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A COMPLETE CONVERGENCE AND STABILITY THEORY FOR A GENERALIZED RICHARDSON EXTRAPOLATION PROCESS*

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Dedicated to the memory of Professor Ivor M. Longman (1923–1993).

Abstract. Let $A(y) \sim A + \sum_{k=1}^{\infty} Q_k(\log y) y^{\sigma_k}$ as $y \to 0+$, where y is a discrete or continuous variable and $Q_k(\xi)$ are polynomials in ξ. It is assumed that σ_k and the degree of $Q_k(\xi)$ or an upper bound for it are known for each k, and that $A(y)$ is known for all possible $y \in (0, b]$. The aim is to find A, whether it is the limit or antilimit of $A(y)$ for $y \to 0+$. A very effective way of doing this is by the generalized Richardson extrapolation. In this paper this procedure is described and a very efficient recursive algorithm for its implementation is given when the set of extrapolation points is $\{y_l = y_0\omega^l,\ l = 0, 1, \ldots\}$ for some $\omega \in (0, 1)$. In addition, a complete theory of convergence and stability for the columns and the diagonals of the corresponding extrapolation table is provided. Finally, two applications are considered in detail, one of which is to generalized Romberg integration of functions with algebraic and logarithmic endpoint singularities.

Key words. acceleration of convergence, generalized Richardson extrapolation

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1. Introduction.

Let $A(y)$ be a function of a discrete or continuous variable y, defined for $0 < y \le b < \infty$, and satisfying

$$A(y) \sim A + \sum_{k=1}^{\infty} Q_k(\log y)\, y^{\sigma_k} \quad \text{as } y \to 0+, \tag{1.1}$$

where $Q_k(\xi)$ are polynomials given as

$$Q_k(\xi) = \sum_{i=0}^{q_k} \alpha_{ki}\xi^i \quad \text{for some integer } q_k \ge 0, \tag{1.2}$$

and

$$\text{Re } \sigma_1 \le \text{ Re } \sigma_2 \le \text{ Re } \sigma_3 \le \ldots; \quad \sigma_k \ne 0 \text{ for all } k; \ \lim_{k\to\infty} \text{ Re } \sigma_k = +\infty. \tag{1.3}$$

Thus there can be only a finite number of σ_k with equal real parts.

Note that if Re $\sigma_1 > 0$, then $\lim_{y\to 0+} A(y)$ exists and is equal to A. When Re $\sigma_1 \le 0$ and $Q_1(\xi) \not\equiv 0$, however, $\lim_{y\to 0+} A(y)$ does not exist, and A in this case is said to be the antilimit of $A(y)$ as $y \to 0+$.

We assume that $A(y)$ is known (computable) for all possible $y > 0$ and that the σ_k and q_k are known as well. Note that q_k is an *upper bound* for ∂Q_k, the degree of $Q_k(\xi)$, and that ∂Q_k need not be known exactly. We assume that A and α_{ki} are not necessarily known. Our purpose is to find (or approximate) A, whether A is the limit

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or antilimit of $A(y)$. One very effective way of doing this is through the generalized Richardson extrapolation.

To simplify the discussion of the generalized Richardson extrapolation process in the context of the present work, let us order the functions $(\log y)^i y^{\sigma_k}$ as follows:

$$
\begin{aligned}
\phi_i(y) &= (\log y)^{i-1} y^{\sigma_1}, \qquad 1 \le i \le \nu_1 \equiv q_1 + 1, \\
\phi_{\nu_1+i}(y) &= (\log y)^{i-1} y^{\sigma_2}, \qquad 1 \le i \le \nu_2 \equiv q_2 + 1, \\
\phi_{\nu_1+\nu_2+i}(y) &= (\log y)^{i-1} y^{\sigma_3}, \qquad 1 \le i \le \nu_3 \equiv q_3 + 1,
\end{aligned} \tag{1.4}
$$

and so on.

Let us now pick a decreasing sequence $\{y_l\}_{l=0}^{\infty}$ such that $y_l \in (0, b]$, and $\lim_{l\to\infty} y_l = 0$. Then we define the generalized Richardson extrapolation process through the linear systems of $p+1$ equations

$$
A(y_l) = A_p^j + \sum_{k=1}^{p} \bar{\alpha}_k \phi_k(y_l), \qquad j \le l \le j + p, \tag{1.5}
$$

for each pair (j, p) of nonnegative integers. Here A_p^j and $\bar{\alpha}_k$, $1 \le k \le p$, are the $p+1$ unknowns, A_p^j being the approximation to A.

The approximations A_p^j to A can be arranged in a two-dimensional table in the form

$$
\begin{array}{ccccc}
p=0 & p=1 & p=2 & p=3 & \cdots \\
A_0^0 & & & & \\
A_0^1 & A_1^0 & & & \\
A_0^2 & A_1^1 & A_2^0 & & \\
A_0^3 & A_1^2 & A_2^1 & A_3^0 & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\qquad A_0^j = A(y_j), \ j = 0, 1, \ldots \ . \tag{1.6}
$$

As shown in numerous places (see, e.g., [Sc] and [Si1]) A_p^j can be expressed in the form

$$
A_p^j = \sum_{i=0}^{p} \gamma_{p,i}^j A(y_{j+i}), \tag{1.7}
$$

where $\gamma_{p,i}^j$ are scalars that satisfy

$$
\sum_{i=0}^{p} \gamma_{p,i}^j = 1 \tag{1.8}
$$

and also

$$
\sum_{i=0}^{p} \gamma_{p,i}^j \phi_k(y_{j+i}) = 0, \qquad k = 1, 2, \ldots, p. \tag{1.9}
$$

In fact, $\gamma_{p,i}^j$ are *determined* by the linear equations in (1.8) and (1.9). What is implied by (1.9) is that the extrapolation process eliminates $\phi_1(y), \ldots, \phi_p(y)$ from the asymptotic expansion of $A(y) - A$ for $y \to 0+$. All of these facts will be used in what follows.

In the present work we pick y_l such that

$$
y_l = y_0 \omega^l, \ l = 1, 2, \ldots, \quad \text{for some } y_0 \in (0, b] \quad \text{and } \omega \in (0, 1). \tag{1.10}
$$

Obviously, $\{y_l\}_{l=0}^{\infty}$ is a decreasing sequence and $\lim_{l\to\infty} y_l = 0$.

Our purpose is to derive a simple and efficient recursive algorithm for computing A_p^j and to give a complete convergence and stability analysis for the columns and diagonals of the extrapolation table in (1.6). All of this has been done in [BuSt] for the special case in which $q_k = 0$, $k = 1, 2, \ldots$, and σ_k are real satisfying $0 < \sigma_1 < \sigma_2 < \sigma_3 < \cdots$, and the results of the present work reduce precisely to those of [BuSt] for this case.

Functions $A(y)$ of the type discussed in the present work arise very naturally as Euler–Maclaurin expansions in the trapezoidal rule approximations of integrals whose integrands have algebraic and logarithmic singularities (at the endpoints in one-dimensional integration and at corners and/or boundaries in multidimensional integration). One way of computing such integrals is by applying the generalized Romberg integration to appropriate sequences of trapezoidal rule approximations. The generalized Romberg integration in these cases is thus precisely the generalized Richardson extrapolation method we have described above. It also falls in the category of the generalized Richardson extrapolation process (GREP) discussed in [Si1] and is one of the examples of GREP there. See also the survey paper [Si3]. Even though this approach has been used successfully in many cases, its theoretical analysis has not been published before. It is hoped that the analysis presented here will contribute to our understanding of the properties of this useful and practical approach. When $q_k = q$ and $\sigma_{k+1} - \sigma_k = \rho$, $k = 1, 2, \ldots$, for arbitrary q, ρ, and σ_1, the extrapolation method described by (1.1)–(1.9) can be shown to be a GREP. The method in this case can be implemented very efficiently by the $W^{(m)}$-algorithm of [FSi], and with *arbitrary* y_l.

In the next section we concentrate on the algebraic aspects of the extrapolation process. In Theorem 2.1 we give a closed form expression for the polynomial $\sum_{i=0}^{p} \gamma_{p,i}^j \lambda^i$, showing at the same time that $\gamma_{p,i}^j$ are independent of j. In Theorem 2.2 we derive a very efficient recursive algorithm for A_p^j, which we denote the SGRom-algorithm for short. Finally, in Theorem 2.3 we provide a simple upper bound on the quantity $\sum_{i=0}^{p} |\gamma_{p,i}^j|$ that controls the numerical stability of A_p^j in the presence of roundoff. All of these results turn out to be very crucial in the rest of the developments of sections 3 and 4.

In section 3 we analyze the convergence and stability of the columns of the extrapolation table. In particular, in Theorem 3.1 we provide a complete asymptotic expansion of the error $A_p^j - A$ for $j \to \infty$, in Theorem 3.2 we provide the most dominant terms in this error for $j \to \infty$, and in Theorem 3.3 we prove that the columns are stable.

In section 4 we analyze the convergence and stability of the diagonals of the extrapolation table. In Theorem 4.1 we provide a very realistic upper bound on $\left|A_p^j - A\right|$. Under additional conditions on σ_k and q_k, we use this upper bound to prove a powerful convergence result in Theorem 4.2. In Theorem 4.3 we show that the diagonals are stable just as the columns are. One of the important contributions of this section is the formulation of the conditions on σ_k and q_k that guarantee both convergence and stability in their general form.

In section 5 we apply the theory of sections 3 and 4 to two problems, one being a generalized Romberg integration scheme for integrals with algebraic and logarithmic singularities. The generalized Romberg integration scheme for this case is an extrapolation method precisely of the form described above, as we mentioned earlier. For this example q_k are uniformly bounded in k. In the second example q_k increase polynomially in k, and this case has apparently not received much attention before in the context of generalized Richardson extrapolation.

In section 6 we consider briefly the case in which the existence of (1.1) is known, but σ_k are not available. In this case the generalized Richardson extrapolation process is not applicable, but the ϵ-algorithm can be used successfully.

We note that the recursive implementation of our generalized Richardson extrapolation process through the SGRom-algorithm is made possible by the choice of y_l given in (1.10). For arbitrary y_l (and arbitrary q_k and σ_k) we do not have an implementation as efficient as the SGRom-algorithm. In this case we can use the algorithm of [FSi, Appendix A] that we shall now denote the FS-algorithm for short. The FS-algorithm is summarized also in [Si3, section 2] and [Si4, section 1]. We can also use the E-algorithm of [Sc], different derivations for which can also be found in [H] and [Br]. We note, however, that the FS-algorithm requires about two thirds of the computation that is required by the E-algorithm and may thus be preferable. When y_l in (1.5) are as in (1.10), the SGRom-algorithm is superior to both the E-algorithm and FS-algorithm. A quantitative discussion of this point is provided following the proof of Theorem 2.2.

Finally, we would like to comment that the order in which the functions $(\log y)^i\, y^{\sigma_k}$ are eliminated in the extrapolation procedure of the present work is not the conventional one. In the present work we eliminate these functions in the order $i = 0, 1, 2, \ldots, q_k$, $k = 1, 2, \ldots$, so that $\phi_k(y)$ in (1.4) do *not* satisfy $\phi_{k+1}(y) = O(\phi_k(y))$ as $y \to 0+$ for all $k = 1, 2, \ldots$. In the conventional order, however, $i = q_k$, $q_k - 1, \ldots, 1, 0$, $k = 1, 2, \ldots$, to achieve $\phi_{k+1}(y) = O(\phi_k(y))$ as $y \to 0+$ for all $k = 1, 2, \ldots$. As a result, intuition would suggest that those columns of the extrapolation table of (1.6) for which $\sum_{i=1}^{t} \nu_i < p < \sum_{i=1}^{t+1} \nu_i$, $t = 0, 1, \ldots$, may not enjoy any acceleration property. The delicate analysis of Theorem 3.2 reveals, however, that this acceleration property is preserved under our ordering in (1.4). Furthermore, this ordering enables the recursive SGRom-algorithm of Theorem 2.2 as well. Our efforts to obtain a good recursive algorithm with the conventional order were not successful.

2. Algebraic properties of $\gamma_{p,i}^{j}$ and recursive computation of A_p^{j}. Let us first note that for any integer $p \geq 0$ there exist unique integers t and s such that $t \geq 0$ and $0 \leq s \leq \nu_{t+1} - 1$ and

$$p = \sum_{k=1}^{t} \nu_k + s, \tag{2.1}$$

where $\nu_k \equiv q_k + 1$, $k = 1, 2, \ldots$, as already defined in (1.4). For $t = 0$ we take $\sum_{k=1}^{t} \nu_k$ to mean zero.

Next, with p, t, and s as above, we also define the sets of integer pairs S_p and T_p by

$$\begin{aligned} S_p &= \{(k,r) : 1 \leq k \leq t \text{ and } 0 \leq r \leq q_k\} \cup \{(k,r) : k = t+1 \text{ and } 0 \leq r \leq s-1\}, \\ T_p &= \{(k,r) : k \geq 1 \text{ and } 0 \leq r \leq q_k\} \backslash S_p. \end{aligned} \tag{2.2}$$

In S_p, (i) when $t = 0$, we can have only $k = 1$ and $0 \leq r \leq s - 1$, and (ii) when $s = 0$, we can have only $1 \leq k \leq t$ and $0 \leq r \leq q_k$.

Note that with $\phi_k(y)$ as defined in (1.4), the sets of functions $\{\phi_k(y)\}_{k=1}^{p}$ and $\{(\log y)^r\, y^{\sigma_k}\}_{(k,r)\in S_p}$ are identical.

In Theorem 2.1 below we show that when y_l are as given in (1.10), $\gamma_{p,i}^{j}$ are coefficients of a simple known polynomial. This result is somewhat surprising in view of the fact that the functions $\phi_k(y)$ in the linear systems of (1.5) that define the extrapolation procedure are not simple at all.

THEOREM 2.1. *Given the integer $p \geq 0$, let $t \geq 0$ and $0 \leq s \leq \nu_{t+1} - 1$ be the unique integers for which p is as given in* (2.1)*. Then, with y_l as given in* (1.10)*, $\gamma_{p,i}^j$ turn out to be independent of j. Let us denote $\gamma_{p,i}^j = \gamma_{p,i}$. Then $\gamma_{p,i}$ satisfy*

$$\sum_{i=0}^{p} \gamma_{p,i} \lambda^i = \left[\prod_{i=1}^{t} \left(\frac{\lambda - c_i}{1 - c_i} \right)^{\nu_i} \right] \left(\frac{\lambda - c_{t+1}}{1 - c_{t+1}} \right)^s \equiv U_p(\lambda), \tag{2.3}$$

where

$$c_k = \omega^{\sigma_k}, \qquad k = 1, 2, \dots . \tag{2.4}$$

Proof. With y_l as given in (1.10) and with $\phi_k(y)$ as defined in (1.4), equation (1.9) becomes

$$\sum_{i=0}^{p} \gamma_{p,i}^j \left[\log y_0 + (j+i) \log \omega \right]^r \left(y_0 \omega^{j+i} \right)^{\sigma_k} = 0, \qquad (k, r) \in S_p. \tag{2.5}$$

Analyzing these equations in the order $r = 0, 1, \dots, q_k$ when $1 \leq k \leq t$, and in the order $r = 0, 1, \dots, s-1$ when $k = t+1$, we can see that they are equivalent to

$$\sum_{i=0}^{p} \gamma_{p,i}^j (j+i)^r c_k^{j+i} = 0, \qquad (k, r) \in S_p. \tag{2.6}$$

But

$$\sum_{i=0}^{p} \gamma_{p,i}^j (j+i)^r c_k^{j+i} = \left(\lambda \frac{d}{d\lambda} \right)^r \left(\sum_{i=0}^{p} \gamma_{p,i}^j \lambda^{j+i} \right) \Bigg|_{\lambda = c_k}. \tag{2.7}$$

Thus, combining (2.6) and (2.7), we obtain

$$\left(\frac{d}{d\lambda} \right)^r \left(\sum_{i=0}^{p} \gamma_{p,i}^j \lambda^{j+i} \right) \Bigg|_{\lambda = c_k} = 0, \qquad (k, r) \in S_p. \tag{2.8}$$

It is obvious from (2.8) that the polynomial $\sum_{i=0}^{p} \gamma_{p,i}^j \lambda^i$ has a zero of order ν_k at c_k, $1 \leq k \leq t$, and a zero of order s at c_{t+1}. Also, the sum of the multiplicities of these zeros is simply $\sum_{k=1}^{t} \nu_k + s$, which is p by (2.1). This, together with (1.8), results in (2.3). □

Note. It must be emphasized that the fact that $\gamma_{p,i}^j$ are independent of j is a consequence of the choice $y_l = y_0 \omega^l$ in (1.10).

With the help of Theorem 2.1 we prove in Theorem 2.2 below that A_p^j can be computed by a very simple recursion relation. As this recursion relation reduces to the Romberg algorithm when $q_k = 0$ and $\sigma_k = k\delta$ for some $\delta \neq 0$, $k = 1, 2, \dots$, we shall call it the generalized Romberg algorithm and denote it the SGRom-algorithm for short.

THEOREM 2.2 (SGRom-algorithm). *Let us define (cf.* (1.4)*)*

$$\begin{aligned} \lambda_i &= c_1, & 1 \leq i \leq \nu_1, \\ \lambda_{\nu_1 + i} &= c_2, & 1 \leq i \leq \nu_2, \\ \lambda_{\nu_1 + \nu_2 + i} &= c_3, & 1 \leq i \leq \nu_3, \end{aligned} \tag{2.9}$$

and so on. Then A_p^j can be computed recursively from

$$A_0^j = A\left(y_j\right), \qquad j = 0, 1, \ldots, \tag{2.10}$$
$$A_p^j = \frac{A_{p-1}^{j+1} - \lambda_p A_{p-1}^j}{1 - \lambda_p}, \quad j = 0, 1, \ldots, \quad p = 1, 2, \ldots .$$

Proof. By (2.9), the polynomial $U_p\left(\lambda\right)$ in (2.3) becomes simply

$$U_p\left(\lambda\right) = \prod_{i=1}^{p} \frac{\lambda - \lambda_i}{1 - \lambda_i}. \tag{2.11}$$

Thus

$$U_p\left(\lambda\right) = \frac{\lambda - \lambda_p}{1 - \lambda_p} U_{p-1}\left(\lambda\right) = \frac{\lambda U_{p-1}\left(\lambda\right) - \lambda_p U_{p-1}\left(\lambda\right)}{1 - \lambda_p}. \tag{2.12}$$

Consequently, with $\gamma_{k,i} = 0$ when $i > k$ or $i < 0$, we have

$$\gamma_{p,i} = \frac{\gamma_{p-1,i-1} - \lambda_p \gamma_{p-1,i}}{1 - \lambda_p}, \qquad 0 \le i \le p. \tag{2.13}$$

The result in (2.10) now follows by substituting (2.13) in (1.7). □

Given $A(y_l)$, $l = 0, 1, ..., N$, the generalized Richardson extrapolation process produces the approximations A_p^j, $0 \le j + p \le N$. The SGRom-algorithm computes all these A_p^j in $O(N^2)$ arithmetic operations as is clear from (2.10). The FS- and E-algorithms, on the other hand, need $O(N^3)$ arithmetic operations for the same task, the FS-algorithm being the more efficient of the two. The latter two also require more storage than the SGRom-algorithm.

Next, we give a simple result concerning $|\gamma_{p,i}^j| = |\gamma_{p,i}|$.

THEOREM 2.3. *Under the conditions of Theorem* 2.1 $|\gamma_{p,i}^j| = |\gamma_{p,i}|$ *satisfy*

$$\sum_{i=0}^{p} \left|\gamma_{p,i}^j\right| |z|^i \le \left[\prod_{i=1}^{t} \left(\frac{|z| + |c_i|}{|1 - c_i|}\right)^{\nu_i}\right] \left(\frac{|z| + |c_{t+1}|}{|1 - c_{t+1}|}\right)^s. \tag{2.14}$$

In particular, we have

$$\sum_{i=0}^{p} \left|\gamma_{p,i}^j\right| \le \left[\prod_{i=1}^{t} \left(\frac{1 + |c_i|}{|1 - c_i|}\right)^{\nu_i}\right] \left(\frac{1 + |c_{t+1}|}{|1 - c_{t+1}|}\right)^s . \tag{2.15}$$

If c_i, $1 \le i \le t+1$, all have the same phase, then equality holds both in (2.14) *and* (2.15)*. This holds, in particular, when c_i, $1 \le i \le t+1$, are all real positive or all real negative. Furthermore, we have $\sum_{i=0}^{p} |\gamma_{p,i}^j| = 1$ for the case $c_i < 0$, $1 \le i \le t+1$.*

Proof. Let $Q\left(z\right) = \sum_{i=0}^{n} a_i z^i$, $a_n = 1$, and denote its zeros by $z_1, \ldots, z_n$. Then $\left(-1\right)^i a_{n-i} = \sum_{k_1 < \cdots < k_i} \prod_{s=1}^{i} z_{k_s}$, $i = 1, 2, \ldots, n$. Thus $|a_{n-i}| \le \sum_{k_1 < \cdots < k_i} \prod_{s=1}^{i} |z_{k_s}| \equiv \tilde{a}_{n-i}$, $i = 1, 2, \ldots, n$. Set $\tilde{a}_n = 1$. Consequently,

$$\sum_{i=0}^{p} |a_i| \, |z|^i \le \sum_{i=0}^{p} \tilde{a}_i \, |z|^i = \prod_{i=1}^{n} \left(|z| + |z_i|\right),$$

whether a_i and/or z_i are real or complex.

Applying this result in conjunction with Theorem 2.1, we obtain (2.14) and hence (2.15). The rest follows from the observation that $|a_{n-i}| = \tilde{a}_{n-i}$, $i = 1, 2, \ldots, n$, when the z_i all have the same phase. □

It is important to mention that to a large extent the quantity $\sum_{i=0}^{p} |\gamma_{p,i}^{j}|$ controls the numerical stability of A_p^j with respect to roundoff. The upper bound on $\sum_{i=0}^{p} |\gamma_{p,i}^{j}|$ in (2.15) thus gives very reliable information on the numerical quality of A_p^j in floating point arithmetic. For details see [Si1].

We would like to note that both Theorems 2.1 and 2.3 are of critical importance in the convergence and stability analyses that we provide in the next two sections.

Finally, we mention that with y_l as in (1.10), (1.1) and (1.2) give the asymptotic expansion

$$A\left(y_n\right) \sim A + \sum_{k=1}^{\infty} \left(\sum_{i=0}^{q_k} \beta_{ki} n^i \right) c_k^n \quad \text{as } n \to \infty \;, \tag{2.16}$$

where β_{ki} depend linearly on α_{kr}, $i \le r \le q_k$, and $\beta_{kq_k} = \alpha_{kq_k} y_0^{\sigma_k} \left(\log \omega\right)^{q_k}$, and c_k is as given in (2.4).

Consequently, the extrapolation process, the SGRom-algorithm, and all of the theoretical developments of this paper directly apply to *sequences* $\{S_n\}_{n=0}^{\infty}$ satisfying

$$S_n \sim S + \sum_{k=1}^{\infty} \left(\sum_{i=0}^{q_k} \beta_{ki} n^i \right) c_k^n \quad \text{as } n \to \infty, \tag{2.17}$$

as well.

3. Convergence and stability of columns. In this section we will be concerned with the problems of convergence and stability of the sequences $\left\{A_p^j\right\}_{j=0}^{\infty}$, where p is held fixed. These appear as columns of the extrapolation table in (1.6). In Theorem 3.1 below we give a complete asymptotic expansion of $A_p^j - A$ for $j \to \infty$. In Theorem 3.2 we analyze the dominant terms in this expansion and provide both quantitative and qualitative results for $A_p^j - A$ as $j \to \infty$.

As c_k, rather than σ_k, are involved in the analysis of this section as well as the next one, it is important to make the following observations about c_k:

(i) $|c_1| \ge |c_2| \ge |c_3| \ge \cdots$, and $|c_i| = |c_j|$ if and only if $\mathrm{Re}\ \sigma_i = \mathrm{Re}\ \sigma_j$;

(ii) $c_k \ne 1$, $k = 1, 2, \ldots$, and $\lim_{k\to\infty} c_k = 0$.

These follow from (1.3), (2.4), and from $|c_k| = \omega^{\mathrm{Re}\ \sigma_k}$.

THEOREM 3.1. *Given the integer p, let the integers* $t \ge 0$ *and* $0 \le s \le \nu_{t+1} - 1$ *and the set* T_p *be as in* (2.1) *and* (2.2)*, respectively. Then, with the polynomial* $U_p(\lambda)$ *as defined in* (2.3) *of Theorem* 2.1*, we have the asymptotic expansion*

$$A_p^j - A \sim \sum_{(k,r)\in T_p} \beta_{kr} \left\{ \left(\lambda \frac{d}{d\lambda} \right)^r \left[\lambda^j U_p\left(\lambda\right) \right] \Big|_{\lambda = c_k} \right\} \ \text{as } j \to \infty. \tag{3.1}$$

Proof. From (1.7), (1.8), and (2.16) we have

$$\begin{aligned} A_p^j - A &= \sum_{i=0}^{p} \gamma_{p,i} \left[A\left(y_{j+i}\right) - A \right] \\ &\sim \sum_{i=0}^{p} \gamma_{p,i} \sum_{k=1}^{\infty} \left[\sum_{r=0}^{q_k} \beta_{kr} \left(j+i\right)^r \right] c_k^{j+i} \quad \text{as } j \to \infty. \end{aligned} \tag{3.2}$$

Interchanging the order of summation, and invoking (2.6), we can rewrite (3.2) in the form

$$A_p^j - A \sim \sum_{(k,r)\in T_p} \beta_{kr} \left[\sum_{i=0}^{p} \gamma_{p,i} \left(j+i\right)^r c_k^{j+i}\right] \quad \text{as } j \to \infty. \tag{3.3}$$

The result in (3.1) now follows by invoking (2.7) in (3.3). ◻

A cursory look at the asymptotic expansion given in (3.1) shows that the error $A_p^j - A$ is at worst $O(j^{\hat{q}} c_{t+1}^j)$ as $j \to \infty$, where $\hat{q} = \max\{q_k : |c_k| = |c_{t+1}|, k \geq t+1\}$. This is due to the fact that $(t+1, q_{t+1}) \in T_p$. This result is not the best possible, however, and can be improved upon by a more careful analysis of (3.1). We do this in Theorem 3.2 below, in which we also provide the dominant terms in (3.1) explicitly.

THEOREM 3.2. *With $p, t,$ and s as in Theorem 3.1, let μ be that integer for which*

$$|c_{t+1}| = \cdots = |c_{t+\mu}| > |c_{t+\mu+1}|. \tag{3.4}$$

(i) *When $s = 0$, A_p^j satisfies*

$$A_p^j - A = \sum_{k=t+1}^{t+\mu} \left\{\beta_{kq_k} U_p\left(c_k\right) c_k^j j^{q_k} + o\left(c_k^j j^{q_k}\right)\right\} \quad \text{as } j \to \infty. \tag{3.5}$$

(ii) *When $0 < s \leq \nu_{t+1} - 1$, A_p^j satisfies*

$$\begin{aligned} A_p^j - A = {} & \beta_{t+1,q_{t+1}} \binom{q_{t+1}}{s} U_p^{(s)}\left(c_{t+1}\right) c_{t+1}^{j+s} j^{q_{t+1}-s} + o\left(c_{t+1}^j j^{q_{t+1}-s}\right) \\ & + \sum_{k=t+2}^{t+\mu} \left\{\beta_{kq_k} U_p\left(c_k\right) c_k^j j^{q_k} + o\left(c_k^j j^{q_k}\right)\right\} \quad \text{as } j \to \infty. \end{aligned} \tag{3.6}$$

(iii) *As a consequence of* (i) *and* (ii), *for all s, $0 \leq s \leq \nu_{t+1} - 1$, we have*

$$A_p^j - A = O\left(\left|c_{t+1}\right|^j j^{\bar{q}}\right) \quad \text{as } j \to \infty, \tag{3.7}$$

where

$$\bar{q} = \max\left(q_{t+1} - s,\ q_{t+2}, \ldots, q_{t+\mu}\right). \tag{3.8}$$

Proof. We start by observing that the dominant terms in the asymptotic expansion given in (3.1) are those with $k = t+1, \ldots, t+\mu$. This can be seen very easily from the identical asymptotic expansion given in (3.3).

To prove (i) we first note that when $s = 0$, $U_p\left(c_k\right) \neq 0$ for all $k \geq t+1$. Thus, for all $(k,r) \in T_p$,

$$\begin{aligned} \sum_{i=0}^{p} \gamma_{p,i} \left(j+i\right)^r c_k^{j+i} & \sim \sum_{i=0}^{p} \gamma_{p,i} j^r c_k^{j+i} \quad \text{as } j \to \infty \\ & \sim U_p\left(c_k\right) c_k^j j^r \quad \text{as } j \to \infty. \end{aligned} \tag{3.9}$$

It follows from (3.9) that, for any $k \geq t+1$, of all the terms with $0 \leq r \leq q_k$ in (3.1), the one with $r = q_k$ is the most dominant. With this the proof of (3.5) can now be completed.

The proof of (ii) proceeds along the same lines as that of (i). As in the previous case we have $U_p(c_k) \neq 0$ for $k \geq t+2$, from which we deduce the validity of the summation on the right-hand side of (3.6). For $k = t+1$, however, we have $U_p(c_{t+1}) = 0$, hence (3.9) is not valid for this case, and we need a more detailed analysis. To this effect we observe that for any function $f(\lambda)$

$$\left(\lambda \frac{d}{d\lambda}\right)^r \left[\lambda^j f(\lambda)\right] = \sum_{i=0}^{r} \upsilon_{ri}(j)\, \lambda^{j+i} f^{(i)}(\lambda), \tag{3.10}$$

where $\upsilon_{r0}(j) = j^r$ and $\upsilon_{rr}(j) = 1$, and

$$\upsilon_{ri}(j) \sim \binom{r}{i} j^{r-i} \quad \text{as } j \to \infty, \ \ 0 \leq i \leq r. \tag{3.11}$$

The proof of (3.10) and (3.11) can be achieved by induction, and we leave its details to the interested reader. By (3.10) and the fact that $U_p^{(i)}(c_{t+1}) = 0$, $0 \leq i \leq s-1$, we have for $r \geq s$

$$\left(\lambda \frac{d}{d\lambda}\right)^r \left[\lambda^j U_p(\lambda)\right]\bigg|_{\lambda = c_{t+1}} = \sum_{i=s}^{r} \upsilon_{ri}(j)\, c_{t+1}^{j+i} U_p^{(i)}(c_{t+1}). \tag{3.12}$$

For $j \to \infty$ the most dominant term in this summation is that with $i = s$ and it is of order $c_{t+1}^j j^{r-s}$. As $s \leq r \leq q_{t+1}$ when $k = t+1$ in (3.12), we therefore have that the most dominant of the terms with $k = t+1$ and $s \leq r \leq q_{t+1}$ in (3.1) is that with $r = q_{t+1}$. By (3.12) and (3.11) this term is asymptotically equivalent to

$$\beta_{t+1,q_{t+1}} \binom{q_{t+1}}{s} U_p^{(s)}(c_{t+1})\, c_{t+1}^{j+s} j^{q_{t+1}-s}$$

as $j \to \infty$. The proof of the result in (3.6) can now be completed. ∎

COROLLARY. *Provided* $\mu = 1$ *and* $\beta_{t+1,q_{t+1}} \neq 0$ *in Theorem* 3.2, *for any* $s = 0, 1, \ldots, \nu_{t+1} - 1$ *we have*

$$A_p^j - A \sim \beta_{t+1,q_{t+1}} \binom{q_{t+1}}{s} U_p^{(s)}(c_{t+1})\, c_{t+1}^{j+s} j^{q_{t+1}-s} \quad \text{as } j \to \infty. \tag{3.13}$$

This is the case when $|c_1| > |c_2| > |c_3| > \cdots$, *for example. Consequently, in case* $|c_{t+1}| < 1$, $\lim_{j\to\infty} A_p^j = A$ *for all* s, $0 \leq s < \nu_{t+1} - 1$, *and the column for which* s *is larger converges more quickly than the preceding ones.*

In connection with Theorem 3.2 and its corollary we note that $\beta_{kq_k} \neq 0$ in (2.16) if and only if $\alpha_{kq_k} \neq 0$ in (1.2).

Note. The results of Theorems 3.1 and 3.2 and its corollary are valid whether $\lim_{j\to\infty} A_p^j$ exists or not. Obviously, $\lim_{j\to\infty} A_p^j = A$ when $|c_{t+1}| < 1$, i.e., when $\operatorname{Re} \sigma_{t+1} > 0$, even when some or all of $|c_1|, \ldots, |c_t|$ may be greater than or equal to unity.

With the question of convergence of columns of the extrapolation table in (1.6) settled, we now turn to the question of stability.

THEOREM 3.3. *The extrapolation process that generates the sequences* $\left\{A_p^j\right\}_{j=0}^{\infty}$ *with* p *fixed is stable in the sense that*

$$\sup_j \sum_{i=0}^{p} \left|\gamma_{p,i}^j\right| < \infty. \tag{3.14}$$

Proof. The result follows from the fact that the $\gamma_{p,i}^j$ are all independent of j as proved in Theorem 2.1. ∎

4. Convergence and stability of diagonals. In this section we will be concerned with the problems of convergence and stability of the sequences $\left\{A_p^j\right\}_{p=0}^{\infty}$, where j is held fixed. These appear as the diagonals of the extrapolation table in (1.6). In Theorem 4.1 we give an upper bound on $\left|A_p^j - A\right|$ that is suitable for analysis as $p \to \infty$. In Theorem 4.2 we provide this analysis under additional realistic assumptions. In Theorem 4.3 we give the corresponding stability result.

First, with p, t, s as in the first paragraph of section 2, and with the set S_p as in the second paragraph there, we define $R_p(y)$ by

$$A(y) - A = \sum_{(k,r)\in S_p} \alpha_{kr} (\log y)^r y^{\sigma_k} + R_p(y). \tag{4.1}$$

Let us set

$$\hat{q} = \max\{q_k : |c_k| = |c_{t+1}|, \, k \ge t+1\}. \tag{4.2}$$

Then we can see from (1.1)–(1.3) that

$$|R_p(y)| \le \hat{\alpha}_p |\log y|^{\hat{q}} y^{\mathrm{Re}\ \sigma_{t+1}} \text{ for some constant } \hat{\alpha}_p > 0. \tag{4.3}$$

From this we also have

$$|R_p(y_n)| \le \hat{\beta}_p n^{\hat{q}} |c_{t+1}|^n, \tag{4.4}$$

with

$$\hat{\beta}_p = \hat{\alpha}_p \left(|\log y_0| + |\log \omega|\right)^{\hat{q}} y_0^{\mathrm{Re}\ \sigma_{t+1}}. \tag{4.5}$$

THEOREM 4.1. *With $p, t, s, \hat{q}$, and $\hat{\beta}_q$ as above, we have*

$$\left|A_p^j - A\right| \le \hat{\beta}_p (j+p)^{\hat{q}} |c_{t+1}|^j \left[\prod_{i=1}^{t} \left(\frac{|c_{t+1}| + |c_i|}{|1-c_i|}\right)^{\nu_i}\right] \left(\frac{2|c_{t+1}|}{|1-c_{t+1}|}\right)^s. \tag{4.6}$$

Proof. Substituting (4.1) in the equality in (3.2), and invoking (1.9), we obtain

$$A_p^j - A = \sum_{i=0}^{p} \gamma_{p,i} R_p(y_{j+i}), \tag{4.7}$$

which by (4.4) becomes

$$\begin{aligned} \left|A_p^j - A\right| &\le \hat{\beta}_p \sum_{i=0}^{p} |\gamma_{p,i}| (j+i)^{\hat{q}} |c_{t+1}|^{j+i} \\ &\le \hat{\beta}_p (j+p)^{\hat{q}} |c_{t+1}|^j \sum_{i=0}^{p} |\gamma_{p,i}| \, |c_{t+1}|^i. \end{aligned} \tag{4.8}$$

The result in (4.6) now follows by invoking (2.14) of Theorem 2.3 in the last summation of (4.8). □

Let $1 = k_1 < k_2 < k_3 < \cdots$ be the (smallest) positive integers for which

$$\mathrm{Re}\ \sigma_{k_i} < \mathrm{Re}\ \sigma_{k_{i+1}} \text{ and } \mathrm{Re}\ \sigma_m = \mathrm{Re}\ \sigma_{k_i}, \ k_i \le m < k_{i+1}, \ i = 1, 2, \ldots . \tag{4.9}$$

By the fact that $|c_i| = \omega^{\mathrm{Re}\ \sigma_i}$, (4.9) is equivalent to

$$|c_{k_i}| > |c_{k_{i+1}}| \text{ and } |c_m| = |c_{k_i}|,\ k_i \le m < k_{i+1},\ i = 1, 2, \dots\ . \tag{4.10}$$

Let us now define

$$\mu_i = k_{i+1} - k_i \text{ and } N_i = \sum_{m=0}^{\mu_i - 1} \nu_{k_i+m},\ i = 1, 2, \dots\ . \tag{4.11}$$

Thus the number of σ_m whose real parts are equal to $\mathrm{Re}\sigma_{k_i}$ or, equivalently, the number of c_m whose moduli are equal to $|c_{k_i}|$ is μ_i, and the sum of their respective multiplicities ν_m is N_i.

THEOREM 4.2. *Assume that σ_k are such that*

$$Re\left(\sigma_{k_{i+1}} - \sigma_{k_i}\right) \ge M > 0, \qquad i = 1, 2, \dots, \textit{for some fixed } M > 0. \tag{4.12}$$

Assume also that N_i satisfy

$$\liminf_{i\to\infty} N_i/i^a = D \quad \textit{and} \quad \limsup_{i\to\infty} N_i/i^b = E \quad \textit{for some } D > 0,\quad E > 0,\ \textit{and} \tag{4.13}$$
$$0 \le a \le b \textit{ with } a + 2 > b.$$

If, for $k_i \le k < k_{i+1}$, all of the α_{km} in (1.2) *grow at most like B^{i^u} for $B > 1$ and $u < a + 2$, and* Re $\sigma_{k_i} = O(i^{u'})$ *as $i \to \infty$ with $u' < a + 2$ when $y_0 > 1$, then $\lim_{p\to\infty} A_p^j = A$ whether $\lim_{y\to 0+} A(y)$ exists or not.*

Let $\tilde{K} = \omega^{DM/(a+2)}$. Let also $k_{r+1} \le t + 1 < k_{r+2}$ for some r. Then, for any $\varepsilon > 0$ for which $\tilde{K} + \varepsilon < 1$, there exists a positive integer p_0 such that

$$\left|A_p^j - A\right| \le \left(\tilde{K} + \varepsilon\right)^{r^{a+2}} \textit{ for all } p \ge p_0. \tag{4.14}$$

Note that r is uniquely determined by t, and t is uniquely determined by p from (2.1)*. Also, $r = O\left(p^{1/(a+1)}\right)$ as $p \to \infty$, so that $r \to \infty$ as $p \to \infty$. The result in* (4.14) *can also be expressed as*

$$\left|A_p^j - A\right| \le \left(\tilde{L} + \varepsilon\right)^{p^{(a+2)/(b+1)}} \quad \textit{for all } p \ge p_0, \tag{4.15}$$

where $\tilde{L} = \omega^\tau < 1$, $\tau = \frac{DM}{a+2}\left(\frac{b+1}{E}\right)^{(a+2)/(b+1)}$.

Proof. Let us rewrite (4.6) as

$$\left|A_p^j - A\right| \le \left\{\left[\prod_{i=1}^{t}\left(\frac{1 + |c_{t+1}/c_i|}{|1 - c_i|}\right)^{\nu_i}\right]\left(\frac{2}{|1 - c_{t+1}|}\right)^s\right\}\hat{\beta}_p\left|\prod_{i=1}^{t} c_i^{\nu_i}\right|\left|c_{t+1}^{s+j}\right|(j+p)^{\hat{q}}. \tag{4.16}$$

(i) First, we show that the infinite products $\prod_{i=1}^\infty |1 - c_i|^{\nu_i}$ and $\prod_{i=1}^\infty (1 + |c_i|)^{\nu_i}$ converge under (4.12) and (4.13). To show this it is sufficient to demonstrate that $\sum_{i=1}^\infty \nu_i |c_i|$ converges. We have

$$\sum_{i=1}^\infty \nu_i |c_i| = \sum_{i=1}^\infty N_i |c_{k_i}| \le E_1 \sum_{i=1}^\infty i^b |c_{k_i}|, \tag{4.17}$$

where we have used the fact that $N_i \leq E_1 i^b$, $i = 1, 2, \ldots$, for some $E_1 > E$ that follows from (4.13). The result follows from the convergence of $\sum_{i=1}^{\infty} i^b |c_{k_i}|$, which can be verified by the ratio test upon invoking

$$\left|\frac{c_{k_{i+1}}}{c_{k_i}}\right| \leq \omega^M < 1, \qquad i = 1, 2, \ldots, \tag{4.18}$$

that follows from (4.12).

The fact that $\prod_{i=1}^{\infty} |1 - c_i|^{\nu_i}$ converges implies that $(\prod_{i=1}^{t} |1 - c_i|^{\nu_i})|1 - c_{t+1}|^s$ in (4.16) is bounded away from zero for all p.

(ii) Next, we show that

$$W_p = 2^s \prod_{i=1}^{t} (1 + |c_{t+1}/c_i|)^{\nu_i} \leq H^{r^b} \text{ for some } H > 1. \tag{4.19}$$

From $s < \nu_{t+1}$, $N_i \leq E_1 i^b$, $i = 1, 2, \ldots$, and $|c_{t+1}| = |c_{k_{r+1}}|$ which follows from $k_{r+1} \leq t + 1 < k_{r+2}$, it follows that

$$\begin{aligned} W_p &\leq \prod_{i=1}^{t+1} (1 + |c_{t+1}/c_i|)^{\nu_i} \leq \prod_{i=1}^{k_{r+2}-1} (1 + |c_{t+1}/c_i|)^{\nu_i} \\ &\leq \prod_{i=1}^{r+1} \left(1 + |c_{k_{r+1}}/c_{k_i}|\right)^{N_i} \leq \left[\prod_{i=1}^{r+1} \left(1 + |c_{k_{r+1}}/c_{k_i}|\right)\right]^{E_2 r^b} \quad \text{for some } E_2 > E_1. \end{aligned} \tag{4.20}$$

From (4.18)

$$\prod_{i=1}^{r+1} \left(1 + |c_{k_{r+1}}/c_{k_i}|\right) \leq \prod_{i=1}^{r+1} \left(1 + K^{r+1-i}\right) = \prod_{i=0}^{r} \left(1 + K^i\right) \tag{4.21}$$

with $K = \omega^M < 1$. Since $\prod_{i=0}^{\infty} \left(1 + K^i\right)$ converges, $\prod_{i=0}^{r} \left(1 + K^i\right)$ is bounded for all r, say, by $H^{1/E_2} > 1$. The result in (4.19) now follows.

(iii) Next, we prove that for all large p, and for $\epsilon > 0$ but arbitrary,

$$V_p^j = \left|\prod_{i=1}^{t} c_i^{\nu_i}\right| \left|c_{t+1}^{s+j}\right| \leq (\tilde{K} + \epsilon)^{r^{a+2}[1+\eta(r)]},\ \eta(r) = O\left(r^{-1}\right) \text{ as } r \to \infty. \tag{4.22}$$

First, we observe that $\sum_{i=1}^{t} \nu_i \leq p \leq \sum_{i=1}^{t+1} \nu_i$ so that $p \to \infty$ implies $t \to \infty$ and vice versa. Also, from the fact that $\sum_{i=1}^{r} N_i \leq p \leq \sum_{i=1}^{r+1} N_i$ and from (4.13), we have that $p \to \infty$ implies $r \to \infty$ and vice versa, and that $p = O\left(r^{b+1}\right)$ as $r \to \infty$ and $r = O\left(p^{1/(a+1)}\right)$ as $p \to \infty$. From $\lim_{k\to\infty} c_k = 0$ it follows that $|c_k| \leq 1$ for all $k \geq m+1$, m being a fixed nonnegative integer. Thus, for p sufficiently large, we have $|c_k| \leq 1$ for $k \geq k_{r+1}$. With this and with $|c_{t+1}| = \left|c_{k_{r+1}}\right|$ we have

$$V_p^j \leq \left|\prod_{i=1}^{t} c_i^{\nu_i}\right| \leq \left|\prod_{i=1}^{k_{r+1}-1} c_i^{\nu_i}\right| = \left|\prod_{i=1}^{r} c_{k_i}^{N_i}\right| = \omega^{X_r}, \tag{4.23}$$

where

$$X_r = \sum_{i=1}^{r} N_i \text{ Re } \sigma_{k_i}. \tag{4.24}$$

But from (4.12) we have $\text{Re}\ \sigma_{k_i} \geq \text{Re}\ \sigma_1 + (i-1)\,M$, $i = 1, 2, \ldots$. Substituting this in (4.24), and using the fact that, given $\varepsilon > 0$ arbitrary and sufficiently close to zero, there exists a positive integer i_0 such that $N_i > (D - \varepsilon)\, i^a$ for $i > i_0$, we obtain

$$X_r \geq \sum_{i=1}^{r} N_i \left[\text{Re}\ \sigma_1 + (i-1)\,M\right] > \frac{(D-\varepsilon)\,M}{a+2} r^{a+2} \left[1 + \eta\,(r)\right], \quad \eta\,(r) = O\left(r^{-1}\right) \text{ as } r \to \infty. \tag{4.25}$$

With this the proof of (4.22) is now complete.

(iv) Next, we have also

$$(j+p)^{\hat{q}} = O\left(r^{(b+1)E_1 r^b}\right) \text{ as } r \to \infty. \tag{4.26}$$

This follows from $p = O\left(r^{b+1}\right)$ as $r \to \infty$ and from $\hat{q} \leq N_{r+1}$ and from (4.13).

(v) The growth condition on the α_{km} and on $\text{Re}\ \sigma_{k_i}$, together with the connection between the α_{km}, $k_{r+1} \leq k < k_{r+1}$, and $\hat{\alpha}_p$, and the connection between $\hat{\beta}_p$ and $\hat{\alpha}_p$ given in (4.5), suggest that $\hat{\beta}_p = O\left(B_1^{r^{u_1}}\right)$ as $r \to \infty$ for some $B_1 > 1$ and $u_1 < a+2$.

Finally, by combining the results in (i)–(v), we obtain the result given in (4.14). The result in (4.15) can be obtained by using the fact that $\limsup_{p\to\infty} \left(p/r^{b+1}\right) \leq E/\,(b+1)$, which follows from (4.13) and $\sum_{i=1}^{r} N_i \leq p \leq \sum_{i=1}^{r+1} N_i$. $\square$

Remarks.

(1) The conditions that are imposed on σ_k, N_i, and α_{km} in Theorem 4.2 may seem to be arbitrary at first, but they are, in fact, naturally satisfied in many cases of practical interest. In addition to guaranteeing quick convergence, these conditions also guarantee stability for the diagonals, as we show in Theorem 4.3 below. The condition in (4.13) that is imposed on N_i can be achieved when $Fk^{\theta_1} \leq \nu_k \leq F'k^{\theta_1'}$ for all k and $Gi^{\theta_2} \leq \mu_i \leq G'i^{\theta_2'}$ for all i, for $0 \leq \theta_1 \leq \theta_1'$ and for $0 \leq \theta_2 \leq \theta_2'$. In connection with the growth condition on the α_{km}, it is worth mentioning that this condition is a mild and rather comprehensive one; it includes, for example, $\alpha_{km} = O\left((di)!\right)$ as $i \to \infty$, for any $d > 0$, where $k_{i+1} \leq k < k_{i+2}$.

(2) When $a \neq b$ and/or $D \neq E$ in (4.13), the results in (4.14) and (4.15) are not equivalent. Although (4.14) implies (4.15), the opposite is not always valid. Only when $a = b$ and $D = E$ does (4.15) imply (4.14). Thus (4.14) is the stronger of the two results in general. We have included (4.15) since we would also like to have a bound on $\left|A_p^j - A\right|$ involving p itself. The situation in which $a = b$ and $D = E$ can be achieved when $N_i \sim Ci^a$ as $i \to \infty$, for then $D = E = C$. This prevails when $\nu_k \sim Fk^{\theta_1}$ as $k \to \infty$ and $\mu_i \sim Gi^{\theta_2}$ as $i \to \infty$ for some $\theta_1 \geq 0$ and $\theta_2 \geq 0$. For example, for the case treated in [BuSt] we have, for all k and i, $\nu_k = 1$, $\mu_i = 1$, and hence $N_i = 1$, which implies $D = E = 1$ and $a = b = 0$; consequently, $|A_p^j - A|$ is practically $O(\omega^{Mp^2/2})$ as $p \to \infty$, and this is precisely what is given in [BuSt].

(3) From the proof of Theorem 4.2 we see that what determines the issues of convergence and rate of convergence of A_p^j for $p \to \infty$ is the factor $|\prod_{i=1}^{t} c_i^{\nu_i}|$ that behaves at worst like $\tilde{L}^{p^{(a+2)/(b+1)}}$ under the given conditions. It is this factor that explains the remarkably quick convergence of the sequences $\left\{A_p^j\right\}_{p=0}^{\infty}$.

With the issue of convergence settled, we now go on to investigate the issue of stability.

THEOREM 4.3. *Under the conditions of Theorem* 4.1 $\gamma_{p,i}^j = \gamma_{p,i}$ *satisfy*

$$\limsup_{p\to\infty} \sum_{i=0}^{p} \left|\gamma_{p,i}^j\right| \leq \prod_{i=1}^{\infty} \left(\frac{1+|c_i|}{|1-c_i|}\right)^{\nu_i} < \infty. \tag{4.27}$$

As a result, the extrapolation process that provides the sequences $\left\{A_p^j\right\}_{p=0}^{\infty}$ with j fixed is stable in the sense that

$$\sup_p \sum_{i=0}^{p} \left|\gamma_{p,i}^j\right| < \infty. \tag{4.28}$$

Furthermore, when c_i, $i = 1, 2, \ldots$, all have the same phase, $\limsup$ *and "$\leq$" in* (4.27) *are to be replaced by* $\lim$ *and "$=$", respectively. This holds, in particular, when c_i, $i = 1, 2, \ldots$, are all real positive or all real negative. When $c_i < 0$, $i = 1, 2, \ldots$, we have $\sum_{i=0}^{p} |\gamma_{p,i}^j| = 1$ for all p.*

Proof. The inequalities in (4.27) follow from (2.15) and from the convergence of the infinite products $\prod_{i=1}^{\infty}(1 + |c_i|)^{\nu_i}$ and $\prod_{i=1}^{\infty}|1 - c_i|^{\nu_i}$ that was proved in part (i) of the proof of Theorem 4.2. The inequality in (4.28) follows directly from (4.27). The rest follows from the last part of Theorem 2.3. ∎

5. Examples. We now demonstrate the results of the previous sections with two examples. The first one comes from the numerical integration of a function having a logarithmic endpoint singularity by the trapezoidal rule. In this example ν_k, μ_i, and hence N_i (cf. (4.11)) are all bounded, i.e., $a = b = 0$ in (4.13). In the second example ν_k and hence N_i are unbounded with $a = b = 1$ in (4.13). To the best of our knowledge cases with unbounded ν_k have not received enough attention in the context of generalized Richardson extrapolation before.

The computations reported in this section were carried out in double precision arithmetic on an IBM-370 computer at the University of Connecticut in Storrs, Connecticut.

Example 5.1. Consider the integral

$$I_q = \int_0^1 (\log x)^q\, x^\alpha g(x)\, dx, \; q = 0, 1, \ldots, \quad \operatorname{Re}\alpha > -1, \; g(x) \in C^\infty[0,1], \tag{5.1}$$

and the (modified) trapezoidal rule approximation to it

$$\begin{aligned} T_q(h) &= h \sum_{i=1}^{m-1} G_q(ih) + \frac{h}{2} G_q(1); \\ G_q(x) &\equiv (\log x)^q\, x^\alpha g(x) \text{ and } h = 1/m, \; m = 1, 2, \ldots \,. \end{aligned} \tag{5.2}$$

Note that $T_q(h)$ does *not* include $G_q(0)$ which is usually undefined for $q > 0$ and/or $\operatorname{Re}\alpha < 0$. Also, note that $G_q(1) = 0$ for $q > 0$.

Theorem 5.1 below gives the Euler–Maclaurin expansion for the error $T_q(h) - I_q$ as $h \to 0$.

THEOREM 5.1. *The approximation $T_q(h)$ satisfies*

$$T_q(h) - I_q \sim \sum_{j=1}^{\infty} a_j^{(q)} h^{2j} + \sum_{j=0}^{\infty} \left[\sum_{i=0}^{q} b_{ji}^{(q)} (\log h)^i\right] h^{\alpha+j+1} \text{ as } h \to 0 \tag{5.3}$$

for some constants $a_j^{(q)}$ and $b_{ji}^{(q)}$ that are independent of h. Actually,

$$\begin{aligned} a_j^{(q)} &= \frac{B_{2j}}{(2j)!} G_q^{(2j-1)}(1), \qquad j = 1, 2, \ldots, \\ b_{ji}^{(q)} &= \binom{q}{i} \left[\frac{d^{q-i}}{d\alpha^{q-i}} \zeta(-\alpha - j)\right] \frac{g^{(j)}(0)}{j!}, \quad 0 \leq i \leq q, \quad j = 0, 1, \ldots, \end{aligned} \tag{5.4}$$

where B_i are the Bernoulli numbers and $\zeta(z)$ is the Riemann zeta function.

Proof. The result in (5.3) and (5.4) when $q = 0$ is a special case of that given in [Na1]. The result for $q = 1$ is similarly a special case of that given in [Na2], and it is obtained by differentiating both sides of (5.3) (with $q = 0$ there) once with respect to α. Applying this technique of differentiation with respect to α q times on both sides of (5.3) (with $q = 0$ there), we obtain the required result. ∎

We observe that for all values of Re $\alpha > -1$, whether α is integral or not, and for all integers $q \geq 0$, $T_q(h)$ in (5.2) is precisely of the form described in (1.1) and (1.2), with $q_k \leq q$, $k = 1, 2, \ldots$, and $0 < \text{Re } \sigma_1 < \text{Re } \sigma_2 < \cdots$, such that $\lim_{k\to\infty} \text{Re } \sigma_k = +\infty$; cf. (1.3).

Letting now $h = h_n = 2^{-n}$ in (5.2), and denoting $S_n = T_q(h_n)$, $n = 0, 1, \ldots$, and $S = I_q$, after some manipulation (5.3) becomes

$$S_n \sim S + \sum_{j=1}^{\infty} \tilde{a}_j \rho_j^n + \sum_{j=0}^{\infty} \left(\sum_{i=0}^{q} \tilde{b}_{ji} n^i \right) \tau_j^n \text{ as } n \to \infty, \tag{5.5}$$

where $\rho_j = 2^{-2j}$, $\tau_j = 2^{-\alpha-j-1}$, and $\tilde{a}_j = a_j^{(q)}$, $\tilde{b}_{ji} = (-\log 2)^i \, b_{ji}^{(q)}$. Of course, (5.5) is of the form given in (2.16).

When $-1 < \text{Re } \alpha < 0$, we have

$$\begin{aligned} c_{3k+1} &= \tau_{2k}, & q_{3k+1} &= q, \\ c_{3k+2} &= \tau_{2k+1}, & q_{3k+2} &= q, \quad k = 0, 1, \ldots, \\ c_{3k+3} &= \rho_{k+1}, & q_{3k+3} &= 0, \end{aligned} \tag{5.6}$$

and $|c_1| > |c_2| > |c_3| > \cdots$.

When $\alpha = 0$, we have

$$c_k = \tau_{k-1} = 2^{-k}, \qquad k = 1, 2, \ldots, \tag{5.7}$$

and thus $c_1 > c_2 > c_3 > \cdots$. Also,

$$q_1 = q; \quad q_{2k} = q, \quad q_{2k+1} = q - 1, \quad k = 1, 2, \ldots, \tag{5.8}$$

since $\zeta(0) \neq 0$ and $\zeta(-2m) = 0$, $\zeta(-2m+1) \neq 0$, $m = 1, 2, \ldots$.

In all cases Theorem 3.2 applies and all of the columns of the extrapolation table converge, the rates of convergence being given by (3.13) in the corollary to Theorem 3.2. Of course, this is subject to (5.6) when $-1 < \text{Re } \alpha < 0$ and subject to (5.7) and (5.8) when $\alpha = 0$.

Also, the additional conditions of Theorem 4.2 that are imposed on σ_k and q_k are automatically satisfied with $a = b = 0$ in (4.13). If also the function $g(x)$ is such that $\max_{0 \leq x \leq 1} \left| g^{(m)}(x) \right| = O((dm)!)$ as $m \to \infty$ for an arbitrary constant d , then the constants $\tilde{a}_m$ and $\tilde{b}_{mi}$, $i = 0, 1, \ldots$, in (5.5) are at worst $O((d'm)!)$ as $m \to \infty$ for some constant d'. In proving this we make use of the facts that $B_{2m}/(2m)! = O((2\pi)^{-2m})$ as $m \to \infty$ and $\zeta(-2m+1) = O((2m-1)!(2\pi)^{-2m})$ as $m \to \infty$. As a result, Theorem 4.2 applies, and all of the diagonals of the extrapolation table converge.

Finally, both the columns and the diagonals are stable as implied by Theorems 3.3 and 4.3.

Before closing this section we mention that the results of [Na1] and [Na2] have been obtained in [LNi] by using entirely different techniques. Generalizations of the expansion in Theorem 5.1 to multidimensional integrals of singular functions have been given in [L], [LM], [ML], and [Si2].

TABLE 5.1
Relative errors in A_p^j for the integral $I_1 = \int_0^1 \log x/(1+x)^2\,dx = -\log 2$ of Example 5.1.

j	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
0	1.0D+00								
1	7.8D−01	5.6D−01							
2	5.4D−01	2.9D−01	2.9D−02						
3	3.4D−01	1.4D−01	7.4D−03	2.0D−02					
4	2.0D−01	6.8D−02	5.6D−03	5.0D−03	2.0D−04				
5	1.2D−01	3.3D−02	2.2D−03	1.1D−03	2.1D−04	2.7D−04			
6	6.7D−02	1.6D−02	7.4D−04	2.5D−04	3.9D−05	1.4D−05	3.5D−06		
7	3.8D−02	8.0D−03	2.3D−04	5.8D−05	5.0D−06	1.4D−07	7.6D−07	5.8D−07	
8	2.1D−02	4.0D−03	6.8D−05	1.4D−05	5.9D−07	3.5D−08	4.6D−08	1.5D−09	2.0D−08
9	1.1D−02	2.0D−03	2.0D−05	3.4D−06	7.0D−08	4.4D−09	2.4D−09	5.0D−10	4.7D−10
10	6.2D−03	9.8D−04	5.5D−06	8.5D−07	8.4D−09	4.0D−10	1.3D−10	2.0D−11	3.9D−12

We have applied the SGRom-algorithm of Theorem 2.2 to the integral

$$\int_0^1 \frac{\log x}{(1+x)^2}dx = -\log 2,$$

precisely as described above. We thus have $c_k = 2^{-k}$, $k = 1, 2, \ldots$, and $q_1 = 1$, $q_{2k} = 1$, $q_{2k+1} = 0$, $k = 1, 2, \ldots$. As a result, $N_i = \nu_i = q_i + 1$, so that $1 \le N_i \le 2$ for all i, giving $D = 1$, $E = 2$, and $a = b = 0$ in (4.13). Also, $M = 1$ in (4.12). Finally, by the analyticity of $1/(1+x)^2$ at $x = 0$ and of $\log x/(1+x)^2$ at $x = 1$, it turns out that the growth condition on α_{km} in Theorem 4.2 is also satisfied. Therefore, Theorems 4.2 and 4.3 apply, and we have $|A_p^j - A| \le (\omega^{1/8} + \varepsilon)^{p^2}$.

Table 5.1 shows the relative errors in A_p^j. Note the absolute stability of A_p^j both in columns and diagonals.

Example 5.2. $A(y) = (1 - y\log y)/\left(1 - 2y\log y + y^2\right)$. This $A(y)$ has the asymptotic expansion

$$A(y) \sim 1 + \sum_{k=1}^{\infty} y^k T_k(\log y) \text{ as } y \to 0+,$$

where $T_k(z)$ are the Chebyshev polynomials of the first kind. Thus $\sigma_k = k$ and $q_k = k$, $k = 1, 2, \ldots$. The validity of this asymptotic expansion can be shown as follows: we first have the identity

$$\frac{1 - xz}{1 - 2xz + z^2} = \sum_{k=0}^{N-1} z^k T_k(x) + z^N \frac{T_N(x) - zT_{N-1}(x)}{1 - 2xz + z^2}$$

that is valid for all x and z as long as $1 - 2xz + z^2 \neq 0$. Now let $z = y$ and $x = \log y$. The left-hand side of the identity above becomes $A(y)$, the summation on the right-hand side becomes $\sum_{k=0}^{N-1} y^k T_k(\log y)$, and the remaining term is precisely $O\left(y^N T_N(\log y)\right)$ as $y \to 0+$.

We have applied the SGRom-algorithm of Theorem 2.2 to this example, taking $y_n = 2^{-n}$, $n = 0, 1, \ldots$. We thus have $c_k = 2^{-k}$, $q_k = k$, $k = 1, 2, \ldots$. As a result $N_i = i + 1$, $i = 1, 2, \ldots$, giving $D = E = 1$ and $a = b = 1$ in (4.13). Also, $M = 1$ in (4.12). Finally, a straightforward analysis of the coefficients of the Chebyshev polynomial $T_k(x)$ reveals that the growth condition imposed on the α_{km} in Theorem 4.2 is satisfied. Therefore, Theorems 4.2 and 4.3 apply.

Table 5.2 shows the relative errors in A_p^j. Note the absolute numerical stability of A_p^j both in columns and in diagonals.

TABLE 5.2
Relative errors in A_p^j for Example 5.2.

j	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
0	5.0D−01								
1	3.1D−01	1.1D−01							
2	2.3D−01	1.6D−01	2.0D−01						
3	1.8D−01	1.3D−01	9.3D−02	5.6D−02					
4	1.3D−01	8.3D−02	4.0D−02	2.2D−02	1.1D−02				
5	9.0D−02	4.8D−02	1.4D−02	4.8D−03	1.0D−03	5.1D−03			
6	5.8D−02	2.6D−02	3.0D−03	4.8D−04	2.3D−03	2.7D−03	2.3D−03		
7	3.5D−02	1.3D−02	6.4D−05	9.3D−04	1.1D−03	6.9D−04	4.0D−04	1.3D−04	
8	2.1D−02	6.3D−03	3.4D−04	4.8D−04	3.3D−04	8.1D−05	5.8D−06	6.4D−05	9.1D−05
9	1.2D−02	3.0D−03	2.2D−04	1.8D−04	7.9D−05	4.7D−06	1.7D−05	1.9D−05	1.2D−05
10	6.7D−03	1.5D−03	9.8D−05	5.8D−05	1.7D−05	3.9D−06	3.8D−06	1.9D−06	4.7D−07
11	3.7D−03	7.1D−04	3.7D−05	1.7D−05	3.5D−06	8.9D−07	4.6D−07	2.1D−08	3.0D−07
12	2.0D−03	3.5D−04	1.3D−05	4.9D−06	7.9D−07	1.3D−07	2.4D−08	3.7D−08	3.9D−08
13	1.1D−03	1.7D−04	4.3D−06	1.4D−06	1.9D−07	1.4D−08	3.4D−09	7.4D−09	3.2D−09
14	5.9D−04	8.6D−05	1.3D−06	3.7D−07	4.6D−08	4.7D−10	1.4D−09	1.1D−09	2.1D−10
15	3.2D−04	4.3D−05	4.1D−07	1.0D−07	1.2D−08	2.1D−10	3.1D−10	1.5D−10	1.7D−11
16	1.7D−04	2.1D−05	1.2D−07	2.8D−08	3.0D−09	7.6D−11	5.7D−11	2.1D−11	2.2D−12
17	9.0D−05	1.1D−05	3.7D−08	7.5D−09	7.6D−10	1.8D−11	9.7D−12	2.9D−12	3.3D−13
18	4.8D−05	5.3D−06	1.1D−08	2.0D−09	1.9D−10	3.6D−12	1.6D−12	4.0D−13	4.3D−14
19	2.5D−05	2.6D−06	3.1D−09	5.4D−10	4.9D−11	6.7D−13	2.5D−13	6.0D−14	1.1D−14
20	1.3D−05	1.3D−06	8.8D−10	1.4D−10	1.2D−11	1.2D−13	3.7D−14	7.1D−15	3.9D−16

6. Concluding remarks.

As mentioned in section 1, our generalized Richardson extrapolation process needs the integers q_k, the (real or complex) numbers σ_k, and y_l in (1.10) as input. This means, obviously, that we need to know that $A(y)$ is of the form given in (1.1) and (1.2), and we need to know σ_k in (1.1) and q_k in (1.2) as well.

When we know that $A(y)$ is of the form given in (1.1) and (1.2), but we have no knowledge of σ_k, the generalized Richardson extrapolation above cannot be applied. Instead, the Shanks transformation of [Sh] or the equivalent ε-algorithm of [W] can be applied to the sequence $\{A(y_n)\}_{n=0}^{\infty}$ with y_l as in (1.10). The convergence of the columns of the epsilon table on sequences $\{S_n\}_{n=0}^{\infty}$ with S_n as in (2.17) has been analyzed in great detail in the recent work [Si6]. Recall that, with y_l as in (1.10), the sequence $\{A(y_n)\}_{n=0}^{\infty}$ is exactly of the form given in (2.17). Furthermore, the c_k in (2.17) (hence the σ_k in (1.1)) and the precise degrees of the $Q_k(\xi)$ in (1.2) can be obtained from the denominators of the Padé approximants associated with the formal power series $f(z) := S_0 + \sum_{i=1}^{\infty}(S_i - S_{i-1})z^i$. In fact, the reciprocals of the poles of these Padé approximants approximate c_k under certain mild conditions. This approach has been proposed in [Si5, section 7], where the construction of good approximations to the c_k is described and a detailed convergence analysis is given.

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