t)\,dt$ converges or not, $\lim_{n\to\infty} \bar{D}_n^{(j)} = I$. In fact, we have the following powerful result:*

$$\bar{D}_n^{(j)} - I = O(n^{-\mu}) \quad \text{as } n \to \infty \text{ for any } \mu > 0. \tag{2.11}$$

In addition, the computation of the $\bar{D}_n^{(j)}$ is completely stable in the sense that errors in the $F(x_l)$ are not magnified with increasing j or n.

The results above hold also for the approximations produced by the mW-transformation with the x_l being those utilized by the $\bar{D}$-transformation.

For both the $\bar{D}$- and the mW-transformations, with all three choices of the x_l, the proof of Theorem 2.4 can be achieved by the techniques in [10, Section 6;14, Section 3.2]. We leave the details to the reader. For other more refined results for specific examples; see [15].

Table 1
Relative errors in $\bar{D}_n^{(0)}$ and $W_n^{(0)}$ for the integral in (3.1) with $y = 1$ and $a = 4$ obtained using the zeros of $J_\nu(x)$. Here $\bar{D}_n^{(0)}$ are produced by the $\bar{D}$-transformation while $W_n^{(0)}$ are produced by the mW-transformation

	$\nu = 0$		$\nu = 10$		$\nu = 100$	
n	$\lvert(\bar{D}_n^{(0)} - I)/I\rvert$	$\lvert(W_n^{(0)} - I)/I\rvert$	$\lvert(\bar{D}_n^{(0)} - I)/I\rvert$	$\lvert(W_n^{(0)} - I)/I\rvert$	$\lvert(\bar{D}_n^{(0)} - I)/I\rvert$	$\lvert(W_n^{(0)} - I)/I\rvert$
0	8.56D-03	8.40D-03	1.91D-02	5.62D-03	4.50D-02	6.68D-03
1	5.11D-04	1.61D-03	2.22D-03	3.54D-04	9.34D-03	1.03D-03
2	8.34D-06	1.61D-04	1.74D-04	1.41D-05	1.92D-03	1.48D-04
3	1.14D-05	5.71D-06	1.20D-05	8.52D-07	3.84D-04	1.93D-05
4	7.53D-07	4.60D-07	5.99D-07	1.40D-07	7.43D-05	2.07D-06
5	3.28D-08	6.74D-08	2.47D-08	1.06D-08	1.39D-05	1.27D-07
6	5.67D-09	3.09D-09	6.75D-10	3.34D-10	2.51D-06	1.66D-08
7	1.54D-10	5.36D-11	1.42D-11	2.76D-12	4.39D-07	8.69D-09
8	1.35D-11	1.57D-11	1.09D-13	1.23D-12	7.42D-08	2.28D-09
9	1.27D-12	8.59D-13	3.00D-16	6.78D-14	1.22D-08	4.81D-10
10	2.57D-14	3.21D-16	1.05D-15	1.35D-15	1.93D-09	8.91D-11
11	2.51D-15	2.41D-15	8.99D-16	1.05D-15	2.98D-10	1.50D-11
12	7.48D-16	8.02D-16	1.05D-15	8.99D-16	4.47D-11	2.34D-12
13	6.95D-16	4.28D-16	8.99D-16	8.99D-16	6.50D-12	3.36D-13
14	5.88D-16	1.60D-16	1.05D-15	1.05D-15	9.19D-13	4.41D-14
15	1.07D-16	5.35D-17	1.20D-15	1.20D-15	1.26D-13	5.30D-15

3. A numerical example

We have compared the $\bar{D}$- and mW-transformations applied to all three choices of the x_l, and found that they perform very similarly. When comparing the two transformations we should recall that $\bar{D}_n^{(0)}$, the approximation from the $\bar{D}$-transformation, is obtained using the $n+2$ integrals $F(x_l)$, $0 \leqslant l \leqslant n+1$, while $W_n^{(0)}$, the corresponding approximation from the mW-transformation, is obtained using the $n+3$ integrals $F(x_l)$, $0 \leqslant l \leqslant n+2$.

We have applied the two methods to the integral; see [1, p. 681, Formula 6.552],

$$I = \int_0^{\infty} J_\nu(xy) \frac{\mathrm{d}x}{(x^2+c^2)^{1/2}} = I_{\nu/2}\left(\tfrac{1}{2}cy\right) K_{\nu/2}\left(\tfrac{1}{2}cy\right), \quad y>0,\ \mathscr{R}c>0,\ \mathscr{R}\nu>-1, \tag{3.1}$$

with $y=1$ and $c=4$, and for $\nu=0, 10, 100$. The computations have been carried out on an IBM-370 machine in double precision arithmetic. The results obtained by using the zeros of $J_\nu(x)$ are given in Table 1.

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