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# Further convergence and stability results for the generalized Richardson extrapolation process GREP$^{(1)}$ with an application to the $D^{(1)}$-transformation for infinite integrals

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**Abstract**

Let $a(t) \sim A + \varphi(t)\sum_{i=0}^{\infty}\beta_i t^i$ as $t \to 0+$, where $a(t)$ and $\varphi(t)$ are known for $0 < t \leqslant c$ for some $c > 0$, but $A$ and the $\beta_i$ are not known. The generalized Richardson extrapolation process GREP$^{(1)}$ is used in obtaining good approximations to $A$, the limit or antilimit of $a(t)$ as $t \to 0+$. The convergence and stability properties of GREP$^{(1)}$ for the case in which $\varphi(t) \sim \alpha t^{\delta}$ as $t \to 0+$, $\delta \neq 0, -1, -2, \ldots$, have been studied to a large extent in a recent work by the author. In the present work, we continue this study for the case in which $\delta$ is complex when the set of extrapolation points is $\{t_i = t_0\omega^i,\ i = 0, 1, \ldots\}$ with $\omega \in (0, 1)$. We give a complete convergence and stability analysis under very weak assumptions on $\varphi(t)$. We show that this analysis applies to the Levin–Sidi $D^{(1)}$-transformation that is a GREP$^{(1)}$, as this transformation is used for computing both convergent and divergent infinite-range integrals of functions $f(x)$ that essentially satisfy $f(x) \sim \nu x^{-\delta-1}$ as $x \to \infty$, with $\delta$ as above. In case of divergence, we show that the $D^{(1)}$-transformation produces approximations to the associated Hadamard finite parts. We append numerical examples that demonstrate the theory.


## 1. Introduction

In this work we continue the convergence and stability analysis of the generalized Richardson extrapolation process GREP$^{(1)}$ due to the author [6] that was begun in the recent paper [8].

GREP$^{(1)}$ is a very effective extrapolation procedure that is used in accelerating the convergence of a large family of infinite sequences that arise from and/or can be identified with functions $A(y)$

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that belong to a certain set denoted by $\mathrm{F}^{(1)}$. For future reference we give below the definitions of $\mathrm{F}^{(1)}$ and GREP$^{(1)}$. This will also establish much of the notation that we use in this work.

**Definition 1.1.** We shall say that a function $A(y)$, defined for $0 < y \leqslant b$, for some $b > 0$, where $y$ can be a discrete or continuous variable, belongs to the set $\mathrm{F}^{(1)}$, if there exist functions $\phi(y)$ and $\beta(y)$ and a constant $A$ such that

$$A(y) = A + \phi(y)\beta(y), \tag{1.1}$$

where $\beta(\xi)$, as a function of the continuous variable $\xi$ and for some $\hat{\xi} \leqslant b$, is continuous in $[0, \hat{\xi}]$, and for some constant $r > 0$, has a Poincaré-type asymptotic expansion of the form

$$\beta(\xi) \sim \sum_{i=0}^{\infty} \beta_i \xi^{ir} \quad \text{as } \xi \to 0+. \tag{1.2}$$

If, in addition, the function $B(t) \equiv \beta(t^{1/r})$, as a function of the continuous variable $t$, is infinitely differentiable for $0 \leqslant t \leqslant \hat{\xi}^r$, we shall say that $A(y)$ belongs to the set $\mathrm{F}^{(1)}_{\infty}$. Note that $\mathrm{F}^{(1)}_{\infty} \subset \mathrm{F}^{(1)}$.

**Remark.** We have $A = \lim_{y\to 0+} A(y)$ whenever this limit exists, in which case $\lim_{y\to 0+} \phi(y) = 0$. If $\lim_{y\to 0+} A(y)$ does not exist, then $A$ is said to be the antilimit of $A(y)$ as $y \to 0+$. In this case, $\lim_{y\to 0+} \phi(y)$ does not exist, as is obvious from (1.1) and (1.2).

It is assumed that the functions $A(y)$ and $\phi(y)$ are computable for $0 < y \leqslant b$ (keeping in mind that $y$ may be discrete or continuous depending on the situation) and that the constant $r$ is known. The constants $A$ and $\beta_i$ are not assumed to be known. In attempting to accelerate the convergence of a sequence that can be identified with $A(y)$, the idea, thus the problem, is to find (or approximate) $A$ whether it is the limit or the antilimit of $A(y)$ as $y \to 0+$, and GREP$^{(1)}$, the extrapolation procedure that corresponds to $\mathrm{F}^{(1)}$, is designed to tackle precisely this problem. The $\beta_i$ are not required in most cases of interest, although GREP$^{(1)}$ produces approximations (usually not very good ones) to them as well.

**Definition 1.2.** Let $A(y) \in \mathrm{F}^{(1)}$, with $\phi(y)$, $\beta(y)$, $A$, and $r$ being as in Definition 1.1. Pick $y_l \in (0, b]$, $l = 0, 1, 2, \ldots$, such that $y_0 > y_1 > y_2 > \cdots$, and $\lim_{l\to\infty} y_l = 0$. Then $A_n^{(j)}$, the approximation to $A$, and the parameters $\bar{\beta}_i$, $i = 0, 1, \ldots, n-1$, are defined to be the solution of the system of $n+1$ linear equations

$$A(y_l) = A_n^{(j)} + \phi(y_l) \sum_{i=0}^{n-1} \bar{\beta}_i y_l^{ir}, \quad j \leqslant l \leqslant j+n, \tag{1.3}$$

provided the matrix of this system is nonsingular. It is this process that generates the approximations $A_n^{(j)}$ that we call GREP$^{(1)}$.

Note that the equations in (1.3) are derived from (1.1) with (1.2).

As is seen, GREP$^{(1)}$ produces a two-dimensional table of approximations of the form

$$\begin{array}{lllll} n=0, & n=1, & n=2, & n=3, & \dots \\ A_0^{(0)} & & & & \\ A_0^{(1)} & A_1^{(0)} & & & \\ A_0^{(2)} & A_1^{(1)} & A_2^{(0)} & & \\ A_0^{(3)} & A_1^{(2)} & A_2^{(1)} & A_3^{(0)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}, \qquad A_0^{(j)} \equiv A(y_j),\ j=0,1,\dots. \tag{1.4}$$

Numerical experiments and the theory that exists for some cases suggest that when $\lim_{y\to 0+} A(y)$ exists, the columns of this table converge, each column converging at least as quickly as those preceding it, while the diagonals converge more quickly than the columns.

Going down a column corresponds to letting $j\to\infty$ while $n$ is being held fixed in $A_n^{(j)}$, and this limiting process is called Process I. Going along a diagonal corresponds to letting $n\to\infty$ while $j$ is being held fixed in $A_n^{(j)}$, and this limiting process is called Process II.

Before going on, we shall let $t=y^r$ and $t_l=y_l^r$, $l=0,1,\dots,$ and define $a(t)\equiv A(y)$ and $\varphi(t)\equiv\phi(y)$. Then the equations in (1.3) take on the more convenient form

$$a(t_l) = A_n^{(j)} + \varphi(t_l)\sum_{i=0}^{n-1} \bar{\beta}_i t_l^i,\ j\leqslant l\leqslant j+n. \tag{1.5}$$

Let us denote by $D_n^{(j)}$ the divided difference operator of order $n$ over the set of points $t_j, t_{j+1},\dots,t_{j+n}$. Then the action of $D_n^{(j)}$ on an arbitrary function $g(t)$ is described by

$$D_n^{(j)}\{g(t)\} = g[t_j, t_{j+1},\dots,t_{j+n}] = \sum_{i=0}^{n} c_{ni}^{(j)} g(t_{j+i}), \tag{1.6}$$

where

$$c_{ni}^{(j)} = \prod_{\substack{s=0\\ s\neq i}}^{n} (t_{j+i}-t_{j+s})^{-1}, \quad i=0,1,\dots,n. \tag{1.7}$$

The solution of the equations in (1.5) can be expressed with the help of the divided difference operator $D_n^{(j)}$ as in

$$A_n^{(j)} = \frac{D_n^{(j)}\{a(t)/\varphi(t)\}}{D_n^{(j)}\{1/\varphi(t)\}}. \tag{1.8}$$

From (1.6) and (1.8) it is obvious that $\varphi(t_l)\neq 0$, $l=0,1,2,\dots,$ must be assumed throughout. $\varphi(t)$ may vanish at other points, however. From (1.6) and (1.8) we also notice that $A_n^{(j)}$ can be expressed as in

$$A_n^{(j)} = \sum_{i=0}^{n} \gamma_{ni}^{(j)}\, a(t_{j+i}), \tag{1.9}$$

where

$$\gamma_{ni}^{(j)} = \frac{c_{ni}^{(j)}/\varphi(t_{j+i})}{D_n^{(j)}\{1/\varphi(t)\}}, \quad i = 0, 1, \ldots, n, \tag{1.10}$$

and thus

$$\sum_{i=0}^{n} \gamma_{ni}^{(j)} = 1. \tag{1.11}$$

As has been described in [6, Section 6], the propagation of the errors (roundoff or other) in the $a(t_l)$ into $A_n^{(j)}$ is controlled by the quantity $\Gamma_n^{(j)}$ defined by

$$\Gamma_n^{(j)} = \sum_{i=0}^{n} |\gamma_{ni}^{(j)}|, \tag{1.12}$$

which can be expressed with the help of $D_n^{(j)}$ (see [8, Theorem 3.3]) as in

$$\Gamma_n^{(j)} = \frac{|D_n^{(j)}\{P(t)\}|}{|D_n^{(j)}\{1/\varphi(t)\}|}, \tag{1.13}$$

where $P(t)$ takes on arbitrary values for $t \neq t_i$, $i = 0, 1, \ldots,$ and

$$P(t_i) = (-1)^i/|\varphi(t_i)|, \; i = 0, 1, \ldots \; . \tag{1.14}$$

More specifically, if $\varepsilon_l$ is the absolute error in the input $a(t_l)$, $l = 0, 1, \ldots,$ and if $\bar{A}_n^{(j)}$ are the entries in the extrapolation table (1.4) computed with the erroneous $a(t_l)$, then

$$|A_n^{(j)} - \bar{A}_n^{(j)}| \leqslant \Gamma_n^{(j)} \left( \max_{j \leqslant l \leqslant j+n} |\varepsilon_l| \right) \tag{1.15}$$

for each $j$ and $n$. Thus the larger $\Gamma_n^{(j)}$, the worse the error propagation is expected to be. (Obviously, $\Gamma_n^{(j)} \geqslant 1$ from (1.11) and (1.12).) On the basis of this we say that Process I that generates the column sequence $\{A_n^{(j)}\}_{j=0}^{\infty}$ with $n$ fixed is stable provided $\sup_j \Gamma_n^{(j)} < \infty$. Similarly, we say that Process II that generates the diagonal sequence $\{A_n^{(j)}\}_{n=0}^{\infty}$ with $j$ fixed is stable provided $\sup_n \Gamma_n^{(j)} < \infty$. Here it is worth recalling that the error in $\bar{A}_n^{(j)}$ satisfies

$$|\bar{A}_n^{(j)} - A| \leqslant |A_n^{(j)} - A| + |A_n^{(j)} - \bar{A}_n^{(j)}|. \tag{1.16}$$

Both $A_n^{(j)}$ and $\Gamma_n^{(j)}$ can be computed very efficiently by the $W$-algorithm of the author [7]. See also Theorem 3.3 and what follows that in [8]. Here are the steps of the $W$-algorithm:

1. For $j = 0, 1, \ldots,$ set

$$M_0^{(j)} = \frac{a(t_j)}{\varphi(t_j)}, \quad N_0^{(j)} = \frac{1}{\varphi(t_j)}, \quad H_0^{(j)} = \frac{(-1)^j}{|\varphi(t_j)|}.$$

2. For $j = 0, 1, \ldots,$ and $n = 1, 2, \ldots,$ compute recursively

$$M_n^{(j)} = \frac{M_{n-1}^{(j+1)} - M_{n-1}^{(j)}}{t_{j+n} - t_j}, \quad N_n^{(j)} = \frac{N_{n-1}^{(j+1)} - N_{n-1}^{(j)}}{t_{j+n} - t_j}, \quad H_n^{(j)} = \frac{H_{n-1}^{(j+1)} - H_{n-1}^{(j)}}{t_{j+n} - t_j}.$$

3. For all $j$ and $n$, set

$$A_n^{(j)} = \frac{M_n^{(j)}}{N_n^{(j)}}, \quad \Gamma_n^{(j)} = \frac{|H_n^{(j)}|}{|N_n^{(j)}|}.$$

The study of GREP$^{(1)}$ carried out in [8] concerned predominantly the case $\phi(y) \equiv \varphi(t) \sim \alpha t^{\delta}$ as $t \to 0+$, where $\delta$ is in general complex and $\delta \neq 0, -1, -2, \ldots$. Functions $A(y) \equiv a(t)$ with this property are related to infinite sequences that converge logarithmically and to their divergent extensions, as shown in [8]. By refining the techniques in [8], in the present work we continue and complete the study of this case with the choice $t_i = t_0 \omega^i$, $i = 1, 2, \ldots$, for some $t_0 > 0$ and some $\omega \in (0, 1)$. We improve substantially on the convergence and stability results in [8] and prove new ones as well. An important feature of the results of the present work is that they are all achieved by assuming conditions on $\varphi(t)$ and $B(t)$ that are much weaker than the ones in [8]. The analytical tools developed in [8], including those in the appendix in [8], turn out to be useful in this work too.

The main result on stability is Theorem 2.1 in Section 2. This result essentially says that in case $\varphi(t) = \alpha t^{\delta} + O(t^{\delta+\theta})$ as $t \to 0+$, where $\theta > 0$, both Processes I and II are stable in the sense described earlier, namely, $\sup_j \Gamma_n^{(j)} < \infty$ and $\sup_n \Gamma_n^{(j)} < \infty$. It also gives $\lim_{j\to\infty} \Gamma_n^{(j)}$ and $\lim_{n\to\infty} \Gamma_n^{(j)}$ exactly, showing at the same time that both limits exist and depend only on $\delta$, and that the latter is independent of $j$ as well. The main results on the convergence of Processes I and II are Theorems 3.1–3.3 in Section 3. These results are obtained under the same (weak) assumptions on $\varphi(t)$ imposed in Section 2. In addition, they employ assumptions on $B(t)$ that are weaker than those employed in [8].

In Section 4 we show that the results of Sections 2 and 3 are directly applicable to the $D^{(1)}$-transformation of Levin and Sidi [4] that is a GREP$^{(1)}$, when this transformation is used for computing infinite-range integrals $\int_a^{\infty} f(t)\,\mathrm{d}t$ of functions $f(x)$ that essentially satisfy $f(x) \sim v x^{-\delta-1}$ as $x \to \infty$. Now when $\Re\delta > 0$ these integrals converge, and the $D^{(1)}$-transformation gives approximations to their values. When $\Re\delta \leqslant 0$, however, they diverge, and the $D^{(1)}$-transformation produces approximations to their Hadamard finite parts (HFPs) provided also that $\delta \neq 0, -1, -2, \ldots$, as we show in Theorem 4.1. As far as is known to us, application of extrapolation methods for computing HFPs of divergent infinite-range integrals has not been given in the literature before. In the appendix to this work we discuss the connection of these HFPs to analytic continuation, the main result there being Theorem A.1.

Now the $D^{(1)}$-transformation is a special case of the general $D^{(m)}$-transformation that has been used very effectively in accelerating the convergence of a large class of infinite-range integrals. Although the theory of Sections 2 and 3 apply to the $D^{(1)}$-transformation only, based on ample numerical evidence, we believe that a very similar theory under similar conditions exists for the $D^{(m)}$-transformation. In this sense the approach of the present work might be a first step towards the analytical treatment of the $D^{(m)}$-transformation for arbitrary $m$. The definition of the $D^{(m)}$-transformation is visited briefly at the end of Section 4.

In Section 5 we demonstrate the results of Sections 2–4 by applying the $D^{(1)}$-transformation to convergent and divergent infinite-range integrals of the type discussed in Section 4. We also include examples for which the $D^{(2)}$-transformation produces excellent approximations but the $D^{(1)}$-transformation is not effective.

## 2. Stability

Throughout the remainder of this work we will assume that

$$\varphi(t) = \alpha t^{\delta} + \mathrm{O}(t^{\delta+\theta}) \text{ as } t \to 0+ \quad \text{for some } \alpha \neq 0 \text{ and } \theta > 0. \tag{2.1}$$

Thus, when $\Re\delta > 0$, $A = \lim_{t\to 0+} a(t)$. In case $\lim_{t\to 0+} a(t)$ does not exist, $A$ is the antilimit of $a(t)$ as $t \to 0+$, and $\Re\delta \leqslant 0$ for this case necessarily. (Note that the condition on $\varphi(t)$ given in (2.1) is already much weaker than those imposed on $\varphi(t)$ in Section 3 in [8].)

We also recall that $t_i$ are picked as in

$$t_i = t_0\omega^i, \quad i = 1, 2, \ldots, \text{ for some } t_0 > 0 \text{ and } \omega \in (0, 1). \tag{2.2}$$

As described earlier, the stability analysis of GREP$^{(1)}$ revolves around the study of $\Gamma_n^{(j)}$ defined in (1.12), which in turn is based on the expression for $\Gamma_n^{(j)}$ given in (1.13) with (1.14). We begin this study by reminding ourselves that $D_n^{(j)}$ is a divided difference operator, as a consequence of which we have the well-known recursion relation

$$D_n^{(j)}\{g(t)\} = \frac{D_{n-1}^{(j+1)}\{g(t)\} - D_{n-1}^{(j)}\{g(t)\}}{t_{j+n} - t_j}. \tag{2.3}$$

Employing (2.3) we first prove the following lemma that will be very useful in the sequel.

**Lemma 2.1.** *Let $v(t;\delta) = t^{-\delta}$, where $\delta$ is in general complex and $\delta \neq 0, -1, -2, \ldots$. Define also $w(t_i;\delta) = (-1)^i|v(t_i;\delta)|$, $i = 0, 1, \ldots$, cf. (1.14), $w(t;\delta)$ being arbitrary for all other values of $t$. Then, for all $j$ and $n$,*

$$D_n^{(j)}\{v(t;\delta)\} = \frac{(-1)^n}{\omega^{\delta n + (n^2-n)/2}\, t_j^{\delta+n}} \prod_{i=1}^{n} \frac{1-\omega^{\delta+i-1}}{1-\omega^i} \tag{2.4}$$

and

$$D_n^{(j)}\{w(t;\delta)\} = \frac{(-1)^j}{\omega^{(\Re\delta)n + (n^2-n)/2}\, t_j^{\Re\delta+n}} \prod_{i=1}^{n} \frac{1+\omega^{\Re\delta+i-1}}{1-\omega^i}. \tag{2.5}$$

**Proof.** The proofs of (2.4) and (2.5) can be achieved by invoking (2.3) and using induction. □

We note that (2.4) is already given by Lemma A.3 in [8]. The result in (2.5) is new, however. With the help of Lemma 2.2 we next prove a result on $D_n^{(j)}\{1/\varphi(t)\}$ and $D_n^{(j)}\{P(t)\}$.

**Lemma 2.2.** *With $\varphi(t)$ and the $t_i$ as in (2.1) and (2.2) respectively, we have*

$$D_n^{(j)}\{1/\varphi(t)\} = \alpha^{-1} D_n^{(j)}\{v(t;\delta)\}[1 + Q_n^{(j)}\omega^{\theta n} t_j^{\theta}] \tag{2.6}$$

*and*

$$|D_n^{(j)}\{P(t)\}| = |\alpha|^{-1} |D_n^{(j)}\{w(t;\delta)\}|[1 + R_n^{(j)}\omega^{\theta n} t_j^{\theta}], \tag{2.7}$$

*where $Q_n^{(j)}$ and $R_n^{(j)}$ are constants that satisfy $\sup_{j,n}|Q_n^{(j)}| < \infty$ and $\sup_{j,n}|R_n^{(j)}| < \infty$.*

**Proof.** We start by analyzing $D_n^{(j)}\{1/\varphi(t)\}$. By (1.6) we have

$$D_n^{(j)}\{1/\varphi(t)\} = \sum_{i=0}^{n} c_{ni}^{(j)}/\varphi(t_{j+i}). \tag{2.8}$$

Let us define the function $N(t)$ through

$$1/\varphi(t) = \alpha^{-1}t^{-\delta} + N(t). \tag{2.9}$$

Now (2.1) implies that there exists $\hat{t} \in (0, b^r]$ such that $\varphi(t) \neq 0$ for $0 < t \leqslant \hat{t}$. Consequently, we have

$$|N(t)| = \left|1/\varphi(t) - \alpha^{-1}t^{-\delta}\right| \leqslant K_1 t^{-\Re\delta+\theta} \quad \text{for } t \in (0, \hat{t}], \tag{2.10}$$

where $K_1$ is some positive constant. Recalling that $1/\varphi(t_l) \neq 0$, $l = 0, 1, 2, \ldots$, we have that $N(t_l)$ are all defined. Noting also that the number of the $t_l$ in $X = (\hat{t}, b^r]$ is *finite*, we have

$$\max_{t_l \in X} |N(t_l)t_l^{\delta-\theta}| \equiv K_2 < \infty. \tag{2.11}$$

Combining (2.10) and (2.11), and letting $K = \max(K_1, K_2)$, we have

$$|N(t_l)| \leqslant K t_l^{-\Re\delta+\theta}, \quad l = 0, 1, 2, \ldots\ . \tag{2.12}$$

Substituting (2.9) in (2.8), we obtain

$$D_n^{(j)}\{1/\varphi(t)\} = \alpha^{-1} D_n^{(j)}\{v(t;\delta)\} + D_n^{(j)}\{N(t)\}. \tag{2.13}$$

But, by (2.12),

$$|D_n^{(j)}\{N(t)\}| \leqslant K \sum_{i=0}^{n} |c_{ni}^{(j)}|\, t_{j+i}^{-\Re\delta+\theta} = K|D_n^{(j)}\{w(t;\delta-\theta)\}|. \tag{2.14}$$

Invoking Lemma 2.1 in (2.14), we obtain

$$\frac{|D_n^{(j)}\{N(t)\}|}{|D_n^{(j)}\{v(t;\delta)\}|} \leqslant K \left( \prod_{i=1}^{n} \frac{1+\omega^{\Re\delta-\theta+i-1}}{|1-\omega^{\delta+i-1}|} \right) \omega^{\theta n} t_j^{\theta}. \tag{2.15}$$

Now, by the fact that $\omega \in (0, 1)$, the infinite products $\prod_{i=1}^{\infty} |1 - \omega^{\delta+i-1}|$ and $\prod_{i=1}^{\infty}(1 + \omega^{\Re\delta-\theta+i-1})$ converge and, therefore, have nonzero limits. By combining (2.13)–(2.15), the result in (2.6) now follows.

Similarly, by (1.6), (1.7), and (1.14), we have

$$|D_n^{(j)}\{P(t)\}| = \sum_{i=0}^{n} |c_{ni}^{(j)}|/|\varphi(t_{j+i})|. \tag{2.16}$$

We also have

$$|1/\varphi(t)| = |\alpha^{-1}|\, t^{-\Re\delta} + M(t) \quad \text{with } |M(t)| \leqslant |N(t)|. \tag{2.17}$$

Substituting (2.17) in (2.16), we obtain

$$|D_n^{(j)}\{P(t)\}| = |\alpha^{-1}||D_n^{(j)}\{w(t;\delta)\}| + E_n^{(j)}\{M(t)\} \tag{2.18}$$

with $E_n^{(j)}\{M(t)\} = \sum_{i=0}^{n} |c_{ni}^{(j)}|M(t_{j+i})$, so that

$$|E_n^{(j)}\{M(t)\}| \leqslant \sum_{i=0}^{n} |c_{ni}^{(j)}|\ |M(t_{j+i})| \leqslant \sum_{i=0}^{n} |c_{ni}^{(j)}|\ |N(t_{j+i})| \leqslant K|D_n^{(j)}\{w(t;\delta-\theta)\}|. \tag{2.19}$$

Invoking Lemma 2.1 in (2.19), we obtain

$$\frac{|E_n^{(j)}\{M(t)\}|}{|D_n^{(j)}\{w(t;\delta)\}|} \leqslant K \left( \prod_{i=1}^{n} \frac{1+\omega^{\Re\delta-\theta+i-1}}{1+\omega^{\Re\delta+i-1}} \right) \omega^{\theta n} t_j^{\theta}. \tag{2.20}$$

The result in (2.7) now follows by combining (2.18)– (2.20) and by the fact that the infinite products $\prod_{i=1}^{\infty}(1+\omega^{\Re\delta+i-1})$ and $\prod_{i=1}^{\infty}(1+\omega^{\Re\delta-\theta+i-1})$ converge and, therefore, have nonzero limits. □

We now turn to the main stability result of this work. The quantity $\hat{\Gamma}_n(\delta;\omega)$ that is defined with the help of Lemma 2.1 and via

$$\frac{|D_n^{(j)}\{w(t;\delta)\}|}{|D_n^{(j)}\{v(t;\delta)\}|} = \prod_{i=1}^{n} \frac{1+\omega^{\Re\delta+i-1}}{|1-\omega^{\delta+i-1}|} \equiv \hat{\Gamma}_n(\delta;\omega) \tag{2.21}$$

is of relevance to this result.

**Theorem 2.1.** *With $\varphi(t)$ and the $t_i$ as in* (2.1) *and* (2.2) *respectively, we have*

$$\Gamma_n^{(j)} = \hat{\Gamma}_n(\delta;\omega) \quad \textit{for all } j \textit{ and } n \textit{ if } \varphi(t)=\alpha t^{\delta} \tag{2.22}$$

*and*

$$\lim_{j\to\infty} \Gamma_n^{(j)} = \hat{\Gamma}_n(\delta;\omega) \quad \textit{and} \quad \lim_{n\to\infty} \Gamma_n^{(j)} = \hat{\Gamma}_{\infty}(\delta;\omega) = \prod_{i=1}^{\infty} \frac{1+\omega^{\Re\delta+i-1}}{|1-\omega^{\delta+i-1}|} \tag{2.23}$$

*whether $\varphi(t)=\alpha t^{\delta}$ or not.*

**Proof.** When $\varphi(t)=\alpha t^{\delta}$, we have $1/\varphi(t)=\alpha^{-1}v(t;\delta)$ and $P(t)=|\alpha|^{-1}w(t;\delta)$ in (1.13). The result in (2.22) now follows by invoking (2.21). In general, we have

$$\Gamma_n^{(j)} = \hat{\Gamma}_n(\delta;\omega)[1+\mathrm{O}(\omega^{\theta j})] \quad \text{as } j\to\infty \quad (n \text{ fixed}). \tag{2.24}$$

and

$$\Gamma_n^{(j)} = \hat{\Gamma}_n(\delta;\omega)[1+\mathrm{O}(\omega^{\theta n})] \quad \text{as } n\to\infty \quad (j \text{ fixed}), \tag{2.25}$$

both of which are obtained by substituting (2.6) and (2.7) in (1.13), invoking (2.21), and, finally, letting $j\to\infty$ for (2.24) and $n\to\infty$ for (2.25). □

An important consequence of Theorem 2.1 is that Processes I and II are both stable. Theorem 2.1 also states that $\lim_{j\to\infty}\Gamma_n^{(j)}$ and $\lim_{n\to\infty}\Gamma_n^{(j)}$ depend only on $\delta$. In other words, they are both determined by the dominant asymptotic behavior of $\varphi(t)$ for $t\to 0+$, the details of the remaining part of $\varphi(t)$ being irrelevant. This, we believe, is a very interesting and surprising result especially since $A_n^{(j)}$ and hence $\Gamma_n^{(j)}$ in Process II are determined from $\varphi(t)$ for $t\in(0,t_j]$ for $j$ *fixed*. Finally, another surprising result is that $\lim_{n\to\infty}\Gamma_n^{(j)}$ is independent of $j$.

**Theorem 2.2.** *With $\varphi(t)$ and the $t_i$* as in (2.1) and (2.2) respectively, we have

$$\lim_{n\to\infty} \gamma_{ni}^{(j)} = 0 \quad \textit{for each fixed } i. \tag{2.26}$$

**Proof.** From (1.7) and (2.2) we have

$$c_{ni}^{(j)} = (-1)^i [t_j^n \omega^{in-(i^2+i)/2} C_i C_{n-i}]^{-1}, \tag{2.27}$$

where $C_k = \prod_{s=1}^{k}(1-\omega^s)$, $k = 1, 2, \ldots$, and $C_0 \equiv 1$. Substituting (2.27) and (2.6) in (1.10), and invoking (2.4), we obtain

$$\gamma_{ni}^{(j)} = \mathrm{O}(\omega^{n^2/2+d_i n}) \quad \text{as } n \to \infty, \text{ some constant } d_i. \tag{2.28}$$

The result now follows. □

Combining Theorems 2.1 and 2.2, and (1.11), we now state the following regularity theorem for Processes I and II of this work.

**Theorem 2.3.** *Process I that generates the column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$, $n = 0, 1, 2, \ldots$, and Process II that generates the diagonal sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$, $j = 0, 1, 2, \ldots$, are both regular summability methods.*

We leave the details of the proof to the interested reader. For summability methods and their regularity, see, e.g., [3] or [5]. See also [6, Section 4].

*Important Note.* As is clear from (1.15), when $A(y)$ is bounded as $y \to 0+$ and $A(y_l)$ are computed to maximum possible accuracy in finite precision arithmetic, the sequence $\{\varepsilon_l\}$ is bounded. As a result, the stability of Processes I and II guarantees that the sequences $\{A_n^{(j)} - \bar{A}_n^{(j)}\}_{j=0}^{\infty}$ ($n$ fixed) or $\{A_n^{(j)} - \bar{A}_n^{(j)}\}_{n=0}^{\infty}$ ($j$ fixed) are bounded. Note that $A(y)$ is bounded as $y \to 0+$ whenever $\Re\delta > 0$ (in which case $A = \lim_{y\to 0+} A(y)$) or $\Re\delta = 0$ (in which case $A$ is the antilimit of $A(y)$ as $y \to 0+$). When $A(y)$ is unbounded as $y \to 0+$ (which occurs for $\Re\delta < 0$) and $A(y_l)$ are computed to maximum possible accuracy in finite precision arithmetic, the sequence $\{\varepsilon_l\}$ is unbounded. In fact, $\varepsilon_l$ generally behaves like $\mathbf{u}A(y_l)$ in this case. Here $\mathbf{u}$ is the unit roundoff of the arithmetic used. As a result, even though both Processes I and II are stable in the sense described above, the sequences $\{A_n^{(j)} - \bar{A}_n^{(j)}\}_{j=0}^{\infty}$ ($n$ fixed) and $\{A_n^{(j)} - \bar{A}_n^{(j)}\}_{n=0}^{\infty}$ ($j$ fixed) are now unbounded. In this situation the hope is that the convergence rates of the sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$ and $\{A_n^{(j)}\}_{n=0}^{\infty}$ are much greater than the divergence rate of $\{\varepsilon_l\}$ such that sufficient accuracy is achieved by $A_n^{(j)}$ before $|A_n^{(j)} - \bar{A}_n^{(j)}|$ grows too much. Naturally, we expect less and less accuracy for sequences $\{A(y_l)\}$ that grow faster and faster. Here we have recalled (1.16).

## 3. Convergence

With the stability problem completely settled, we now state the relevant convergence results for $A_n^{(j)}$. Some of these results are directly based on [8], and we leave their verification to the interested reader. In addition, all our results are valid whether $A$ is the limit or the antilimit of $A(y)$ for $y \to 0+$.

Theorem 3.1 below relates to Process I in which $n$ is held fixed and $j \to \infty$, and its result is best possible asymptotically. Theorems 3.2 and 3.3, on the other hand, deal with Process II in which $j$

is held fixed and $n \to \infty$. Note that in Theorems 3.1, 3.2, and part (i) of Theorem 3.3 we require only that $A(y) \in \mathrm{F}^{(1)}$, which is a weaker requirement than $A(y) \in \mathrm{F}^{(1)}_{\infty}$.

**Theorem 3.1.** *Let $B(t)$ be as in Definition* 1.1 *and let $\varphi(t)$ and the $t_i$ be as in* (2.1) and (2.2), *respectively. Define $c_k = \omega^{\delta+k-1}$, $k = 1, 2, \ldots$. Then, for any fixed $n$, we have*

$$A_n^{(j)} - A \sim \alpha \beta_{n+\mu} \left( \prod_{i=1}^{n} \frac{c_{n+\mu+1} - c_i}{1 - c_i} \right) t_j^{\delta+n+\mu} \quad \text{as } j \to \infty, \tag{3.1}$$

*where $\beta_{n+\mu}$ is the first nonzero $\beta_i$ with $i \geqslant n$.*

It follows from Theorem 3.1 that the column sequence $\{A_n^{(j)}\}_{j=0}^{\infty}$ converges to $A$ provided $n > -\Re\delta - \mu$, and diverges otherwise. It also follows that if $\{A_{n-1}^{(j)}\}_{j=0}^{\infty}$ converges, $\{A_n^{(j)}\}_{j=0}^{\infty}$ converges at least as quickly, and if $\{A_n^{(j)}\}_{j=0}^{\infty}$ diverges, it diverges at worst as quickly as $\{A_{n-1}^{(j)}\}_{j=0}^{\infty}$. In other words, the sequence $\{A_n^{(j)}\}_{j=0}^{\infty}$ has convergence properties at least as good as those of $\{A_{n-1}^{(j)}\}_{j=0}^{\infty}$.

Lemma 3.1 below gives upper bounds on $|A_n^{(j)} - A|$ that are valid for all $j$ and $n$. These bounds are crucial in obtaining convergence results relevant to Process II.

**Lemma 3.1.** *Let $B(t)$ be as in Definition* 1.1, *and let $\varphi(t)$ and the $t_i$ be as in* (2.1) *and* (2.2), *respectively. Then there exist positive constants $\hat{\alpha}$ and $\hat{\beta}_s$, $s = 0, 1, 2, \ldots$, all independent of $j$ and $n$, such that, for each $s$, $s = 0, 1, \ldots, n$,*

$$|A_n^{(j)} - A| \leqslant \hat{\alpha}\hat{\beta}_s \frac{|D_n^{(j)}\{w(t; -s)\}|}{|D_n^{(j)}\{v(t; \delta)\}|} = \hat{\alpha}\hat{\beta}_s \left( \prod_{i=1}^{n} \frac{1 + \omega^{-s+i-1}}{|1 - \omega^{\delta+i-1}|} \right) \omega^{(\Re\delta+s)n} t_j^{\Re\delta+s}. \tag{3.2}$$

*The nature of the constants $\hat{\alpha}$ and $\hat{\beta}_s$, $s = 0, 1, \ldots$, will be discussed in the proof below.*

**Proof.** From [8] we have

$$A_n^{(j)} - A = \frac{D_n^{(j)}\{B(t)\}}{D_n^{(j)}\{1/\varphi(t)\}}. \tag{3.3}$$

Due to the fact that $D_n^{(j)}\{g(t)\} = 0$ when $g(t)$ is a polynomial in $t$ of degree at most $n-1$, (3.3) is equivalent to

$$A_n^{(j)} - A = \frac{D_n^{(j)}\{B(t) - \sum_{i=0}^{s-1} \beta_i t^i\}}{D_n^{(j)}\{1/\varphi(t)\}}, \quad s = 0, 1, \ldots, n. \tag{3.4}$$

As we have shown in the proof of Lemma 2.2, there exists $\hat{t} \in (0, b^r]$ such that $\varphi(t) \neq 0$ for $t \in (0, \hat{t}]$. This together with the observation that $B(t) = [a(t) - A]/\varphi(t)$, cf. (1.1), implies that $B(t)$ is well defined for $t \in [0, \hat{t}]$, but may not necessarily be continuous for $t \in X = (\hat{t}, b^r]$. From this and from $B(t) \sim \sum_{i=0}^{\infty} \beta_i t^i$ as $t \to 0+$, cf. (1.2), we deduce that there exist positive constants $\hat{\beta}_{s,1}$ defined as in

$$\hat{\beta}_{s,1} = \max_{0 \leqslant t \leqslant \hat{t}} \left( \left| B(t) - \sum_{i=0}^{s-1} \beta_i t^i \right| \Big/ t^s \right), \quad s = 0, 1, 2, \ldots . \tag{3.5}$$

Similarly, by the fact that the interval $X$ contains a finite number of the $t_l$, there exist additional positive constants $\hat{\beta}_{s,2}$ defined by

$$\hat{\beta}_{s,2} = \max_{t_l \in X} \left( \left| B(t_l) - \sum_{i=0}^{s-1} \beta_i t_l^i \right| \Big/ t_l^s \right), \quad s = 0, 1, 2, \ldots \; . \tag{3.6}$$

Combining (3.5) and (3.6), we see that there exist positive constants $\hat{\beta}_s$ defined as in

$$\hat{\beta}_s = \max_{l} \left( \left| B(t_l) - \sum_{i=0}^{s-1} \beta_i t_l^i \right| \Big/ t_l^s \right), \quad s = 0, 1, 2, \ldots, \tag{3.7}$$

that satisfy

$$|\beta_s| \leqslant \hat{\beta}_s \leqslant \max(\hat{\beta}_{s,1}, \hat{\beta}_{s,2}), \quad s = 0, 1, 2, \ldots \; . \tag{3.8}$$

Invoking (1.6) in the numerator on the right-hand side of (3.4), taking moduli, and, finally, using (3.7), we obtain for each $s$, $s = 0, 1, \ldots, n$,

$$\left| D_n^{(j)} \left\{ B(t) - \sum_{i=0}^{s-1} \beta_i t^i \right\} \right| \leqslant \hat{\beta}_s \sum_{i=0}^{n} |c_{ni}^{(j)}| t_{j+i}^s = \hat{\beta}_s |D_n^{(j)}\{w(t; -s)\}|. \tag{3.9}$$

By (2.25) there exists a positive constant $\hat{\alpha} \geqslant |\alpha|$ independent of $j$ and $n$ such that

$$\frac{1}{|D_n^{(j)}\{1/\varphi(t)\}|} \leqslant \frac{\hat{\alpha}}{|D_n^{(j)}\{v(t; \delta)\}|}, \quad n = 0, 1, 2, \ldots \; . \tag{3.10}$$

(Obviously, $\hat{\alpha} = |\alpha|$ when $\varphi(t) = \alpha t^\delta$.) The result in (3.2) now follows by combining (3.9) and (3.10) in (3.4) and by invoking Lemma 2.1. □

**Theorem 3.2.** *Assume all the conditions of Lemma* 3.1. *Then, for any fixed j, the diagonal sequence* $\{A_n^{(j)}\}_{n=0}^\infty$ *converges to A. The nature of this convergence is at worst as in*

$$A_n^{(j)} - A = O(\omega^{cn}) \quad \text{as } n \to \infty \quad \text{for any } c > 0. \tag{3.11}$$

**Proof.** From (3.2) we have that

$$A_n^{(j)} - A = \mathrm{O}(\omega^{(\Re\delta + s)n}) \quad \text{as } n \to \infty \quad \text{for fixed } s. \tag{3.12}$$

The result follows from the fact that, as $n \to \infty$, $s$ takes on all positive integer values. □

By comparing Theorem 3.1 with Theorem 3.2 we can see very clearly that a diagonal sequence $\{A_n^{(j)}\}_{n=0}^\infty$ converges at a much greater rate than any of the column sequences $\{A_p^{(s)}\}_{s=0}^\infty$, $p = 1, 2, \ldots$ .

Theorem 3.2 actually states that, whether $\lim_{t \to 0+} a(t)$ exists or not, $|A_n^{(j)} - A|$ tends to zero as $n \to \infty$ faster than $\exp(-\lambda n)$ for any $\lambda > 0$. By imposing suitable growth conditions on the $\hat{\beta}_s$ it is possible to show that $|A_n^{(j)} - A|$ tends to zero as $n \to \infty$ at worst like $\exp(-\kappa n^2)$ for some $\kappa > 0$. This is done in Theorem 3.3 below, following Lemma 3.2 that provides suitable bounds for $|A_n^{(j)} - A|$.

**Lemma 3.2.** *Assume that the conditions of Lemma* 3.1 *are satisfied.*

(i) *With* $\hat{\alpha}$ *and* $\hat{\beta}_s$ *as in Lemma* 3.1, *we have*

$$|A_n^{(j)} - A| \leqslant \hat{\alpha}\hat{\beta}_n \left(\prod_{i=1}^{n} \frac{1+\omega^i}{|1-\omega^{\delta+i-1}|}\right) t_j^{\Re\delta+n} \omega^{(\Re\delta)n+(n^2-n)/2}. \tag{3.13}$$

(ii) *If we assume, in addition, that* $A(y) \in F_\infty^{(1)}$ *with* $B(t) \in C^\infty[0,\hat{t}]$ *and* $\hat{t} \geqslant t_j$, *then* (3.13) *can be improved to read*

$$|A_n^{(j)} - A| \leqslant \hat{\alpha}\hat{\beta}_{n,3} \left(\prod_{i=1}^{n} \frac{1-\omega^i}{|1-\omega^{\delta+i-1}|}\right) t_j^{\Re\delta+n} \omega^{(\Re\delta)n+(n^2-n)/2}, \tag{3.14}$$

*where*

$$\hat{\beta}_{s,3} = \max_{0 \leqslant t \leqslant \hat{t}} |B^{(s)}(t)/s!|, \quad s = 0,1,2,\ldots, \tag{3.15}$$

*and* $\hat{\alpha}$ *is as in Lemma* 3.1.

**Proof.** Eq. (3.13) is obtained from (3.2) by letting $s = n$ on the right-hand side of the latter and by realizing that

$$\prod_{i=1}^{n}(1+\omega^{-n+i-1}) = \omega^{-(n^2+n)/2} \prod_{i=1}^{n}(1+\omega^i).$$

The proof of (3.14) is achieved by recalling that

$$D_n^{(j)}\{B(t)\} = B^{(n)}(\xi)/n! \quad \text{for some } \xi \in (t_{j+n}, t_j) \tag{3.16}$$

when $B(t) \in C^\infty[0,\hat{t}]$ and $\hat{t} \geqslant t_j$, and by substituting (3.16) in (3.3) and invoking (3.10) as well. □

**Theorem 3.3.** *Assume that the conditions of Lemma* 3.1 *are satisfied. Assume also that*

(i) $\hat{\beta}_{n,1}$ *defined as in* (3.5) *grows at worst like* $\exp(\gamma n^{1+\tau})$ *with increasing* $n$, *or that*

(ii) $B(t) \in C^\infty[0,\hat{t}]$ *for some* $\hat{t} \in (0, b^r]$ *and that* $\hat{\beta}_{n,3}$ *defined as in* (3.15) *grows at worst like* $\exp(\gamma n^{1+\tau})$ *with increasing* $n$,

*where* $\gamma > 0$ *and* $\tau \in (0,1)$ *are some constants. Then, for any* $\varepsilon > 0$ *such that* $\omega + \varepsilon < 1$, *there exists a positive integer* $n_0$ *for which*

$$|A_n^{(j)} - A| \leqslant (\omega+\varepsilon)^{n^2/2} \text{ when } n \geqslant n_0. \tag{3.17}$$

**Proof.** *Case* (i): By the fact that $\beta_s = \lim_{t\to 0+}([B(t) - \sum_{i=0}^{s-1} \beta_i t^i]/t^s)$, $s = 0,1,2,\ldots,$ and by (3.5), we have that $|\beta_s| \leqslant \hat{\beta}_{s,1}$, $s = 0,1,2,\ldots$ . Using this in (3.6), we obtain

$$\hat{\beta}_{n,2} \leqslant \left(\max_{t_l \in X} |B(t_l)|\right) \hat{t}^{-n} + \left(\max_{0 \leqslant i \leqslant n-1} \hat{\beta}_{i,1}\right) \sum_{i=1}^{n} \hat{t}^{-i}. \tag{3.18}$$

Finally, by the assumption that, with increasing $n, \hat{\beta}_{n,1}$ grows at most like $\exp(\gamma n^{1+\tau})$ for some $\gamma > 0$ and $\tau \in (0,1)$, (3.8) and (3.18) together imply that $\hat{\beta}_n$ grows as most like $\exp(\gamma' n^{1+\tau})$ for some $\gamma' > \gamma$. Combining this with (3.13) we obtain (3.17).

*Case* (ii): When $B(t)\in C^{\infty}[0,\hat{t}]$, the fact that $\sum_{i=0}^{\infty}\beta_i t^i$ represents $B(t)$ asymptotically for $t\to 0+$, implies that $\beta_i = B^{(i)}(0)/i!,\ i=0,1,\ldots,$ and hence, for $t\in[0,\hat{t}]$,

$$B(t)-\sum_{i=0}^{s-1}\beta_i t^i = \frac{B^{(s)}(\xi(t))}{s!}t^s \quad \text{for some } \xi(t)\in(0,t), \tag{3.19}$$

from which we have $\hat{\beta}_{s,1}\leqslant\hat{\beta}_{s,3}$, and also $|\beta_s|\leqslant\hat{\beta}_{s,3},\ s=0,1,2,\ldots$ . Using these facts in (3.6), we obtain, analogously to (3.18),

$$\hat{\beta}_{n,2}\leqslant\left(\max_{t_l\in X}|B(t_l)|\right)\hat{t}^{-n}+\left(\max_{0\leqslant i\leqslant n-1}\hat{\beta}_{i,3}\right)\sum_{i=1}^{n}\hat{t}^{-i}. \tag{3.20}$$

The rest of the proof can be completed as that of case (i). □

Note that the growth condition on $\hat{\beta}_{n,1}$ in case (i) and on $\hat{\beta}_{n,3}$ in case (ii) of Theorem 3.3 is very liberal and holds in most practical situations. Consequently, the result in (3.17) of Theorem 3.3 captures the true nature of the convergence of Process II. In particular, this growth condition on $\hat{\beta}_{n,3}$ is automatically satisfied when $B(t)$ is analytic on $[0,\hat{t}]$. In this case, $\hat{\beta}_{n,1}=\mathrm{O}(R^n)$ as $n\to\infty$ for some $R>0$.

Before we end this section we would also like to mention the convergence results that are relevant to the case in which $B(t)$ does *not* have an (infinite) asymptotic expansion for $t\to 0+$, but it satisfies

$$B(t)=\sum_{i=0}^{s-1}\beta_i t^i+\mathrm{O}(t^s) \quad \text{as } t\to 0+ \tag{3.21}$$

for some finite and fixed integer $s$. Thus, $A(y)$ in this case is *not* in $F^{(1)}$.

Theorem 3.1 is modified as follows: The asymptotic equivalence in (3.1) is valid for $n+\mu<s$ there. Otherwise, we have $A_n^{(j)}-A=\mathrm{O}(t_j^{\delta+s})$ as $j\to\infty$, as follows from (3.2). A less refined but inclusive result is as follows:

$$A_n^{(j)}-A=\mathrm{O}(\omega^{(\delta+\hat{n})j}) \quad \text{as } j\to\infty \quad \text{where } \hat{n}=\min(n,s). \tag{3.22}$$

Theorem 3.2 is modified as follows: (3.11) is replaced by

$$A_n^{(j)}-A=\mathrm{O}(\omega^{(\delta+s)n}) \quad \text{as } n\to\infty. \tag{3.23}$$

Thus, convergence takes place in Process II provided $\Re\delta+s>0$.

## 4. Application to the $D^{(1)}$-transformation for infinite integrals

Consider the function $f(x)$ that is integrable in $[a,T]$ for any $T>a\geqslant 0$. Assume that $f(x)$ has the asymptotic expansion

$$f(x)\sim x^{-\delta-1}\sum_{i=0}^{\infty}v_i x^{-i} \quad \text{as } x\to\infty, \quad v_0\neq 0, \tag{4.1}$$

for some $\delta$ that is in general complex and satisfies $\delta\neq 0,-1,-2,\ldots$ . Only when $\Re\delta>0$, $\int_a^{\infty}f(t)\,\mathrm{d}t$ exists as an improper integral and its value is $\lim_{x\to\infty}F(x)$, where

$$F(x)=\int_a^x f(t)\,\mathrm{d}t. \tag{4.2}$$

When $\Re\delta \leqslant 0$, $\lim_{x\to\infty} F(x)$ does not exist, and hence $\int_a^\infty f(t)\,\mathrm{d}t$ does not exist as an improper integral, but it has a Hadamard finite part (HFP) that is well defined, as is shown in Theorem 4.1 below. We are interested in computing the value of $\int_a^\infty f(t)\,\mathrm{d}t$ when $\Re\delta > 0$, and its HFP when $\Re\delta \leqslant 0$. The Levin–Sidi $D^{(1)}$-transformation is the appropriate extrapolation method through which this can be achieved very efficiently, as we show below. As will become clear, the $D^{(1)}$-transformation is a GREP$^{(1)}$, and the theory of the previous sections is valid for the $D^{(1)}$-transformation as this is applied to the integral $\int_a^\infty f(t)\,\mathrm{d}t$ with $f(x)$ as above.

In Theorem 4.1 below we derive the asymptotic expansion of $F(x)$ as $x \to \infty$.

**Theorem 4.1.** *With $f(x)$ as described above, $F(x)$ satisfies*

$$F(x) = I[f] + xf(x)g(x), \tag{4.3}$$

*where the function $g(x)$ has the asymptotic expansion*

$$g(x) \sim \sum_{i=0}^{\infty} \beta_i x^{-i} \quad \text{as } x \to \infty, \tag{4.4}$$

*for some constants $\beta_i$. Here $I[f] = \lim_{x\to\infty} F(x)$ when $\Re\delta > 0$ and $I[f]$ is the HFP of $\int_a^\infty f(t)\,\mathrm{d}t$ when $\Re\delta \leqslant 0$. In (4.4) $\beta_0 = -1/\delta$.*

*From (4.1) there exists $\hat{x} \in (a,\infty)$ such that $f(x) \neq 0$ for $\hat{x} \leqslant x < \infty$. If we assume, in addition, that $f(x)$ is in $C^\infty(a,\infty)$ and that its derivatives have asymptotic expansions for $x \to \infty$ that are obtained by differentiating (4.1) formally term by term, then the function $B(\xi) = g(\xi^{-1})$ is in $C^\infty[0,\hat{x}^{-1}]$.*

**Proof.** Let us first consider the case $\Re\delta > 0$. Then

$$F(x) = \int_a^\infty f(t)\,\mathrm{d}t - \int_x^\infty f(t)\,\mathrm{d}t, \tag{4.5}$$

since $\int_a^\infty f(t)\,\mathrm{d}t$ exists as an improper integral. Invoking (4.1), it can be shown that $\int_x^\infty f(t)\,\mathrm{d}t$ has the asymptotic expansion

$$\int_x^\infty f(t)\,\mathrm{d}t \sim x^{-\delta} \sum_{i=0}^{\infty} \frac{v_i}{\delta + i} x^{-i} \quad \text{as } x \to \infty. \tag{4.6}$$

Substituting (4.6) in (4.5), and using the fact that $\lim_{x\to\infty} F(x) = \int_a^\infty f(x)\,\mathrm{d}x$ when $\Re\delta > 0$, we obtain

$$F(x) \sim I[f] - x^{-\delta} \sum_{i=0}^{\infty} \frac{v_i}{\delta + i} x^{-i} \quad \text{as } x \to \infty. \tag{4.7}$$

Next, let us consider the case $\Re\delta \leqslant 0$. Let us now define

$$\hat{f}(x) = f(x) - \sum_{i=0}^{N-1} v_i x^{-\delta-i-1} \tag{4.8}$$

for some integer $N$ that satisfies $\Re\delta + N > 0$. Therefore, $\hat{f}(x) = O(x^{-\delta-N-1})$ as $x \to \infty$ so that $\int_x^\infty \hat{f}(t)\,dt$ exists as an improper integral. As a result, we can write for arbitrary $u > 0$

$$F(x) = \int_a^u f(t)\,dt + \int_u^x \left(\sum_{i=0}^{N-1} v_i t^{-\delta-i-1}\right) dt + \int_u^\infty \hat{f}(t)\,dt - \int_x^\infty \hat{f}(t)\,dt. \tag{4.9}$$

Similarly to the previous case, cf. (4.6), we have

$$\int_x^\infty \hat{f}(t)\,dt \sim \sum_{i=N}^{\infty} \frac{v_i}{\delta+i} x^{-\delta-i} \quad \text{as } x \to \infty. \tag{4.10}$$

Substituting (4.10) in (4.9), we obtain (4.7), with $I[f]$ this time given by

$$I[f] = \int_a^u f(t)\,dt + \sum_{i=0}^{N-1} \frac{v_i}{\delta+i} u^{-\delta-i} + \int_u^\infty \hat{f}(t)\,dt, \tag{4.11}$$

which is nothing but the HFP of the divergent integral $\int_a^\infty f(t)\,dt$.

Finally, (4.3) and (4.4) follow from (4.1) and (4.7), the $\beta_i$ in (4.4) being uniquely defined by

$$\left(\sum_{i=0}^{\infty} v_i x^{-i}\right)\left(\sum_{i=0}^{\infty} \beta_i x^{-i}\right) = -\sum_{i=0}^{\infty} \frac{v_i}{\delta+i} x^{-i} \quad \text{as } x \to \infty. \tag{4.12}$$

That $B(\xi) = g(\xi^{-1})$ is in $C^\infty[0,\hat{x}^{-1}]$ under the additional differentiability conditions on $f(x)$ follows from the fact that $B(\xi)$ satisfies the ordinary differential equation $\xi(d/d\xi)B + q(\xi)B = -1$ with $q(\xi) = \xi(d/d\xi)\log[\xi^{-1} f(\xi^{-1})] \sim \sum_{i=0}^\infty q_i \xi^i$ as $\xi \to 0+$, $q_0 = \delta \neq 0$, which is obtained by differentiating (4.3) and invoking (4.1). We leave the details to the reader. □

*Note*. From Theorem A.1 in the appendix it follows that if we denote by $\Phi(\delta)$ the convergent integral $\int_a^\infty f(t)\,dt$ when $\Re\delta > 0$, then $\Phi(\delta)$ is analytic in $\delta$ for $\Re\delta > 0$, and the HFP of the divergent integral $\int_a^\infty f(t)\,dt$ when $\Re\delta \leqslant 0$ is the analytic continuation of $\Phi(\delta)$ into the left half of the $\delta$-plane. Also this analytic continuation is a meromorphic function with a simple pole at $\delta = 0$ with residue $v_0$ and has additional simple poles at $\delta = -i$ with residue $v_i$ whenever $v_i \neq 0$, $i = 1, 2, \ldots$ .

Comparing the asymptotic expansion of $F(x)$ given in (4.3) and (4.4) with (1.1) and (1.2), and drawing the analogy $F(x) \leftrightarrow A(y)$, $x^{-1} \leftrightarrow y$, $xf(x) \leftrightarrow \phi(y)$, $r = 1$, and $I[f] \leftrightarrow A$, we realize that the function $F(y^{-1})$ is in $\mathrm{F}^{(1)}$, in general, and it is in $\mathrm{F}^{(1)}_\infty$ with $B(t) \in C^\infty[0,\hat{x}^{-1}]$ if $f(x)$ satisfies the additional differentiability conditions for $x \to \infty$, whether $\int_a^\infty f(t)\,dt$ exists as an improper integral or not. Thus $I[f]$ can be approximated very efficiently by applying GREP$^{(1)}$. But GREP$^{(1)}$ for this case is nothing but the $D^{(1)}$-transformation, and the relevant approximations $A_n^{(j)}$ to $I[f]$ are now denoted by $D_n^{(1,j)}$.

Specifically, the sequence of approximations $D_n^{(1,j)}$ to $I[f]$ generated by the $D^{(1)}$-transformation are defined by the linear equations

$$F(x_l) = D_n^{(1,j)} + x_l f(x_l) \sum_{i=0}^{n-1} \bar{\beta}_i x_l^{-i}, \quad j \leqslant l \leqslant j+n, \tag{4.13}$$

derived from (4.3) with (4.4), where the $x_l$ are picked such that $a < x_0 < x_1 < x_2 < \ldots$, and $\lim_{l\to\infty} x_l = \infty$. Here the finite integrals $F(x_l)$ are computed numerically. One good and economical way of doing this is by computing the integrals $\nabla F(x_l)=\int_{x_{l-1}}^{x_l} f(t)\,dt$, $l=0,1,\ldots,$ with $x_{-1}\equiv a$, by a fixed- and low-order Gauss–Legendre quadrature formula, and then by using the fact that $F(x_l)=\sum_{i=0}^{l}\nabla F(x_i)$.

A very useful property of the $D^{(1)}$-transformation that transpires from the defining equations in (4.13) is that the user need not know the exact value of $\delta$. The only input that he needs is a procedure for computing $f(x)$.

We note that the $D^{(1)}$-transformation was originally designed for convergent integrals. That it can be used for computing the HFP of divergent integrals was not known previously; it is based directly on Theorem 4.1.

So far the $x_i$ in (4.13) are arbitrary. If we pick them as in

$$x_i=x_0/\omega^i,\quad i=1,2,\ldots,\ \text{for some } x_0>a \text{ and some } \omega\in(0,1), \tag{4.14}$$

then all of the results of Section 2 pertaining to stability hold. Similarly, Theorems 3.1 and 3.2 hold. By imposing suitable growth conditions on the derivatives of $x^{\delta+1}f(x)$ at infinity, we can cause the constants $\hat{\beta}_{n,3}$ in (3.15) to grow as in case (ii) of Theorem 3.3. In particular, if $f(x)$ is nonvanishing for $x\geqslant\hat{x}$ and analytic on $[\hat{x},\infty)$ including $x=\infty$, then $B(\xi)$ is analytic on $[0,\hat{x}^{-1}]$, which implies that $\max_{0\leqslant\xi\leqslant\hat{x}^{-1}}|B^{(n)}(\xi)/n!|=O(R^n)$ as $n\to\infty$ for some $R>0$. Thus $\hat{\beta}_{n,3}$ grows at a rate much smaller than that considered in Theorem 3.3, ensuring that Theorem 3.3 holds as well.

With the $x_l$ fixed as in (4.14), the $D^{(1)}$-transformation can be modified by replacing the terms $x_lf(x_l)$ in the defining equations (4.13) by $\nabla F(x_l)=\int_{x_{l-1}}^{x_l} f(t)\,dt$. This is possible since $F(x_l)$ has the asymptotic expansion

$$F(x_l)\sim I[f]+\nabla F(x_l)\sum_{i=0}^{\infty}\beta_i' x_l^{-i}\quad \text{as } l\to\infty, \tag{4.15}$$

where $\beta_i'$ are some constants. The validity of (4.15) can be shown by combining

$$\nabla F(x_l)\sim x_l^{-\delta}\sum_{i=0}^{\infty}\frac{v_i}{\delta+i}(\omega^{-\delta-i}-1)x_l^{-i}\quad \text{as } l\to\infty, \tag{4.16}$$

that follows from (4.7), with (4.7) itself. This modification of the $D^{(1)}$-transformation, just as the $D^{(1)}$-transformation, does not require any knowledge of $\delta$.

When the exact value of $\delta$ is known, another modification of the $D^{(1)}$-transformation can be given in which the terms $x_lf(x_l)$ in (4.13) are now replaced by $x_l^{-\delta}$. The new set of defining equations are, of course, derived from the asymptotic expansion in (4.7).

For completeness we note that the $D^{(1)}$-transformation and its two modifications can be implemented via the $W$-algorithm by replacing $t_l$ and $a(t_l)$ by $x_l^{-1}$ and $F(x_l)$ respectively, and $\varphi(t_l)$ by $x_lf(x_l)$ for the $D^{(1)}$-transformation and by $\nabla F(x_l)$ and $x_l^{-\delta}$ for the modifications of the $D^{(1)}$-transformation.

Before we end this section we would like to recall that the $D^{(1)}$-transformation is the simplest form of the Levin–Sidi $D^{(m)}$-transformation that is defined via the linear systems of equations

$$F(x_l)=D_n^{(m,j)}+\sum_{k=1}^{m}x_l^kf^{(k-1)}(x_l)\sum_{i=0}^{n_k-1}\bar{\beta}_{ki}x_l^{-i},\quad j\leqslant l\leqslant j+N, \tag{4.17}$$

Table 1a
Numerical results for the integral $\int_1^\infty f(t)\,dt$ of Example 5.1 with $\delta = 0.5 + 10\,i$, obtained via the $D^{(1)}$-transformation with $x_l = 2^{l+1},\ l = 0, 1, \ldots$. Here $\hat{\Gamma}_n \equiv \hat{\Gamma}_n(\delta; \omega)$

| $n$ | $x_n$ | $\lvert F(x_n) - 1\rvert$ | $\lvert D_n^{(1,0)} - 1\rvert$ | $\Gamma_n^{(0)}$ | $\lvert \Gamma_n^{(0)} - \hat{\Gamma}_n\rvert / \hat{\Gamma}_n$ |
|---|---|---|---|---|---|
| 1 | 4.00D+00 | 9.43D−01 | 1.54D−01 | 3.00D+00 | 7.11D−02 |
| 2 | 8.00D+00 | 8.00D−01 | 5.23D−02 | 5.92D+00 | 1.72D−01 |
| 3 | 1.60D+01 | 6.29D−01 | 8.58D−03 | 7.70D+00 | 1.21D−01 |
| 4 | 3.20D+01 | 4.71D−01 | 7.21D−04 | 8.58D+00 | 6.89D−02 |
| 5 | 6.40D+01 | 3.43D−01 | 3.10D−05 | 9.00D+00 | 3.65D−02 |
| 6 | 1.28D+02 | 2.46D−01 | 6.76D−07 | 9.20D+00 | 1.88D−02 |
| 7 | 2.56D+02 | 1.75D−01 | 7.41D−09 | 9.30D+00 | 9.51D−03 |
| 8 | 5.12D+02 | 1.25D−01 | 4.08D−11 | 9.35D+00 | 4.79D−03 |
| 9 | 1.02D+03 | 8.82D−02 | 1.12D−13 | 9.38D+00 | 2.40D−03 |
| 10 | 2.05D+03 | 6.24D−02 | 1.55D−16 | 9.39D+00 | 1.20D−03 |
| 11 | 4.10D+03 | 4.42D−02 | 1.07D−19 | 9.39D+00 | 6.02D−04 |
| 12 | 8.19D+03 | 3.12D−02 | 3.70D−23 | 9.40D+00 | 3.01D−04 |
| 13 | 1.64D+04 | 2.21D−02 | 6.38D−27 | 9.40D+00 | 1.51D−04 |
| 14 | 3.28D+04 | 1.56D−02 | 5.51D−31 | 9.40D+00 | 7.53D−05 |
| 15 | 6.55D+04 | 1.10D−02 | 1.84D−33 | 9.40D+00 | 3.77D−05 |
| 16 | 1.31D+05 | 7.81D−03 | 8.84D−34 | 9.40D+00 | 1.88D−05 |
| 17 | 2.62D+05 | 5.52D−03 | 7.72D−34 | 9.40D+00 | 9.42D−06 |
| 18 | 5.24D+05 | 3.91D−03 | 6.39D−35 | 9.40D+00 | 4.71D−06 |
| 19 | 1.05D+06 | 2.76D−03 | 3.77D−34 | 9.40D+00 | 2.35D−06 |
| 20 | 2.10D+06 | 1.95D−03 | 5.26D−34 | 9.40D+00 | 1.18D−06 |

where $D_n^{(m,j)}$ is the approximation to $I[f]$ and $\beta_{ki}$ are additional unknowns, $n$ stands for the integer vector $(n_1, n_2, \ldots, n_m)$, and $N = \sum_{k=1}^m n_k$. (The definition of the $D^{(m)}$-transformation through (4.17) is slightly different from the original definition in [4] and more user-friendly as well.) Note that the only input necessary for this transformation is $f^{(i)}(x),\ i = 0, 1, \ldots, m-1$. The $D^{(m)}$-transformation has proved to be one of the most effective acceleration methods for computing infinite-range integrals of varying degrees of complexity. Although it has been mainly used to accelerate the convergence of infinite-range convergent integrals, numerical experiments done by the author have shown that the $D^{(m)}$-transformation with $m \geqslant 2$ is capable of producing very good approximations to the HFPs of divergent integrals that can be much more complicated than the ones treated in this section. In the next section we provide examples to which the $D^{(2)}$-transformation is applicable but the $D^{(1)}$-transformation is not.

## 5. Numerical examples

We now give two sets of examples to which the $D^{(1)}$- and $D^{(2)}$-transformations are applicable. To keep things simple we have picked the functions $f(x)$ to be the derivatives of some easily computable functions so that $F(x) = \int_a^x f(t)\,dt$ is known analytically. The computations for this section were carried out in quadruple precision arithmetic.

Table 1b
Numerical results for the integral $\int_1^\infty f(t)\,\mathrm{d}t$ of Example 5.1 with $\delta = 10\,\mathrm{i}$, obtained via the $D^{(1)}$-transformation with $x_l = 2^{l+1},\ l = 0, 1, \ldots$ . Here $\hat{\Gamma}_n \equiv \hat{\Gamma}_n(\delta;\omega)$

| $n$ | $x_n$ | $\lvert F(x_n) - 1\rvert$ | $\lvert D_n^{(1,0)} - 1\rvert$ | $\Gamma_n^{(0)}$ | $\lvert\Gamma_n^{(0)} - \hat{\Gamma}_n\rvert/\hat{\Gamma}_n$ |
|---|---|---|---|---|---|
| 1 | 4.00D+00 | 1.33D+00 | 1.82D−01 | 2.98D+00 | 5.20D−02 |
| 2 | 8.00D+00 | 1.60D+00 | 1.11D−01 | 8.27D+00 | 1.82D−01 |
| 3 | 1.60D+01 | 1.78D+00 | 2.82D−02 | 1.25D+01 | 1.64D−01 |
| 4 | 3.20D+01 | 1.88D+00 | 3.43D−03 | 1.47D+01 | 9.60D−02 |
| 5 | 6.40D+01 | 1.94D+00 | 2.10D−04 | 1.57D+01 | 5.12D−02 |
| 6 | 1.28D+02 | 1.97D+00 | 6.51D−06 | 1.62D+01 | 2.64D−02 |
| 7 | 2.56D+02 | 1.98D+00 | 1.01D−07 | 1.65D+01 | 1.34D−02 |
| 8 | 5.12D+02 | 1.99D+00 | 7.88D−10 | 1.66D+01 | 6.74D−03 |
| 9 | 1.02D+03 | 2.00D+00 | 3.08D−12 | 1.66D+01 | 3.38D−03 |
| 10 | 2.05D+03 | 2.00D+00 | 6.00D−15 | 1.67D+01 | 1.69D−03 |
| 11 | 4.10D+03 | 2.00D−00 | 5.86D−18 | 1.67D+01 | 8.47D−04 |
| 12 | 8.19D+03 | 2.00D+00 | 2.86D−21 | 1.67D+01 | 4.24D−04 |
| 13 | 1.64D+04 | 2.00D+00 | 6.99D−25 | 1.67D+01 | 2.12D−04 |
| 14 | 3.28D+04 | 2.00D+00 | 8.53D−29 | 1.67D+01 | 1.06D−04 |
| 15 | 6.55D+04 | 2.00D+00 | 3.27D−31 | 1.67D+01 | 5.30D−05 |
| 16 | 1.31D+05 | 2.00D+00 | 6.79D−32 | 1.67D+01 | 2.65D−05 |
| 17 | 2.62D+05 | 2.00D+00 | 2.24D−31 | 1.67D+01 | 1.33D−05 |
| 18 | 5.24D+05 | 2.00D+00 | 2.85D−31 | 1.67D+01 | 6.63D−06 |
| 19 | 1.05D+06 | 2.00D+00 | 3.00D−32 | 1.67D+01 | 3.31D−06 |
| 20 | 2.10D+06 | 2.00D+00 | 4.83D−31 | 1.67D+01 | 1.66D−06 |

**Example 5.1.** Let us consider the integrals $\int_a^\infty f(t)\,\mathrm{d}t$ with

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}[x^{-\delta}v(x)] = x^{-\delta-1}[-\delta v(x) + xv'(x)], \tag{5.1}$$

where $v(x)$ is an arbitrary differentiable function that has an asymptotic expansion of the form

$$v(x) \sim \sum_{i=0}^{\infty} v_i x^{-i} \quad \text{as } x \to \infty,\ v_0 \neq 0, \tag{5.2}$$

and $v'(x)$ has an asymptotic expansion obtained by differentiating (5.2) term by term. From (5.1) we immediately have

$$F(x) = \int_a^x f(t)\,\mathrm{d}t = x^{-\delta}v(x) - a^{-\delta}v(a). \tag{5.3}$$

Furthermore, $f(x)$ satisfies all the conditions of Theorem 4.1 and thus we have

$$I[f] = -a^{-\delta}v(a) \quad \text{for } \delta \neq 0, -1, -2, \ldots\ . \tag{5.4}$$

In the computations that we report below we took $a = 1$ and $v(x) = -2x/(1+x)$, so that $I[f] = 1$ for all values of $\delta \neq 0, -1, -2, \ldots$ .

In Tables 1a–1c we give $x_n$, $\lvert F(x_n) - 1\rvert$, $\lvert D_n^{(1,0)} - 1\rvert$, $\Gamma_n^{(0)}$, and $\lvert\Gamma_n^{(0)} - \hat{\Gamma}_n(\delta;\omega)\rvert/\hat{\Gamma}_n(\delta;\omega)$, for $\delta = 0.5 + 10\,\mathrm{i}$, $\delta = 10\,\mathrm{i}$, and $\delta = -0.5$ respectively, obtained with $\omega = 0.5$. In the first case the integral $\int_a^\infty f(t)\,\mathrm{d}t$ exists as an improper integral, while in the second and third cases it diverges with

Table 1c
Numerical results for the integral $\int_1^\infty f(t)\,\mathrm{d}t$ of Example 5.1 with $\delta=-0.5$, obtained via the $D^{(1)}$-transformation with $x_l=2^{l+1},\ l=0,1,\ldots$. Here $\hat{\Gamma}_n\equiv\hat{\Gamma}_n(\delta;\omega)$

| $n$ | $x_n$ | $\lvert F(x_n)-1\rvert$ | $\lvert D_n^{(1,0)}-1\rvert$ | $\Gamma_n^{(0)}$ | $\lvert\Gamma_n^{(0)}-\hat{\Gamma}_n\rvert/\hat{\Gamma}_n$ |
|---|---|---|---|---|---|
| 1 | 4.00D+00 | 1.89D+00 | 1.08D+00 | 5.70D+00 | 2.20D−02 |
| 2 | 8.00D+00 | 3.20D+00 | 3.72D+00 | 5.05D+01 | 4.86D−01 |
| 3 | 1.60D+01 | 5.03D+00 | 1.43D+00 | 6.15D+01 | 1.35D−01 |
| 4 | 3.20D+01 | 7.53D+00 | 4.10D−01 | 8.71D+01 | 1.43D−01 |
| 5 | 6.40D+01 | 1.10D+01 | 6.05D−02 | 1.11D+02 | 8.23D−02 |
| 6 | 1.28D+02 | 1.58D+01 | 4.27D−03 | 1.27D+02 | 4.16D−02 |
| 7 | 2.56D+02 | 2.25D+01 | 1.46D−04 | 1.36D+02 | 2.11D−02 |
| 8 | 5.12D+02 | 3.19D+01 | 2.47D−06 | 1.40D+02 | 1.08D−02 |
| 9 | 1.02D+03 | 4.52D+01 | 2.07D−08 | 1.43D+02 | 5.45D−03 |
| 10 | 2.05D+03 | 6.39D+01 | 8.65D−11 | 1.44D+02 | 2.74D−03 |
| 11 | 4.10D+03 | 9.05D+01 | 1.80D−13 | 1.44D+02 | 1.38D−03 |
| 12 | 8.19D+03 | 1.28D+02 | 1.87D−16 | 1.45D+02 | 6.89D−04 |
| 13 | 1.64D+04 | 1.81D+02 | 9.70D−20 | 1.45D+02 | 3.45D−04 |
| 14 | 3.28D+04 | 2.56D+02 | 2.51D−23 | 1.45D+02 | 1.73D−04 |
| 15 | 6.55D+04 | 3.62D+02 | 3.26D−27 | 1.45D+02 | 8.63D−05 |
| 16 | 1.31D+05 | 5.12D+02 | 2.80D−29 | 1.45D+02 | 4.32D−05 |
| 17 | 2.62D+05 | 7.24D+02 | 5.47D−29 | 1.45D+02 | 2.16D−05 |
| 18 | 5.24D+05 | 1.02D+03 | 3.23D−29 | 1.45D+02 | 1.08D−05 |
| 19 | 1.05D+06 | 1.45D+03 | 6.41D−30 | 1.45D+02 | 5.39D−06 |
| 20 | 2.10D+06 | 2.05D+03 | 6.81D−29 | 1.45D+02 | 2.70D−06 |

HFP equal to 1. Although what we report here is only the results obtained by using the (original) $D^{(1)}$-transformation, the two modifications perform almost identically.

When $\delta=0.5+10\,\mathrm{i}$, $\{F(x_n)\}$ converges to $I[f]=1$. When $\delta=10\,\mathrm{i}$, $\{F(x_n)\}$ does not converge but is bounded. Finally, when $\delta=-0.5$, $\{F(x_n)\}$ diverges and is unbounded.

**Example 5.2.** Let us now consider the integrals $\int_a^\infty f(t)\,\mathrm{d}t$ with

$$f(x)=\frac{\mathrm{d}}{\mathrm{d}x}[x^{-\delta}\log(1+x)v(x)], \tag{5.5}$$

where $v(x)$ is exactly as in Example 5.1. Thus we have

$$F(x)=\int_a^x f(t)\,\mathrm{d}t=x^{-\delta}\log(1+x)v(x)-a^{-\delta}\log(1+a)v(a). \tag{5.6}$$

Clearly, $\lim_{x\to\infty}F(x)$ exists only when $\Re\delta>0$, and

$$I[f]=-a^{-\delta}\log(1+a)v(a)\ \text{for}\ \delta\neq 0,-1,-2,\ldots, \tag{5.7}$$

is $\lim_{x\to\infty}F(x)$ when $\Re\delta>0$ and the antilimit of $F(x)$ for $x\to\infty$ otherwise. (The antilimit is the HFP of $\int_a^\infty f(t)\,\mathrm{d}t$ in this example too.)

By expanding $F(x)$ given in (5.6) for $x\to\infty$, we realize that its asymptotic behavior is more complicated than that given in Theorem 4.1, cf. especially (4.7). Therefore, we do not expect the $D^{(1)}$-transformation to be effective in this example. The $D^{(2)}$-transformation, with the $x_l$ picked as in (4.14), is very effective, however.

Table 2
Numerical results for the integral $\int_1^\infty f(t)\,dt$ of Example 5.2 with $\delta=0.5$ and $\delta=-0.5$, obtained via the $D^{(2)}$-transformation with $x_l=2^{l+1},\ l=0,1,\ldots$

| $v$ | $x_{2v}$ | $\delta=0.5$ $\lvert F(x_{2v})-1\rvert$ | $\lvert D^{(2,0)}_{(v,v)}-1\rvert$ | $\delta=-0.5$ $\lvert F(x_{2v})-1\rvert$ | $\lvert D^{(2,0)}_{(v,v)}-1\rvert$ |
|---|---|---|---|---|---|
| 1 | 8.00D+00 | 4.98D−01 | 4.03D−01 | 3.98D+00 | 7.54D−01 |
| 2 | 3.20D+01 | 4.32D−01 | 1.97D−02 | 1.38D+01 | 3.98D+00 |
| 3 | 1.28D+02 | 3.07D−01 | 9.28D−04 | 3.94D+01 | 3.38D+00 |
| 4 | 5.12D+02 | 1.99D−01 | 3.59D−06 | 1.02D+02 | 5.10D+00 |
| 5 | 2.05D+03 | 1.21D−01 | 8.48D−09 | 2.49D+02 | 1.01D−01 |
| 6 | 8.19D+03 | 7.18D−02 | 2.52D−12 | 5.88D+02 | 1.10D−03 |
| 7 | 3.28D+04 | 4.14D−02 | 3.57D−16 | 1.36D+03 | 2.90D−06 |
| 8 | 1.31D+05 | 2.35D−02 | 7.01D−21 | 3.08D+03 | 1.89D−09 |
| 9 | 5.24D+05 | 1.31D−02 | 6.13D−26 | 6.88D+03 | 3.07D−13 |
| 10 | 2.10D+06 | 7.25D−03 | 7.45D−32 | 1.52D+04 | 1.24D−17 |
| 11 | 8.39D+06 | 3.97D−03 | 6.77D−33 | 3.33D+04 | 1.25D−22 |
| 12 | 3.36D+07 | 2.16D−03 | 1.18D−32 | 7.24D+04 | 3.68D−25 |
| 13 | 1.34D+08 | 1.17D−03 | 4.65D−33 | 1.56D+05 | 6.26D−26 |
| 14 | 5.37D+08 | 6.26D−04 | 1.01D−33 | 3.36D+05 | 1.87D−24 |

In the computations that we report below we took $a=1$ and $v(x)=Kx/(1+x)$, where $K=-2/\log 2$, so that $I[f]=1$ for all values of $\delta\neq 0,-1,-2,\ldots$ .

In Table 2 we give the $x_{2v}$, $|F(x_{2v})-1|$, and $|D^{(2,0)}_{(v,v)}-1|$ for $\delta=0.5$ and $\delta=-0.5$, obtained with $\omega=0.5$ in (4.14).

The sequence $\{F(x_l)\}$ converges for $\delta=0.5$ and diverges and is unbounded for $\delta=-0.5$. The computations also show that the extrapolation procedure is numerically very stable even though we have no theory of stability for the $D^{(2)}$-transformation.

## Acknowledgements

The author is grateful to one of the referees for his remarks on the stability issue and for drawing his attention to the reference [2].

## Appendix. HFP as analytic continuation

**Theorem A.1.** *Let $H(x)$ be defined on $[a,\infty)$ for some $a>0$ and satisfy*

$$H(x)=\sum_{i=0}^{N-1} v_i x^{-\sigma_i}+O(x^{-\sigma_N})\quad \text{as } x\to\infty, \tag{A.1}$$

*where $N$ is some positive integer and*

$$v_i\neq 0,\quad i=0,1,\ldots,\quad \text{and}\quad 0=\sigma_0<\Re\sigma_1<\Re\sigma_2<\cdots . \tag{A.2}$$

*Define the function* $\Phi(\delta)$ *via the convergent integral*

$$\Phi(\delta) = \int_a^\infty x^{-\delta-1} H(x)\,\mathrm{d}x, \quad \Re\delta > 0. \tag{A.3}$$

*Then the following assertions are true.*

(i) $\Phi(\delta)$ *is an analytic function of* $\delta$ *for* $\Re\delta > 0$.

(ii) $\Phi(\delta)$ *has an analytic continuation for* $\Re\delta > -\Re\sigma_N$ *that has simple poles at* $\delta = 0, -\sigma_1, -\sigma_2, \ldots, -\sigma_{N-1}$, *with corresponding residues* $v_0, v_1, v_2, \ldots, v_{N-1}$.

(iii) *When* $-\Re\sigma_N < \Re\delta \leqslant 0$ *but* $\delta \neq 0, -\sigma_1, -\sigma_2, \ldots$, *the HFP of the divergent integral* $\int_a^\infty x^{-\delta-1} H(x)\,\mathrm{d}x$ *is nothing but the analytic continuation of* $\Phi(\delta)$.

**Proof.** From (A.1) and (A.2) it is clear that $H(x) = \mathrm{O}(1)$ as $x \to \infty$. Making the change of variable of integration $x = a\mathrm{e}^t$, (A.3) can be expressed via the Laplace transform

$$\Phi(\delta) = a^{-\delta} \int_0^\infty \mathrm{e}^{-\delta t} H(a\mathrm{e}^t)\,\mathrm{d}t, \quad \Re\delta > 0. \tag{A.4}$$

Applying the standard theory of Laplace transforms to (A.4), see, e.g., [1, p. 265], part (i) of the theorem follows.

Substituting (A.1) into (A.3) and again making the change of variables $x = a\mathrm{e}^t$, we next obtain

$$\Phi(\delta) = \sum_{i=0}^{N-1} v_i \frac{a^{-\delta-\sigma_i}}{\delta+\sigma_i} + a^{-(\delta+\sigma_N)} \int_0^\infty \mathrm{e}^{-(\delta+\sigma_N)t} R_N(a\mathrm{e}^t)\,\mathrm{d}t, \quad \Re\delta > 0, \tag{A.5}$$

where we have defined

$$R_N(x) = \left[ H(x) - \sum_{i=0}^{N-1} v_i x^{-\sigma_i} \right] x^{\sigma_N}. \tag{A.6}$$

Now the summation on the right hand side of (A.5) is analytic everywhere in the $\delta$-plane except at $\delta = -\sigma_i$ where it has a simple pole with residue $v_i$, $i = 0, 1, \ldots, N-1$. As for the integral on the right-hand side of (A.5), again, by the standard theory of Laplace transforms, and also by the fact that $R_N(x) = \mathrm{O}(1)$ as $x \to \infty$, it is an analytic function of $\delta$ for $\Re\delta > -\Re\sigma_N$. Thus the right-hand side of (A.5) is a meromorphic function of $\delta$ for $\Re\delta > -\Re\sigma_N$, while its left-hand side is analytic for $\Re\delta > 0$. Consequently, the right-hand side of (A.5) is the analytic continuation of $\Phi(\delta)$ defined by (A.3) to the half plane $\Re\delta > -\Re\sigma_N$. This proves part (ii).

Part (iii) follows by realizing that the right-hand side of (A.5) is also the HFP of the divergent integral $\int_a^\infty x^{-\delta-1} H(x)\,\mathrm{d}x$ when $-\Re\sigma_N < \Re\delta \leqslant 0$ but $\delta \neq 0, -\sigma_1, -\sigma_2, \ldots$ . □

Obviously, the integrals of Example 5.1 in Section 5 of the present work are covered by this theorem with $\sigma_i = i$, $i = 0, 1, 2, \ldots$ .

Note that Theorem A.1 can be generalized in a straightforward manner to the case in which $v_i$ in (A.1) is replaced by a polynomial in $\log x$ of some arbitrary order $k_i - 1$ for each $i$ and the condition on the $\sigma_i$ in (A.2) is replaced by the weaker condition $0 = \sigma_0 \leqslant \Re\sigma_1 \leqslant \Re\sigma_2 \leqslant \cdots \leqslant \Re\sigma_{N-1} < \Re\sigma_N$. Under these conditions parts (i) and (iii) of the theorem do not change. Part (ii) is modified in that the analytic continuation of $\Phi(\delta)$ now has poles at $0, -\sigma_1, \ldots, -\sigma_{N-1}$, of respective multiplicities $k_0, k_1, \ldots, k_{N-1}$. The integrals of Example 5.2 in Section 5 are covered by this generalization of Theorem A.1.

**Remark.** A result similar to Theorem A.1 is proved in [2, Chapter 3, pp. 48–49] for integrals of the form $\int_0^\infty x^\lambda \varphi(x)\,\mathrm{d}x$, where $\varphi \in C^{(-\infty,+\infty)}$ and has bounded support. Our result is different from that of [2] in that $H(x)$ has an asymptotic expansion in arbitrary powers of $x$ as $x \to \infty$ and is not required to be differentiable at $x = \infty$, whereas the analogous $\varphi(x)$ of [2] is assumed to be differentiable for all finite $x$. (Note that $x = \infty$ in Theorem A.1 is analogous to $x = 0$ in [2].) Besides, our conditions on $H(x)$ can be generalized further as mentioned in the previous paragraph.

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