where $C(x) = 1 - 3x + x^2/2 + x^3/30$. Note that $C'(x)$ is positive for $x > 3$ and $C(9/2) = 53/80$. Hence $C(x)$ is positive for $x > 9/2$ and the bounds are established. We can now write

$$F(z) > z\int_0^2 e^{-zt}dt + z\int_2^\infty 2t^2 e^{-(z+1)t}dt,$$

and thus

$$F(z) > 1 + e^{-2z-2}(w(z) - e^2),$$

where

$$w(z) = 8 - \frac{4}{(z+1)^2} - \frac{4}{(z+1)^3}.$$

Note that $w(z)$ increases from 200/27 at $z = 2$ to 8 at $z = \infty$. Since $200/27 > e^2$, we are done.

REFERENCES

[1] R. Bellman, *A Brief Introduction to Theta Functions*, Holt, Rinehart and Winston, New York, 1961.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, Vol. 2, Bateman Manuscript Project, McGraw–Hill, New York, 1954.

A Family of Matrix Problems

Problem 97-11*, *by* Dan Givoli (Technion, Haifa, Israel).

The following family of matrix problems arises in the design of some high-order local non-reflecting boundary conditions [1,2,3]:

$$\begin{pmatrix} 1^0 & 1^2 & 1^4 & \cdots & 1^{2(N-1)} \\ 2^0 & 2^2 & 2^4 & \cdots & 2^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N^0 & N^2 & N^4 & \cdots & N^{2(N-1)} \end{pmatrix} \left\{ \begin{array}{c} \alpha_1^{(N)} \\ \alpha_2^{(N)} \\ \vdots \\ \alpha_N^{(N)} \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ 2 \\ \vdots \\ N \end{array} \right\}.$$

For example, the solutions of this system for $N = 1, 2, 3$ are:

$$N = 1: \quad \alpha_1^{(1)} = 1$$

$$N = 2: \quad \alpha_1^{(2)} = 2/3, \quad \alpha_2^{(2)} = 1/3,$$

$$N = 3: \quad \alpha_1^{(3)} = 3/5, \quad \alpha_2^{(3)} = 5/12, \quad \alpha_3^{(3)} = -1/60.$$

1. Numerical solution for various values of N shows that $\alpha_m^{(N)}$ is positive for even m and for $m = 1$, and is negative for odd $m \neq 1$. Thus, the pattern of the signs of the solutions $\alpha_m^{(N)}$ is $+, +, -, +, -, +, -, +$, and so on. (This has a bearing on the stability of the non-reflecting boundary condition.) Prove that this is indeed the case for all N.
2. Estimate the condition number of the matrix as a function of N.

3. Give an asymptotic approximation for the solution of the system for large N.

REFERENCES

[1] D. Givoli and J. B. Keller, *Non-reflecting boundary conditions for elastic waves*, Wave Motion, 12 (1990), pp. 261–279.
[2] I. Patlashenko and D. Givoli, *Non-local and local artificial boundary conditions for two-dimensional flow in an infinite channel*, Internat. J. Numer. Methods Heat Fluid Flow, 6 (1996), pp. 47–62.
[3] D. Givoli, I. Patlashenko, and J. B. Keller, *High-order boundary conditions and finite elements for infinite domains*, to appear in Comput. Methods Appl. Mech. Engrg.

Solution by A. Sidi (Technion, Haifa, Israel).

Part 1. We prove part 1 for the more general problem

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{pmatrix} \begin{pmatrix} \alpha_1^{(N)} \\ \alpha_2^{(N)} \\ \vdots \\ \alpha_N^{(N)} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{pmatrix}, \tag{1}$$

where $0 < x_1 < x_2 < \cdots < x_N < X$ for some $X > 0$ and

(i) $f \in C[0, X]$ and $f(0) \geq 0$, and

(ii) $f \in C^N(0, X)$ and $(-1)^{j-1} f^{(j)}(x) > 0$ for $x \in (0, X)$, $j = 1, 2, \dots, N$.

Thus, the original problem is a special case of this one, with $f(x) = \sqrt{x}$ and $x_k = k^2$, $k = 1, \dots, N$. Obviously, $Q_N(x) = \sum_{i=1}^{N} \alpha_i^{(N)} x^{i-1}$ is the polynomial of interpolation to $f(x)$ at the points $x_1, x_2, \dots, x_N$.

Substituting $x = 0$ in the well-known error formula

(2)
$$f(x) - Q_N(x) = \frac{f^{(N)}(\xi(x))}{N!} \prod_{i=1}^{N} (x - x_i), \quad \text{for some } \xi(x) \in (\min\{x, x_1\}, \max\{x, x_N\}),$$

that is valid also for $x = 0$ even though $f(x)$ is not necessarily differentiable there, we obtain

$$\alpha_1^{(N)} = Q_N(0) = f(0) + (-1)^{N+1} \frac{f^{(N)}(\xi(0))}{N!} \prod_{i=1}^{N} x_i, \quad \text{with} \quad \xi(0) \in (0, x_N). \tag{3}$$

By the assumptions that $f(0) \geq 0$ and $(-1)^{N-1} f^{(N)}(x) > 0$ for $x \in (0, X)$, (3) gives $\alpha_1^{(N)} > 0$.

Next, let us look at the Newton form of $Q_N(x)$, namely,

$$Q_N(x) = f(x_1) + \sum_{k=2}^{N} f[x_1, \dots, x_k](x - x_1) \cdots (x - x_{k-1}). \tag{4}$$

Here $f[x_1, \dots, x_k]$ are the divided differences of $f(x)$, and we know that they satisfy

$$f[x_1, \dots, x_k] = \frac{f^{(k-1)}(\xi_k)}{(k-1)!}, \quad \text{for some} \quad \xi_k \in (x_1, x_k), \quad k = 2, \dots, N. \tag{5}$$

We also observe that $x_i > 0$ for all i implies that

$$(6)\qquad (x - x_1)\cdots(x - x_{k-1}) = \sum_{i=1}^{k}(-1)^{k-i}C_{ki}x^{i-1}, \quad k = 2, 3, \ldots,$$

for some *positive* constants C_{ki}. From

$$(7)\qquad \alpha_i^{(N)} = \sum_{k=i}^{N} f[x_1, \ldots, x_k](-1)^{k-i}C_{ki}, \quad i = 2, \ldots, N,$$

that follows from (4) and (6), from (5), and from the assumption that $(-1)^{j-1}f^{(j)}(x) > 0$ for $x \in (0, X),\ j = 1, 2, \ldots, N$, we obtain $(-1)^i\alpha_i^{(N)} > 0,\ i = 2, \ldots, N$. (Note that even though (7) holds also for $i = 1$, it can not be used to show that $\alpha_1^{(N)} > 0$. This is the reason we have treated $\alpha_1^{(N)}$ separately.)

Now apply the above to $f(x) = \sqrt{x}$ with $x_k = k^2$, $k = 1, 2, \ldots$, and with arbitrary N.

(It is interesting to note that if $(-1)^{(j)}f^{(j)}(x) > 0$ for $x \in (0, X),\ j = 0, 1, \ldots, N$, then with the help of (4)–(7) we can show that $(-1)^{i-1}\alpha_i^{(N)} > 0,\ i = 1, 2, \ldots, N$. This is the case for $f(x) = x^a$ with $a < 0$, for example.)

Part 2. Let us denote the matrix of the linear system in (1) by A_N. Then the l_1 condition number $\kappa_1(A_N)$ of AN is $\kappa_1(A_N) = \|A_N\|_1\,\|A_N^{-1}\|_1$, where

$$(8)\qquad \|A_N\|_1 = \max_{1\le j\le N}\left(\sum_{i=1}^{N} x_i^{j-1}\right) \quad\text{and}\quad \|A_N^{-1}\|_1 = \max_{1\le k\le N}\left(\prod_{\substack{i=1\\ i\ne k}}^{N}\frac{1+x_i}{|x_k - x_i|}\right).$$

(for $\|A_N^{-1}\|_1$, see Gautschi [1]).

For the problem at hand, we have $x_k = k^2,\ k = 1, 2, \ldots.$ Thus

$$(9)\qquad \|A_N\|_1 = \sum_{i=1}^{N} i^{2(N-1)} \in (N^{2N-2}, N^{2N-1})$$

and

$$(10)\qquad \|A_N^{-1}\|_1 = \frac{\Pi_N}{\binom{2N}{N}}\max_{1\le k\le N}\left[\frac{2k^2}{1+k^2}\binom{2N}{N+k}\right] \in \left(\frac{N\Pi_N}{N+1}, \frac{2N^3\Pi_N}{(N+1)(N^2+1)}\right)$$

where $\Pi_N \equiv \prod_{i=1}^{N}(1 + i^{-2})$. Consequently,

$$(11)\qquad \frac{N}{N+1}\Pi_N N^{2N-2} < \kappa_1(A_N) < \frac{2N^3}{(N+1)(N^2+1)}\Pi_N N^{2N-1}.$$

In other words, $\kappa_1(A_N)$ is at best $O(N^{2N-2})$ and at worst $O(N^{2N-1})$, as $N \to \infty$. (Here we have used the fact that $\lim_{N\to\infty}\Pi_N$ exists and is finite.)

Part 3. Using the well-known recursion relation that is used in defining the divided differences, we can show by induction that when $f(x) = \sqrt{x}$ and $x_k = k^2,\ k = 0, 1, 2, \ldots,$

$$(12)\qquad f[x_i, x_{i+1}, \ldots, x_{i+k}] = (-4)^{k-1}\frac{(\frac{1}{2})_{k-1}}{k!}\frac{(2i)!}{(2i+2k-1)!}, \quad i = 0, 1, \ldots, \quad k = 1, 2, \ldots.$$

(Note that $x_0 \equiv 0$.) We thus have

(13)
$$\alpha_N^{(N)} = f[x_1, x_2, \ldots, x_N] = \frac{2(-4)^{N-2}(\frac{1}{2})_{N-2}}{(N-1)!(2N-1)!} \sim (-1)^N \frac{4^{N-1}}{\sqrt{\pi N}(2N)!} \quad \text{as } N \to \infty.$$

Also, substituting $x = x_0 = 0$ in the divided difference form of the error

(14)
$$f(x) - Q_N(x) = f[x, x_1, x_2, \ldots, x_N] \prod_{i=1}^{N} (x - x_i),$$

we obtain

(15)
$$\alpha_1^{(N)} = Q_N(0) = f(0) - f[x_0, x_1, x_2, \ldots, x_N](-1)^N \prod_{i=1}^{N} x_i,$$

which, by (12), becomes

(16)
$$\alpha_1^{(N)} = \frac{4^{N-1}(\frac{1}{2})_{N-1}(N!)}{(2N-1)!} \sim \frac{1}{2} \quad \text{as } N \to \infty.$$

In obtaining the asymptotic behaviors of $\alpha_N^{(N)}$ and $\alpha_1^{(N)}$ for $N \to \infty$ in (13) and (16) we have made use of the Stirling formula for the gamma function.

We next discuss the asymptotic behaviors of the C_{Ni} as these are important in determining the asymptotic behaviors of the $\alpha_i^{(N)}$. We first note that

(17)
$$C_{ki} = C_{k-1,i-1} + x_{k-1}C_{k-1,i}, \quad i = 1, 2, \ldots, k,$$

with $C_{kk} = 1$ and $C_{k0} = C_{k,k+1} = 0$ for all k. (From this we obtain $C_{k1} = \prod_{i=1}^{k-1} x_i$ and $C_{k,k-1} = \sum_{i=1}^{k-1} x_i$ for all k, which are, of course, true.) We look at two different cases.

(i) Letting $C_{ki} = C_{k1}D_{ki}$, $i = 1, 2, \ldots$, with $D_{k1} = 1$, we rewrite (17) in the form of a difference equation as in

(18)
$$D_{ki} - D_{k-1,i} = \frac{1}{x_{k-1}} D_{k-1,i-1}, \quad i = 2, 3, \ldots.$$

We can now show that $D_{k2} = \sum_{i=1}^{k-1} 1/x_i = \sum_{i=1}^{k-1} i^{-2} \sim \zeta(2)$ as $k \to \infty$. With this knowledge, we can next show that

$$D_{k3} = \sum_{1 \le i < j \le k-1} \frac{1}{x_i x_j} = \frac{1}{2} \left\{ \left(\sum_{i=1}^{k-1} \frac{1}{x_i} \right)^2 - \sum_{i=1}^{k-1} \frac{1}{x_i^2} \right\} \sim \frac{1}{2} \{ [\zeta(2)]^2 - \zeta(4) \} \text{ as } k \to \infty.$$

(Here $\zeta(z)$ is the Riemann zeta function.) In general, by induction, we obtain from (18) that

(19)
$$D_{ki} \sim \hat{D}_i \quad \text{as} \quad k \to \infty, \quad i = 1, 2, \ldots, \quad i \text{ fixed},$$

for some constants $\hat{D}_i$ that are independent of k. As a result,

(20)
$$C_{ki} \sim \hat{D}_i [(k-1)!]^2 \quad \text{as} \quad k \to \infty, \quad i = 1, 2, \ldots, \quad i \text{ fixed}.$$

(ii) Replacing i in (17) by $k-s$, we obtain the difference equation

$$(21) \qquad C_{k,k-s} - C_{k-1,(k-1)-s} = x_{k-1} C_{k-1,(k-1)-(s-1)}, \quad s = 0, 1, \ldots, k-1.$$

For $s = 1$ we obtain from (21) that $C_{k,k-1} = \sum_{i=1}^{k-1} x_i = \sum_{i=1}^{k-1} i^2 = (k-1)k(2k-1)/6 \sim k^3/3$ as $k \to \infty$. Continuing with $s = 2, 3, \ldots$, and realizing that $C_{k,k-s}$ are polynomials in k whose degrees depend on s, we obtain

$$(22) \qquad C_{k,k-s} \sim \frac{k^{3s}}{3^s (s!)} \text{ as } k \to \infty, \quad s = 0, 1, 2, \ldots, \quad s \text{ fixed}.$$

We now proceed to the asymptotic behavior of the $\alpha_i^{(N)}$. Again, we look at two different cases.

(i) From (7) and from the fact that for fixed i

$$(23) \qquad (-1)^{k-i} f[x_1, \ldots, x_k] C_{ki} \sim (-1)^i \frac{\hat{D}_i}{4k^2} \text{ as } k \to \infty,$$

that follows from (13) and (19), we see that, for $i = 1, 2, \ldots$, and i fixed,

$$(24) \qquad \lim_{N \to \infty} \alpha_i^{(N)} = \hat{\alpha}_i \text{ for some constant } \hat{\alpha}_i.$$

(ii) Let us replace i in (7) by $N - s$. Then the summation there has only $s+1$ terms independently of N. The asymptotic behavior for $N \to \infty$ of this summation is determined solely by its last term, namely, by $(-1)^s f[x_1, \ldots, x_N] C_{N,N-s}$, since

$$(25) \qquad \frac{f[x_1, \ldots, x_N] C_{N,N-s}}{f[x_1, \ldots, x_{N-1}] C_{N-1,N-s}} \sim -\frac{1}{3s} N \text{ as } N \to \infty.$$

Thus, for $s = 0, 1, \ldots$, and s fixed,

$$(26) \quad \alpha_{N-s}^{(N)} \sim (-1)^s f[x_1, \ldots, x_N] C_{N,N-s} \sim (-1)^{N-s} \frac{4^{N-1}}{\sqrt{\pi N}(2N)!} \frac{N^{3s}}{3^s (s!)} \text{ as } N \to \infty.$$

That is to say, $\alpha_{N-s}^{(N)}$ tends to 0 as $N \to \infty$ like $4^N N^{3s-\frac{1}{2}}/(2N)!$

Note that the result in (24) is not contained in that given in (26) and vice versa. Note also that neither (24) nor (26) covers $\alpha_i^{(N)}$, $i = 1, \ldots, N$, uniformly in i. It is, however, possible to show that $\alpha_i^{(N)}$ are *uniformly bounded* both in N and in i. To do this we first show that $C_{ki} \le \Pi_{k-1}[(k-1)!]^2$ for all k and i, with $\Pi_m = \prod_{i=1}^m (1+i^{-2})$ as before. This is achieved by using induction in (17). Consequently, $C_{ki} < \Pi_\infty [(k-1)!]^2$ for all k and i, since $\Pi_\infty = \lim_{m \to \infty} \Pi_m$ exists. Next, we substitute this upper bound on C_{ki} in (7) to obtain

$$(27) \qquad \left| \alpha_i^{(N)} \right| \le \Pi_\infty \sum_{k=i}^{N} |f[x_1, \ldots, x_k]| \, [(k-1)!]^2, \quad i = 1, 2, \ldots, N.$$

Invoking (12) in (27) and using Stirling's formula, we can show that $|f[x_1, \ldots, x_k]| \, [(k-1)!]^2 = O(k^{-2})$ as $k \to \infty$. The result now follows.

REFERENCE

[1] W. GAUTSCHI, *Norm estimates for inverses of Vandermonde matrices*, Numer. Math., 23 1975, pp. 337–347.

Also partially solved by C. C. GROSJEAN *and* H. E. DE MEYER (University of Ghent, Ghent, Belgium).

A Laplace Transform from a Diffusion Problem

Problem 97-12*, *by* M. L. GLASSER (Clarkson University).

In an investigation [1] of the scaling properties of diffusion in a space of dimensionality d, the authors require knowledge of the Laplace transform

$$\phi(s) = \int_0^\infty e^{-st} \sin^{-1}\left(\operatorname{sech}^{d/2}(t/2)\right) dt.$$

1. Show that for $d = 1, 2$ it is possible to express $\phi(s)$ in closed form in terms of the digamma function.
2. Can this be done for $d = 3, 4$?

REFERENCE

[1] S. N. MAJUMDAR, C. SIRE, A. J. BRAY, AND S. J. CORNELL, *Nontrivial exponent for simple diffusion*, Phys. Rev. Lett., 77 (1996), pp. 2867–2870.

Solution by CARL C. GROSJEAN (University of Ghent, Ghent, Belgium).

The given integral is convergent for $\operatorname{Re}(s) > -d/4$ since the integrand approximately behaves like $2^{d/2}e^{-(s+d/4)t}$ as $t \to +\infty$. For $s \neq 0$, integration by parts can be carried out as follows:

$$\phi(s) = -\frac{1}{s}\int_0^\infty \sin^{-1}\left(\operatorname{sech}^{d/2}(t/2)\right) de^{-st}$$

$$= \frac{\pi}{2s} - \frac{d}{4s}\int_0^\infty e^{-st}\frac{\sinh(t/2)}{\cosh(t/2)[\cosh^d(t/2) - 1]^{1/2}}\, dt.$$

For $s \to 0$, this right-hand side tends to

$$\frac{d}{4}\int_0^\infty \frac{t\sinh(t/2)}{\cosh(t/2)[\cosh^d(t/2) - 1]^{1/2}}\, dt$$

representing $\phi(0)$. Note that, with the substitution of a new integration variable, $x^2 = \cosh^d(t/2) - 1$,

$$\frac{d}{4}\int_0^\infty \frac{\sinh(t/2)}{\cosh(t/2)[\cosh^d(t/2) - 1]^{1/2}}\, dt = \frac{\pi}{2},$$

as required.

The simplest case is that of $d = 2$. For $s \neq 0$, we have

$$\phi_2(s) = \frac{\pi}{2s} - \frac{1}{2s}\int_0^\infty \frac{e^{-st}}{\cosh(t/2)}\, dt \tag{1}$$

$$= \frac{\pi}{2s} - \frac{1}{s}\int_0^\infty \frac{e^{-(s+1/2)t}}{1 + e^{-t}}\, dt$$

$$= \frac{\pi}{2s} - \frac{1}{s}\left[\frac{1}{s+1/2} - \frac{1}{s+3/2} + \frac{1}{s+5/2} - \frac{1}{s+7/2} + \cdots\right].$$