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THE RICHARDSON EXTRAPOLATION PROCESS WITH A HARMONIC SEQUENCE OF COLLOCATION POINTS*

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Abstract. Let $A(y) \sim A + \sum_{k=1}^{\infty} \alpha_k y^{\sigma_k}$ as $y \to 0+$, where y is a discrete or continuous variable. Assume that σ_k are known numbers that may be complex in general and that $A(y)$ is known for $y \in (0, b]$ for some $b > 0$. The aim is to find or approximate A, the limit or antilimit of $A(y)$ as $y \to 0+$. One very effective way of approximating A is by the Richardson extrapolation process that is defined via the linear systems $A(y_l) = A_n^{(j)} + \sum_{k=1}^{n} \overline{\alpha}_k y_l^{\sigma_k},\ j \le l \le j+n$. Here $A_n^{(j)}$ are approximations to A and $\overline{\alpha}_k$ are additional unknowns. The y_l are picked such that $y_0 > y_1 > y_2 > \cdots > 0$ and $\lim_{l\to\infty} y_l = 0$. In this paper we give a detailed analysis of the convergence and stability of the column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$ with n fixed, when $y_l = c/(l+\eta)^q$ for some positive c, η, and q. Specifically, we prove that convergence takes place as $j \to \infty$ and give the precise rate at which it does. We also prove that the process is unstable and quantify its instability asymptotically. This instability may be dealt with numerically by using high-precision floating-point arithmetic.

Key words. Richardson extrapolation, harmonic sequence of collocation points, convergence, stability, asymptotic expansions, numerical quadrature

AMS subject classifications. 40A25, 41A60, 65B05, 65B10, 65D30

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1. Introduction.

1.1. Brief review of the Richardson extrapolation process. Let a function $A(y)$ be known and hence computable for $y \in (0, b]$ with some $b > 0$, the variable y being continuous or discrete. Assume, furthermore, that $A(y)$ has an asymptotic expansion of the form

$$A(y) \sim A + \sum_{k=1}^{\infty} \alpha_k y^{\sigma_k} \quad \text{as } y \to 0+, \tag{1.1}$$

where σ_k are known scalars satisfying

$$\sigma_k \neq 0,\ k = 1, 2, \ldots; \quad \Re\sigma_1 < \Re\sigma_2 < \cdots; \quad \lim_{k\to\infty} \Re\sigma_k = +\infty, \tag{1.2}$$

and A and $\alpha_k,\ k = 1, 2, \ldots$, are constants independent of y that are not necessarily known.

From (1.1) and (1.2) it is clear that $A = \lim_{y\to 0+} A(y)$ when this limit exists. When $\lim_{y\to 0+} A(y)$ does not exist, A is the antilimit of $A(y)$ for $y \to 0+$, and in this case $\Re\sigma_1 \le 0$ necessarily. In any case, A can be approximated very effectively by the Richardson extrapolation process (REP) that is defined via the linear systems of equations

$$A(y_l) = A_n^{(j)} + \sum_{k=1}^{n} \overline{\alpha}_k y_l^{\sigma_k}, \quad j \le l \le j+n, \tag{1.3}$$

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with the y_l picked to satisfy

$$b \geq y_0 > y_1 > y_2 > \cdots > 0 \quad \text{and} \quad \lim_{l\to\infty} y_l = 0. \tag{1.4}$$

Here $A_n^{(j)}$ are the approximations to A and the $\overline{\alpha}_k$ are additional (auxiliary) unknowns.

In this paper we give a detailed analytical study of the convergence and stability of the so-called column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$, with n fixed, when we pick $y_l = c/(l+\eta)^q$, $l = 0, 1, \ldots,$ for some positive constants c, η, and q. That is, we analyze the properties of $A_n^{(j)}$ as $j \to \infty$. Later in this section we shall give a detailed discussion of why the application of REP with this choice of the y_l is of practical importance and deserves to be studied seriously.

As is known, $A_n^{(j)}$ can be expressed in the form

$$A_n^{(j)} = \sum_{i=0}^{n} \gamma_{ni}^{(j)} A(y_{j+i}) \tag{1.5}$$

for some $\gamma_{ni}^{(j)}$ that depend only on the $y_l^{\sigma_k}$ and that satisfy the linear system

$$\sum_{i=0}^{n} \gamma_{ni}^{(j)} = 1 \quad \text{and} \quad \sum_{i=0}^{n} \gamma_{ni}^{(j)} y_{j+i}^{\sigma_k} = 0, \quad k = 1, \ldots, n. \tag{1.6}$$

The quantity $\Gamma_n^{(j)}$ defined by

$$\Gamma_n^{(j)} = \sum_{i=0}^{n} |\gamma_{ni}^{(j)}| \tag{1.7}$$

is a very important constant that controls the numerical stability of $A_n^{(j)}$. It gives a precise measure of sensitivity of $A_n^{(j)}$ to errors (roundoff or other) in the $A(y_l)$. Indeed, practically speaking, the difference between $A_n^{(j)}$ and its computed value is of the order of the largest of the absolute errors in $A(y_l)$, $j \leq l \leq j+n$, multiplied by $\Gamma_n^{(j)}$. Therefore, there is great value to obtaining $\Gamma_n^{(j)}$ or a reasonable estimate of it simultaneously with $A_n^{(j)}$ in order to assess the accuracy of the latter. Obviously, $\Gamma_n^{(j)} \geq 1$ for all j and n, and we would like $\Gamma_n^{(j)}$ not to grow to infinity as $j \to \infty$ or $n \to \infty$. We shall say more on this topic in section 4.

For the general framework above, we refer the reader to Schneider [Sc] and Sidi [Si7]. In particular, the equations in (1.6) were originally given in [Sc]. When the y_l are arbitrary, the $A_n^{(j)}$ can be computed either by direct solution of the linear system in (1.3) or by recursive means. The first recursive algorithm for this problem was derived in [Sc] and, independently and by different methods, in Håvie [H] and in Brezinski [B]. This algorithm was named the E-algorithm in [B]. Another recursive algorithm, the FS-algorithm, was derived more recently in Ford and Sidi [FS]. The FS-algorithm turns out to be more economical computationally than the E-algorithm. Finally, when $\sigma_{k+1} - \sigma_k = d$, $k = 1, 2, \ldots,$ for some fixed d, the W-algorithm of Sidi [Si2] is a most efficient means of computing the $A_n^{(j)}$ in the presence of arbitrary y_l's.

When the y_l are picked such that

$$y_l = y_0 \omega^l, \quad l = 0, 1, \ldots, \quad \text{for some } y_0 \in (0, b] \text{ and } \omega \in (0, 1), \tag{1.8}$$

the $A_n^{(j)}$ can be computed very efficiently by the following algorithm due to Bulirsch and Stoer [BS1]:

$$\begin{aligned} A_0^{(j)} &= A(y_j), \quad j = 0, 1, \dots, \\ A_n^{(j)} &= \frac{A_{n-1}^{(j+1)} - c_n A_{n-1}^{(j)}}{1 - c_n}, \quad j = 0, 1, \dots, \ n = 1, 2, \dots, \end{aligned} \tag{1.9}$$

where we have defined

$$c_n = \omega^{\sigma_n}, \quad n = 1, 2, \dots \ . \tag{1.10}$$

This is also the most extensively studied case of REP. In particular, it is known that, with n fixed,

$$A_n^{(j)} - A \sim \left(\prod_{i=1}^{n} \frac{c_{n+\mu} - c_i}{1 - c_i} \right) \alpha_{n+\mu} y_j^{\sigma_{n+\mu}} \quad \text{as } j \to \infty, \tag{1.11}$$

where $\alpha_{n+\mu}$ is the first nonzero α_{n+i} with $i \geq 1$. As is obvious from (1.11), the sequence $\{A_n^{(j)}\}_{j=0}^{\infty}$ converges (to A) when $\Re\sigma_{n+\mu} > 0$ more quickly than $A(y_l)$, $l = j, j+1, \dots, j+n$, which are used in constructing $A_n^{(j)}$; see (1.3). In addition, the extrapolation process is stable as $\Gamma_n^{(j)}$ is independent of j and hence does not grow with increasing j. We actually have

$$\Gamma_n^{(j)} \leq \prod_{i=1}^{n} \frac{1 + |c_i|}{|1 - c_i|} \quad \text{for all } j \text{ and } n. \tag{1.12}$$

(In (1.12) equality holds when c_i all have the same phase.) Imposing the additional condition that $\Re\sigma_{k+1} - \Re\sigma_k \geq d > 0$, $k = 1, 2, \dots$, for some fixed d, it is possible to prove very powerful convergence and stability results for $A_n^{(j)}$ and $\Gamma_n^{(j)}$ as $n \to \infty$ with j fixed. As we shall not be dealing with this limiting process in this paper, we skip these results and refer the reader to [BS1] for the case involving real σ_k's and to Sidi [Si9] for the case involving complex σ_k's in general. (The functions $A(y)$ treated in [Si9] are actually more general than the one in (1.1).)

It is worth noting that the result in (1.11) remains valid also when the y_l satisfy $\lim_{l\to\infty}(y_{l+1}/y_l) = \omega \in (0, 1)$ instead of (1.8). In this case we also have

$$\lim_{j\to\infty} \Gamma_n^{(j)} \leq \prod_{i=1}^{n} \frac{1 + |c_i|}{|1 - c_i|} \tag{1.13}$$

instead of (1.12). These follow from Theorems 2.2 and 2.4 in [Si7].

Now sequences $\{y_l\}$ that satisfy (1.8) converge to 0 exponentially. This convergence is especially quick when ω is not too close to 1. In most practical situations $\omega = 1/2$ is the common choice. Again in most cases of interest computing $A(y)$ for very small values of y either is very costly or entails a great loss of significance in finite-precision arithmetic. To cope with this problem the extrapolation process can also be carried out with a sequence $\{y_l\}$ that tends to zero less quickly than exponentially. A very common choice used in many instances has been the harmonic sequence $y_l = c/(l+1)$, $l = 0, 1, 2, \dots$. Even though the process is not stable numerically with this sequence of y_l's, it can be used effectively with high-precision arithmetic, i.e., when $A(y_l)$ are computed with significantly high precision.

An area in which this turns out to be important is that of numerical integration of regular or singular functions over a hypercube or hypersimplex in $\mathbb{R}^N$. Here y corresponds to the integration stepsize, $A(y)$ to the (offset) trapezoidal rule approximation, and A to the value of the integral, and the expansion in (1.1) is the generalized Euler–Maclaurin expansion for the corresponding integral. For example, for the one-dimensional integral $I = \int_0^1 x^{\mu}(1-x)^{\nu} f(x)\,dx$, where $\Re\mu > -1$ and $\Re\nu > -1$ and $f \in C^{\infty}[0,1]$, we have, with $h = 1/n$, where n is a positive integer,

$$M(h) \sim I + \sum_{k=1}^{\infty} \alpha_k h^{\mu+k} + \sum_{k=1}^{\infty} \beta_k h^{\nu+k} \quad \text{as } h \to 0 \tag{1.14}$$

for some constants α_k and β_k. Here $M(h)$ may be the midpoint rule approximation to I, for example. Even more involved expansions arise in multidimensional integration of singular functions. Most of the expansions that result from such integrals are of the form given in (1.1) provided the integrand functions have only algebraic singularities at corners, edges, or surfaces of the domain of integration, while more complicated expansions may arise in some cases. For generalized Euler–Maclaurin expansions of one-dimensional singular integrals see Navot [N] and Lyness and Ninham [LN]. For multidimensional integrals with corner singularities see Lyness [Ly], and with line or edge singularities see Sidi [Si3]. For a review of the applications to numerical integration see Davis and Rabinowitz [DR] and Sidi [Si5]. It must be mentioned that, in even moderate dimensions N, the use of sequences $\{y_l\}$ as in (1.8) becomes practically impossible as the number of integrand evaluations entailed in this usage becomes prohibitively large.

To the best of our knowledge, no analysis of $A_n^{(j)}$ when the y_l are as in this paper has been given in the literature for the general case of *arbitrary* σ_k, real or complex. It is the purpose of this paper to present a rigorous convergence and stability theory precisely for this general case that pertains to the column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$ with n fixed. We mention that a theory of the case in which $\sigma_k = k\tau,\ k = 1, 2, \ldots,$ for some arbitrary *real* $\tau > 0$, is already contained in that presented in [BS1] and [BS2]. Actually, the theory of both of these papers considers arbitrary y_l's. We note that the techniques of [BS1] and [BS2] are not applicable to our problem. In particular, they cannot be applied to the numerical integration problems described in the previous paragraph. Convergence and stability results for the case in which $\sigma_k = \sigma_0 + k,\ k = 1, 2, \ldots,$ with arbitrary σ_0 and for the choice $y_l = c/(l+1),\ l = 0, 1, \ldots,$ are contained in the theory of the Levin [L] transformations that was presented in Sidi [Si1]. For a summary of these results see Theorem 4.1 of [Si7].

In the next section, we give important technical preliminaries toward the proofs of the main convergence and stability results. Theorems 2.3 and 2.5, which are the main results of this section, are of interest in themselves.

With the help of the results of section 2, in section 3 we state and prove the main stability and convergence theorems of this work. These theorems are stated in simple and elegant form, despite the fact that the mathematical problems are highly nonlinear and complex. In particular, we show the following:

1. All column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$ are unstable.
2. If $\Re\sigma_{n+1} > 0$, then $\{A_n^{(j)}\}_{j=0}^{\infty}$ converges to A even though the process is unstable.
3. Each column sequence is at least as good as the ones preceding it.

Finally, in section 4 we demonstrate some of the results of section 3 with numerical examples.

It is important to mention that the results of section 2, which form the key to everything, are obtained by employing a very general technique that was originally developed in Sidi, Ford, and Smith [SFS] within the context of vector extrapolation methods and that is of interest in itself. With suitable extensions and refinements this technique was used subsequently in several works by the author and other researchers in vector extrapolation methods (see, e.g., [Si4], [SB], and [Le]), in Krylov subspace methods for eigenvalue problems (see, e.g., [Si8]), and also in Padé approximants (see [Si6]). An important advantage of it is that it enables us to obtain complete asymptotic expansions that can be used to derive theoretical results that are best possible asymptotically. The theorems of section 3 are of this form.

We close this section with the following two lemmas that will be used in our proofs later. The first of these lemmas was stated and proved as Lemma A.1 in the appendix of [SFS] and is an integral part of the technique of [SFS] that we are about to employ.

LEMMA 1.1. *Let $i_1, \ldots, i_k$ be positive integers, and assume that the scalars $v_{i_1,\ldots,i_k}$ are odd under an interchange of any two indices $i_1, \ldots, i_k$. Let $t_{i,j},\ i \geq 1,\ 1 \leq j \leq k$, be scalars. Then*

$$\sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \left(\prod_{p=1}^{k} t_{i_p,p} \right) v_{i_1,\ldots,i_k} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} \begin{vmatrix} t_{i_1,1} & t_{i_2,1} & \cdots & t_{i_k,1} \\ t_{i_1,2} & t_{i_2,2} & \cdots & t_{i_k,2} \\ \vdots & \vdots & & \vdots \\ t_{i_1,k} & t_{i_2,k} & \cdots & t_{i_k,k} \end{vmatrix} v_{i_1,\ldots,i_k} .$$

LEMMA 1.2. *Let $Q_i(x) = \sum_{j=0}^{i} a_{ij} x^j$, with $a_{ii} \neq 0,\ i = 0, 1, \ldots, n$, and let $x_i,\ i = 0, 1, \ldots, n$, be arbitrary points. Then*

$$\begin{vmatrix} Q_0(x_0) & Q_0(x_1) & \cdots & Q_0(x_n) \\ Q_1(x_0) & Q_1(x_1) & \cdots & Q_1(x_n) \\ \vdots & \vdots & & \vdots \\ Q_n(x_0) & Q_n(x_1) & \cdots & Q_n(x_n) \end{vmatrix} = \left(\prod_{i=0}^{n} a_{ii} \right) V(x_0, x_1, \ldots, x_n), \tag{1.15}$$

where $V(x_0, x_1, \ldots, x_n) = \prod_{0 \leq i < j \leq n} (x_j - x_i)$ is a Vandermonde determinant.

Proof. As can easily be seen, it is enough to consider the case $a_{ii} = 1,\ i = 0, 1, \ldots, n$. Let us now perform the following elementary row transformations in the determinant on the left-hand side of (1.15):

for $i = 1, 2, \ldots, n$ do
  for $j = 0, 1, \ldots, i-1$ do
    multiply $(j+1)$st row by a_{ij} and subtract from $(i+1)$st row
  end for
end for

The result is the Vandermonde determinant $V(x_0, x_1, \ldots, x_n)$. ∎

2. Technical preliminaries.

2.1. General framework. As mentioned in the previous section, throughout this work we shall consider the extrapolation process in which the y_l are picked such that

$$y_l = \frac{c}{(l+\eta)^q}, \quad l = 0, 1, 2, \ldots, \quad \text{for some } c > 0,\ \eta > 0, \text{ and } q > 0. \tag{2.1}$$

Obviously, this is slightly more general than the harmonic sequence we mentioned in the previous section. Note also that these y_l's satisfy (1.4) and also $\lim_{l\to\infty}(t_{l+1}/t_l) =$

1. As will become clear soon, the presence of the parameters η and q does not complicate the mathematics in any way. In addition, we will see in the next section that both the convergence and the stability properties of $A_n^{(j)}$ improve with increasing q. Thus, the parameter q may be of significance in practice.

To simplify the analysis as much as possible we will adopt the approach and notation of Sidi [Si7].

Let us define the sequences $g_k \equiv \{g_k(l)\}_{l=0}^{\infty}$ by

$$g_k(l) = y_l^{\sigma_k}, \quad l = 0, 1, 2, \ldots, \tag{2.2}$$

and define also for an arbitrary sequence $b \equiv \{b(l)\}_{l=0}^{\infty}$

$$f_n^{(j)}(b) = \begin{vmatrix} g_1(j) & g_2(j) & \cdots & g_n(j) & b(j) \\ g_1(j+1) & g_2(j+1) & \cdots & g_n(j+1) & b(j+1) \\ \vdots & \vdots & & \vdots & \vdots \\ g_1(j+n) & g_2(j+n) & \cdots & g_n(j+n) & b(j+n) \end{vmatrix}. \tag{2.3}$$

Let us also define the sequences $a \equiv \{a(l)\}_{l=0}^{\infty}$, $I \equiv \{I(l)\}_{l=0}^{\infty}$, and $r \equiv \{r(l)\}_{l=0}^{\infty}$ via, respectively,

$$a(l) = A(y_l), \quad I(l) = 1, \quad \text{and} \quad r(l) = a(l) - A, \quad l = 0, 1, \ldots\ . \tag{2.4}$$

Then, with the help of Cramer's rule, we have from (1.3),

$$A_n^{(j)} = \frac{f_n^{(j)}(a)}{f_n^{(j)}(I)}, \tag{2.5}$$

and, therefore,

$$A_n^{(j)} - A = \frac{f_n^{(j)}(r)}{f_n^{(j)}(I)}. \tag{2.6}$$

Substituting $r(j) = \sum_{k=1}^{s} \alpha_k g_k(j) + \epsilon_s(j)$, where $\epsilon_s(j) = O(g_{s+1}(j))$ as $j \to \infty$, as follows from (1.1) and (1.4), and using the fact that $f_n^{(j)}(g_k) = 0,\ k = 1, 2, \ldots, n$, we have

$$A_n^{(j)} - A = \sum_{k=n+1}^{s} \alpha_k \frac{f_n^{(j)}(g_k)}{f_n^{(j)}(I)} + \frac{f_n^{(j)}(\epsilon_s)}{f_n^{(j)}(I)}. \tag{2.7}$$

Finally, setting for an arbitrary sequence b,

$$\chi_n^{(j)}(b) = \frac{f_n^{(j)}(b)}{f_n^{(j)}(I)} = \sum_{i=0}^{n} \gamma_{ni}^{(j)} b(j+i), \tag{2.8}$$

we reexpress (2.7) in the form

$$A_n^{(j)} - A = \sum_{k=n+1}^{s} \alpha_k \chi_n^{(j)}(g_k) + \chi_n^{(j)}(\epsilon_s). \tag{2.9}$$

Thus, we need to analyze the asymptotic behavior of the quantities $\chi_n^{(j)}(g_k),\ k \geq n+1$, and $\chi_n^{(j)}(\epsilon_s)$ for $j \to \infty$. By (2.8), it is clear that we need to study the behavior of the determinants $f_n^{(j)}(g_k),\ k \geq n+1$, and $f_n^{(j)}(I)$ and $f_n^{(j)}(\epsilon_s)$.

As for $\Gamma_n^{(j)}$, we note that

$$\gamma_{ni}^{(j)} = \frac{N_{ni}^{(j)}}{f_n^{(j)}(I)}, \quad i = 0, 1, \ldots, n, \tag{2.10}$$

where $N_{ni}^{(j)}$ is the cofactor of $b(j+i)$ in the determinant $f_n^{(j)}(b)$. Thus, we need to analyze $\gamma_{ni}^{(j)}$, $i = 0, 1, \ldots, n$, for $j \to \infty$.

As it turns out, the treatments of $\chi_n^{(j)}(\epsilon_s)$ and $\Gamma_n^{(j)}$ can be unified in a simple fashion.

2.2. Analysis of $\chi_n^{(j)}(g_k)$. Let us begin by defining

$$u = \frac{1}{j+\eta} \quad \text{and} \quad \nu_i = -q\sigma_i, \quad i = 1, 2, \ldots . \tag{2.11}$$

Then, for each i,

$$g_i(j+p) = g_i(j)(1+pu)^{\nu_i}, \quad p = 0, 1, 2, \ldots . \tag{2.12}$$

Next, substituting (2.12) in the determinant expression for $f_n^{(j)}(g_k)$ with $k \geq n+1$, we have

(2.13)

$$f_n^{(j)}(g_k) = \begin{vmatrix} g_1(j) & g_2(j) & \cdots & g_n(j) & g_k(j) \\ g_1(j)(1+1u)^{\nu_1} & g_2(j)(1+1u)^{\nu_2} & \cdots & g_n(j)(1+1u)^{\nu_n} & g_k(j)(1+1u)^{\nu_k} \\ g_1(j)(1+2u)^{\nu_1} & g_2(j)(1+2u)^{\nu_2} & \cdots & g_n(j)(1+2u)^{\nu_n} & g_k(j)(1+2u)^{\nu_k} \\ \vdots & \vdots & & \vdots & \vdots \\ g_1(j)(1+nu)^{\nu_1} & g_2(j)(1+nu)^{\nu_2} & \cdots & g_n(j)(1+nu)^{\nu_n} & g_k(j)(1+nu)^{\nu_k} \end{vmatrix}.$$

Factoring out $g_i(j)$ from the ith column, $i = 1, \ldots, n$, and $g_k(j)$ from the last column, and invoking the binomial expansion $(1+z)^\nu = \sum_{i=0}^{\infty} \binom{\nu}{i} z^i$ that converges absolutely and uniformly for $|z| < 1$, we next obtain for $u < 1/n$, hence for $j \geq n$,

(2.14)

$$\frac{f_n^{(j)}(g_k)}{\prod_{i=0}^n g_i(j)} = \begin{vmatrix} \sum_{i_1} \binom{\nu_1}{i_1}(0u)^{i_1} & \sum_{i_2} \binom{\nu_2}{i_2}(0u)^{i_2} & \cdots & \sum_{i_n} \binom{\nu_n}{i_n}(0u)^{i_n} & \sum_{i_0} \binom{\nu_0}{i_0}(0u)^{i_0} \\ \sum_{i_1} \binom{\nu_1}{i_1}(1u)^{i_1} & \sum_{i_2} \binom{\nu_2}{i_2}(1u)^{i_2} & \cdots & \sum_{i_n} \binom{\nu_n}{i_n}(1u)^{i_n} & \sum_{i_0} \binom{\nu_0}{i_0}(1u)^{i_0} \\ \sum_{i_1} \binom{\nu_1}{i_1}(2u)^{i_1} & \sum_{i_2} \binom{\nu_2}{i_2}(2u)^{i_2} & \cdots & \sum_{i_n} \binom{\nu_n}{i_n}(2u)^{i_n} & \sum_{i_0} \binom{\nu_0}{i_0}(2u)^{i_0} \\ \vdots & \vdots & & \vdots & \vdots \\ \sum_{i_1} \binom{\nu_1}{i_1}(nu)^{i_1} & \sum_{i_2} \binom{\nu_2}{i_2}(nu)^{i_2} & \cdots & \sum_{i_n} \binom{\nu_n}{i_n}(nu)^{i_n} & \sum_{i_0} \binom{\nu_0}{i_0}(nu)^{i_0} \end{vmatrix},$$

where $\sum_i$ stands for $\sum_{i=0}^{\infty}$ throughout. Note that $\sum_i \binom{\nu}{i} 0^i = 1$ in the first row with the understanding that $0^0 \equiv 1$ and $0^i = 0$ for $i = 1, 2, \ldots$. Also note that we have written $g_0(j)$ instead of $g_k(j)$ and ν_0 instead of ν_k. This simplifies our notation considerably, as we shall see soon.

By the multilinearity property of determinants, we can move the summations outside the determinant. Factoring out $\binom{\nu_p}{i_p} u^{i_p}$ from the pth column, $p = 1, 2, \ldots, n$, and $\binom{\nu_0}{i_0} u^{i_0}$ from the last column, we have

$$\frac{f_n^{(j)}(g_k)}{\prod_{i=0}^n g_i(j)} = \sum_{i_0} \sum_{i_1} \cdots \sum_{i_n} \left[\prod_{p=0}^n \binom{\nu_p}{i_p} \right] \left(\prod_{p=0}^n u^{i_p} \right) Y_{i_0, i_1, \ldots, i_n}, \tag{2.15}$$

where we have defined

$$Y_{i_0,i_1,\ldots,i_n} = \begin{vmatrix} 0^{i_1} & 0^{i_2} & \cdots & 0^{i_n} & 0^{i_0} \\ 1^{i_1} & 1^{i_2} & \cdots & 1^{i_n} & 1^{i_0} \\ 2^{i_1} & 2^{i_2} & \cdots & 2^{i_n} & 2^{i_0} \\ \vdots & \vdots & & \vdots & \vdots \\ n^{i_1} & n^{i_2} & \cdots & n^{i_n} & n^{i_0} \end{vmatrix}. \tag{2.16}$$

Note that the multiple sum on the right-hand side of (2.15) converges absolutely and uniformly in u for $u < 1/n$, hence uniformly in j for $j \geq n$, since each of the series in (2.14) does. We are now ready to prove the following interesting result on $f_n^{(j)}(g_k)$.

LEMMA 2.1. *With $Y_{i_0,i_1,\ldots,i_n}$ as defined in (2.16), $f_n^{(j)}(g_k)/\left[\prod_{i=0}^n g_i(j)\right]$ can be expanded in a power series in u as in*

$$\frac{f_n^{(j)}(g_k)}{\prod_{i=0}^n g_i(j)} = \sum_{0\leq i_0<i_1<\cdots<i_n}^{\infty} Y_{i_0,i_1,\ldots,i_n} Z_{i_0,i_1,\ldots,i_n}^{\nu_0,\nu_1,\ldots,\nu_n} \left(\prod_{p=0}^n u^{i_p}\right), \tag{2.17}$$

where we have also defined

$$Z_{i_0,i_1,\ldots,i_n}^{\nu_0,\nu_1,\ldots,\nu_n} = \begin{vmatrix} \binom{\nu_0}{i_0} & \binom{\nu_1}{i_0} & \cdots & \binom{\nu_n}{i_0} \\ \binom{\nu_0}{i_1} & \binom{\nu_1}{i_1} & \cdots & \binom{\nu_n}{i_1} \\ \vdots & \vdots & & \vdots \\ \binom{\nu_0}{i_n} & \binom{\nu_1}{i_n} & \cdots & \binom{\nu_n}{i_n} \end{vmatrix}. \tag{2.18}$$

This series converges absolutely and uniformly in u for $u < 1/n$, and hence uniformly in j for $j \geq n$. In addition, this power series is also an asymptotic expansion as $u \to 0$ (equivalently, as $j \to \infty$), whose behavior is given by the asymptotic equality

$$\frac{f_n^{(j)}(g_k)}{\left[\prod_{i=1}^n g_i(j)\right] g_k(j)} \sim V(\nu_1, \nu_2, \ldots, \nu_n, \nu_k) \left(\prod_{p=1}^n j^{-p}\right) \quad \text{as } j \to \infty, \tag{2.19}$$

where $V(\xi_1, \xi_2, \ldots, \xi_n)$ denotes the Vandermonde determinant

$$V(\xi_1, \xi_2, \ldots, \xi_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_n \\ \xi_1^2 & \xi_2^2 & \cdots & \xi_n^2 \\ \vdots & \vdots & & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} = \prod_{1\leq i<j\leq n} (\xi_j - \xi_i). \tag{2.20}$$

Proof. The product $(\prod_{p=0}^n u^{i_p})$ is a symmetric function of $i_0, i_1, \ldots, i_n$. The determinant $Y_{i_0,i_1,\ldots,i_n}$, on the other hand, is odd under an interchange of any two of the indices $i_0, i_1, \ldots, i_n$, since such an interchange is equivalent to an interchange of two columns. Therefore, $(\prod_{p=0}^n u^{i_p}) Y_{i_0,i_1,\ldots,i_n}$ is odd under an interchange of any two of the indices $i_0, i_1, \ldots, i_n$. This being the case, we can apply Lemma 1.1 to (2.15) to obtain (2.17). That the multiple series in (2.17) converges absolutely and uniformly in u for $u < 1/n$, and hence in j for $j \geq n$, is immediate. Let us now recall that a power series $\sum_{n=0}^\infty a_n (z - z_0)^n$ that converges absolutely and uniformly for $|z - z_0| < \rho$ is the Taylor series of a function analytic for $|z - z_0| < \rho$, and that a Taylor series of an analytic function about z_0 is also its asymptotic expansion in powers of $(z-z_0)$ for $z \to z_0$. We therefore conclude that the power series in (2.17) is also an asymptotic expansion of the left-hand side as $u \to 0$, hence as $j \to \infty$. Thus, its behavior as $j \to \infty$ is determined by the terms in which $(\prod_{p=0}^{n} u^{i_p})$ is most dominant as $u \to 0$, or, equivalently, by the terms for which $\sum_{p=0}^{n} i_p$ is smallest possible. It seems this will be achieved only by the term with $i_0 = 0,\ i_1 = 1,\ i_2 = 2, \dots,\ i_n = n$, provided $Y_{0,1,\dots,n}\ Z_{0,1,\dots,n}^{\nu_0,\nu_1,\dots,\nu_n} \neq 0$, in which case

$$\frac{f_n^{(j)}(g_k)}{\prod_{i=0}^{n} g_i(j)} \sim Y_{0,1,\dots,n}\ Z_{0,1,\dots,n}^{\nu_0,\nu_1,\dots,\nu_n} \left(\prod_{p=0}^{n} u^p\right) \quad \text{as } j \to \infty. \tag{2.21}$$

Indeed, recalling our definition that $0^0 \equiv 1$ and $0^i = 0$ for $i = 1, 2, \dots,$ (2.16) gives

$$Y_{0,1,\dots,n} = (-1)^n (n!) V(1, 2, \dots, n) = (-1)^n \left(\prod_{i=1}^{n} i!\right) \neq 0. \tag{2.22}$$

Also, since $\binom{\nu}{i}$ is a polynomial of degree exactly i in ν given by $\binom{\nu}{i} = \nu(\nu-1)\cdots(\nu - i+1)/i!$, Lemma 1.2 applies and we have

$$Z_{0,1,\dots,n}^{\nu_0,\nu_1,\dots,\nu_n} = \left(\prod_{i=1}^{n} i!\right)^{-1} V(\nu_0, \nu_1, \dots, \nu_n), \tag{2.23}$$

and since the ν_i are all distinct, $V(\nu_0, \nu_1, \dots, \nu_n) \neq 0$, as a consequence of which, we have $Z_{0,1,\dots,n}^{\nu_0,\nu_1,\dots,\nu_n} \neq 0$ as well. Combining (2.22) and (2.23) in (2.21), and noting that $u \sim j^{-1}$ as $j \to \infty$, the result in (2.19) follows. ∎

We now turn to $f_n^{(j)}(I)$. Comparing the determinant representation of $f_n^{(j)}(I)$ with that of $f_n^{(j)}(g_k)$, we realize that the former is obtained from (2.13) by replacing $g_k(j)$ by 1 and ν_k by 0 in the last column there, with everything else remaining the same. Realizing also that $\nu_i \neq 0,\ i = 1, 2, \dots,$ we obtain the following result from Lemma 2.1.

LEMMA 2.2. *$f_n^{(j)}(I)/[\prod_{i=1}^{n} g_i(j)]$ can be expanded as a power series in u in the form*

$$\frac{f_n^{(j)}(I)}{\prod_{i=1}^{n} g_i(j)} = \sum_{0 \le i_0 < i_1 \cdots < i_n}^{\infty} Y_{i_0,i_1,\dots,i_n} Z_{i_0,i_1,\dots,i_n}^{0,\nu_1,\dots,\nu_n} \left(\prod_{p=0}^{n} u^{i_p}\right), \tag{2.24}$$

which converges absolutely and uniformly in j for $j \ge n$, and its behavior for large j is given by the asymptotic equality

$$\frac{f_n^{(j)}(I)}{\prod_{i=1}^{n} g_i(j)} \sim V(\nu_1, \dots, \nu_n, 0) \left(\prod_{p=1}^{n} j^{-p}\right) \quad \text{as } j \to \infty. \tag{2.25}$$

The $Y_{i_0,i_1,\dots,i_n}$ and $Z_{i_0,i_1,\dots,i_n}^{0,\nu_1,\dots,\nu_n}$ in (2.24) are precisely as given in (2.16) and (2.18), respectively.

Combining the two lemmas above, we obtain the following theorem on the $\chi_n^{(j)}(g_k)$.

THEOREM 2.3. *For fixed n, $\chi_n^{(j)}(g_k),\ k \ge n+1$, satisfy*

$$\chi_n^{(j)}(g_k) \sim \left(\prod_{i=1}^{n} \frac{\sigma_i - \sigma_k}{\sigma_i}\right) g_k(j) \quad \text{as } j \to \infty. \tag{2.26}$$

Consequently, $\{\chi_n^{(j)}(g_k)\}_{k=n+1}^{\infty}$ *is an asymptotic sequence as* $j \to \infty$*, i.e.,*

$$\lim_{j\to\infty} \frac{\chi_n^{(j)}(g_{k+1})}{\chi_n^{(j)}(g_k)} = 0, \quad k = n+1, n+2, \dots . \tag{2.27}$$

It is interesting to observe from (2.26) that $\chi_n^{(j)}(g_k)$ is proportional to $g_k(j)$ as $j \to \infty$, the constant of proportionality being $[\prod_{i=1}^{n}(\sigma_i - \sigma_k)/\sigma_i]$.

2.3. Analysis of $\gamma_{ni}^{(j)}$ and $\chi_n^{(j)}(\epsilon_s)$. Let us replace the last column of the determinant $f_n^{(j)}(b)$ in (2.3) by an arbitrary fixed vector $(v_0, v_1, \dots, v_n)$ and denote the resulting determinant by $\hat{f}_n^{(j)}(v)$. Next, let us observe that $\hat{f}_n^{(j)}(v)$ is obtained from (2.13) by replacing the last column in (2.13) by the vector $(v_0, v_1, \dots, v_n)^T$. By repeating the steps that lead from (2.13) to (2.16), we obtain

$$\frac{\hat{f}_n^{(j)}(v)}{\prod_{i=1}^{n} g_i(j)} = \sum_{i_1} \cdots \sum_{i_n} \left[\prod_{p=1}^{n} \binom{\nu_p}{i_p}\right] \left(\prod_{p=1}^{n} u^{i_p}\right) \hat{Y}_{i_1,\dots,i_n}(v), \tag{2.28}$$

where we have defined

$$\hat{Y}_{i_1,\dots,i_n}(v) = \begin{vmatrix} 0^{i_1} & 0^{i_2} & \cdots & 0^{i_n} & v_0 \\ 1^{i_1} & 1^{i_2} & \cdots & 1^{i_n} & v_1 \\ 2^{i_1} & 2^{i_2} & \cdots & 2^{i_n} & v_2 \\ \vdots & \vdots & & \vdots & \vdots \\ n^{i_1} & n^{i_2} & \cdots & n^{i_n} & v_n \end{vmatrix}. \tag{2.29}$$

Making use of Lemma 1.1 again, we obtain from (2.29) the power series in u

$$\frac{\hat{f}_n^{(j)}(v)}{\prod_{i=1}^{n} g_i(j)} = \sum_{0\le i_1<\cdots<i_n}^{\infty} \hat{Y}_{i_1,\dots,i_n}(v)\; Z_{i_1,\dots,i_n}^{\nu_1,\dots,\nu_n} \left(\prod_{p=1}^{n} u^{i_p}\right) \tag{2.30}$$

that converges absolutely and uniformly for $u < 1/n$. Since this is also an asymptotic expansion as $u \to 0$, with the dominant term being that for which $i_1 = 0,\ i_2 = 1, \dots,\ i_n = n-1$, we have

$$\frac{\hat{f}_n^{(j)}(v)}{\prod_{i=1}^{n} g_i(j)} \sim \hat{Y}_{0,1,\dots,n-1}(v)\; Z_{0,1,\dots,n-1}^{\nu_1,\nu_2,\dots,\nu_n} \left(\prod_{p=1}^{n-1} u^{p}\right) \quad \text{as } j \to \infty, \tag{2.31}$$

provided the right-hand side is nonzero. Expanding $\hat{Y}_{0,1,\dots,n-1}(v)$ with respect to its last column, we have

$$\begin{aligned} \hat{Y}_{0,1,\dots,n-1}(v) &= \sum_{i=0}^{n} (-1)^{n+i} v_i\, V(0,1,\dots,i-1,i+1,\dots,n) \\ &= \left(\prod_{i=1}^{n-1} i!\right) \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} v_i. \end{aligned} \tag{2.32}$$

Also, analogously to (2.23),

$$Z_{0,1,\dots,n-1}^{\nu_1,\nu_2,\dots,\nu_n} = \left(\prod_{i=1}^{n-1} i!\right)^{-1} V(\nu_1,\nu_2,\dots,\nu_n). \tag{2.33}$$

Combining (2.32) and (2.33) in (2.31) and invoking (2.11) we have the following result on $\hat{f}_n^{(j)}(v)$.

LEMMA 2.4. *Provided* $\sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i}v_i \neq 0$, $\hat{f}_n^{(j)}(v)$ *satisfies the asymptotic equality*

$$\frac{\hat{f}_n^{(j)}(v)}{\prod_{i=1}^{n} g_i(j)} \sim V(\nu_1,\nu_2,\dots,\nu_n)\left[\sum_{i=0}^{n}(-1)^{n+i}\binom{n}{i}v_i\right]\left(\prod_{p=1}^{n-1} j^{-p}\right) \quad \text{as } j\to\infty. \tag{2.34}$$

We now use Lemma 2.4 to state the following result concerning $\gamma_{ni}^{(j)}$ and $\chi_n^{(j)}(\epsilon_s)$.

THEOREM 2.5. (i) *For each* $i=0,1,\dots,n$, $\gamma_{ni}^{(j)}$ *satisfies the asymptotic equality*

$$\gamma_{ni}^{(j)} \sim (-1)^{n-i}\binom{n}{i}\left(\prod_{p=1}^{n}\sigma_p\right)^{-1}\left(\frac{j}{q}\right)^{n} \quad \text{as } j\to\infty. \tag{2.35}$$

(ii) $\chi_n^{(j)}(\epsilon_s)$ *satisfies*

$$\chi_n^{(j)}(\epsilon_s) = O(j^n g_{s+1}(j)) \quad \text{as } j\to\infty. \tag{2.36}$$

Proof. The proof of part (i) follows from the fact that $\hat{f}_n^{(j)}(v)/f_n^{(j)}(I) = \sum_{i=0}^{n}\gamma_{ni}^{(j)}v_i$ and from (2.34) and (2.25). The proof of part (ii) follows from the fact that $\chi_n^{(j)}(\epsilon_s) = \sum_{i=0}^{n}\gamma_{ni}^{(j)}\epsilon_s(j+i)$ and from (2.35) and $g_k(j+i)=O(g_k(j))$ as $j\to\infty$ for each k and for each finite i. □

Note that (2.35) is the best that we can obtain for $\gamma_{ni}^{(j)}$ asymptotically. We believe that (2.36) is also the best that can be obtained for $\chi_n^{(j)}(\epsilon_s)$ under the given conditions.

3. Main results. We now present the main stability and convergence theorems of this work. We only would like to recall that $\lim_{y\to 0+}A(y)$ is not assumed to exist. In other words, A may be the limit or antilimit of $A(y)$ as $y\to 0+$.

THEOREM 3.1. *With the* y_l *as in* (2.1), *the extrapolation process that produces the sequence* $\{A_n^{(j)}\}_{j=0}^{\infty}$ *with* n *fixed is unstable in the sense that* $\sup_j \Gamma_n^{(j)} = \infty$. *Specifically, we have the asymptotic equality*

$$\Gamma_n^{(j)} \sim \left(\prod_{p=1}^{n}|\sigma_p|\right)^{-1}\left(\frac{2j}{q}\right)^{n} \quad \text{as } j\to\infty. \tag{3.1}$$

Proof. The asymptotic equality in (3.1) follows from part (i) of Theorem 2.5 above. □

THEOREM 3.2. *Let* $A(y)$ *be as in the first paragraph of section* 1. *Then* $A_n^{(j)}-A$ *has the genuine asymptotic expansion*

$$A_n^{(j)} - A \sim \sum_{k=n+1}^{\infty}\alpha_k\chi_n^{(j)}(g_k) \quad \text{as } j\to\infty. \tag{3.2}$$

Remark. Comparing (3.2) with (2.9), one may be led to believe erroneously that (3.2) follows from (2.9) in a trivial way by letting $s\to\infty$ in the latter. This is far from being the case as (3.2) needs to be *proved* in a rigorous manner. As we show in the proof below, Theorems 2.3 and 2.5 that were obtained with considerable effort are the key to establishing the validity of (3.2).

Proof. To prove that (3.2) is valid we must first show that its right-hand side makes sense as an asymptotic expansion. That this is indeed the case follows from the fact that $\{\chi_n^{(j)}(g_k)\}_{k=n+1}^{\infty}$ is an asymptotic sequence as $j \to \infty$, which was proved in Theorem 2.3. Next, we must show that, for any integer $N \geq n+1$, there holds

$$A_n^{(j)} - A - \sum_{k=n+1}^{N} \alpha_k \chi_n^{(j)}(g_k) = O(\chi_n^{(j)}(g_{N+1})) \quad \text{as } j \to \infty. \tag{3.3}$$

By the assumption in (1.2) that $\lim_{k\to\infty} \Re\sigma_k = +\infty$, there exists an integer $s > N$ for which

$$\Re\sigma_{s+1} \geq \Re\sigma_{N+1} + \frac{n}{q}. \tag{3.4}$$

Let us rewrite (2.9) in the form

$$A_n^{(j)} - A - \sum_{k=n+1}^{N} \alpha_k \chi_n^{(j)}(g_k) = \sum_{k=N+1}^{s} \alpha_k \chi_n^{(j)}(g_k) + \chi_n^{(j)}(\epsilon_s). \tag{3.5}$$

Now

$$\sum_{k=N+1}^{s} \alpha_k \chi_n^{(j)}(g_k) = O(\chi_n^{(j)}(g_{N+1})) = O(g_{N+1}(j)) \quad \text{as } j \to \infty \tag{3.6}$$

by Theorem 2.3, and

$$\chi_n^{(j)}(\epsilon_s) = O(j^n g_{s+1}(j)) = O(g_{N+1}(j)) = O(\chi_n^{(j)}(g_{N+1})) \quad \text{as } j \to \infty \tag{3.7}$$

by part (ii) of Theorem 2.5, (2.2), (3.4), and Theorem 2.3. Substituting (3.6) and (3.7) in (3.5), we obtain (3.3). This completes the proof. ∎

COROLLARY. *Under the conditions of Theorem* 3.2, $A_n^{(j)} - A$ *satisfies the asymptotic equality*

$$A_n^{(j)} - A \sim \left[\alpha_{n+\mu} \left(\prod_{i=1}^{n} \frac{\sigma_i - \sigma_{n+\mu}}{\sigma_i}\right) c^{\sigma_{n+\mu}}\right] j^{-q\sigma_{n+\mu}} \quad \textit{as } j \to \infty, \tag{3.8}$$

where $\alpha_{n+\mu}$ *is the first nonzero* α_{n+i} *with* $i \geq 1$.

Proof. The proof of (3.8) is achieved by combining (3.2) with (2.26) and then using (2.1). ∎

Before going on we would like to comment on the results above. From Theorem 3.1 it is clear that $\Gamma_n^{(j)}$, being proportional to q^{-n} as $j \to \infty$, will decrease when q is increased, even though $\lim_{j\to\infty} \Gamma_n^{(j)} = \infty$. Thus, by increasing q we can cause the column sequences to have better stability properties. (Numerical experience seems to suggest that this is the case also for the diagonal sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$ with j fixed.)

We next turn to Theorem 3.2 and its corollary. Let us compare the asymptotic expansion of $A_n^{(j)} - A$ that is given in (3.2) with that of $A(y_j) - A$, namely, with

$$A(y_j) - A \sim \sum_{k=1}^{\infty} \alpha_k g_k(j) \quad \text{as } j \to \infty, \tag{3.9}$$

which follows from (1.1) and (2.2). Theorem 3.2 thus tells us that, in generating $A_n^{(j)}$, REP "eliminates" the first n terms, $\alpha_k y^{\sigma_k}$, $k = 1, \ldots, n$, from the asymptotic expansion of $A(y) - A$. The corollary to Theorem 3.2, on the other hand, says that the error $A_n^{(j)} - A$, as $j \to \infty$, is exactly of the order of the first nonzero term following the term $\alpha_n g_n(j)$ in (3.9). Thus, in case $\alpha_{m+1} \neq 0$, $\alpha_{m+2} = \cdots = \alpha_s = 0$, and $\alpha_{s+1} \neq 0$, we have

$$\text{(3.10)} \qquad \begin{aligned} A_n^{(j)} - A &= o(A_m^{(j)} - A) \quad \text{as } j \to \infty, \quad m+1 \le n \le s, \\ A_{s+1}^{(j)} - A &= o(A_s^{(j)} - A) \quad \text{as } j \to \infty. \end{aligned}$$

In addition, the sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$, $m+1 \le n \le s$, all have the same kind of asymptotic behavior as $j \to \infty$ in the sense that

$$\text{(3.11)} \qquad A_n^{(j)} - A \sim \theta_n j^{-q\sigma_{s+1}} \quad \text{as } j \to \infty, \quad m+1 \le n \le s,$$

for some nonzero constants θ_n.

Finally, as both $\Gamma_n^{(j)}$ and $A_n^{(j)} - A$ are directly proportional to the product $\prod_{i=1}^{n} |\sigma_i|^{-1}$, we conclude that it is easier to extrapolate $A(y)$ when σ_k are large. One practical situation in which this becomes relevant is that of $\sigma_k = \sigma_0 + kd$ for some $d > 0$. Here the larger $\Im\sigma_0$ the better the convergence and stability properties of the column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$, despite the fact that $A_n^{(j)} - A = O(y_j^{\Re\sigma_0 + (n+1)d})$ as $j \to \infty$ for any value of $\Im\sigma_0$. (Again, numerical experience seems to suggest that this is the case also for the diagonal sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$ with j fixed.)

Both Theorem 3.2 and its corollary are valid under (1.1). When (1.1) does not hold, but instead we have, for some finite and largest possible integer s,

$$\text{(3.12)} \qquad A(y) = A + \sum_{k=1}^{s} \alpha_k y^{\sigma_k} + O(y^{\sigma_{s+1}}) \quad \text{as } y \to 0+,$$

with (1.2) still valid, substantial changes need to be made in the convergence results for $A_n^{(j)}$. These are summarized in Theorem 3.3, the proof of which is similar to that of Theorem 3.2 and is left to the reader.

THEOREM 3.3. *Assume that $A(y)$ is as in the previous paragraph and define $\hat{\sigma}$ by*

$$\text{(3.13)} \qquad \hat{\sigma} = \min\left\{\Re\sigma_{s+1} - \frac{n}{q}, \Re\sigma_{n+1}\right\}.$$

Then

$$\text{(3.14)} \qquad A_n^{(j)} - A = O(y_j^{\hat{\sigma}}) \quad \textit{as } j \to \infty \quad \textit{for all } n.$$

In addition, when $n < s$ and $\alpha_{n+\mu}$ is the first nonzero α_{n+i} with $i \ge 1$ in (3.12), (3.8) *holds provided*

$$\text{(3.15)} \qquad \Re\sigma_{s+1} > \Re\sigma_{n+\mu} + \frac{n}{q}.$$

We have been informed by one of the referees that in the special case where $\sigma_k = k$, $k = 1, 2, \ldots$, and $y_l = c/(l+1)$, $l = 0, 1, \ldots$, when $n \le s/2$, (3.14) of Theorem 3.3 reduces to Lemma 2.15 on p. 54 of Crouzeix and Mignot [CM]. In this case $\hat{\sigma} = n + 1$ in (3.13).

4. A numerical example. In this section we present a numerical example to demonstrate the results of Theorems 3.1 and 3.2 of the previous section. Before we do this, however, we would like to discuss the subject of stability in some detail. Through this discussion the significance of the quantity $\Gamma_n^{(j)}$ will also become clear.

Let us assume that REP is being applied with the quantities $A(y_l)$ replaced by $\overline{A}_l = A(y_l) + \varepsilon_l$, $l = 0, 1, \ldots$. (That is to say, for each l, ε_l is the error committed in the computation of $A(y_l)$.) Let us denote the resulting approximations by $\overline{A}_n^{(j)}$. Obviously, the errors in $\overline{A}_n^{(j)}$ and $A_n^{(j)}$ are related to each other through the inequality

$$|\overline{A}_n^{(j)} - A| \leq |\overline{A}_n^{(j)} - A_n^{(j)}| + |A_n^{(j)} - A|. \tag{4.1}$$

By the fact that $A_n^{(j)} = \sum_{i=0}^{n} \gamma_{ni}^{(j)} A(y_{j+i})$ and $\overline{A}_n^{(j)} = \sum_{i=0}^{n} \gamma_{ni}^{(j)} \overline{A}_{j+i}$, we also have

$$|\overline{A}_n^{(j)} - A_n^{(j)}| = \left| \sum_{i=0}^{n} \gamma_{ni}^{(j)} \varepsilon_{j+i} \right| \leq \Gamma_n^{(j)} \varepsilon; \quad \varepsilon = \max\{|\varepsilon_l| : \ j \leq l \leq j+n\}. \tag{4.2}$$

Therefore, substituting (4.2) in (4.1), we obtain

$$|\overline{A}_n^{(j)} - A| \leq \Gamma_n^{(j)} \varepsilon + |A_n^{(j)} - A| \tag{4.3}$$

for the absolute error, and when $A \neq 0$,

$$\frac{|\overline{A}_n^{(j)} - A|}{|A|} \leq \Gamma_n^{(j)} \frac{\varepsilon}{|A|} + \frac{|A_n^{(j)} - A|}{|A|} \tag{4.4}$$

for the relative error. The inequality in (4.3) implies that, practically speaking, the absolute error in $\overline{A}_n^{(j)}$ is at least of the order of the corresponding theoretical error in $A_n^{(j)}$, but it may be as large as $\Gamma_n^{(j)} \varepsilon$ if this quantity dominates. (Note that, being the *theoretical* error, $A_n^{(j)} - A$ is not affected by the errors ε_l committed in the computation of the $A(y_l)$ and is expected to tend to 0 as j or $n \to \infty$.)

When $A \neq 0$, the inequality in (4.4) can be used to obtain a good estimate of the relative error in $\overline{A}_n^{(j)}$. Suppose that the $A(y_l)$ have been computed to an accuracy of r significant decimal digits. Thus, ε is of order $10^{-r} E$, where $E = \max\{|A(y_l)|, \ j \leq l \leq j+n\}$. If the extrapolation process converges, then $A \approx A_n^{(j)}$. Combining these, we obtain $10^{-r} E / |A_n^{(j)}|$ as a good estimate of $\varepsilon / |A|$ that can be used in (4.4). In particular, if $\lim_{y \to 0+} A(y) = A \neq 0$, then $\varepsilon / |A|$ is of order 10^{-r} for all practical purposes. If, in addition, $\Gamma_n^{(j)}$ is of order 10^s for some positive integer s, then we see that the relative error in $\overline{A}_n^{(j)}$ is of order 10^{s-r} if $r > s$, i.e., $\overline{A}_n^{(j)}$ and A agree to at most $r - s$ significant decimal digits, even though $|A_n^{(j)} - A| / |A|$ may be very small. In other words, we have lost s of these digits when computing $\overline{A}_n^{(j)}$. Of course, if $r \leq s$, then $\overline{A}_n^{(j)}$ will be completely incorrect. This shows that the quality of $\overline{A}_n^{(j)}$ is poor when r is small and improves as r increases. Thus, if computations in a certain precision do not produce satisfactory results, then we may be able to remedy the situation by doubling the precision.

Let us now look at the problem of summation of the series $\sum_{k=1}^{\infty} k^{-z}$ whether it converges or not. As is well-known, this series converges for $\Re z > 1$ and defines the Riemann zeta function $\zeta(z)$, and $\zeta(z)$ can be continued analytically to the whole z-plane with the exception of the point $z = 1$ where it has a simple pole with residue 1. It is also known that, provided $z \neq 1$,

$$\tilde{S}_n = \sum_{k=1}^{n-1} k^{-z} \sim \zeta(z) + \frac{1}{1-z} \sum_{i=0}^{\infty} \binom{1-z}{i} B_i n^{-z-i+1} \quad \text{as } n \to \infty, \tag{4.5}$$

where B_i are the Bernoulli numbers.

Letting $y = n^{-1}$ and defining $A(y) = \tilde{S}_n$, we thus see that $A(y)$ satisfies (1.1) with $A = \zeta(z)$ and $\sigma_k = z + k - 2,\ k = 1, 2, \ldots$, and that y is a discrete variable that takes on the values $1, 1/2, 1/3, \ldots$. Also, $z \neq 1, 0, -1, -2, \ldots$, in this case. Since $\alpha_k = (1-z)^{-1}\binom{1-z}{k-1} B_{k-1},\ k = 1, 2, \ldots$; $B_{2i+1} = 0,\ i = 1, 2, \ldots$; and $B_i \neq 0$ otherwise, it follows from (4.5) that $\alpha_4 = \alpha_6 = \alpha_8 = \cdots = 0$, while the rest of the α_k are nonzero.

We have applied REP to $A(y)$ with $y_l = 1/(l+1),\ l = 0, 1, \ldots$, and with various values of z for which $\sum_{k=1}^{\infty} k^{-z}$ may be convergent or divergent. Thus, the approximation $A_n^{(j)}$ is obtained from the terms $k^{-z},\ k = 1, 2, \ldots, j+n+1$, i.e., from the first $j + n + 1$ terms of the infinite series.

The results of Table 1 (that have been obtained in quadruple-precision arithmetic) concern the convergence of the column sequences and illustrate Theorem 3.1 and Theorem 3.2. Since $\alpha_6 = 0$ but $\alpha_7 \neq 0$, we have by the corollary to Theorem 3.1 that $\Delta_5^{(j)} \sim C_5 y_j^{\Re\sigma_7}$ as $j \to \infty$, where we have defined $\Delta_n^{(j)} = |A_n^{(j)} - A|/|A|$. Consequently, $\Delta_5^{(j)}/\Delta_5^{(2j)} \sim (y_j/y_{2j})^{\Re\sigma_7} \sim 2^{\Re\sigma_7}$ as $j \to \infty$. Similarly, by Theorem 3.2, we have $\Gamma_5^{(2j)}/\Gamma_5^{(j)} \sim 2^n$ as $j \to \infty$. The numbers in Table 1 can be used to verify these conclusions.

Table 2 concerns the diagonal sequences. In Table 2 we see first that double-precision approximations $\overline{A}_n^{(0)}(\mathrm{d})$ to $A = \zeta(z)$ deteriorate due to the fact that $\Gamma_n^{(0)}$ are becoming exceedingly large. Next we see that quadruple-precision approximations $\overline{A}_n^{(0)}(\mathrm{q})$ do achieve much better accuracy using the same number of terms of the infinite series $\sum_{k=1}^{\infty} k^{-z}$.

Table 3 concerns the diagonal sequences as well. Its results (that have been obtained in double-precision arithmetic) show that better stability and also accuracy prevails when $|\Im\sigma_k|$ are increased, as was mentioned in the previous section.

5. Concluding remarks. In this work we have discussed and analyzed the application of REP to functions $A(y)$ that are of the form given in (1.1) and (1.2), with *arbitrary* and in general *complex* σ_k, to determine (or approximate) A, the limit or antilimit of $A(y)$ as $y \to 0+$. One may wonder whether methods other than REP can be applied to treat the same problem and how they compare with REP. We conclude this paper by commenting on this question.

As far as is known to us, there is one additional method that can be used for this purpose and this method is the Shanks [Sh] transformation (ST) that can best be implemented via the ϵ-algorithm of Wynn [W]. In connection with this the following are true:

1. Only ST can be applied when the σ_k are *not* known, provided we take $y_l = y_0\omega^l$ for some $\omega \in (0, 1)$. If the σ_k *are* known, then REP can also be applied with the same y_l and is much more economical than ST. Both methods are stable with this choice of the y_l.
2. Only REP can be applied with arbitrary $y_l \neq y_0\omega^l$, provided the σ_k are known. In this case REP is defined via the linear systems of (1.3).

TABLE 1

Relative errors $\Delta_5^{(j)} = |A_5^{(j)} - A|/|A|$ *and* $\Gamma_5^{(j)}$, $j = 0(10)190$, *for* $z = 2$ (*convergent* $\{\tilde{S}_n\}$), $z = 1 + 10\,i$ (*divergent but bounded* $\{\tilde{S}_n\}$), *and* $z = 0.5$ (*divergent and unbounded* $\{\tilde{S}_n\}$).

j	$z=2$		$z=1+10\,\mathrm{i}$		$z=0.5$	
	$\Delta_5^{(j)}$	$\Gamma_5^{(j)}$	$\Delta_5^{(j)}$	$\Gamma_5^{(j)}$	$\Delta_5^{(j)}$	$\Gamma_5^{(j)}$
0	$1.05D-05$	$3.02D+02$	$1.96D-04$	$1.40D+00$	$1.78D-04$	$6.94D+03$
10	$1.05D-09$	$1.28D+05$	$4.05D-09$	$1.45D+02$	$9.03D-08$	$4.48D+06$
20	$2.19D-11$	$1.95D+06$	$1.23D-10$	$2.09D+03$	$4.24D-09$	$7.05D+07$
30	$1.83D-12$	$1.14D+07$	$1.41D-11$	$1.20D+04$	$6.02D-10$	$4.13D+08$
40	$2.94D-13$	$4.18D+07$	$2.89D-12$	$4.39D+04$	$1.43D-10$	$1.52D+09$
50	$6.92D-14$	$1.17D+08$	$8.27D-13$	$1.23D+05$	$4.58D-11$	$4.28D+09$
60	$2.09D-14$	$2.76D+08$	$2.94D-13$	$2.89D+05$	$1.78D-11$	$1.01D+10$
70	$7.49D-15$	$5.73D+08$	$1.22D-13$	$5.99D+05$	$7.98D-12$	$2.09D+10$
80	$3.07D-15$	$1.08D+09$	$5.67D-14$	$1.13D+06$	$3.96D-12$	$3.96D+10$
90	$1.39D-15$	$1.91D+09$	$2.87D-14$	$1.99D+06$	$2.12D-12$	$6.97D+10$
100	$6.83D-16$	$3.17D+09$	$1.56D-14$	$3.31D+06$	$1.21D-12$	$1.16D+11$
110	$3.58D-16$	$5.03D+09$	$8.96D-15$	$5.24D+06$	$7.31D-13$	$1.84D+11$
120	$1.98D-16$	$7.67D+09$	$5.40D-15$	$7.99D+06$	$4.60D-13$	$2.80D+11$
130	$1.15D-16$	$1.13D+10$	$3.38D-15$	$1.18D+07$	$3.00D-13$	$4.14D+11$
140	$6.93D-17$	$1.62D+10$	$2.19D-15$	$1.69D+07$	$2.01D-13$	$5.94D+11$
150	$4.32D-17$	$2.27D+10$	$1.46D-15$	$2.37D+07$	$1.39D-13$	$8.31D+11$
160	$2.78D-17$	$3.12D+10$	$1.00D-15$	$3.25D+07$	$9.82D-14$	$1.14D+12$
170	$1.84D-17$	$4.19D+10$	$7.01D-16$	$4.37D+07$	$7.09D-14$	$1.53D+12$
180	$1.24D-17$	$5.55D+10$	$5.01D-16$	$5.78D+07$	$5.21D-14$	$2.03D+12$
190	$8.55D-18$	$7.24D+10$	$3.64D-16$	$7.54D+07$	$3.89D-14$	$2.65D+12$

TABLE 2

Relative (floating-point) errors $\overline{\Delta}_n^{(0)} = |\overline{A}_n^{(0)} - A|/|A|$ *and* $\Gamma_n^{(0)}$, $n = 1(1)20$, *for* $z = 2$ (*convergent* $\{\tilde{S}_n\}$) *and* $z = 0.5$ (*divergent and unbounded* $\{\tilde{S}_n\}$). $\overline{\Delta}_n^{(0)}(\mathrm{d})$ *and* $\overline{\Delta}_n^{(0)}(\mathrm{q})$ *are computed, respectively, in double precision (approximately* 16 *decimal digits) and in quadruple precision (approximately* 35 *decimal digits).*

n	$z=2$			$z=0.5$		
	$\overline{\Delta}_n^{(0)}(\mathrm{d})$	$\overline{\Delta}_n^{(0)}(\mathrm{q})$	$\Gamma_n^{(0)}$	$\overline{\Delta}_n^{(0)}(\mathrm{d})$	$\overline{\Delta}_n^{(0)}(\mathrm{q})$	$\Gamma_n^{(0)}$
1	$2.16D-01$	$2.16D-01$	$3.00D+00$	$6.53D-01$	$6.53D-01$	$5.83D+00$
2	$1.21D-02$	$1.21D-02$	$9.00D+00$	$8.79D-02$	$8.79D-02$	$5.77D+01$
3	$8.61D-04$	$8.61D-04$	$2.83D+01$	$5.95D-03$	$5.95D-03$	$3.21D+02$
4	$1.90D-05$	$1.90D-05$	$9.17D+01$	$6.93D-04$	$6.93D-04$	$1.54D+03$
5	$1.05D-05$	$1.05D-05$	$3.02D+02$	$1.78D-04$	$1.78D-04$	$6.94D+03$
6	$6.80D-07$	$6.80D-07$	$1.01D+03$	$5.10D-06$	$5.10D-06$	$2.99D+04$
7	$7.50D-08$	$7.50D-08$	$3.39D+03$	$2.90D-06$	$2.90D-06$	$1.25D+05$
8	$1.56D-08$	$1.56D-08$	$1.15D+04$	$3.96D-07$	$3.96D-07$	$5.13D+05$
9	$3.64D-10$	$3.65D-10$	$3.93D+04$	$1.46D-08$	$1.38D-08$	$2.07D+06$
10	$1.78D-10$	$1.83D-10$	$1.35D+05$	$5.98D-09$	$8.75D-09$	$8.26D+06$
11	$4.65D-11$	$2.15D-11$	$4.63D+05$	$9.49D-09$	$6.37D-10$	$3.26D+07$
12	$1.04D-10$	$6.38D-13$	$1.60D+06$	$2.22D-08$	$9.32D-11$	$1.28D+08$
13	$3.68D-10$	$3.62D-13$	$5.54D+06$	$2.56D-08$	$2.07D-11$	$4.97D+08$
14	$1.16D-09$	$2.38D-14$	$1.92D+07$	$8.22D-08$	$4.40D-13$	$1.92D+09$
15	$3.21D-09$	$3.03D-15$	$6.69D+07$	$4.48D-07$	$3.18D-13$	$7.38D+09$
16	$7.68D-09$	$6.18D-16$	$2.33D+08$	$2.55D-07$	$3.94D-14$	$2.83D+10$
17	$1.84D-08$	$1.28D-17$	$8.14D+08$	$6.79D-06$	$1.55D-15$	$1.08D+11$
18	$6.30D-08$	$7.78D-18$	$2.85D+09$	$4.69D-05$	$8.29D-16$	$4.10D+11$
19	$2.93D-07$	$9.02D-19$	$9.97D+09$	$2.07D-04$	$5.58D-17$	$1.55D+12$
20	$1.27D-06$	$3.05D-20$	$3.50D+10$	$7.38D-04$	$8.50D-18$	$5.87D+12$

TABLE 3

Relative (double-precision floating-point) errors $\overline{\Delta}_n^{(0)} = |\overline{A}_n^{(0)} - A|/|A|$ *and* $\Gamma_n^{(0)}$, $n = 1(1)20$, *for* $z = 1.5$ *and* $z = 1.5 + 20\,i$ *(convergent* $\{\tilde{S}_n\}$*) and for* $z = -0.5$ *and* $z = -0.5 + 20\,i$ *(divergent* $\{\tilde{S}_n\}$*).*

n	$z = 1.5$		$z = 1.5 + 20\,\mathrm{i}$		$z = -0.5$		$z = -0.5 + 20\,\mathrm{i}$	
	$\overline{\Delta}_n^{(0)}$	$\Gamma_n^{(0)}$	$\overline{\Delta}_n^{(0)}$	$\Gamma_n^{(0)}$	$\overline{\Delta}_n^{(0)}$	$\Gamma_n^{(0)}$	$\overline{\Delta}_n^{(0)}$	$\Gamma_n^{(0)}$
1	$3.07D-01$	$5.83D+00$	$2.24D-01$	$1.62D+00$	$1.63D+00$	$2.09D+00$	$8.90D-01$	$1.40D+00$
2	$2.30D-02$	$2.20D+01$	$2.05D-01$	$1.51D+00$	$3.99D-01$	$1.83D+01$	$7.91D-01$	$2.21D+00$
3	$1.64D-03$	$8.03D+01$	$1.96D-01$	$2.45D+00$	$2.19D-02$	$2.68D+02$	$7.44D-01$	$2.00D+00$
4	$6.99D-05$	$2.91D+02$	$7.58D-02$	$3.09D+00$	$6.75D-03$	$1.99D+03$	$8.58D-01$	$4.49D+00$
5	$2.67D-05$	$1.05D+03$	$8.32D-03$	$1.64D+00$	$1.36D-03$	$1.20D+04$	$1.46D-01$	$2.51D+00$
6	$1.46D-06$	$3.78D+03$	$9.89D-04$	$1.23D+00$	$1.95D-05$	$6.46D+04$	$1.85D-02$	$1.47D+00$
7	$2.55D-07$	$1.36D+04$	$1.13D-04$	$1.08D+00$	$3.73D-05$	$3.24D+05$	$2.44D-03$	$1.17D+00$
8	$4.54D-08$	$4.90D+04$	$1.26D-05$	$1.08D+00$	$3.84D-06$	$1.55D+06$	$3.11D-04$	$1.09D+00$
9	$3.78D-10$	$1.76D+05$	$1.38D-06$	$1.23D+00$	$4.05D-07$	$7.14D+06$	$3.89D-05$	$1.17D+00$
10	$6.79D-10$	$6.34D+05$	$1.51D-07$	$1.54D+00$	$3.62D-07$	$3.20D+07$	$4.80D-06$	$1.43D+00$
11	$4.04D-11$	$2.28D+06$	$1.66D-08$	$2.13D+00$	$7.43D-07$	$1.40D+08$	$5.89D-07$	$1.93D+00$
12	$3.60D-10$	$8.19D+06$	$1.83D-09$	$3.20D+00$	$1.26D-06$	$6.04D+08$	$7.20D-08$	$2.86D+00$
13	$9.47D-10$	$2.94D+07$	$2.02D-10$	$5.16D+00$	$1.74D-06$	$2.56D+09$	$8.79D-09$	$4.62D+00$
14	$1.81D-09$	$1.06D+08$	$2.24D-11$	$8.91D+00$	$4.32D-05$	$1.07D+10$	$1.07D-09$	$8.01D+00$
15	$5.09D-10$	$3.80D+08$	$2.50D-12$	$1.63D+01$	$4.99D-04$	$4.43D+10$	$1.31D-10$	$1.48D+01$
16	$2.10D-08$	$1.36D+09$	$2.80D-13$	$3.15D+01$	$3.49D-03$	$1.82D+11$	$1.61D-11$	$2.91D+01$
17	$1.53D-07$	$4.90D+09$	$3.24D-14$	$6.36D+01$	$2.00D-02$	$7.40D+11$	$1.95D-12$	$6.01D+01$
18	$7.61D-07$	$1.76D+10$	$3.51D-15$	$1.34D+02$	$1.11D-01$	$2.99D+12$	$1.47D-13$	$1.30D+02$
19	$3.14D-06$	$6.33D+10$	$1.06D-14$	$2.95D+02$	$5.82D-01$	$1.19D+13$	$8.22D-14$	$2.93D+02$
20	$1.16D-05$	$2.27D+11$	$3.09D-14$	$6.69D+02$	$2.77D+00$	$4.85D+13$	$1.62D-13$	$6.86D+02$

REFERENCES

[BRS] F.L. Bauer, H. Rutishauser, and E. Stiefel, *New aspects in numerical quadrature*, in Experimental Arithmetic, High Speed Computing, and Mathematics, AMS, Providence, RI, 1963, pp. 199–218.

[B] C. Brezinski, *A general extrapolation algorithm*, Numer. Math., 35 (1980), pp. 175–187.

[BS1] R. Bulirsch and J. Stoer, *Fehlerabschätzungen und Extrapolation mit rationalen Funktionen bei Verfahren vom Richardson-Typus*, Numer. Math., 6 (1964), pp. 413–427.

[BS2] R. Bulirsch and J. Stoer, *Asymptotic upper and lower bounds for results of extrapolation methods*, Numer. Math., 8 (1966), pp. 93–104.

[CM] M. Crouzeix and A.L. Mignot, *Analyse Numérique des Equations Différentielles*, 2nd ed., Masson, Paris, 1989.

[DR] P.J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, 2nd ed., Academic Press, New York, 1984.

[FS] W.F. Ford and A. Sidi, *An algorithm for a generalization of the Richardson extrapolation process*, SIAM J. Numer. Anal., 24 (1987), pp. 1212–1232.

[H] T. Håvie, *Generalized Neville type extrapolation schemes*, BIT, 19 (1979), pp. 204–213.

[Le] H. Le Ferrand, *The quadratic convergence of the topological epsilon algorithm for systems of nonlinear equations*, Numer. Algorithms, 3 (1992), pp. 273–284.

[L] D. Levin, *Development of non-linear transformations for improving convergence of sequences*, Int. J. Comput. Math., B3 (1973), pp. 371–388.

[Ly] J.N. Lyness, *An error functional expansion for N-dimensional quadrature with an integrand function singular at a point*, Math. Comp., 30 (1976), pp. 1–23.

[LN] J.N. Lyness and B.W. Ninham, *Numerical quadrature and asymptotic expansions*, Math. Comp., 21 (1967), pp. 162–178.

[N] I. Navot, *An extension of the Euler-Maclaurin summation formula to functions with a branch singularity*, J. Math. Phys., 40 (1961), pp. 271–276.

[Sc] C. Schneider, *Vereinfachte Rekursionen zur Richardson-Extrapolation in Spezialfällen*, Numer. Math., 24 (1975), pp. 177–184.

[Sh] D. Shanks, *Non-linear transformations of divergent and slowly convergent sequences*, J. Math. Phys., 34 (1955), pp. 1–42.

[Si1] A. Sidi, *Convergence properties of some nonlinear sequence transformations*, Math. Comp., 33 (1979), pp. 315–326.

[Si2] A. Sidi, *An algorithm for a special case of a generalization of the Richardson extrapolation process*, Numer. Math., 38 (1982), pp. 299–307.

[Si3] A. Sidi, *Euler-Maclaurin expansions for integrals over triangles and squares of functions having algebraic/logarithmic singularities along an edge*, J. Approx. Theory, 39 (1983), pp. 39–53.

[Si4] A. Sidi, *Convergence and stability properties of minimal polynomial and reduced rank extrapolation algorithms*, SIAM J. Numer. Anal., 23 (1986), pp. 197–209.

[Si5] A. Sidi, *Generalizations of Richardson extrapolation with applications to numerical integration*, in Numerical Integration III, Internat. Schriftenreihe Numer. Math. 85, H. Brass and G. Hämmerlin, eds., Birkhäuser, Basel, Boston, 1988, pp. 237–250.

[Si6] A. Sidi, *Quantitative and constructive aspects of the generalized Koenig's and de Montessus's theorems for Padé approximants*, J. Comp. Appl. Math., 29 (1990), pp. 257–291.

[Si7] A. Sidi, *On a generalization of the Richardson extrapolation process*, Numer. Math., 57 (1990), pp. 365–377.

[Si8] A. Sidi, *Application of vector-valued rational approximations to the matrix eigenvalue problem and connections with Krylov subspace methods*, SIAM J. Matrix Anal. Appl., 16 (1995), pp. 1341–1369.

[Si9] A. Sidi, *A complete convergence and stability theory for a generalized Richardson extrapolation process*, SIAM J. Numer. Anal., 34 (1997), pp. 1761–1778.

[SB] A. Sidi and J. Bridger, *Convergence and stability analysis for some vector extrapolation methods in the presence of defective iteration matrices*, J. Comp. Appl. Math., 22 (1988), pp. 35–61.

[SFS] A. Sidi, W.F. Ford, and D.A. Smith, *Acceleration of convergence of vector sequences*, SIAM J. Numer. Anal., 23 (1986), pp. 178–196.

[W] P. Wynn, *On a device for computing the $e_m(S_n)$ transformation*, Math. Tables Aids Comput., 10 (1956), pp. 91–96.