IMPROVING THE ACCURACY OF QUADRATURE METHOD SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS THAT ARISE FROM NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper we are concerned with high-accuracy quadrature method solutions of nonlinear Fredholm integral equations of the form

$$y(x) = r(x) + \int_0^1 g(x,t)F(t,y(t))\,dt, \quad 0 \le x \le 1,$$

where the kernel function $g(x,t)$ is continuous, but its partial derivatives have finite jump discontinuities across $x = t$. Such integral equations arise, e.g., when one applies Green's function techniques to nonlinear two-point boundary value problems of the form $y''(x) = f(x,y(x))$, $0 \le x \le 1$, with $y(0) = y_0$ and $y(1) = y_1$, or other linear boundary conditions. A quadrature method that is especially suitable and that has been employed for such equations is one based on the trapezoidal rule that has a low accuracy. By analyzing the corresponding Euler-Maclaurin expansion, we derive suitable correction terms that we add to the trapezoidal rule, thus obtaining new numerical quadrature formulas of arbitrarily high accuracy that we also use in defining quadrature methods for the integral equations above. We prove an existence and uniqueness theorem for the quadrature method solutions, and show that their accuracy is the same as that of the underlying quadrature formula. The solution of the nonlinear systems resulting from the quadrature methods is achieved through successive approximations whose convergence is also proved. The results are demonstrated with numerical examples.

1. Introduction. Consider the Fredholm integral equation of the second kind of the form

$$(1.1) \qquad y(x) = r(x) + \int_0^1 g(x,t)F(t,y(t))\,dt, \quad 0 \le x \le 1,$$

where the function $F(t, w)$ is assumed to be nonlinear in w, in general. Let M be a nonnegative integer and assume the following:

(i) $r \in C^M(I)$, where $I \equiv [0, 1]$.

(ii) $g \in C(\Omega)$, where $\Omega \equiv I \times I$. If $M \geq 1$, then the partial derivatives $(\partial^{j+k}/\partial x^j \partial t^k) g(x, t) \equiv g_{j,k}(x, t)$ with $j \geq 0$, $k \geq 0$, and $1 \leq j + k \leq M$, are all in $PC(\Omega)$. By this we mean that they are continuous in each of the two halves $S_- = \{(x, t) : 0 \leq t \leq x \leq 1\}$ and $S_+ = \{(x, t) : 0 \leq x \leq t \leq 1\}$ of Ω, but they are discontinuous across the diagonal $S_+ \cap S_-$ of Ω, i.e., across $x = t$, where they have finite jump discontinuities. For future reference let us define $\delta_k(x) = g_{0,k}(x, x+) - g_{0,k}(x, x-)$, $k = 1, 2, \ldots, M$. By the assumptions above, $\delta_k(x)$ are continuous on I and thus bounded there.

(iii) $F(t, w) \in C(\Delta)$ and also $F_{0,1}(t, w) \equiv (\partial/\partial w) F(t, w) \in C(\Delta)$, where $\Delta = I \times J$ with $J = [R_1, R_2]$ for some R_1 and R_2 that can be finite or infinite. For $M \geq 3$ we also assume that $F_{j,k}(t, w) \equiv (\partial^{j+k}/\partial t^j \partial w^k) F(t, w)$, with $j + k \leq M - 2$, are all in $C(\Delta)$. (Starting with our discussion of improved quadrature methods in Section 3, we will assume this with $j + k \leq M$ for $M \geq 1$.)

Thus, for each value of M, the assumptions in (i)–(iii) contain those for lower values. In particular, we have $r \in C(I)$, $g \in C(\Omega)$, and $F, F_{0,1} \in C(\Delta)$, for any $M \geq 0$. These minimal smoothness conditions on r, g, and F, along with other conditions not pertaining to smoothness, are sufficient to guarantee the existence and uniqueness of (i) a continuous solution $y(x)$ of (1.1), cf. Theorem 2.1, and (ii) a quadrature method (approximate) solution of (1.1), cf. Theorem 5.1. Theorem 2.1 in the next section states, furthermore, that $y(x) \in C^M(I)$ for each $M \geq 0$ under the conditions of (i)–(iii). In particular, $y \in C^\infty(I)$ when $M = \infty$.

Integral equations of the kind described in this introduction arise, for example, when one applies Green's function techniques to two-point boundary value problems (BVP's) governed by nonlinear ordinary differential equations (ODE's). See, e.g., Courant and Hilbert [**4**], Morse and Feshbach [**9**], Keller [**7**], and Pennline [**10-12**].

To illustrate this point let us consider

$$(1.2) \qquad\qquad y''(x) = f(x, y(x)), \quad 0 \leq x \leq 1,$$

with the inhomogeneous boundary conditions (BC's)

(1.3) $$y(0) = y_0 \quad \text{and} \quad y(1) = y_1.$$

As is shown in [**7**], (1.2)–(1.3) can be converted into the Fredholm integral equation of the second kind

(1.4) $$y(x) = r(x) + \int_0^1 g(x,t)[k^2 y(t) - f(t, y(t))]dt, \quad 0 \le x \le 1,$$

where

(1.5) $$g(x,t) = \frac{1}{k \sinh k} \begin{cases} \sinh kx \sinh k(1-t), & 0 \le x \le t \\ \sinh k(1-x) \sinh kt, & t \le x \le 1 \end{cases}$$

and

(1.6) $$r(x) = \frac{y_0 \sinh k(1-x) + y_1 \sinh kx}{\sinh k}.$$

Here $k > 0$ is a free parameter chosen to guarantee the convergence to the solution $y(x)$ of the sequence of successive approximations $\{y^{(m)}(x)\}_{m=0}^{\infty}$ obtained as in

(1.7) $$y^{(m+1)}(x) = r(x) + \int_0^1 g(x,t)[k^2 y^{(m)}(t) - f(t, y^{(m)}(t))]\, dt,$$
$$m = 0, 1, \dots,$$

with $y^{(0)}(x)$ chosen suitably.

A standard procedure for solving (1.1) numerically is the quadrature method; see, e.g., Baker [**2**, p. 686]. In this method we start with a numerical quadrature formula $I_N[\phi] = \sum_{j=0}^{N} \alpha_j \phi(x_j)$ for the integral $\int_0^1 \phi(t)\, dt$. Here $0 \le x_0 < x_1 < \cdots < x_N \le 1$. Next, we replace the integral $\int_0^1 g(x,t)F(t, y(t))\, dt$ by the corresponding $I_N[g(x, \cdot)F]$. Finally, we collocate the resulting equation at the abscissas x_i, $i = 0, 1, \dots, N$, to obtain the nonlinear system of equations

(1.8) $$y_i = r(x_i) + \sum_{j=0}^{N} \alpha_j g(x_i, x_j)F(x_j, y_j), \quad i = 0, 1, \dots, N,$$

where, for each i, y_i is the approximation to $y(x_i)$.

Subsequently, this system may be solved, e.g., by successive approximations as in

$$y_i^{(0)} = y^{(0)}(x_i), \quad i = 0, 1, \ldots, N,$$

$$(1.9) \qquad y_i^{(m+1)} = r(x_i) + \sum_{j=0}^{N} \alpha_j g(x_i, x_j) F(x_j, y_j^{(m)}),$$

$$i = 0, 1, \ldots, N; \quad m = 0, 1, \ldots .$$

One can also use Newton's method for solving the system in (1.8), but this requires the computation of the Jacobian matrix and the solution of a linear system of $N+1$ equations at each iteration, which may make the solution very expensive computationally. See, e.g., [7] and [2]. We shall come back to this subject in Section 8, where we will discuss other options as well.

In general, the accuracy of the y_i in (1.8) is that provided by the numerical quadrature formula $I_N[g(x, \cdot)F]$, subject to the condition that $g(x, t)F(t, y(t))$ is sufficiently smooth for $t \in I$. For the case considered in this work, however, $g(x, t)F(t, y(t))$ is *not* continuously differentiable for $t \in I$, but only continuous there. This is so since $g_{0,1}(x, t) = (\partial/\partial t)g(x, t)$ has a (finite) jump discontinuity for $t = x$. Therefore, we cannot expect to obtain a high-accuracy numerical solution by using a high-accuracy numerical quadrature formula such as a Gaussian formula. For this reason, the trapezoidal rule that has a low accuracy of $O(N^{-2})$ has been used in previous work, [7].

When the approach above, with I_N taken as the trapezoidal rule, is applied to the integral equation (1.4)–(1.6), the resulting y_i have errors of order $O(N^{-2})$ as shown in [7], provided that $y \in C^2(I)$ and $\{y^{(m)}(x)\}_{m=0}^{\infty}$ defined by (1.7) is a contractive sequence. The same approach was used also in [10–12].

In the present work we propose to improve the accuracy from $O(N^{-2})$ to $O(N^{-2p})$ for arbitrary integers $p \geq 2$, by replacing the trapezoidal rule by "numerical quadrature formulas" that have higher accuracy in the presence of the nonsmooth kernels $g(x, t)$ that we consider here. Specifically, these formulas are obtained by adding suitable correction terms to the trapezoidal rule approximations at the endpoints $t = 0$ and $t = 1$ and also at $t = x$, the point where $g(x, t)$ fails to be smooth.

These terms are derived from a careful analysis of the Euler-Maclaurin expansion associated with the error in the trapezoidal rule. Due to the nature of the correction terms, what we obtain are not real numerical quadrature formulas in the sense described in the paragraph following (1.7).

An important point that will be seen later is that, given N, the amount of computational work per iteration is practically independent of the order of accuracy N^{-2p} of the quadrature formula used. This means we can increase the order of accuracy by keeping the cost per iteration almost the same.

An approach similar in spirit to the one here was taken by Sidi and Israeli [**14**] in the quadrature method solution of *periodic* Fredholm integral equations with weakly singular kernels that have algebraic/logarithmic singularities along the line $t = x$. In [**14**] too the Euler-Maclaurin expansion of the trapezoidal rule plays a crucial role in the development of new numerical quadrature formulas of high-order accuracy. Only there the periodic nature of the kernel and the solution enables one to propose *extrapolated* (Romberg-type) formulas to replace the trapezoidal rule. In the present case, however, we do not have any periodicity either in the kernel or in the solution, and, therefore, we cannot use extrapolated integration formulas. Instead, we use *corrected* formulas to replace the trapezoidal rule. Thus, the approach, methods, and results of the present work are quite different from those of [**14**].

The existence and uniqueness of the solution to the nonlinear system in (1.8) as well as the solution to the integral equation in (1.1) has been discussed in [**7**, Chapter 4] in the context of two-point BVP's described above. Keller's results are obtained under the condition that $F_{0,1}(t, w)$ is continuous and bounded for $t \in I$ and for *all $w \in (-\infty, +\infty)$*. This is a very severe restriction on F, however. Most problems of engineering interest do not satisfy this restriction. In many applications physical considerations lead one to conclude that the solution is restricted to some *finite* interval. This suggests that it may be feasible to state existence and uniqueness theorems in which $F_{0,1}(t, w)$ is continuous and, therefore, also bounded for $t \in I$ and $w \in J = [R_1, R_2]$ for some *finite R_1 and R_2*, the solution satisfying $y(x) \in J$ for $x \in I$ as well. This view is taken in the series of papers by Pennline, who establishes several existence and uniqueness theorems in the context of two-point BVP's.

Pennline also shows how these theorems apply to various problems that arise in certain engineering applications. R_1 and R_2 are assumed finite also in the present paper.

Both in [**7**] and in [**10–12**] the existence and uniqueness of the solution to (1.1) is proved by establishing that a sequence $\{y^{(m)}(x)\}_{m=0}^{\infty}$ of successive approximations from (1.7) contracts and thus converges to the solution $y(x)$ of (1.1) uniformly on I. Before this can be done, however, one has to show that, if the initial approximation $y^{(0)}(x)$ satisfies $y^{(0)}(x) \in J$ for $x \in I$, where J is the *finite* interval mentioned in the previous paragraph, then so do all the other $y^{(m)}(x)$. (This is *not* necessary when J is $(-\infty, \infty)$.) When analyzing the existence and uniqueness of the numerical solution defined by the quadrature methods in (1.8), one would like to adopt the same approach. That is to say, we would like to be able to show first that the successive approximations $y_i^{(m)}$ in (1.5) satisfy $y_i^{(m)} \in J$, $i = 0, 1, \ldots, N$, for all $m = 1, 2, \ldots$, and use this to establish that $\{y_i^{(m)}, \ i = 0, 1, \ldots, N\}_{m=0}^{\infty}$ contracts and thus has a limit $\{y_i, \ i = 0, 1, \ldots, N\}$ that is the unique solution to (1.8). Although $y^{(0)}(x) \in J$ may imply that $y^{(1)}(x) \in J$, $y_i^{(0)} = y^{(0)}(x_i) \in J$ may not guarantee that $y_i^{(1)} \in J$, due to the error in the numerical quadrature formula $\sum_{j=0}^{N} \alpha_j g(x_i, x_j) F(x_j, y^{(0)}(x_j))$ for $\int_0^1 g(x_i, t) F(t, y^{(0)}(t)) \, dt$. Similarly, the $y_i^{(2)}$ that are obtained from the $y_i^{(1)}$ and the subsequent $y_i^{(m)}$ may not all lie in J. In short, the analysis of the nonlinear system in (1.8) seems to become rather complicated when J is a finite interval. Simply, the conditions $r \in C(I)$, $g \in C(\Omega)$ and $F, F_{0,1} \in C(\Delta)$, for which we are able to state an existence and uniqueness theorem for the solution of (1.1) do not seem to suffice for a corresponding theorem for the approximate solution defined by (1.8).

In this paper we consider this problem in detail and prove an existence and uniqueness theorem for the numerical solution by extending the condition $F, F_{0,1} \in C(\Delta)$ slightly to read $F, F_{0,1} \in C(\Delta')$, where $\Delta' = I \times J'$, where $J' = [R_1 - \eta, R_2 + \eta] \supset J$, $\eta > 0$ being arbitrarily small. A useful feature of our proof technique is the use of the modulus of continuity in many places. This enables us to carry out the analysis without resorting to the ε-δ formalism that would have to be used otherwise. We believe that the idea of employing the modulus of continuity may be applicable in other problems of numerical analysis as well.

For an existence and uniqueness theorem under assumptions that are of a different nature, see [**2**, pp. 689–691].

The plan of this paper is as follows.

In the next section we consider the convergence of the method of successive approximations for (1.1) and state an existence and uniqueness theorem for the solution of (1.1) that relies on successive approximations. We also derive an equicontinuity result for the successive approximations $y^{(m)}(x)$ that we use later.

In Section 3 we derive our higher-order "numerical quadrature formulas" that we use in the quadrature method by correcting the trapezoidal rule appropriately. In Section 4 we derive error bounds for the trapezoidal rule and its modifications that are expressed in terms of moduli of continuity and thus are uniform in the x_i. These bounds form an essential part of the analyses given in Sections 5 and 6. With the slight extension $F, F_{0,1} \in C(\Delta')$ that we mentioned above, in Section 5 we prove the convergence of the sequences $\{y_i^{(m)}\}_{m=0}^{\infty}$ for all i, thereby establishing the existence and uniqueness of the numerical solution y_i, $i = 0, 1, \dots, N$, as well. With the same extension, in Section 6 we analyze the errors in the y_i as functions of N. We do this analysis both for the trapezoidal rule and for its modifications. We give uniform bounds on $|y_i - y(x_i)|$ for all $M \geq 0$. In particular, the bounds for $M = 0$ and $M = 1$ are of forms not encountered before. One of the conclusions that can be drawn from this analysis is that if $y \in C^M(I)$, $M \geq 1$, then by using the appropriate modified trapezoidal rule we can achieve an error of order h^M, where $h = 1/N$. Finally, in Section 7 we illustrate the new quadrature method and the accompanying theory with specific nonlinear two-point BVP's.

As far as is known to us, the quadrature methods proposed in this work and their accompanying theory on existence, uniqueness, and convergence of numerical solutions have not been published elsewhere previously.

We mentioned above that nonlinear two-point BVP's can be formulated as Fredholm integral equations of the second kind of the type treated in this work. Thus, the methods of this work can also be used for solving numerically two-point BVP's. Here it can be argued that solving the associated ODE's by finite differences may be less expensive than solving the corresponding integral equations as the differ-

ence equations that are formed have banded Jacobian matrices and hence may be solved efficiently by Newton's method. The band size increases as the accuracy of the solution is increased, however. It is also known that the finite difference approach has great difficulties in treating BVP's with solutions $y(x)$ that vary rapidly on $(0, 1)$, which may occur, for example, in the form of very thin boundary layers. The integral equation approach does not seem to have problems in producing numerical solutions in a stable manner also for such BVP's. In addition, the accuracy of the numerical solution of the integral equation approach can be increased arbitrarily, practically with no extra computational cost. See, e.g., Example 2 in Section 7. Finally, in case $y'(0)$ or $y'(1)$ or both are present in the boundary conditions, they have to be discretized with suitable accuracy when solving the ODE's, whereas in the integral equation approach boundary conditions are built right into the associated integral equations and require no discretization.

2. Existence and uniqueness for solution of (1.1). Let us pick a function $y^{(0)}(x)$ and generate the functions $y^{(m)}(x)$, $m = 1, 2, \ldots$, by the method of successive approximations as in

$$(2.1) \quad y^{(m+1)}(x) = r(x) + \int_0^1 g(x, t) F(t, y^{(m)}(t)) \, dt, \quad m = 0, 1, \ldots.$$

The following theorem gives a set of sufficient conditions for $\{y^{(m)}(x)\}_{m=0}^\infty$ to converge, establishing the existence and uniqueness of a continuous solution to (1.1) at the same time.

Theorem 2.1. *Denote*

$$(2.2) \qquad \Psi[u](x) = r(x) + \int_0^1 g(x, t) F(t, u(t)) \, dt,$$

and assume that

$$(2.3) \quad u(x) \in J \quad for \quad x \in I \quad implies \quad \Psi[u](x) \in J \quad for \quad x \in I.$$

Assume also that $r \in C(I)$, $g \in C(\Omega)$ and $F, F_{0,1} \in C(\Delta)$. Denote the operator L_∞-norm of $g(x, t)$ on Ω and the L_∞-norm of $F_{0,1}(t, w)$ on Δ by $[\![g]\!]$ and $\|F_{0,1}\|$, respectively, i.e.,

$$(2.4) \quad [\![g]\!] = \max_{x \in I} \int_0^1 |g(x, t)| \, dt \quad and \quad \|F_{0,1}\| = \max_{(t,w) \in \Delta} |F_{0,1}(t, w)|.$$

Then, provided that

(2.5)
$$\mu \equiv [\![g]\!]\|F_{0,1}\| < 1$$

and

(2.6)
$$y^{(0)} \in C(I) \quad and \quad y^{(0)}(x) \in J \quad for \quad x \in I,$$

the following hold:

(i) $y^{(m)} \in C(I)$ *and* $y^{(m)}(x) \in J$ *for* $x \in I$, $m = 1, 2, \dots$.

(ii) $\{y^{(m)}(x)\}_{m=0}^{\infty}$ *converges uniformly on I to a function $y(x)$ such that $y \in C(I)$ and $y(x) \in J$ for $x \in I$.*

(iii) $y(x)$ *is the unique solution of* (1.1).

(iv) *If, in addition, $r(x), g(x,t)$, and $F(t,w)$ are as described in* (i)–(iii) *of the first paragraph of Section* 1 *with arbitrary M, then* $y \in C^M(I)$.

The proof of parts (i)–(iii) of this theorem are almost identical to that of Theorem 4.1.2 in [**7**, pp. 108–109], provided suitable additions and modifications are made in the latter.

The result of part (iv) can be verified by splitting the integral $\int_0^1$ in (1.1) into the sum $\int_0^x + \int_x^1$ and then differentiating under the integral sign and using induction on M. (The case $M = 0$ is already covered in parts (i)–(iii).) In the course of the proof, it also becomes clear that only those $g_{j,k}(x,t)$ for which $j \geq 1$ and $j + k \leq M - 1$ and $g_{M,0}(x,t)$ are required to be in $PC(\Omega)$ for $M \geq 1$.

Our next theorem essentially states that, under the conditions of Theorem 2.1, the sequence $\{y^{(m)}(x)\}_{m=0}^{\infty}$ is equicontinuous on I. We state it in terms of the moduli of continuity of r and $y^{(m)}$ on I and of g on Ω. For the sake of completeness we give the precise definition of this concept.

Definition. Let $Y(\vec{\xi}) \equiv Y(\xi_1, \dots, \xi_n)$ be defined on a subset X of $\mathbf{R}^n$. Then its *modulus of continuity* ω_Y on X is defined as

(2.7)
$$\omega_Y(h) \equiv \sup\{|Y(\vec{\xi}) - Y(\vec{\xi'})| : \vec{\xi}, \vec{\xi'} \in X, \ |\xi_i - \xi_i'| \leq h, \ i = 1, \dots, n\}.$$

It is known that if X is a compact set and $Y(\vec{\xi})$ is continuous on X, therefore, uniformly continuous there, $\omega_Y(h) \to 0$ *monotonically* as $h \to 0$. We refer the reader to Cheney [**3**] for this and other details on moduli of continuity.

Theorem 2.2. *Define*

$$(2.8) \qquad \|F\| = \max_{(t,w)\in\Delta} |F(t,w)|,$$

and let ω_r, ω_g, and $\omega_{y^{(m)}}$ denote the moduli of continuity of $r(x)$ on I, of $g(x,t)$ on Ω, and of $y^{(m)}(x)$ on I, respectively. Then, for any $M \geq 0$, we have

$$(2.9) \qquad \omega_{y^{(m)}}(h) \leq \omega_r(h) + \|F\|\,\omega_g(h), \quad m = 1, 2, \ldots,$$

and thus $\omega_{y^{(m)}}(h) \to 0$ as $h \to 0$ uniformly in m.

Proof. From (2.1) we have, for $m = 1, 2, \ldots,$

$$(2.10) \quad y^{(m)}(x) - y^{(m)}(x')$$
$$= [r(x) - r(x')] + \int_0^1 [g(x,t) - g(x',t)]F(t, y^{(m-1)}(t))\,dt.$$

The result in (2.9) now follows by taking absolute values on both sides of (2.10) and invoking (2.7) and (2.8) along with the result that $y^{(m)}(x) \in J$ for $x \in I$. The rest follows from the fact that $r \in C(I)$, $y^{(m)} \in C(I)$, $m = 0, 1, \ldots$ and $g \in C(\Omega)$. $\square$

3. Derivation of the improved quadrature formulas. Let us denote $\phi(t) = g(x,t)F(t,y(t))$ with x being held *fixed*. Let us also assume that, in case $M \geq 1$, $F_{j,k}(t,w) \in C(\Delta)$ for $j + k \leq M$, instead of $j + k \leq M - 2$ for $M \geq 3$. Here $y(x)$ is the unique solution of (1.1) in the sense of Theorem 2.1, i.e., $y \in C^M(I)$ and $y(x) \in J$ when $x \in I$. Thus, we are assuming that the conditions (i)–(iii) of Section 1 and the conditions (2.3) and (2.5) of Theorem 2.1 are satisfied. We will retain all these assumptions throughout the remainder of this work. We conclude that $\phi(t)$ is continuous for $t \in I$, but not continuously differentiable.

We also conclude that $\phi^{(k)}(t)$, $k = 1, \ldots, M$, are continuous in each of the intervals $[0, x]$ and $[x, 1]$, but have finite jump discontinuities at $t = x$ when $x \in (0, 1)$.

Let $h = 1/N$, where N is some positive integer, and let $x_i = ih$, $i = 0, 1, \ldots, N$. Assume now that the point x mentioned in the previous paragraph is equal to x_i for some *fixed* i. Let us consider the trapezoidal rule approximations $T_-(h)$ for $\int_0^x \phi(t)\, dt$ and $T_+(h)$ for $\int_x^1 \phi(t)\, dt$ that are given by

$$(3.1) \qquad T_-(h) = h \sum_{j=0}^{i}{}'' \phi(x_j) \quad \text{and} \quad T_+(h) = h \sum_{j=i}^{N}{}'' \phi(x_j),$$

where $\sum_{j=r}^{s}{}'' \mu_j = (1/2)(\mu_r + \mu_s) + \sum_{j=r+1}^{s-1} \mu_j$ if $r < s$ and $\sum_{j=r}^{s}{}'' \mu_j = 0$ if $r = s$. Obviously,

$$(3.2) \qquad T_-(h) + T_+(h) = h \sum_{j=0}^{N}{}'' \phi(x_j) = T(h),$$

where $T(h)$ is the trapezoidal rule approximation for $\int_0^1 \phi(t)\, dt$.

3.1. *Euler-Maclaurin expansions for $T(h)$.* Let us first consider the cases $i = 1, \ldots, N - 1$. For each such case $x = x_i \in (0, 1)$, and we have the following (Euler-Maclaurin) expansions for $T_-(h)$ and $T_+(h)$:

$$
\begin{aligned}
\int_0^x \phi(t)\, dt &= T_-(h) - \sum_{s=1}^{p-1} \frac{B_{2s}}{(2s)!} [\phi^{(2s-1)}(x-) - \phi^{(2s-1)}(0)] h^{2s} \\
&\qquad + E_p^{(-)}(h; x),
\end{aligned}
$$
$$(3.3)$$
$$E_p^{(-)}(h; x) = -x \frac{B_{2p}}{(2p)!} \phi^{(2p)}(\xi_-) h^{2p}, \quad \text{for some} \quad \xi_- \in (0, x),$$

and

$$
\begin{aligned}
\int_x^1 \phi(t)\, dt &= T_+(h) - \sum_{s=1}^{p-1} \frac{B_{2s}}{(2s)!} [\phi^{(2s-1)}(1) - \phi^{(2s-1)}(x+)] h^{2s} \\
&\qquad + E_p^{(+)}(h; x),
\end{aligned}
$$
$$(3.4)$$
$$E_p^{(+)}(h; x) = -(1 - x) \frac{B_{2p}}{(2p)!} \phi^{(2p)}(\xi_+) h^{2p}, \quad \text{for some} \quad \xi_+ \in (x, 1).$$

In (3.3) and (3.4) B_r are the Bernoulli numbers. Combining (3.3) and (3.4), we have

$$\int_0^1 \phi(t)\,dt = T(h) - \sum_{s=1}^{p-1} \frac{B_{2s}}{(2s)!}[\phi^{(2s-1)}(1) - \phi^{(2s-1)}(0)]h^{2s}$$

(3.5)
$$+ \sum_{s=1}^{p-1} \frac{B_{2s}}{(2s)!}[\phi^{(2s-1)}(x+) - \phi^{(2s-1)}(x-)]h^{2s}$$
$$+ E_p(h;x),$$
$$E_p(h;x) = E_p^{(-)}(h;x) + E_p^{(+)}(h;x)$$
$$= -\frac{B_{2p}}{(2p)!}[x\phi^{(2p)}(\xi_-) + (1-x)\phi^{(2p)}(\xi_+)]h^{2p}.$$

Let us now turn to the cases $i = 0$ and $i = N$. When $x = x_0 = 0$, we have simply $T(h) = T_+(h)$. Therefore, the Euler-Maclaurin expansion of $T(h)$ now is given by (3.4) with x there replaced by 0. Similarly, when $x = x_N = 1$, $T(h) = T_-(h)$, and the Euler-Maclaurin expansion of $T(h)$ is given by (3.3) with x there replaced by 1. In other words, $T(h)$ satisfies (3.5) also when $x = x_0 = 0$ and $x = x_N = 1$, with the second summation involving $[\phi^{(2s-1)}(x+) - \phi^{(2s-1)}(x-)]$ being absent from (3.5) in both cases.

Notice that $E_p(h;x) = O(h^{2p})$ as $h \to 0$ *uniformly* in $x = x_i$, $i = 0, 1, \ldots, N$, and in N. Also in (3.3)–(3.5) we have assumed that $M \geq 2p$, $p \geq 1$. We shall make this assumption throughout the remainder of this section even though it does not cover all possible cases. (We will consider the remaining cases following Theorem 6.7 in Section 6.)

The combined Euler-Maclaurin expansion of (3.5) guides the derivation of the improved numerical quadrature formulas below. For a discussion of the Euler-Maclaurin expansion see, e.g., Davis and Rabinowitz [5].

3.2. *Corrections to the trapezoidal rule.* It is clear from the Euler-Maclaurin expansions given in (3.3)–(3.5) that $\int_0^1 \phi(t)dt - T(h) = O(h^2)$ as $h \to 0$, for *all* $x = x_i$, $i = 0, 1, \ldots, N$. While for $x = x_0 = 0$ and $x = x_N = 1$ this result is immediate, for $x = x_i \in (0,1)$ it comes somewhat as a surprise, as $\phi'(t)$ is not continuous on $[0,1]$ for such

values of x. We now aim at improving the accuracy of $T(h)$ by taking its Euler-Maclaurin expansion into account.

To motivate our approach let us take $x = x_i \in (0,1)$. Then $T(h)$ satisfies (3.5). Next, if $\phi^{(2s-1)}(0)$, $\phi^{(2s-1)}(1)$ and $[\phi^{(2s-1)}(x+) - \phi^{(2s-1)}(x-)]$, $s = 1, \ldots, p-1$, are available, then the numerical quadrature formula

$$
\begin{aligned}
T_p(h) = T(h) &- \sum_{s=1}^{p-1} \frac{B_{2s}}{(2s)!}[\phi^{(2s-1)}(1) - \phi^{(2s-1)}(0)]h^{2s} \\
&+ \sum_{s=1}^{p-1} \frac{B_{2s}}{(2s)!}[\phi^{(2s-1)}(x+) - \phi^{(2s-1)}(x-)]h^{2s}
\end{aligned}
$$

(3.6)

will satisfy $\int_0^1 \phi(t)\,dt - T_p(h) = E_p(h; x) = O(h^{2p})$ as $h \to 0$. Obviously, $T_p(h)$ with $p \geq 2$ cannot be used as part of a quadrature method for integral equations as in (1.8), since $(d^k/dt^k)F(t, y(t))$ and hence $\phi^{(k)}(t)$, $k \geq 1$, are not known. We, therefore, modify $T_p(h)$ by using suitable approximations for the $\phi^{(k)}(t)$. Below we illustrate this approach in detail for $p = 2$.

3.2.1. *Modification of $T_2(h)$.* We again start by taking $x = x_i \in (0,1)$. Letting $p = 2$ in (3.6), we thus have

$$
(3.7) \quad T_2(h) = T(h) - \frac{B_2}{2}\{[\phi'(1) - \phi'(0)] - [\phi'(x+) - \phi'(x-)]\}h^2,
$$

and, therefore, $\int_0^1 \phi(t)dt - T_2(h) = O(h^4)$ as $h \to 0$. We can maintain an error of the order of h^4 by approximating the quantity inside the curly brackets on the right-hand side of (3.7) with an error of h^2. As we want to be able to preserve the form of the equations in (1.8), we need to express the relevant approximations solely in terms of the $F(x_j, y(x_j))$, $j = 0, 1, \ldots, N$. Although this can be achieved in various ways, we suggest the following route that seems to be the simplest mathematically and also very effective computationally.

We start by breaking up $\phi'(t)$ in the form

$$
(3.8) \quad \phi'(t) = g_{0,1}(x, t)F(t, y(t)) + g(x, t)\frac{d}{dt}F(t, y(t)).
$$

We compute $g(x, t)$ and $g_{0,1}(x, t)$ exactly since $g(x, t)$ is given. Thus, only $(d/dt)F(t, y(t))$ remains to be approximated.

Approximations to $(d/dt)F(t, y(t))$ at $t = 0$ and $t = 1$ are provided by the one-sided three-point differentiation formulas

(3.9)
$$Q'(0) = \frac{1}{2h}[-3Q(0) + 4Q(h) - Q(2h)] + \frac{1}{3}Q'''(\xi)h^2,$$
$$0 < \xi < 2h,$$

and

(3.10)
$$Q'(1) = \frac{1}{2h}[3Q(1) - 4Q(1 - h) + Q(1 - 2h)] + \frac{1}{3}Q'''(\xi)h^2,$$
$$1 - 2h < \xi < 1,$$

and we use these in the approximations for $\phi'(0)$ and $\phi'(1)$. For a detailed discussion of differentiation formulas see, e.g., Hildebrand [6].

As for the term $[\phi'(x+) - \phi'(x-)]$, we have

(3.11)
$$\phi'(x+) - \phi'(x-) = [g_{0,1}(x, x+) - g_{0,1}(x, x-)]F(x, y(x))$$
$$= \delta_1(x)F(x, y(x)).$$

Note that $(d/dt)F(t, y(t))|_{t=x}$ is absent from (3.11) since $g(x, t)$ is continuous at $t = x$.

Combining all the above, we obtain the "numerical quadrature formula" $\hat{T}_2(h)$ given by

(3.12)
$$\hat{T}_2(h) = T(h) - \frac{h}{24}\{g(x_i, 1)(3F_N - 4F_{N-1} + F_{N-2})$$
$$- g(x_i, 0)(-3F_0 + 4F_1 - F_2)\}$$
$$- \frac{h^2}{12}\{g_{0,1}(x_i, 1)F_N - g_{0,1}(x_i, 0)F_0\}$$
$$+ \frac{h^2}{12}\delta_1(x_i)F_i \quad \text{for} \quad x = x_i \in (0, 1),$$

where $F_j \equiv F(x_j, y(x_j))$ for short, and we have used the fact that $B_2 = 1/6$. This completes the treatment for $x = x_i \in (0, 1)$.

Remark. One might think that the break-down of $\phi'(t)$ as in (3.8) in order to apply the differentiation formulas of (3.9) and (3.10) to

$(d/dt)F(t, y(t))$ is redundant, and that these formulas can be directly applied to $\phi'(t)$. While this is true for $x = x_i$, $i = 2, 3, \ldots, N-2$, it fails to be true for $x = x_1$ and $x = x_{N-1}$. The reason for this failure is that when $x = x_1 = h$ or $x = x_{N-1} = 1 - h$, $g_{0,1}(x, x)$ does not exist, hence $\phi(t)$ is not differentiable on $(0, 2h)$ or $(1 - 2h, 1)$, respectively. Thus, the approximations to $\phi'(0)$ and $\phi'(1)$ by (3.9) and (3.10), respectively, cannot have errors of the order of h^2. (Actually, the errors are $O(1)$ as $h \to 0$, at best.) Finally, the simplicity of the correction term in (3.11) coming from the point $t = x$ is also a consequence of (3.8).

When $x = x_0 = 0$ and $x = x_N = 1$, the integrand $\phi(t)$ is M times continuously differentiable for $t \in I$, hence $\phi'(x+) - \phi'(x-) = 0$ in (3.7). Consequently, (3.12) is now modified to read

$$\hat{T}_2(h) = T(h) - \frac{h}{24}\{g(0, 1)(3F_N - 4F_{N-1} + F_{N-2})$$

(3.13)
$$- g(0, 0)(-3F_0 + 4F_1 - F_2)\}$$

$$- \frac{h^2}{12}\{g_{0,1}(0, 1)F_N - g_{0,1}(0, 0+)F_0\} \quad \text{for} \quad x = x_0 = 0$$

and

$$\hat{T}_2(h) = T(h) - \frac{h}{24}\{g(1, 1)(3F_N - 4F_{N-1} + F_{N-2})$$

(3.14)
$$- g(1, 0)(-3F_0 + 4F_1 - F_2)\}$$

$$- \frac{h^2}{12}\{g_{0,1}(1, 1-)F_N - g_{0,1}(1, 0)F_0\} \quad \text{for} \quad x = x_N = 1.$$

The "numerical quadrature formula" that is defined through (3.12)–(3.14) thus satisfies $\int_0^1 \phi(t)\, dt - \hat{T}_2(h) = O(h^4)$ as $h \to 0$, *uniformly* in the x_i and N (if $M \geq 4$).

3.3. *Modification of $T_p(h)$, $p \geq 3$.*   Again let us begin by taking $x = x_i \in (0, 1)$, and consider $T_p(h)$ in (3.6). It is sufficient to replace the coefficients of h^{2s} in the two summations there by approximations

whose errors are of order h^{2p-2s}, $s = 1, \ldots, p-1$. Then, the resulting modified $T_p(h)$, which we call $\hat{T}_p(h)$, will maintain an error of order h^{2p}. We do this as follows. First, we break up $\phi^{(2s-1)}(t)$ in the form

$$(3.15) \qquad \phi^{(2s-1)}(t) = \sum_{\mu=0}^{2s-1} \binom{2s-1}{\mu} g_{0,2s-1-\mu}(x,t) \frac{d^\mu}{dt^\mu} F(t, y(t)).$$

Next, we approximate $(d^\mu/dt^\mu)F(t, y(t))$, $\mu = 1, \ldots, 2s-1$, at $t = 0$ and $t = 1$, by one-sided $(2p - 2s + \mu)$-point differentiation formulas, involving x_j, $0 \leq j \leq 2p - 2s + \mu - 1$, when $t = 0$, and x_j, $N - (2p - 2s + \mu) + 1 \leq j \leq N$, when $t = 1$. All of the $g_{0,2s-1-\mu}(x,t)$ at $t = 0$ and $t = 1$ are computed exactly.

As for the term $[\phi^{(2s-1)}(x+) - \phi^{(2s-1)}(x-)]$, we have from (3.15) and from the assumption that $g(x,t)$ is continuous for $t \in I$, that

$$(3.16) \quad \phi^{(2s-1)}(x+) - \phi^{(2s-1)}(x-)$$
$$= \sum_{\mu=0}^{2s-2} \binom{2s-1}{\mu} \delta_{2s-1-\mu}(x) \frac{d^\mu}{dt^\mu} F(t, y(t))|_{t=x}.$$

We approximate $(d^\mu/dt^\mu)F(t, y(t))|_{t=x}$, $\mu = 1, \ldots, 2s-2$, by $(2p - 2s + \mu)$-point differentiation formulas in which x is in the center of the point set or as close to the center as possible. All of the $\delta_{2s-1-\mu}(x)$ are computed exactly.

Note that all of the differentiation formulas above will have errors of order h^{2p-2s}, $s = 1, \ldots, p-1$, under the assumption that $M \geq 2p - 1$, as can easily be shown.

The "numerical quadrature formula" $\hat{T}_p(h)$ that is obtained by the approximation procedures above obviously satisfies $\int_0^1 \phi(t)dt - \hat{T}_p(h) = O(h^{2p})$ as $h \to 0$, *uniformly* in the x_i and N, if $M \geq 2p$.

In the next sections we shall refer to $\hat{T}_p(h)$ as a numerical quadrature formula even though it is not one in the true sense of the expression.

3.4. *The new quadrature method from $\hat{T}_2(h)$.* We close this section by giving the new quadrature method for (1.1). It is defined through

the following system of $N+1$ equations

$$y_i = r(x_i) + h \sum_{j=0}^{N}{}'' g(x_i, x_j) F_j$$

$$- \frac{h}{24} \{ g(x_i, 1)(3F_N - 4F_{N-1} + F_{N-2})$$

$$- g(x_i, 0)(-3F_0 + 4F_1 - F_2) \}$$

$$- \frac{h^2}{12} \{ g_{0,1}(x_i, 1) F_N - g_{0,1}(x_i, 0) F_0 \}$$

$$+ \frac{h^2}{12} \delta_1(x_i) F_i, \quad i = 1, 2, \dots, N-1,$$

$$y_0 = r(x_0) + h \sum_{j=0}^{N}{}'' g(x_0, x_j) F_j$$

(3.17)
$$- \frac{h}{24} \{ g(0, 1)(3F_N - 4F_{N-1} + F_{N-2})$$

$$- g(0, 0)(-3F_0 + 4F_1 - F_2) \}$$

$$- \frac{h^2}{12} \{ g_{0,1}(0, 1) F_N - g_{0,1}(0, 0+) F_0 \}$$

$$y_N = r(x_N) + h \sum_{j=0}^{N}{}'' g(x_N, x_j) F_j$$

$$- \frac{h}{24} \{ g(1, 1)(3F_N - 4F_{N-1} + F_{N-2})$$

$$- g(1, 0)(-3F_0 + 4F_1 - F_2) \}$$

$$- \frac{h^2}{12} \{ g_{0,1}(1, 1-) F_N - g_{0,1}(1, 0) F_0 \}.$$

Here $F_i \equiv F(x_i, y_i)$ and y_i is the approximation to $y(x_i)$.

If we now write the system in (3.17) as

(3.18) $\qquad y_i = \Phi_i(y_0, y_1, ..., y_N), \quad i = 0, 1, \dots, N,$

then the method of successive approximations takes the form

$$y_i^{(0)} = y^{(0)}(x_i), \quad i = 0, 1, \dots, N,$$

(3.19)
$$y_i^{(m+1)} = \Phi_i(y_0^{(m)}, y_1^{(m)}, \dots, y_N^{(m)}),$$

$$i = 0, 1, \dots, N; \quad m = 0, 1, 2, \dots.$$

4. Preliminary results on $T(h)$ and $\hat{T}_p(h)$.

4.1. *Bound on trapezoidal rule error.* The next theorem bounds the error in the trapezoidal rule in terms of the modulus of continuity of the integrand.

Theorem 4.1. *Assume that $Q(t)$ is integrable on I, and denote by $T^Q(h)$ the trapezoidal rule approximation to $\int_0^1 Q(t)\,dt$. Then*

$$(4.1) \qquad |E^Q(h)| \equiv \left| \int_0^1 Q(t)\,dt - T^Q(h) \right| \leq \omega_Q(h),$$

where ω_Q is the modulus of continuity of $Q(t)$ on I.

Proof. We have

$$(4.2) \qquad \int_0^1 Q(t)\,dt = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} Q(t)\,dt$$

and

$$(4.3) \qquad T^Q(h) = \sum_{i=0}^{N-1} \frac{1}{2} \int_{x_i}^{x_{i+1}} [Q(x_i) + Q(x_{i+1})]\,dt.$$

Subtracting (4.3) from (4.4), we obtain

$$(4.4) \quad E^Q(h) = \sum_{i=0}^{N-1} \frac{1}{2}\Bigg\{ \int_{x_i}^{x_{i+1}} [Q(t) - Q(x_i)]\,dt \\ + \int_{x_i}^{x_{i+1}} [Q(t) - Q(x_{i+1})]\,dt \Bigg\}.$$

Taking absolute values on both sides of (4.4), we next obtain

$$(4.5) \quad |E^Q(h)| \leq \sum_{i=0}^{N-1} \frac{1}{2}\Bigg\{ \int_{x_i}^{x_{i+1}} |Q(t) - Q(x_i)|\,dt \\ + \int_{x_i}^{x_{i+1}} |Q(t) - Q(x_{i+1})|\,dt \Bigg\}.$$

The result in (4.1) now follows from (2.7) and from the fact that $x_{i+1} - x_i = h = 1/N$. $\quad\square$

Our next result is an application of Theorem 4.1 with $Q(t) = g(x, t)F(t, u(t))$.

Theorem 4.2. *Assume that $g \in C(\Omega)$ and that $F, F_{0,1} \in C(\Delta)$, and define $G(x, t) = g(x, t)F(t, u(t))$. Assume also that $u(t)$ is such that $u(t) \in J$ for $t \in I$, and $G(x, t)$, as a function of t, is integrable on I when $x \in I$. Denote by $T^G(h; x)$ the trapezoidal rule approximation for $\int_0^1 G(x, t)\, dt$. Then*

(4.6)
$$|E^G(h; x)| \equiv \left| \int_0^1 G(x, t)\, dt - T^G(h; x) \right|$$
$$\leq \|F\|\, \omega_g(h) + \|g\|[\omega_F(h) + \|F_{0,1}\|\, \omega_u(h)],$$

uniformly in $x \in I$. Here $\|g\|$ is the L_∞-norm of $g(x, t)$ on Ω defined by $\|g\| = \max_{(x,t)\in\Omega} |g(x, t)|$.

Proof. From Theorem 4.1

(4.7)
$$|E^G(h; x)| \leq \sup\{|G(x, t) - G(x, t')| : t, t' \in I \quad \text{and} \quad |t - t'| \leq h\}.$$

Now

(4.8)
$$G(x, t) - G(x, t') = F(t, u(t))[g(x, t) - g(x, t')]$$
$$+ g(x, t')[F(t, u(t)) - F(t', u(t'))]$$

and

(4.9)
$$F(t, u(t)) - F(t', u(t')) = [F(t, u(t)) - F(t, u(t'))]$$
$$+ [F(t, u(t')) - F(t', u(t'))]$$

and, finally, by the mean value theorem,

(4.10)
$$F(t, u(t)) - F(t, u(t')) = F_{0,1}(t, \tilde{w})[u(t) - u(t')]$$
$$\text{for some} \quad \tilde{w} \in J.$$

The result now follows by taking absolute values in (4.8)–(4.10) and maximizing over I, Ω, and Δ. We leave the details to the reader. □

4.2. *Bound on error in $\hat{T}_2(h)$.*   We now proceed to the corrected rules $\hat{T}_p^G(h; x_i)$ with $G(x, t) = g(x, t)F(t, u(t))$ that are obtained from the trapezoidal rule $T^G(h; x_i)$ for $\int_0^1 G(x_i, t)\, dt$ of Theorem 4.2 exactly as described in the previous section. It is sufficient to examine the details of the case $p = 2$ as all other cases are treated in exactly the same way and the conclusions are the same for all values of p.

From (3.12)–(3.14) it is clear that the correction to $T^G(h; x_i)$, $i = 0, 1, \dots, N$, is of the form

$$\hat{T}_2^G(h; x_i) - T^G(h; x_i) = h \sum_{j=0}^{2} [\alpha_{ij} F(x_j, u(x_j)) + \beta_{ij} F(x_{N-j}, u(x_{N-j}))]$$

$$\text{(4.11)} \qquad\qquad + h^2 [\gamma_{i0} F(x_0, u(x_0)) + \gamma_{iN} F(x_N, u(x_N))$$

$$+ \gamma_{ii} F(x_i, u(x_i))].$$

There are two important points to be noted here: (i) The number of function values $F(x_j, u(x_j))$ in this correction is *fixed* and is thus *independent of* N. (ii) The coefficients α_{ij}, β_{ij}, and γ_{ij} are some constant multiples of $g(x_i, 0)$, $g(x_i, 1)$, $g_{0,1}(x_i, 0)$, $g_{0,1}(x_i, 1)$, and $\delta_1(x_i)$, and thus are *uniformly bounded* in i, j, and N if $g \in C(\Omega)$ and $g_{0,1} \in PC(\Omega)$. Thus, if $F \in C(\Delta)$, we have for all i

$$\text{(4.12)} \qquad |\hat{T}_2^G(h; x_i) - T^G(h; x_i)| \leq (C_1^{(2)} h + C_2^{(2)} h^2)\|F\|$$

for some positive constants $C_1^{(2)}$ and $C_2^{(2)}$ that depend on g but are independent of F, i and h.

Combining all this with Theorem 4.2, we have the following important convergence result on $\hat{T}_2^G(h; x_i)$.

Theorem 4.3.   *Assume that $g \in C(\Omega)$ and that $g_{0,1} \in PC(\Omega)$. Assume also that $F, F_{0,1} \in C(\Delta)$. Let $u \in C(I)$ and $u(t) \in J$ for $t \in I$. Then, for $i = 0, 1, \dots, N$, we have*

$$\text{(4.13)} \quad |\hat{E}_2^G(h; x_i)| = \left| \int_0^1 G(x_i, t)\, dt - \hat{T}_2^G(h; x_i) \right| \leq D_2(h) + \sigma \omega_u(h),$$

where $\sigma > 0$ is a constant and $D_2(h)$ is a function that goes to 0 monotonically as $h \to 0$, and both are independent of i, N, and $u(t)$.

The result of Theorem 4.3 is important since it states that $\hat{T}_2^G(h; x_i) \to \int_0^1 G(x_i, t) \, dt$ as $h \to 0$ *uniformly* in $i = 0, 1, \ldots, N$. (For σ and $D_2(h)$, see Theorem 4.4 below.)

4.3. Bound on error in $\hat{T}_p(h)$ for arbitrary p. So far we have analyzed the properties of $\hat{T}_2^G(h; x_i)$. We would now like to summarize the properties of $\hat{T}_p^G(h; x_i)$ for arbitrary $p \geq 1$. Note that $\hat{T}_1^G(h; x_i)$ is simply the trapezoidal rule $T^G(h; x_i)$ throughout.

Theorem 4.4. *Provided $g \in C(\Omega)$ and $g_{0,i} \in PC(\Omega)$, $1 \leq i \leq 2p-3$, and $u \in C(I)$, we can construct the corrected trapezoidal rule $\hat{T}_p^G(h; x_i)$, which is of the form*

$$(4.14) \qquad \hat{T}_p^G(h; x_i) = T^G(h; x_i) + \sum_{j=0}^{N} \left(\sum_{k=1}^{2p-2} A_{ijk}^{(p)} h^k \right) F(x_j, u(x_j)).$$

Here the $A_{ijk}^{(p)}$ depend on g but not on F, and can be bounded independently of i, j, and N, and the number of the nonzero $A_{ijk}^{(p)}$ is fixed and thus independent of N. As a result, under the conditions that $u(t) \in J$ for $t \in I$, and $F, F_{0,1} \in C(\Delta)$, (4.13) can be generalized to

$$(4.15) \quad |\hat{E}_p^G(h; x_i)| = \left| \int_0^1 G(x_i, t) \, dt - \hat{T}_p^G(h; x_i) \right| \leq D_p(h) + \sigma \omega_u(h),$$

where $\sigma > 0$ is a constant and $D_p(h)$ is a function that goes to 0 monotonically as $h \to 0$, and both are independent of i, N, and $u(t)$. While σ is the same for all p, $D_p(h)$ depends on p. Specifically,

(4.16)

$$\sigma = \|g\| \, \|F_{0,1}\| \quad and \quad D_p(h) = \|F\| \left[\omega_g(h) + \sum_{k=1}^{2p-2} C_k^{(p)} h^k \right] + \|g\| \, \omega_F(h),$$

where, for each k, $C_k^{(p)}$ is an upper bound on $\sum_{j=0}^{N} |A_{ijk}^{(p)}|$ that depends on g but is independent of F, i, and N. (If $p = 1$, then the $A_{ijk}^{(p)}$ and hence $C_k^{(p)}$ are all zero.)

5. Existence and uniqueness of numerical solution. In order to solve the problem of existence and uniqueness of the numerical solution we need to enlarge the set $J = [R_1, R_2]$ by an arbitrarily small amount $\eta > 0$ that will be fixed later. Hence, we define $J' = [R'_1, R'_2] = [R_1 - \eta, R_2 + \eta]$ and $\Delta' = I \times J'$. We also define

$$\|F_{0,1}\|' = \max_{(t,w)\in\Delta'} |F_{0,1}(t, w)|.$$

We start with the following lemma.

Lemma 5.1. *Assume that $g(x, t)$ is as in Theorem 4.4 so that we can define $\hat{T}^G_p(h; x_i)$ as in (4.14). Define also*

(5.1)
$$Z_{p,i}(h; \{w_k\}) = h \sum_{j=0}^{N}{}'' g(x_i, x_j) F(x_j, w_j)$$
$$+ \sum_{j=0}^{N} \left(\sum_{k=1}^{2p-2} A^{(p)}_{ijk} h^k \right) F(x_j, w_j).$$

Then, provided that $F, F_{0,1} \in C(\Delta')$ and $u_j, v_j \in J'$, $j = 0, 1, \ldots, N$, we have

(5.2) $\quad |Z_{p,i}(h; \{u_k\}) - Z_{p,i}(h; \{v_k\})| \leq \mu(h)\|u - v\|, \quad i = 0, 1, \ldots, N,$

where

(5.3)
$$\mu(h) = \left([\![g]\!] + \omega_{|g|}(h) + \sum_{k=1}^{2p-2} C^{(p)}_k h^k \right) \|F_{0,1}\|',$$

where $[\![g]\!]$ is as defined by (2.4) and $C^{(p)}_k$ are as described in Theorem 4.4, and

(5.4)
$$\|u - v\| = \max_{0 \leq j \leq N} |u_j - v_j|.$$

Remark. Note that $Z_{p,i}(h; \{u(x_k)\}) = \hat{T}^G_p(h; x_i)$ when $G(x, t) = g(x, t)F(t, u(t))$ for an arbitrary *function* $u(t)$.

Proof. The difference $W_{p,i} = Z_{p,i}(h; \{u_k\}) - Z_{p,i}(h; \{v_k\})$ can be expressed as

$$W_{p,i} = h \sum_{j=0}^{N} {}'' g(x_i, x_j)[F(x_j, u_j) - F(x_j, v_j)]$$
$$+ \sum_{j=0}^{N} \left(\sum_{k=1}^{2p-2} A_{ijk}^{(p)} h^k \right) [F(x_j, u_j) - F(x_j, v_j)],$$

which, upon applying the mean value theorem, becomes

$$W_{p,i} = h \sum_{j=0}^{N} {}'' g(x_i, x_j) F_{0,1}(x_j, w_j)(u_j - v_j)$$
$$+ \sum_{j=0}^{N} \left(\sum_{k=1}^{2p-2} A_{ijk}^{(p)} h^k \right) F_{0,1}(x_j, w_j)(u_j - v_j),$$

where w_j is between u_j and v_j for each j, and hence $w_j \in J'$, $j = 0, 1, \ldots, N$. Taking absolute values, and maximizing appropriately, we obtain from this

$$|W_{p,i}| \le \left[h \sum_{j=0}^{N} {}'' |g(x_i, x_j)| + \sum_{k=1}^{2p-2} C_k^{(p)} h^k \right] \|F_{0,1}\|' \, \|u - v\|.$$

Now $h \sum_{j=0}^{N} {}'' |g(x_i, x_j)|$ is the trapezoidal rule approximation for $\int_0^1 |g(x_i, t)| \, dt$, and Theorem 4.1 applies to it. Invoking also the definition of $[\![g]\!]$ we thus have

$$h \sum_{j=0}^{N} {}'' |g(x_i, x_j)| \le [\![g]\!] + \omega_{|g|}(h).$$

The result now follows. $\quad \square$

We now consider the nonlinear system

$$(5.5) \qquad y_i = r(x_i) + Z_{p,i}(h; \{y_k\}), \quad i = 0, 1, \ldots, N,$$

that results from applying the corrected trapezoidal rule $\hat{T}_p^G(h; x_i)$ to (1.1). In the next theorem we show that there exists a unique solution for the y_i, $i = 0, 1, \ldots, N$, for all large N, under suitable conditions. When $p = 1$, $\hat{T}_1^G(h; x_i) = T^G(h; x_i)$ and these conditions are almost the same as those given in Theorem 2.1. For $p \geq 2$ we should impose sufficient differentiability conditions on $g(x, t)$ so that $\hat{T}_p^G$ can be defined, as mentioned earlier.

Theorem 5.1. *Assume that all the conditions of Theorem* 2.1 *concerning* $r(x)$, $g(x, t)$, *and* $F(t, w)$ *hold. Assume, in addition, that* $g(x, t)$ *satisfies the differentiability conditions of Theorem* 4.4 *so that* $\hat{T}_p^G$ *is defined. Assume, by extension, that* $F, F_{0,1} \in C(\Delta_0)$, *where* $\Delta_0 = I \times [R_1 - \eta_0, R_2 + \eta_0]$ *for some* $\eta_0 > 0$ *and thus* $\Delta_0 \supset \Delta$. *Let the sequence of successive approximations* $\{y_i^{(m)}, \ i = 0, 1, \ldots, N\}_{m=0}^{\infty}$ *be generated from* (5.5) *according to*

(5.6)
$$y_i^{(0)} = y^{(0)}(x_i), \quad i = 0, 1, \ldots, N,$$
$$y_i^{(m+1)} = r(x_i) + Z_{p,i}(h; \{y_k^{(m)}\}),$$
$$i = 0, 1, \ldots, N; \quad m = 0, 1, \ldots.$$

(Here we recall that the function $y^{(0)}(x)$ *is the initial approximation in* (2.6).) *Then there exists a constant* $\eta \in (0, \eta_0)$ *and a positive integer* N_0, *such that the following hold*:

(i) $y_i^{(m)} \in J' = [R_1 - \eta, R_2 + \eta]$, $i = 0, 1, \ldots, N$, $m = 0, 1, \ldots$, *for each* $N \geq N_0$.

(ii) $\lim_{m \to \infty} y_i^{(m)} = y_i \in J'$, $i = 0, 1, \ldots, N$, *exist for each* $N \geq N_0$.

(iii) y_i, $i = 0, 1, \ldots, N$, *is the unique solution to* (5.5), *for each* $N \geq N_0$.

Remark. As will follow from the proof below, such an η can be picked and once this is done any smaller η will render the theorem valid. Thus, η can be picked arbitrarily small.

Proof. We start by picking $\eta \in (0, \eta_0)$ such that $[\![g]\!] \|F_{0,1}\|' < 1$, which is possible from the assumption that $F_{0,1} \in C(\Delta_0)$ and from (2.5) in Theorem 2.1. Thus the choice of η is independent of N.

With η thus fixed, we next pick a positive integer $\overline{N}$, thus $\bar{h} = 1/\overline{N}$, for which $\mu(\bar{h}) < 1$, with $\mu(h)$ as defined in (5.3). This is possible as $\lim_{h \to 0} \mu(h) = [\![g]\!] \|F_{0,1}\|'$ by the assumption that $g \in C(\Omega)$ that implies that $|g| \in C(\Omega)$ so that $\lim_{h \to 0} \omega_{|g|}(h) = 0$. Moreover, we have $\mu(h) < \mu(\bar{h}) < 1$ for all $h < \bar{h}$ or all $N > \overline{N}$.

From Theorem 2.2 we have

$$(5.7) \qquad \omega_{y^{(m)}}(h) \leq \max\{\omega_{y^{(0)}}(h), \omega_r(h) + \|F\| \omega_g(h)\} \equiv \rho(h),$$
$$m = 0, 1, \ldots,$$

and $\rho(h) \to 0$ monotonically as $h \to 0$. Define now

$$(5.8) \qquad \varepsilon(h) = D_p(h) + \sigma\rho(h),$$

with $D_p(h)$ and σ as in Theorem 4.4. Thus $\varepsilon(h) \to 0$ monotonically as $h \to 0$ as well. Therefore, there exists a positive integer $N_0 \geq \overline{N}$, $1/N_0 = h_0 \leq \bar{h}$, for which

$$(5.9) \qquad \varepsilon(h_0) < \eta[1 - \mu(\bar{h})] \quad \Longleftrightarrow \quad \frac{\varepsilon(h_0)}{1 - \mu(\bar{h})} < \eta.$$

Obviously, with this h_0 we have

$$(5.10) \qquad \varepsilon(h) \sum_{q=0}^{m} [\mu(h)]^q \leq \frac{\varepsilon(h)}{1 - \mu(h)} \leq \frac{\varepsilon(h_0)}{1 - \mu(h_0)} \leq \frac{\varepsilon(h_0)}{1 - \mu(\bar{h})} < \eta,$$
$$\text{for } h \leq h_0 \leq \bar{h}, \quad m = 0, 1, 2, \ldots.$$

At this point it is worth recalling that $Z_{p,i}(h; \{w(x_k)\})$ is the corrected trapezoidal rule $\hat{T}_p^G(h; x_i)$ for the integral $\int_0^1 g(x_i, t) F(t, w(t)) \, dt$.

Let us set $m = 0$ in (5.6) and (2.1). Upon subtraction we obtain

$$(5.11) \quad y_i^{(1)} - y^{(1)}(x_i) = Z_{p,i}(h; \{y^{(0)}(x_k)\}) - \int_0^1 g(x_i, t) F(t, y^{(0)}(t)) \, dt,$$

and, by (4.15), (5.7), (5.8) and (5.10), this gives

$$(5.12) \qquad \|e^{(1)}\| \leq \varepsilon(h) < \eta,$$

where we have defined
(5.13)
$$e_i^{(m)} = y_i^{(m)} - y^{(m)}(x_i), \quad i = 0, 1, \dots, N, \quad \text{and} \quad \|e^{(m)}\| = \max_i |e_i^{(m)}|.$$

By (5.12) and the assumption that $y^{(1)}(x) \in J$ for $x \in I$, it is clear that $y_i^{(1)} \in J'$, $i = 0, 1, \dots, N$.

Let us next set $m = 1$ in (5.6) and (2.1). Upon subtraction we obtain

$$y_i^{(2)} - y^{(2)}(x_i) = \left[Z_{p,i}(h; \{y_k^{(1)}\}) - Z_{p,i}(h; \{y^{(1)}(x_k)\}) \right]$$
$$(5.14) \qquad + \left[Z_{p,i}(h; \{y^{(1)}(x_k)\}) - \int_0^1 g(x_i, t) F(t, y^{(1)}(t)) \, dt \right].$$

Applying Lemma 5.1 to the first brackets, and (4.15) to the second ones, and also invoking (5.7), (5.8) and (5.12), and, following that, (5.10), we obtain

$$(5.15) \qquad \|e^{(2)}\| \le \varepsilon(h) + \mu(h)\|e^{(1)}\| \le \varepsilon(h)[1 + \mu(h)] < \eta.$$

Therefore, $y_i^{(2)} \in J'$, $i = 0, 1, \dots, N$, too. Proceeding by induction, we can show in general that

$$(5.16) \qquad \|e^{(m)}\| \le \varepsilon(h) + \mu(h)\|e^{(m-1)}\| \le \varepsilon(h) \sum_{q=0}^{m-1} [\mu(h)]^q < \eta,$$
$$m = 1, 2, \dots,$$

as a result of which $y_i^{(m)} \in J'$, $i = 0, 1, \dots, N$, for all m. This proves part (i) of the theorem.

For part (ii) we proceed in the standard way. From (5.6) we have

$$(5.17) \qquad y_i^{(m+1)} - y_i^{(m)} = Z_{p,i}(h; \{y_k^{(m)}\}) - Z_{p,i}(h; \{y_k^{(m-1)}\}),$$
$$m = 1, 2, \dots.$$

Since $y_i^{(m)} \in J'$ for all i and m, Lemma 5.1 applies and we have

$$(5.18) \quad \|y^{(m+1)} - y^{(m)}\| \le \mu(h)\|y^{(m)} - y^{(m-1)}\|, \quad m = 1, 2, \dots.$$

The result now follows by the fact that $\mu(h) < 1$ since $h \leq h_0$, which implies that $\{y_i^{(m)}, i = 0, 1, \ldots, N\}_{m=0}^{\infty}$ is a contractive sequence and thus has a limit.

The proof of part (iii) also follows from the fact that $\mu(h) < 1$ when $h \leq h_0$. □

It is worth pointing out the similarity between (5.18) for the numerical solution and

$$(5.19) \quad \max_{x \in I} |y^{(m+1)}(x) - y^{(m)}(x)| \leq \mu \max_{x \in I} |y^{(m)}(x) - y^{(m-1)}(x)|,$$
$$m = 1, 2, \ldots,$$

for the analytical solution, with μ as in (2.5).

Since we can pick η arbitrarily close to 0 and thus $\|F_{0,1}\|'$ arbitrarily close to $\|F_{0,1}\|$, we see that, for very large N, hence very small h, $\mu(h)$ in (5.3) is arbitrarily close to μ in (2.5). That is to say, the discrete successive approximation procedure converges practically at the same rate as the continuous one does.

6. Accuracy of numerical solution. With the existence and uniqueness questions resolved, we now turn to that of the accuracy of the numerical solution y_i, $i = 0, 1, \ldots, N$ of (5.5). Our proof proceeds along the same lines as that of Theorem 4.2.1 in [**7**, pp. 114–115].

Theorem 6.1. *Under the conditions of Theorem* 5.1*, the numerical solution with* $N \geq N_0$ *satisfies*

$$(6.1) \qquad \|e\| = \max_{0 \leq i \leq N} |y_i - y(x_i)| \leq \frac{1}{1 - \mu(h)} \max_{0 \leq i \leq N} |\hat{E}_p^G(h; x_i)|,$$

where $G(x, t) = g(x, t)F(t, y(t))$.

Proof. Subtracting (1.1) with $x = x_i$ from (5.5), we can write

$$(6.2) \quad \begin{aligned} y_i - y(x_i) &= [Z_{p,i}(h; \{y_k\}) - Z_{p,i}(h; \{y(x_k)\})] \\ &\quad + \left[Z_{p,i}(h; \{y(x_k)\}) - \int_0^1 g(x_i, t)F(t, y(t)) \, dt \right]. \end{aligned}$$

Since $y(x) \in J$ for $x \in I$ and $y_i \in J'$, $i = 0, 1, \ldots, N$, Lemma 5.1 applies to the expression in the first brackets. The expression in the second brackets is nothing but $\hat{E}_p^G(h; x_i)$, the error in $\hat{T}_p^G(h; x_i)$. Thus, taking absolute values, we have

$$(6.3) \qquad |e_i| = |y_i - y(x_i)| \leq \mu(h)\|e\| + |\hat{E}_p^G(h; x_i)|.$$

The result in (6.1) follows by maximizing both sides of (6.3) and by using the fact that $\mu(h) < 1$. We leave the details to the reader.   □

Since for all $N \geq N_0$ we have $\mu(h) \leq \mu(h_0) < 1$ and thus $1/(1 - \mu(h)) \leq 1/(1 - \mu(h_0))$, we realize from Theorem 6.1 that the accuracy of the numerical approximations y_i is determined strictly by that of the numerical quadrature formula underlying the quadrature method.

In subsection 3.1 on Euler-Maclaurin expansions we proved that $E_p(h; x_i) = O(h^{2p})$ as $h \to 0$, *uniformly* in i and N, under the assumption that $M \geq 2p$, $p \geq 1$. This also produces the result that $\hat{E}_p^G(h; x_i) = O(h^{2p})$ as $h \to 0$, uniformly in i and N, whenever $M \geq 2p$, $p \geq 1$, as we have already shown. This result covers all cases except some in which M is an odd integer. In case $M = 2p - 1$ ($p \geq 1$), we have $E_p(h; x_i) = O(h^{2p-1})$ as $h \to 0$, again *uniformly* in i and N. (This time $E_p(h; x_i)$ involves $\phi^{(2p-1)}(t) \equiv (\partial^{2p-1}/\partial t^{2p-1})G(x_i, t)$ but is given by an integral representation involving periodic Bernoullian functions that we will skip.) This produces the result that $\hat{E}_p^G(h; x_i) = O(h^{2p-1})$ as $h \to 0$, uniformly in i and N, whenever $M = 2p - 1$, $p \geq 1$. This then covers *all* possible cases that can occur. Both bounds on $\hat{E}_p(h; x_i)$ will be of use below.

Also note that for arbitrary M the rules $\hat{T}_q^G$, $q = 1, \ldots, \lfloor (M + 3)/2 \rfloor$, are all well defined. Using the facts mentioned in the previous paragraph, it can be shown that $\hat{T}_{\lfloor (M+1)/2 \rfloor}^G$ and $\hat{T}_{\lfloor (M+3)/2 \rfloor}^G$ have errors of the same order, namely, $O(h^M)$ as $h \to 0$. Thus, there is no advantage to the rule $\hat{T}_{\lfloor (M+3)/2 \rfloor}^G$ when we know that $y \in C^M(I)$, and we shall not consider it in what follows.

In light of the contents of the previous two paragraphs we now discuss the various possible cases to which Theorem 6.1 applies.

1. *The case $M = 0$.* Here $r \in C(I)$, $g \in C(\Omega)$, and $F, F_{0,1} \in C(\Delta')$, and no other differentiability properties for r, g and F are given. From

Theorem 2.1 $y \in C(I)$ only. Thus the quadrature rule that can be used for this case is only $\hat{T}_1(h) = T(h)$, namely, the trapezoidal rule itself. Applying Theorem 4.2, we obtain

$$(6.4) \quad \|e\| \leq B_1^{(0)} \, \omega_g(h) + B_2^{(0)} \, \omega_F(h) + B_3^{(0)} \, \omega_y(h) = o(1) \quad \text{as} \quad h \to 0.$$

2. *The case $M = 1$.* Here $r \in C^1(I)$, $g \in C(\Omega)$ and $g_{1,0}, g_{0,1} \in PC(\Omega)$, and $F, F_{1,0}, F_{0,1} \in C(\Delta')$. From Theorem 2.1 $y \in C^1(I)$ only. The quadrature rule that can be used for this case is $\hat{T}_1(h) = T(h)$, the trapezoidal rule. (As we mentioned above, we disregard $\hat{T}_2(h)$ even though it is well defined.) Now the error in the trapezoidal rule is of order h uniformly in i. Hence we have for this case

$$(6.5) \qquad\qquad \|e\| \leq B_1^{(1)} h \quad \text{for} \quad \hat{T}_1(h).$$

3. *The case $M = 2$.* Here $r \in C^2(I)$, $g \in C(\Omega)$ and $g_{j,k} \in PC(\Omega)$, $j+k \leq 2$ and F and $F_{j,k}$, $j+k \leq 2$, are all in $C(\Delta')$. From Theorem 2.1 $y \in C^2(I)$ only. In this case too we can use $\hat{T}_1(h) = T(h)$. (Again, we disregard $\hat{T}_2(h)$ even though it is well defined.) The error in the trapezoidal rule now is of order h^2 uniformly in i. Thus, we have for this case

$$(6.6) \qquad\qquad \|e\| \leq B_1^{(2)} h^2 \quad \text{for} \quad \hat{T}_1(h).$$

4. *The case $M \geq 3$.* Here $r \in C^M(I)$, $g \in C(\Omega)$ and $g_{j,k} \in PC(\Omega)$, $j + k \leq M$, and F and $F_{j,k}$, $j + k \leq M$, are all in $C(\Delta')$. From Theorem 2.1 $y \in C^M(I)$ only. In this case we can use the rules $\hat{T}_p(h)$, $p = 1, 2, \ldots, \lfloor (M + 1)/2 \rfloor$. (We disregard $\hat{T}_{\lfloor (M+3)/2 \rfloor}$ even though it is well defined.) We then have

$$(6.7) \qquad \|e\| \leq \begin{cases} B_p^{(M)} h^{2p}, & p = 1, \ldots, \lfloor (M - 1)/2 \rfloor, \\ B_p^{(M)} h^M, & p = \lfloor (M + 1)/2 \rfloor. \end{cases}$$

Thus the maximum accuracy that can be achieved is determined by the differentiability properties of the exact solution $y(x)$, which, in turn, are determined by those of r, g and F.

7. Applications to two-point boundary value problems.

Example 1. Consider the two-point BVP with

$$y'' = 2y^3, \quad 0 \le x \le 1,$$
$$y(0) = 2 \quad \text{and} \quad y(1) = 2/3.$$

The exact solution to this problem is

$$y(x) = \frac{1}{x + 1/2}.$$

General problems with ODE's of the form $y'' = ay^n$ with $a > 0$ and $n \ge 1$ occur in nth order reaction kinetics, see Aries [**1**]. We note that Dirichlet BC's are not the standard BC's associated with reaction kinetics problems (normally, $y'(0) = 0$ is assigned at $x = 0$). We use Dirichlet BC's in our example, as this enables us to determine the exact solution by which we can demonstrate the accuracy of the corrected trapezoidal rule quadrature methods rigorously.

We observe that, for $x \in [0, 1]$ and $y \in [0, 2]$, $f(x, y) = 2y^3$ satisfies $0 \le (\partial f / \partial y) \le 24$ and $0 \le f(x, y) \le 8y$. Therefore, Theorem 1 in [**11**] applies, and we conclude that (i) a unique solution $y(x) \in [0, 2]$ exists, and (ii) with $k^2 = 12$ and $y^{(0)}(x) = r(x)$ in (1.4)–(1.7), $y(x) = \lim_{m \to \infty} y^{(m)}(x)$ uniformly in $[0, 1]$. In turn, all the conditions of our Theorem 2.1 and hence of Theorem 5.1 as well are satisfied, and the quadrature method solutions via the trapezoidal rule and its modifications exist and are unique for all large N.

From (5.18) and (5.3), it is clear that the contraction parameter $\mu(h)$ of the sequence of successive approximations $\{y_i^{(m)}, \ i = 0, 1, \ldots, N\}_{m=0}^{\infty}$ is practically the same as μ, the contraction parameter of the sequence $\{y^{(m)}(x)\}_{m=0}^{\infty}$, that is given in (2.5). Consequently, we can conclude that the sequences $\{y_i^{(m)}, i = 0, 1, \ldots, N\}_{m=0}^{\infty}$ will converge to a prescribed accuracy in the same number of iterations independently of N. We have verified this conclusion by solving the problem above with different values of N.

We have applied the quadrature method via the trapezoidal rule $T(h)$ and also via the corrected trapezoidal rule $\hat{T}_2(h)$ with $N = 100$ and $N = 200$. The results of the computations are given in Tables 1a and

1b, respectively. The last columns of these tables demonstrate very clearly the orders of accuracy of h^2 and h^4. It seems from these tables that if the quadrature formula $\hat{T}_3(h)$, whose order of accuracy is h^6, were used, then we would be able to achieve errors of order 10^{-12} with $N = 100$. We also note again that the computational cost per iteration of all three quadrature methods is practically the same, and this makes the high-accuracy methods practical.

Example 2. In a problem concerning the analysis of heat and mass transfer in a porous catalyst, see Kubecek and Hlavacek [**8**], the following two-point BVP is obtained:

$$y'' = \alpha y \exp\left(\frac{\gamma\beta(1-y)}{1+\beta(1-y)}\right), \quad 0 \le x \le 1,$$
$$y'(0) = 0 \quad \text{and} \quad y(1) = 1.$$

The quantities γ, β and α are positive constants representative of dimensionless energy of activation, heat evolution, and Thiele's modulus, respectively.

The solution $y(x)$ can again be shown to satisfy a Fredholm integral equation of the form (1.4) with

$$g(x,t) = \frac{1}{k\cosh k}\begin{cases} \cosh kx \sinh k(1-t), & 0 \le x \le t \\ \sinh k(1-x)\cosh kt, & t \le x \le 1 \end{cases}$$

and

$$r(x) = \frac{\cosh kx}{\cosh k}.$$

The existence and uniqueness of a solution $y(x)$ satisfying $0 \le y(x) \le 1$ when $0 \le x \le 1$ was proved in [**12**]. There it is shown that $f(x,y) = \alpha y \exp(\gamma\beta(1-y)/(1+\beta(1-y)))$ satisfies $0 \le \partial f/\partial y \le 2\alpha e^{\gamma\beta}$ and $0 \le f(x,y) \le \alpha e^{\gamma\beta}y$ for $0 \le y \le 1$, provided $\gamma\beta \le 1$. Consequently, Theorem 2B in [**12**] applies, and the sequence of successive approximations $\{y^{(m)}(x)\}_{m=0}^{\infty}$ with $y^{(0)}(x) = r(x)$ converges to the unique solution when we pick $k^2 = \alpha e^{\gamma\beta}$. Again, all the conditions of our Theorem 2.1, and hence of Theorem 5.1 as well, are satisfied, and the quadrature method solutions via the trapezoidal rule and its modifications exist and are unique for all large N.

TABLE 1a. Results from quadrature method solution via the trapezoidal
rule for Example 1. Here $e^{(N)}(x)$ stands for $|y_i - y(x_i)|$, where $x = x_i = i/N$
for some i and y_i is the approximate solution with $h = 1/N$.

x	$e^{(100)}(x)$	$e^{(200)}(x)$	$e^{(100)}(x)/e^{(200)}(x)$
0.1	$4.347D - 05$	$1.087D - 05$	$3.999D + 00$
0.2	$6.508D - 05$	$1.627D - 05$	$3.999D + 00$
0.3	$7.671D - 05$	$1.918D - 05$	$4.000D + 00$
0.4	$8.206D - 05$	$2.052D - 05$	$4.000D + 00$
0.5	$8.218D - 05$	$2.055D - 05$	$4.000D + 00$
0.6	$7.721D - 05$	$1.930D - 05$	$4.000D + 00$
0.7	$6.691D - 05$	$1.673D - 05$	$4.000D + 00$
0.8	$5.091D - 05$	$1.273D - 05$	$4.000D + 00$
0.9	$2.876D - 05$	$7.189D - 06$	$4.000D + 00$

TABLE 1b. Results from quadrature method solution via the corrected
trapezoidal rule $\hat{T}_2(h)$ for Example 1. Here $e^{(N)}(x)$ stands for $|y_i - y(x_i)|$,
where $x = x_i = i/N$ for some i and y_i is the approximate solution with $h = 1/N$.

x	$e^{(100)}(x)$	$e^{(200)}(x)$	$e^{(100)}(x)/e^{(200)}(x)$
0.1	$9.571D - 09$	$5.985D - 10$	$1.599D + 01$
0.2	$1.056D - 08$	$6.601D - 10$	$1.599D + 01$
0.3	$9.611D - 09$	$6.010D - 10$	$1.599D + 01$
0.4	$8.291D - 09$	$5.184D - 10$	$1.599D + 01$
0.5	$6.956D - 09$	$4.349D - 10$	$1.599D + 01$
0.6	$5.652D - 09$	$3.534D - 10$	$1.599D + 01$
0.7	$4.350D - 09$	$2.720D - 10$	$1.599D + 01$
0.8	$3.003D - 09$	$1.877D - 10$	$1.599D + 01$
0.9	$1.566D - 09$	$9.790D - 11$	$1.599D + 01$

We have applied the quadrature method via the corrected trapezoidal
rule $\hat{T}_2(h)$ with $N = 50, 100$, and 200. This enables us to verify

numerically the h^4 behavior of the quadrature method error even though we do not have the exact solution against which to compare our approximate solution. If we let $y_{N,i}$ stand for the approximate solution at $x = i/N$, then, for each $i = 0, 1, \ldots, N$, the ratios $(y_{N,i} - y_{2N,2i})/(y_{2N,2i} - y_{4N,4i})$ should approach $2^4 = 16$ as N becomes large. This was seen to be the case in the numerous computations that we performed.

We have also observed that, for $\gamma\beta$ fixed, as α becomes large, the ODE has the characteristic of a singularly perturbed problem $\varepsilon y'' = \hat{f}(x, y)$, where $\varepsilon = 1/\alpha$ and $\hat{f}(x, y)$ is independent of ε. Our computations suggest the existence of a boundary layer near $x = 1$ for large values of α. Even in such cases our quadrature method seems to be producing very smooth and accurate approximations to $y(x)$ everywhere in $[0, 1]$. This is true, in particular, near $x = 1$ where $y(x)$ has a boundary layer and hence is varying rapidly.

In Tables 2a and 2b we give some numerical results obtained for the cases (a) $\alpha = 10$, $\beta = 0.5$, $\gamma = 2$ and (b) $\alpha = 100$, $\beta = 0.5$, $\gamma = 2$, respectively. These tables show the numerical solution with $N = 200$, the differences $d_i^{(50)} = |y_{50,i} - y_{100,2i}|$ and $d_i^{(100)} = |y_{100,2i} - y_{200,4i}|$ and their ratios $d_i^{(50)}/d_i^{(100)}$. In the absence of knowledge of the exact solution, and making the reasonable assumption that $y_{2N,2i}$ is a better approximation than $y_{N,i}$, we can say that $d_i^{(50)}$ and $d_i^{(100)}$ are almost identical to the absolute errors in $y_{50,i}$ and $y_{100,2i}$, respectively. In addition, since the corrected trapezoidal rule $\hat{T}_2(h)$ has error of order h^4, $d_i^{(N)}/d_i^{(2N)}$ must approach $2^4 = 16$ as N becomes large. This is observed in the last columns of our tables.

8. Summary and concluding remarks. In this work we have considered the quadrature method solution (via the trapezoidal rule) of Fredholm integral equations of the second kind described by (1.1) and (i)–(iii) in the first paragraph of Section 1. Exploiting the known singularity structure of the kernel function $g(x, t)$, we have designed a class of corrected quadrature formulas for the integral $\int_0^1 g(x, t)F(t, u(t))dt$ with corrections derived from an analysis of the Euler-Maclaurin expansion associated with the trapezoidal rule. These new quadrature formulas allow us to improve the order of accuracy in the standard trapezoidal

TABLE 2a. Results from quadrature method solution via the corrected
trapezoidal rule $\hat{T}_2(h)$ for Example 2 with $\alpha = 10$, $\beta = 0.5$, and $\gamma = 2$.
Here $y_{200}(x)$ is the numerical solution with $N = 200$, $d^{(50)}(x) = |y_{50}(x) - y_{100}(x)|$,
and $d^{(100)}(x) = |y_{100}(x) - y_{200}(x)|$.

x	$y_{200}(x)$	$d^{(50)}(x)$	$d^{(100)}(x)$	$d^{(50)}(x)/d^{(100)}(x)$
0.0	$2.902507D - 02$	$2.008D - 10$	$6.044D - 11$	$3.322D + 00$
0.1	$3.185952D - 02$	$1.026D - 09$	$9.504D - 11$	$1.079D + 01$
0.2	$4.090769D - 02$	$2.274D - 09$	$1.620D - 10$	$1.403D + 01$
0.3	$5.789865D - 02$	$4.142D - 09$	$2.715D - 10$	$1.525D + 01$
0.4	$8.603731D - 02$	$6.813D - 09$	$4.336D - 10$	$1.571D + 01$
0.5	$1.305101D - 01$	$1.027D - 08$	$6.461D - 10$	$1.589D + 01$
0.6	$1.992036D - 01$	$1.381D - 08$	$8.647D - 10$	$1.597D + 01$
0.7	$3.035426D - 01$	$1.519D - 08$	$9.475D - 10$	$1.603D + 01$
0.8	$4.590925D - 01$	$9.880D - 09$	$6.124D - 10$	$1.613D + 01$
0.9	$6.849658D - 01$	$4.638D - 09$	$2.973D - 10$	$1.560D + 01$

TABLE 2b. Results from quadrature method solution via the corrected
trapezoidal rule $\hat{T}_2(h)$ for Example 2 with $\alpha = 100$, $\beta = 0.5$, and $\gamma = 2$.
Here $y_{200}(x)$ is the numerical solution with $N = 200$, $d^{(50)}(x) = |y_{50}(x) - y_{100}(x)|$,
and $d^{(100)}(x) = |y_{100}(x) - y_{200}(x)|$.

x	$y_{200}(x)$	$d^{(50)}(x)$	$d^{(100)}(x)$	$d^{(50)}(x)/d^{(100)}(x)$
0.0	$2.072705D - 06$	$6.755D - 11$	$3.368D - 12$	$2.005D + 01$
0.1	$4.440899D - 06$	$7.612D - 11$	$4.622D - 12$	$1.647D + 01$
0.2	$1.695708D - 05$	$1.954D - 10$	$1.245D - 11$	$1.569D + 01$
0.3	$6.822186D - 05$	$5.196D - 10$	$3.355D - 11$	$1.549D + 01$
0.4	$2.753740D - 04$	$1.061D - 09$	$7.027D - 11$	$1.510D + 01$
0.5	$1.111655D - 03$	$1.345D - 10$	$2.261D - 11$	$5.948D + 00$
0.6	$4.485999D - 03$	$1.580D - 08$	$9.371D - 10$	$1.686D + 01$
0.7	$1.807543D - 02$	$1.231D - 07$	$7.511D - 09$	$1.639D + 01$
0.8	$7.238558D - 02$	$6.288D - 07$	$3.858D - 08$	$1.630D + 01$
0.9	$2.827760D - 01$	$1.614D - 06$	$9.721D - 08$	$1.660D + 01$

rule from N^{-2} to N^{-2p} for arbitrary $p \geq 2$, where $N+1$ is the number of points in the discrete approximation. We have also shown that the accuracy of the quadrature method solution y_i, $i = 0, 1, \ldots, N$, is the same as that of the underlying numerical quadrature formula.

One can also achieve an increase in accuracy by extrapolation provided an asymptotic expansion for the error involving negative powers of N is known. However, for one extrapolation the problem will have to be solved for a given N and then again for $2N$. The improvement will only be able to be achieved on the course grid at an expense that is almost four times that of the improved quadrature. With the improved quadrature we do not need to obtain another approximation with twice the number of points.

Finally, we would like to comment on the solution of the nonlinear system of equations (5.5) for the discrete approximation y_i, $i = 0, 1, \ldots, N$. The reader is aware that throughout the paper we have emphasized the method of successive approximations as given in (5.6) for solving this system. Actually, successive approximations have served as an indispensable tool in the theoretical study of the new methods proposed here. In particular, the proof of Theorem 5.1 on existence and uniqueness of the y_i, $i = 0, 1, \ldots, N$, and that of Theorem 6.1 concerning the error in the y_i rely entirely on successive approximations.

In addition to being a theoretical tool, the method of successive approximations has something to offer as a practical numerical tool for actually solving the system in (5.5). First, it is an extremely easy method to implement on a computer. Next, as we have shown in the course of Sections 5 and 6, the convergence of successive approximations in our problem has the nice property that the associated contraction parameter $\mu(h)$ given in (5.3) is practically independent of η, of h, and of which quadrature formula $\hat{T}_p^G$ is being used, since $\eta > 0$ is arbitrarily close to zero and N is sufficiently large, and thus $\lim_{\eta \to 0} (\lim_{h \to 0} \mu(h)) = \mu$ with μ as in (2.5). This implies that the number of iterations to reach convergence is nearly independent of N and of the accuracy of the quadrature formula. Consequently, with N fixed, the cost of the solution is practically the same for all accuracies. For these reasons the method of successive approximations may be a very efficient numerical tool for obtaining the y_i when the contraction parameter μ is sufficiently smaller than 1, as it will require a small number of iterations to reach convergence.

Of course, when the contraction parameter μ is too close to unity, successive approximations converge very slowly and hence become quite expensive. In such a case Newton's method may be very efficient as it has quadratic convergence and thus may produce the y_i with high accuracy at the cost of a small number of iterations. Newton's method may be more efficient than successive approximations in such cases despite the fact that each of its iterations has a large computational cost. Now we need a reasonable initial approximation for Newton's method in order to reduce the number of iterations and hence the cost. Again, successive approximations can be used to produce such an initial guess. Thus, this kind of a combination of successive approximations and Newton's method may be a good way for determining the y_i from (5.5).

Whatever the value of μ, we can also employ vector extrapolation methods, such as the minimal polynomial extrapolation (MPE) or the reduced rank extrapolation (RRE), to accelerate the convergence of the sequences of successive approximations from (5.6). As no Jacobian matrices need to be computed and no large scale linear systems need to be solved in applying MPE or RRE, this approach to the solution of the nonlinear equations in (5.5) via successive approximations and vector extrapolation methods may turn out to be more economical than that of Newton's method, at least in some cases. For the subject of vector extrapolation methods we refer the reader, for example, to the review paper by Smith, Ford, and Sidi [15], and to Sidi [13], where a FORTRAN program that implements MPE and RRE in a numerically stable way is also given. More references to developments pertaining to MPE and RRE can be found in these two papers.

Clearly, the problem of actual solution of (5.5) is of importance in itself and should be the subject of a separate publication.

One last remark that we would like to make is that the approach of the present work can be applied to *systems* of nonlinear Fredholm integral equations, and hence to *systems* of nonlinear two-point BVP's, almost with no modification.

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