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Extrapolation Methods for Infinite Multiple Series and Integrals

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Abstract: Methods for the numerical evaluation of multi-dimensional infinite-range integrals and infinite series are reviewed. Particular emphasis is put on those methods that are based on the generalized Richardson extrapolation (GREP) of Sidi and the Levin-Sidi D- and d-transformations for one-dimensional integrals and series respectively. After summarizing in detail the essentials of the D- and d-transformations, the paper first discusses an approach due to Sidi that makes sequential use of the D- and d-transformations in multidimensional problems. Next a multi-dimensional version of GREP is introduced and recent generalizations of the D- and d-transformations that are due to Greif and Levin are discussed. All these are based on a careful analysis of the asymptotic expansions of suitably defined remainders.

Keywords: Convergence acceleration, multiple series, multi-dimensional integrals, multivariate extrapolation.

Mathematics Subject Classification: 40A05, 40A10, 40A25, 40A30, 40B05, 41A60, 41A63, 65B05, 65B10.

1 Introduction

The problem of accelerating the convergence of infinite multiple series and integrals by extrapolation methods has been of some interest recently.

The first work in multiple series acceleration was published by Chisholm [1]. In this work Chisholm defines the diagonal Padé approximants to double series of the form $f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}x^iy^j$. The 'diagonal' approximants considered in [1] are of the form $[n/n]_f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n} u_{ij}x^iy^j / \sum_{i=0}^{n}\sum_{j=0}^{n} v_{ij}x^iy^j$.

The nondiagonal approximants $[m/n]_f(x,y)$ were later defined by Graves-Morris, Hughes Jones, and Makinson [7]. The diagonal approximants of [1] were generalized to power series in N variables by Chisholm and McEwan [2] and the nondiagonal ones of [7] were generalized to N variables by

Hughes Jones [9]. General order Padé approximants for multiple power series were defined by Levin [11] and further developed by Cuyt [3], [4], and [5].

A general discussion on accelerating the convergence of infinite double series and integrals was presented in Levin [12]. A recent paper by Greif and Levin [8] combines the general idea in [12] with the discussion, together with an approach based on the D-transformation for 1-D infinite integrals and the d-transformation for 1-D infinite series due to Levin and Sidi [13]. Earlier Sidi [15] proposed an approach in which one uses of the d-transformation sequentially in the summation of multiple series. The same approach can be used in the computation of multiple infinite-range integrals.

In this paper we give a brief survey of some of these methods. We also give some of the details of an approach based on the asymptotic expansions and the generalized Richardson extrapolation process which lead to the D- and d-transformations [13].

2 GREP: A Generalized Richardson Extrapolation Process

As we would like to consider extensions to N dimensions of the D-transformation for 1-D infinite-range integrals and the d-transformation for 1-D infinite series, and as both of these methods are generalized Richardson extrapolation processes of the type that was defined in Sidi [16] and denoted GREP there, we start with a short review of GREP. In the following A represents the quantity to be estimated, i.e., the infinite sum of a series or the infinite integral; $A(y)$ stands for the computable quantities as the partial sums of the series or the finite parts of the infinite integral; E.g., for 1-D series, $A(y) = S_n$, the sum of the first n terms of the series, $y = 1/n$, and we are interested in the behavior of the remainder $A - A(y)$ as $y \to 0$.

Definition 2.1 We shall say that a function $A(y)$, defined for $y \in (0, b]$, for some $b > 0$, where y is a discrete or continuous variable, belongs to the set $\mathbf{F}^{(m)}$ for some positive integer m, if there exist functions $\phi_k(y)$ and $\beta_k(y)$, $k = 1, 2, \dots, m$, and a constant A, such that

$$A(y) = A + \sum_{k=1}^{m} \phi_k(y)\beta_k(y), \tag{2.1}$$

where the functions $\phi_k(y)$ are defined for $y \in (0, b]$, and $\beta_k(\xi)$, as functions of the continuous variable ξ, are continuous in $[0, \hat{\xi}]$ for some $\hat{\xi} \le b$, and for some constants $r_k > 0$, have Poincaré-type asymptotic expansions of the form

$$\beta_k(\xi) \sim \sum_{i=0}^{\infty} \beta_{ki}\xi^{ir_k} \quad \text{as } \xi \to 0+, \quad k = 1, \dots, m. \tag{2.2}$$

If, in addition, $B_k(t) \equiv \beta_k(t^{1/r_k})$, as a function of the continuous variable t, is infinitely differentiable in $[0, \hat{\xi}^{r_k}]$, $k = 1, \dots, m$, we shall say that $A(y)$ belongs to the set $\mathbf{F}^{(m)}_{\infty}$. (Thus, $\mathbf{F}^{(m)}_{\infty} \subset \mathbf{F}^{(m)}$.)

Combining (2.1) and (2.2), we see that $A(y)$ has an asymptotic expansion of the form

$$A(y) \sim A + \sum_{k=1}^{m} \phi_k(y) \sum_{i=0}^{\infty} \beta_{ki} y^{ir_k} \quad \text{as } y \to 0+ . \tag{2.3}$$

Obviously, when $\lim_{y\to 0+} \phi_k(y) = 0$, $k = 1, \dots, m$, we also have $\lim_{y\to 0+} A(y) = A$. When $\lim_{y\to 0+} A(y)$ does not exist, A is said to be the antilimit of $A(y)$ as $y \to 0+$. In case $\lim_{y\to 0+} A(y)$ does not exist, it is clear that $\lim_{y\to 0+} \phi_k(y)$ does not exist for at least one value of k.

We assume that $A(y)$ and the $\phi_k(y)$ are all known (or computable) for all possible values that y is allowed to assume in $(0, b]$, and that the r_k are known as well. We do not assume that the constants β_{ki} are known. We do not assume the functions $\phi_k(y)$ to have any particular structure. Nor do we assume that they satisfy $\phi_{k+1}(y) = o(\phi_k(y))$ as $y \to 0+$. Finally, we are interested in finding (or approximating) A, whether it is the limit or the antilimit of $A(y)$ as $y \to 0+$. We achieve this via the generalized Richardson extrapolation process that we denote GREP$^{(m)}$ and that we define next.

Definition 2.2 Let $A(y)$ belong to $\mathbf{F}^{(m)}$ with the notation of Definition 2.1. Pick a decreasing positive sequence $\{y_l\} \subset (0, b]$ such that $\lim_{l\to\infty} y_l = 0$. Let $n \equiv (n_1, n_2, \ldots, n_m)$, where $n_1, \ldots, n_m$ are nonnegative integers. Then the approximation $A_n^{(m,j)}$ to A, whether A is the limit or the antilimit of $A(y)$ as $y \to 0+$, is defined through the linear system

$$A(y_l) = A_n^{(m,j)} + \sum_{k=1}^{m} \phi_k(y_l) \sum_{i=0}^{n_k-1} \bar{\beta}_{ki} y_l^{ir_k}, \ j \le l \le j+N; \ N = \sum_{k=1}^{m} n_k, \tag{2.4}$$

$\bar{\beta}_{ki}$ being the additional (auxiliary) N unknowns. In (2.4), $\sum_{i=0}^{-1} c_i \equiv 0$ so that $A_{(0,\ldots,0)}^{(m,j)} = A(y_j)$ for all j. This generalization of the Richardson extrapolation process that generates the $A_n^{(m,j)}$ will be denoted GREP$^{(m)}$. When there is no room for confusion, we will write GREP instead of GREP$^{(m)}$ for short.

Comparing the equations in (2.4) with the asymptotic expansion of $A(y)$ that is given in (2.3), we realize that the former are obtained by truncating the latter at the terms $\beta_{k,n_k-1} y^{(n_k-1)r_k}$, $k = 1, \ldots, m$, replacing the β_{ki} by $\bar{\beta}_{ki}$, and finally collocating at $y = y_l$, $j \le l \le j + N$, where $N = \sum_{k=1}^{m} n_k$. Figuratively speaking, in producing the approximation $A_n^{(m,j)}$, GREP$^{(m)}$ "eliminates" the terms y^{ir_k}, $0 \le i \le n_k - 1$, $1 \le k \le m$, from the asymptotic expansion in (2.3).

Two limiting processes for the $A_n^{(m,j)}$ have been considered: (i) Process I, in which n_k are held fixed and $j \to \infty$, and (ii) Process II, in which j is held fixed and all $n_k \to \infty$ simultaneously. All of our numerical experience and theoretical results indicate that Process II has better convergence properties. In particular, the sequence of approximations $\{A_{(\nu,\nu,\ldots,\nu)}^{(m,j)}\}_{\nu=0}^{\infty}$ has excellent convergence. When $m = 1$, the $A_n^{(1,j)}$ can all be computed by the W-algorithm of Sidi [18]. When $m > 1$ and the r_k are all the same, the W$^{(m)}$-algorithm of Ford and Sidi [6] produces the sequence $\{A_{(n_1,\ldots,n_m)}^{(m,j)} : n_k = \lfloor (p-k)/m \rfloor, \ k = 1, \ldots, m\}_{p=0}^{\infty}$, which has $\{A_{(\nu,\nu,\ldots,\nu)}^{(m,j)}\}_{\nu=0}^{\infty}$ as a subsequence. Both algorithms are recursive and have very low computational cost and storage requirements. A FORTRAN 77 program that implements the W$^{(m)}$-algorithm is given in [6, Appendix B], and can be used with both the D- and d-transformations, for which r_k are all equal to 1, as we will see in the next two sections.

3 The $D^{(m)}$-Transformation for 1-D Infinite-Range Integrals

We now discuss the D-transformation of [13] for infinite-range integrals. We start by defining two classes of functions that we denote $\mathbf{A}^{(\gamma)}$ and $\mathbf{B}^{(m)}$.

Definition 3.1 A function $\alpha(x)$ belongs to the set $\mathbf{A}^{(\gamma)}$ if it is infinitely differentiable for all $x \ge a$ and has a Poincaré-type asymptotic expansion of the form

$$\alpha(x) \sim \sum_{i=0}^{\infty} \alpha_i x^{\gamma-i} \quad \text{as } x \to \infty, \tag{3.1}$$

and its derivatives have Poincaré-type asymptotic expansions obtained by differentiating that in (3.1) formally term by term. If, in addition, $\alpha_0 \neq 0$ in (3.1), then $\alpha(x)$ is said to belong to $\mathbf{A}^{(\gamma)}$ strictly. Here γ is complex in general.

Definition 3.2 A function $f(x)$ that is infinitely differentiable on (a, ∞) belongs to the set $\mathbf{B}^{(m)}$ if it satisfies a linear homogeneous ordinary differential equation (ODE) of order m of the form

$$f(x) = \sum_{k=1}^{m} p_k(x) f^{(k)}(x), \tag{3.2}$$

where $p_k \in \mathbf{A}^{(k)},\ k = 1, \ldots, m$.

Theorem 3.3 below was given in [13] and forms the basis of the D-transformation.

Theorem 3.3 *Let $f(x)$ be a function in $\mathbf{B}^{(m)}$ that is also integrable at infinity. Assume, in addition, that*

$$\lim_{x\to\infty} p_k^{(j-1)}(x) f^{(k-j)}(x) = 0, \quad k = j, j+1, \ldots, m,\ j = 1, 2, \ldots, m, \tag{3.3}$$

and that

$$\sum_{k=1}^{m} l(l-1)\cdots(l-k+1)\bar{p}_k \neq 1, \quad l = \pm 1, 2, 3, \ldots, \tag{3.4}$$

where

$$\bar{p}_k = \lim_{x\to\infty} x^{-k} p_k(x), \quad k = 1, \ldots, m. \tag{3.5}$$

Define

$$I[f] = \int_a^\infty f(t)\,dt \quad \text{and} \quad F(x) = \int_a^x f(t)\,dt. \tag{3.6}$$

Then

$$F(x) = I[f] + \sum_{k=0}^{m-1} x^{\rho_k} f^{(k)}(x) g_k(x) \tag{3.7}$$

for some integers $\rho_k \leq k+1$ and functions $g_k \in \mathbf{A}^{(0)},\ k = 0, 1, \ldots, m-1$. Actually, if $p_k \in \mathbf{A}^{(i_k)}$ strictly for some integer $i_k \leq k,\ k = 1, \ldots, m$, then

$$\rho_k \leq \bar{\rho}_k \equiv \max(i_{k+1}, i_{k+2} - 1, \ldots, i_m - m + k + 1) \leq k+1,\ k = 0, 1, \ldots, m-1. \tag{3.8}$$

Equality holds in (3.8) when the integers whose maximum is being considered are distinct. Finally, being in $\mathbf{A}^{(0)}$, the functions $g_k(x)$ have asymptotic expansions of the form

$$g_k(x) \sim \sum_{i=0}^{\infty} g_{ki} x^{-i} \quad \text{as } x \to \infty. \tag{3.9}$$

Remarks.

1. By (3.5), $\bar{p}_k \neq 0$ if and only if $p_k \in \mathbf{A}^{(k)}$ strictly. Thus, whenever $p_k \in \mathbf{A}^{(i_k)}$ with $i_k < k$, we have $\bar{p}_k = 0$. This implies that whenever $i_k < k$, $k = 1, \ldots, m$, we have $\bar{p}_k = 0$, $k = 1, \ldots, m$, and the condition in (3.4) is automatically satisfied as the left-hand side of the inequality there is zero for all values of l.

2. It follows from (3.8) that $\rho_{m-1} = i_m$ always.

3. Similarly, for $m = 1$ we have $\rho_0 = i_1$ precisely.

4. For numerous examples that we have treated equality seems to hold in (3.8) for all $k = 1, \ldots, m$.

5. The integers ρ_k and the functions $g_k(x)$ in (3.7) depend only on the functions $p_k(x)$ in the ODE (3.2). This being the case, they are the same for all solutions $f(x)$ of (3.2) that are integrable at infinity and that satisfy (3.3).

6. From (3.3) and (3.8) we also have that $\lim_{x\to\infty} x^{\bar{\rho}_k} f^{(k)}(x) = 0,\ k = 0, 1, \ldots, m-1$. Thus, $\lim_{x\to\infty} x^{\rho_k} f^{(k)}(x) = 0,\ k = 0, 1, \ldots, m-1$, as well.

Making the analogy $F(x) \leftrightarrow A(y),\ x^{-1} \leftrightarrow y,\ x^{\rho_{k-1}} f^{(k-1)}(x) \leftrightarrow \phi_k(y)$ and $r_k = 1,\ k = 1, \ldots, m$, and $I[f] \leftrightarrow A$, we realize that $A(y)$ is in $\mathbf{F}^{(m)}$. Actually, $A(y)$ is even in $\mathbf{F}_\infty^{(m)}$ because of the differentiability conditions imposed on $f(x)$ and the $p_k(x)$. Finally, the variable y is continuous for this case.

The examples that we have studied seem to indicate that the requirement that $f(x)$ be in $\mathbf{B}^{(m)}$ for some m is the most crucial of the conditions in Theorem 3.3. The rest of the conditions, namely, (3.3)–(3.5), seem to be satisfied automatically. Therefore, in order to decide whether $A(y) \equiv F(x)$, where $y = x^{-1}$, is in $\mathbf{F}^{(m)}$ for some m it is practically sufficient to check whether $f(x)$ is in $\mathbf{B}^{(m)}$.

Finally, even though Theorem 3.3 is stated for functions $f \in \mathbf{B}^{(m)}$ that are integrable at infinity, $F(x)$ may satisfy (3.7)–(3.9) also when $f \in \mathbf{B}^{(m)}$ but is not integrable at infinity, at least in some cases. In such a case the constant $I[f]$ in (3.7) will, of course, be the antilimit of $F(x)$ as $x \to \infty$. In the papers Sidi [19], [20], and [25] it is shown that (3.7)–(3.9) hold (i) for *all* functions $f(x)$ in $\mathbf{B}^{(1)}$ that are either (i) integrable at infinity, or (ii) not integrable there but for which the antilimit $I[f]$ is either the Abel sum or the Hadamard Finite Part of the divergent integral $\int_a^\infty f(t)\,dt$ (such functions grow at most like a power of x as $x \to \infty$).

Replacing each ρ_k in (3.7) by its upper bound $k+1$, which is allowed mathematically, and applying the formalism of Definition 2.2 on GREP, we can now define the D-transformation.

Definition 3.4 Pick an increasing positive sequence $\{x_l\} \subset (a, \infty)$ such that $\lim_{l\to\infty} x_l = \infty$. Let $n \equiv (n_1, n_2, \ldots, n_m)$, where $n_1, \ldots, n_m$ are nonnegative integers. Then the approximation $D_n^{(m,j)}$ to $I[f]$ is defined through the linear system

$$F(x_l) = D_n^{(m,j)} + \sum_{k=1}^{m} x_l^k f^{(k-1)}(x_l) \sum_{i=0}^{n_k-1} \frac{\bar{\beta}_{ki}}{x_l^i}, \quad j \le l \le j+N;\ N = \sum_{k=1}^{m} n_k, \tag{3.10}$$

$\bar{\beta}_{ki}$ being the additional (auxiliary) N unknowns. In (3.10), $\sum_{i=0}^{-1} c_i \equiv 0$ so that $D_{(0,\ldots,0)}^{(m,j)} = F(x_j)$ for all j. We will call this GREP that generates the $D_n^{(m,j)}$ the $D^{(m)}$-transformation. When there is no room for confusion, we will call it the D-transformation for short.

This definition of the D-transformation was given in [25], and differs from the original definition of [13] in that we have replaced the ρ_k by their known upper bounds $k+1$. As it does not necessitate knowledge of the ρ_k, it is also more user-friendly. Of course, if we are aware of the precise values of the $\bar{\rho}_k$ or some upper bounds for them, we should use those and replace the $x_l^k f^{(k-1)}(x_l)$ in (3.10) by $x_l^{\bar{\rho}_{k-1}} f^{(k-1)}(x_l)$, as this reduces the numerical cost for a given required level of accuracy. In some important cases involving integral transforms the $\bar{\rho}_k$ are readily available, see Sidi [17] and [19].

To apply the $D^{(m)}$-transformation we have to decide what m is. This can be done in one of two ways. (i) By trial and error. We start with $m = 1$, and increase m if necessary until good convergence acceleration is achieved. (ii) By estimating m or an upper bound for it mathematically. Here we can use the rules of thumb that if $u \in \mathbf{B}^{(r)}$ and $v \in \mathbf{B}^{(s)}$, then (a) $uv \in \mathbf{B}^{(m)}$ with $m \leq rs$, and (b) $u+v \in \mathbf{B}^{(m)}$ with $m \leq r+s$.

In case $f(x)$ and/or some its derivatives vanish an infinite number of times at infinity, we can choose the x_l appropriately to eliminate some of the terms $x^{\rho_k} f^{(k)}(x) g_k(x)$ from (3.7). This reduces the computational cost and improves numerical stability. This approach was proposed in Sidi [17]. The resulting methods are denoted the $\bar{D}$- and $\tilde{D}$-transformations. For the application of the $\bar{D}$-transformation to Bessel function integrals see also Sidi [24]. A different approach along similar lines and its accompanying method, the mW-transformation, were later proposed in Sidi [21]. The mW-transformation seems to be one of the most effective methods for computing oscillatory infinite-range integrals.

4 The $d^{(m)}$-Transformation for 1-D Infinite Series

We next discuss the $d^{(m)}$-Transformation of [13] for infinite series. We start with a set of functions $\mathbf{A}_0^{(\gamma)}$.

Definition 4.1 A function $\alpha(x)$ defined for all $x \geq a$ with some $a \geq 0$ is in the set $\mathbf{A}_0^{(\gamma)}$ if it has a Poincaré-type asymptotic expansion of the form

$$\alpha(x) \sim \sum_{i=0}^{\infty} \alpha_i x^{\gamma - i} \quad \text{as } x \to \infty. \tag{4.1}$$

If, in addition, $\alpha_0 \neq 0$ in (4.1), then $\alpha(x)$ is said to belong to $\mathbf{A}_0^{(\gamma)}$ strictly. Here γ is complex in general.

Note that we have not required the functions in $\mathbf{A}_0^{(\gamma)}$ to have any differentiability properties. Therefore, $\mathbf{A}_0^{(\gamma)} \supset \mathbf{A}^{(\gamma)}$.

We next define a family of sequences $\mathbf{b}^{(m)}$ that is a discrete counterpart of $\mathbf{B}^{(m)}$.

Definition 4.2 A sequence $\{a_n\}$ belongs to the set $\mathbf{b}^{(m)}$ if it satisfies a linear homogeneous difference equation of order m of the form

$$a_n = \sum_{k=1}^{m} p_k(n) \Delta^k a_n, \tag{4.2}$$

where $p_k \in \mathbf{A}_0^{(k)}$, $k = 1, \ldots, m$. Here $\Delta^0 a_n = a_n$, $\Delta^1 a_n = \Delta a_n = a_{n+1} - a_n$, and $\Delta^k a_n = \Delta(\Delta^{k-1} a_n)$, $k = 2, 3, \ldots$.

The next theorem was stated in [13] and is a discrete counterpart of Theorem 3.3.

Theorem 4.3 *Let the sequence $\{a_n\}$ be in $\mathbf{b}^{(m)}$ and let $\sum_{k=1}^{\infty} a_k$ be a convergent series. Assume, in addition, that*

$$\lim_{n\to\infty} \left(\Delta^{j-1} p_k(n)\right)\left(\Delta^{k-j} a_n\right) = 0, \quad k = j, j+1, \ldots, m, \quad j = 1, 2, \ldots, m, \tag{4.3}$$

and that

$$\sum_{k=1}^{m} l(l-1)\cdots(l-k+1)\bar{p}_k \neq 1, \quad l = \pm 1, 2, 3, \ldots, \tag{4.4}$$

where

$$\bar{p}_k = \lim_{n\to\infty} n^{-k} p_k(n), \quad k = 1, \ldots, m. \tag{4.5}$$

Define

$$S(\{a_k\}) = \sum_{k=1}^{\infty} a_k \quad \textit{and} \quad A_n = \sum_{k=1}^{n} a_k, \quad n = 1, 2, \ldots\ . \tag{4.6}$$

Then

$$A_{n-1} = S(\{a_k\}) + \sum_{k=0}^{m-1} n^{\rho_k} (\Delta^k a_n) g_k(n) \tag{4.7}$$

for some integers $\rho_k \le k+1$, and functions $g_k \in \mathbf{A}_0^{(0)},\ k = 0, 1, \ldots, m-1$. Actually, if $p_k \in \mathbf{A}_0^{(i_k)}$ strictly for some integer $i_k \le k,\ k = 1, \ldots, m$, then

$$\rho_k \le \bar{\rho}_k \equiv \max(i_{k+1}, i_{k+2} - 1, \ldots, i_m - m + k + 1) \le k+1, \quad k = 0, 1, \ldots, m-1. \tag{4.8}$$

Equality holds in (4.8) when the integers whose maximum is being considered are distinct. Finally, being in $\mathbf{A}_0^{(0)}$, the functions $g_k(n)$ have asymptotic expansions of the form

$$g_k(n) \sim \sum_{i=0}^{\infty} g_{ki} n^{-i} \quad \textit{as } n \to \infty. \tag{4.9}$$

Remarks.

1. By (4.5), $\bar{p}_k \ne 0$ if and only if $p_k \in \mathbf{A}_0^{(k)}$ strictly. Thus, whenever $p_k \in \mathbf{A}_0^{(i_k)}$ with $i_k < k$, we have $\bar{p}_k = 0$. This implies that whenever $i_k < k,\ k = 1, \ldots, m$, we have $\bar{p}_k = 0,\ k = 1, \ldots, m$, and the condition in (4.4) is automatically satisfied.
2. It follows from (4.8) that $\rho_{m-1} = i_m$ always.
3. Similarly, for $m = 1$ we have $\rho_0 = i_1$ precisely.
4. For numerous examples that we have treated equality seems to hold in (4.8) for all $k = 1, \ldots, m$.
5. The integers ρ_k and the functions $g_k(n)$ in (4.7) depend only on the functions $p_k(n)$ in the difference equation in (4.2). This being the case, they are the same for all solutions a_n of (4.2) that satisfy (4.3) and for which $\sum_{k=1}^{\infty} a_k$ converges.
6. From (4.3) and (4.8) we also have that $\lim_{n\to\infty} n^{\bar{\rho}_k} \Delta^k a_n = 0,\ k = 0, 1, \ldots, m-1$, as well.

Making the analogy $A_{n-1} \leftrightarrow A(y),\ n^{-1} \leftrightarrow y,\ n^{\rho_{k-1}} \Delta^{k-1} a_n \leftrightarrow \phi_k(y)$ and $r_k = 1,\ k = 1, \ldots, m$, and $S(\{a_k\}) \leftrightarrow A$, we realize that $A(y)$ is in $\mathbf{F}^{(m)}$. Finally, the variable y is discrete for this case and assumes the values $1, 1/2, 1/3, \ldots$.

The examples that we have studied seem to indicate that the requirement that $\{a_n\}$ be in $\mathbf{b}^{(m)}$ for some m is the most crucial of the conditions in Theorem 4.3. The rest of the conditions, namely, (4.3)–(4.5), seem to be satisfied automatically. Therefore, in order to decide whether $A(y) \equiv A_{n-1}$, where $y = n^{-1}$, is in $\mathbf{F}^{(m)}$ for some m it is practically sufficient to check whether $\{a_n\}$ is in $\mathbf{b}^{(m)}$.

Finally, even though Theorem 4.3 is stated for sequences $\{a_n\} \in \mathbf{b}^{(m)}$ for which $\sum_{k=1}^{\infty} a_k$ converges, A_n may satisfy (4.7)–(4.9) also when $\{a_n\} \in \mathbf{b}^{(m)}$ but $\sum_{k=1}^{\infty} a_k$ diverges, at least in some cases. In such a case the constant $S(\{a_k\})$ in (4.7) will, of course, be the antilimit of A_n as $n \to \infty$. In the papers Sidi [22] and [23] it is shown that (4.7)–(4.9) hold for *all* sequences $\{a_n\}$ in

$\mathbf{b}^{(1)}$ for which either (i) $\sum_{k=1}^{\infty} a_k$ converges, or (ii) $\sum_{k=1}^{\infty} a_k$ diverges, its antilimit $S(\{a_k\})$ being defined in some summability sense (such a_n grow at most like a power of n as $n \to \infty$).

Replacing each ρ_k in (4.7) by its upper bound $k+1$, which is allowed mathematically, by adding a_n to both sides of (4.7), and applying the formalism of Definition 2.2 on GREP, we can now define the d-transformation.

Definition 4.4 Pick a sequence of integers $\{R_l\}_{l=0}^{\infty}$, $1 \le R_0 < R_1 < R_2 < \cdots$. Let $n \equiv (n_1, \ldots, n_m)$, where $n_1, \ldots, n_m$ are nonnegative integers. Then the approximation $d_n^{(m,j)}$ to $S(\{a_k\})$ is defined through the linear system

$$A_{R_l} = d_n^{(m,j)} + \sum_{k=1}^{m} R_l^k (\Delta^{k-1} a_{R_l}) \sum_{i=0}^{n_k - 1} \frac{\bar{\beta}_{ki}}{R_l^i}, \quad j \le l \le j+N; \quad N = \sum_{k=1}^{m} n_k, \tag{4.10}$$

$\bar{\beta}_{ki}$ being the additional (auxiliary) N unknowns. In (4.10) $\sum_{i=0}^{-1} c_i \equiv 0$ so that $d_{(0,\ldots,0)}^{(m,j)} = A_j$ for all j. We call this GREP that generates the $d_n^{(m,j)}$ the $d^{(m)}$-transformation. When there is no room for confusion, we will call it the d-transformation for short.

This definition of the d-transformation was given in [6], and differs from the original definition of [13] in that we have replaced the ρ_k by their known upper bounds $k+1$. As it does not necessitate knowledge of the ρ_k, it is also more user-friendly. Of course, if we are aware of the precise values of the $\bar{\rho}_k$ or some upper bounds for them, we should use those and replace the $R_l^k(\Delta^{k-1} a_{R_l})$ in (4.10) by $R_l^{\bar{\rho}_{k-1}}(\Delta^{k-1} a_{R_l})$, as this reduces the numerical cost for a given required level of accuracy. In some important cases the $\bar{\rho}_k$ are readily available. One such case is that of power series, as discussed in Sidi and Levin [26]. Another is that of Fourier series and generalizations, see Sidi [22]. In both cases $\bar{\rho}_k = 0$ for all k.

Finally, the d-transformation as defined here reduces to the u-transformation of Levin [10] when $m = 1$ and the R_l are chosen as $R_l = l+1$, $l = 0, 1, \ldots$. The u-transformation has been observed to be a most effective convergence acceleration method for infinite series $\sum_{k=1}^{\infty} a_k$ when $\{a_k\} \in \mathbf{b}^{(1)}$, see Smith and Ford [27] and [28].

To apply the $d^{(m)}$-transformation we have to decide what m is. This can be done in one of two ways. (i) By trial and error. We start with $m = 1$, and increase m if necessary until good convergence acceleration is achieved. (ii) By estimating m or an upper bound for it mathematically. Here we can use the rules of thumb that if $\{u_n\} \in \mathbf{b}^{(r)}$ and $\{v_n\} \in \mathbf{b}^{(s)}$, then (a) $\{u_n v_n\} \in \mathbf{b}^{(m)}$ with $m \le rs$, and (b) $\{u_n + v_n\} \in \mathbf{b}^{(m)}$ with $m \le r+s$.

5 Sequential Transformations for s-D Integrals and Series

The computation of multiple integrals and series can be achieved by sequential use of the D- and d-transformations under suitable conditions. This approach was first suggested in Sidi [15, Chapter 4, Section 6] in relation to double infinite series, where it was also justified theoretically and illustrated numerically with convincing examples. In this section we give a brief description of the approach of [15].

To make the presentation simple, and for future use as well, we introduce here some notation:

$$\mathbf{y} = (y_1, \ldots, y_s), \quad \mathbf{0} = (0, \ldots, 0), \quad \mathbf{1} = (1, \ldots, 1),$$

$$\mathbf{u} \ge \mathbf{v} \iff u_j \ge v_j, \quad j = 1, \ldots, s,$$

$$\mathbb{R}_0^s = \{\mathbf{t} \mid \mathbf{t} \ge \mathbf{0}\}, \quad \mathbb{R}_{\mathbf{x}}^s = \{\mathbf{t} \mid \mathbf{t} \ge \mathbf{x}\},$$

$$\mathbb{Z}^s = \{\mathbf{i} = (i_1, \ldots, i_s),\ i_j \text{ integers}\}, \quad \mathbb{Z}_0^s = \{\mathbf{i} \in \mathbb{Z}^s \mid \mathbf{i} \ge \mathbf{0}\},$$

$$\mathbb{Z}_{\mathbf{r}}^s = \{\mathbf{i} \in \mathbb{Z}_0^s \mid \mathbf{i} \ge \mathbf{r}\}, \quad \mathbb{Z}_+^s = \mathbb{Z}_{\mathbf{1}}^s.$$

5.1 Sequential D-Transformation for s-D Integrals

Let us consider the s-D integral $I[f] = \int_{\mathbb{R}_0^s} f(\mathbf{t})\,d\mathbf{t}$, where we have denoted $\mathbf{t} = (t_1, \ldots, t_s)$, and $d\mathbf{t} = \prod_{j=1}^{s} dt_j$, and let us define

$$H_1(t_1, \ldots, t_s) = f(\mathbf{t}) = f(t_1, \ldots, t_s),$$
$$H_{k+1}(t_{k+1}, \ldots, t_s) = \int_0^\infty H_k(t_k, \ldots, t_s)\,dt_k, \quad k = 1, \ldots, s-1.$$

Then $I[f] = \int_0^\infty H_s(t_s)\,dt_s$. Let us now assume that, for each k and for fixed $t_{k+1}, \ldots, t_s$, and as a function of t_k, $H_k(t_k, \ldots, t_s) \in \mathbf{B}^{(m_k)}$ for some integer m_k. (This assumption seems to hold when $f(\mathbf{t})$, as a function of the variable t_k—the rest of the variables being held fixed—is in $\mathbf{B}^{(m_k)}$.) This means that we can compute $H_{k+1}(t_{k+1}, \ldots, t_s)$ by applying the $D^{(m_k)}$-transformation to the integral $\int_0^\infty H_k(t_k, \ldots, t_s)\,dt_k$. The computation of $I[f]$ can thus be completed by applying the $D^{(m_s)}$-transformation to the integral $\int_0^\infty H_s(t_s)\,dt_s$.

It is very easy to see that the assumption above is automatically satisfied when $f(\mathbf{x}) = \prod_{j=1}^{s} f_j(x_j)$, with $f_j \in \mathbf{B}^{(m_j)}$ for some integers m_j. This then serves as the motivation for the sequential use of the D-transformation.

As an example, consider the function $f(x, y) = e^{-ax}u(y)/(x + g(y))$, where a is a constant with $\Re a > 0$, $u(y) \in \mathbf{B}^{(q)}$, and $g(y) \in \mathbf{A}^{(r)}$ for some positive integer r and $g(y) > 0$ for all large y. (We have $q = 2$ when $u(y) = \cos by$ or $u(y) = J_\nu(by)$, for example.) First, $f(x, y)$ is in $\mathbf{B}^{(1)}$ as a function of x (with fixed y) and $f(x, y)$ is in $\mathbf{B}^{(q)}$ as a function of y (with fixed x). Next, invoking the relation $1/c = \int_0^\infty e^{-c\xi}\,d\xi$, $\Re c > 0$, we can show that

$$H_2(y) = \int_0^\infty f(x, y)\,dx = u(y) \int_0^\infty e^{-\xi g(y)}/(a + \xi)\,d\xi.$$

Applying Watson's lemma (see Olver [14]) to this integral, we see that $H_2(y)$ has an asymptotic expansion of the form

$$H_2(y) \sim u(y) \sum_{i=0}^{\infty} \alpha_i [g(y)]^{-i-1} \sim u(y) \sum_{i=0}^{\infty} \delta_i y^{-i-r} \quad \text{as } y \to \infty.$$

This implies that $H_2(y) \in \mathbf{B}^{(q)}$.

5.2 Sequential d-Transformation for s-D Series

The sequential use of the d-transformation for computing s-D infinite series is analogous to the use of the D-transformation for s-D integrals. Let us now consider the s-D infinite series $S(\{a_\mathbf{i}\}) = \sum_{\mathbf{i} \in \mathbb{Z}_+^s}^{\infty} a_\mathbf{i}$, and define

$$L_1(i_1, \ldots, i_s) = a_\mathbf{i} = a_{i_1, \ldots, i_s},$$
$$L_{k+1}(i_{k+1}, \ldots, i_s) = \sum_{i_k=1}^{\infty} L_k(i_k, \ldots, i_s), \quad k = 1, \ldots, s-1.$$

Hence $S(\{a_\mathbf{i}\}) = \sum_{i_s=1}^{\infty} L_s(i_s)$. Let us assume that, for each k and for fixed $i_{k+1}, \ldots, i_s$, the sequence $\{L_k(i_k, \ldots, i_s)\}_{i_k=1}^{\infty}$ is in $\mathbf{b}^{(m_k)}$ for some integer m_k. (This assumption seems to hold when $\{a_\mathbf{i}\}_{i_k=1}^{\infty} \in \mathbf{b}^{(m_k)}$, for each k and for $i_{k+1}, \ldots, i_s$ fixed.) Therefore, we can compute $L_{k+1}(i_{k+1}, \ldots, i_s)$ by applying the $d^{(m_k)}$-transformation to the series $\sum_{i_k=1}^{\infty} L_k(i_k, \ldots, i_s)$, and the computation of $S(\{a_{\mathbf{i}}\})$ can be completed by applying the $d^{(m_s)}$-transformation to the series $\sum_{i_s=1}^{\infty} L_s(i_s)$.

What motivates this approach to the summation of s-D series is the fact that the assumption above is automatically satisfied when $a_{\mathbf{i}} = \prod_{j=1}^{s} a_{i_j}^{(j)}$, with $\{a_i^{(j)}\}_{i=1}^{\infty} \in \mathbf{b}^{(m_j)}$ for some integers m_j.

As an example consider the double series $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}$, where $a_{j,k} = x^j u_k/(j+g(k))$, where $|x|<1$, $\{u_k\} \in \mathbf{b}^{(q)}$, and $g(k) \in \mathbf{A}_0^{(r)}$ for some positive integer r and $g(k) > 0$ for all large k. (We have $q = 2$ when $u_k = \cos k\theta$ or $u_k = P_k(y)$, the kth Legendre polynomial, for example.) First, $\{a_{j,k}\}_{j=1}^{\infty} \in \mathbf{b}^{(1)}$ with fixed k, while $\{a_{j,k}\}_{k=1}^{\infty} \in \mathbf{b}^{(q)}$ with fixed j. Next, invoking the relation $1/c = \int_0^{\infty} e^{-c\xi}\,d\xi$, $\Re c > 0$, we can show that

$$L_2(k) = \sum_{j=1}^{\infty} a_{j,k} = x u_k \int_0^{\infty} e^{-\xi g(k)}/(e^{\xi} - x)\,d\xi.$$

Applying Watson's lemma, we can show that $L_2(k)$ has the asymptotic expansion

$$L_2(k) \sim u_k \sum_{i=0}^{\infty} \alpha_i [g(k)]^{-i-1} \sim u_k \sum_{i=0}^{\infty} \delta_i k^{-i-r} \quad \text{as } k \to \infty.$$

Therefore, $\{L_2(k)\} \in \mathbf{b}^{(q)}$.

The following examples have been taken from Sidi [15]:

Example 5.1 Consider the double power series

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k} x^{j-1} y^{k-1}, \quad c_{j,k} = \frac{1}{j^2 + k^3}.$$

Since $\{c_{j,k}x^{j-1}y^{k-1}\}_{j=1}^{\infty} \in \mathbf{b}^{(1)}$ and $\{c_{j,k}x^{j-1}y^{k-1}\}_{k=1}^{\infty} \in \mathbf{b}^{(1)}$, we apply the sequential d-transformation with $p = 1$ and $q = 1$. Using about 100 terms of the series, in double precision arithmetic (approximately 14 decimal digits), this method produces the results shown in Table 1.

x	y	approximation
-0.5	-0.5	0.3843515211843
-1.0	-1.0	0.3149104237
-1.5	-1.5	0.26744390
-2.0	-2.0	0.2337732
-2.5	-2.5	0.207640

Table 1: Numerical results by the sequential d-transformation on the the double power series of Example 5.1.

Note that the series diverges when $|x| > 1$ or $|y| > 1$, but the method produces its sum very efficiently. (The accuracy decreases as the rate of divergence increases, since the absolute errors in the partial sums of the series increase in finite-precision arithmetic in this case.) The series converges very slowly when $|x| = 1$ or $|y| = 1$, and the method produces very accurate results for such x and y.

Example 5.2 Consider the double Fourier sine series

$$U(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} c_{j,k} \sin\left(\frac{j\pi x}{a}\right) \sin\left(\frac{k\pi y}{b}\right), \quad c_{j,k} = \frac{32}{\pi^4} \cdot \frac{1}{jk(j^2/a^2 + k^2/b^2)}.$$

The function $U(x,y)$ is the solution of the 2-D Poisson equation $\Delta U = -2$ for $(x,y) \in \operatorname{int} R$, where R is the rectangle $R = \{(x,y) : \ 0 < x < a, \ 0 < y < b\}$, with homogeneous boundary conditions on ∂R. Obviously, this double series converges very slowly. It is easy to see that $\{c_{j,k} \sin(j\pi x/a) \sin(k\pi y/b)\}_{j=1}^{\infty} \in \mathbf{b}^{(2)}$ and $\{c_{j,k} \sin(j\pi x/a) \sin(k\pi y/b)\}_{k=1}^{\infty} \in \mathbf{b}^{(2)}$. Therefore, we apply the sequential d-transformation with $p = 2$ and $q = 2$. Using about 400 terms of this series we can obtain its sum to 13 digit accuracy in double precision arithmetic (approximately 14 decimal digits). The exact value of $U(x,y)$ can easily be obtained from the simple series

$$U(x,y) = x(a-x) - \frac{8a^2}{\pi^3} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cosh[n\pi(2y-b)/(2a)]}{n^3 \cosh[n\pi b/(2a)]} \sin\left(\frac{n\pi x}{a}\right),$$

that converges very quickly for $0 < y < b$.

6 Multi-Dimensional Extrapolation

6.1 MGREP: Multi-Dimensional GREP

In order to treat multiple infinite series and integrals by extrapolation we need a multi-dimensional generalization of GREP. The basis of that should be an asymptotic expansion that is a suitable extension of that in (2.3). Thus, we should assume that

$$A(\mathbf{y}) \sim A + \sum_{k=1}^{\mu} \phi_k(\mathbf{y}) \sum_{\mathbf{i} \in \mathbb{Z}_0^s} \beta_{k,\mathbf{i}}\, \mathbf{y}^{\mathbf{i} * \mathbf{r}_k} \quad \text{as } \mathbf{y} \to \mathbf{0}+; \quad \mathbf{y} \in \mathbb{R}_0^s. \tag{6.1}$$

Here

$$\mathbf{y} = (y_1, \ldots, y_s); \quad \mathbf{y}^{\mathbf{i}} = \prod_{j=1}^{s} y_j^{i_j},$$

$$\mathbf{r}_k = (r_{k,1}, \ldots, r_{k,s}), \quad r_{k,j} > 0, \quad j = 1, \ldots, s; \quad \mathbf{y}^{\mathbf{i} * \mathbf{r}_k} = \prod_{j=1}^{s} y_j^{i_j r_{k,j}}.$$

We assume that the functions $A(\mathbf{y})$ and $\phi_k(\mathbf{y})$ are known (computable) for all $\mathbf{y} \neq \mathbf{0}$, and the $\mathbf{r}_k$ are known. Again A is the required limit or antilimit of $A(\mathbf{y})$ as $\mathbf{y} \to \mathbf{0}+$.

Proceeding as was done for GREP, we truncate the kth sum on the right-hand side of (6.1) leaving only $|\mathbf{I}_k|$ of the "dominant" terms with indices in $\mathbf{I}_k \subset \mathbb{Z}_0^s$, $k = 1, \ldots, \mu$, replace A and $\beta_{k,\mathbf{i}}$ by $A_{\mathbf{I},\mathbf{Y}}$ and $\bar{\beta}_{k,\mathbf{i}}$ respectively, and collocate (6.1) at $\mathbf{y} \in \mathbf{Y}$, where $\mathbf{Y}$ is a discrete set of points in $\mathbb{R}_0^s$, such that

$$\mathbf{I} = \{\mathbf{I}_1, \ldots, \mathbf{I}_\mu\}, \quad |\mathbf{Y}| = \sum_{k=1}^{\mu} |\mathbf{I}_k| + 1.$$

This results in the linear $|\mathbf{Y}| \times |\mathbf{Y}|$ system

$$A(\mathbf{y}) = A_{\mathbf{I},\mathbf{Y}} + \sum_{k=1}^{\mu} \phi_k(\mathbf{y}) \sum_{\mathbf{i} \in \mathbf{I}_k} \bar{\beta}_{k,\mathbf{i}}\, \mathbf{y}^{\mathbf{i} * \mathbf{r}_k}, \quad \mathbf{y} \in \mathbf{Y}. \tag{6.2}$$

Of course, here $A_{\mathbf{I},\mathbf{Y}}$ is the desired approximation to A, while the $\bar{\beta}_{k,\mathbf{i}}$ are the additional unknowns.

6.2 MGREP for Multiple Integrals

We now show via s-D infinite-range integrals that the ansatz in (6.1) that leads to the definition of MGREP via (6.2) is a natural one. When doing that we also uncover the form of the relevant $A(\mathbf{y})$.

For a 1-D integral $I[f] = \int_0^\infty f(t)\,dt$ there is only one natural definition of the "tail", i.e., $T_x[f] = \int_x^\infty f(t)\,dt$. This also defines the "partial integral" in the form $F(x) = I[f] - T_x[f] = \int_0^x f(t)\,dt$. For multiple integrals the definition of the tail, and hence the definition of the partial integrals, may take many different forms.

To motivate the "natural" definition that may be suitable for MGREP, let us consider the simple case of an s-D integral $I[f] = \int_{\mathbb{R}_0^s} f(\mathbf{t})\,d\mathbf{t}$, where $f(\mathbf{x}) = \prod_{j=1}^s f_j(x_j)$, with $f_j \in \mathbf{B}^{(m_j)}$ for some integers m_j. (Here $\mathbf{x} = (x_1, \ldots, x_s)$ and $\mathbf{t} = (t_1, \ldots, t_s)$ as usual, and $d\mathbf{t} = \prod_{j=1}^s dt_j$.) Therefore, for each j, $f_j(x)$ satisfies an ODE of the form $f_j(x) = \sum_{k=1}^{m_j} p_{j,k}(x) f_j^{(k)}(x)$, with $p_{j,k} \in \mathbf{A}^{(k)}$, $k = 1, \ldots, m_j$. Consequently, $f(\mathbf{x})$ satisfies the PDE

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbf{Q_m}} p_{\mathbf{k}}(\mathbf{x})\, \mathbf{D}^{\mathbf{k}} f(\mathbf{x}), \tag{6.3}$$

where $\mathbf{m} = (m_1, \ldots, m_s)$, $\mathbf{Q_m} = \{\mathbf{k} \in \mathbb{Z}_0^s \mid \mathbf{1} \le \mathbf{k} \le \mathbf{m}\}$, $p_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^s p_{j,k_j}(x_j)$, and $\mathbf{D}^{\mathbf{k}} \equiv \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \cdots \partial_{x_s}^{k_s}$, with $\mathbf{k} = (k_1, \ldots, k_s)$ and $\mathbf{k} \in \mathbf{Q_m}$.

Under suitable additional conditions, we also have from Theorem 3.3 that

$$T_x[f_j] = I[f_j] - F_j(x) \sim \sum_{k=0}^{m_j - 1} x^{\rho_{j,k}} f_j^{(k)}(x) \sum_{i=0}^{\infty} g_{j,k,i} x^{-i} \quad \text{as } x \to \infty; \quad \rho_{j,k} \le k+1, \text{ integers.} \tag{6.4}$$

Let us now define the tail of $I[f]$ as $T_{\mathbf{x}}[f] = \prod_{i=1}^s T_{x_j}[f_j]$. That is,

$$T_{\mathbf{x}}[f] = \int_{\mathbb{R}_{\mathbf{x}}^s} f(\mathbf{t})\,d\mathbf{t}; \quad \mathbb{R}_{\mathbf{x}}^s = \{\mathbf{t} \mid \mathbf{t} \ge \mathbf{x}\}. \tag{6.5}$$

Invoking (6.4) (with $\rho_{j,k}$ replaced by its upper bound $k+1$ for convenience), this yields

$$T_{\mathbf{x}}[f] \sim \sum_{\mathbf{k} \in \mathbf{Q_m}} \mathbf{x}^{\mathbf{k}} \mathbf{D}^{\mathbf{k}-\mathbf{1}} f(\mathbf{x}) \sum_{\mathbf{i} \in \mathbb{Z}_0^s} \beta_{\mathbf{k},\mathbf{i}} \mathbf{x}^{-\mathbf{i}} \quad \text{as } \mathbf{x} \to \infty. \tag{6.6}$$

Comparing (6.5) with (6.1), we realize that $T_{\mathbf{x}}[f]$ has an asymptotic expansion of exactly the same form as that of $A - A(\mathbf{y})$. This suggest that we should define the partial integral $A_{\mathbf{x}}[f]$ as in

$$A_{\mathbf{x}}[f] = I[f] - T_{\mathbf{x}}[f] = \int_{\mathbb{R}_0^s} f(\mathbf{t})\,d\mathbf{t} - \int_{\mathbb{R}_{\mathbf{x}}^s} f(\mathbf{t})\,d\mathbf{t}. \tag{6.7}$$

Now that we have arrived at the asymptotic expansion

$$A_{\mathbf{x}}[f] \sim I[f] - \sum_{\mathbf{k} \in \mathbf{Q_m}} \mathbf{x}^{\mathbf{k}} \mathbf{D}^{\mathbf{k}-\mathbf{1}} f(\mathbf{x}) \sum_{\mathbf{i} \in \mathbb{Z}_0^s} \beta_{\mathbf{k},\mathbf{i}} \mathbf{x}^{-\mathbf{i}} \quad \text{as } \mathbf{x} \to \infty, \tag{6.8}$$

we proceed to the definition of MGREP for $I[f]$: We truncate the $\mathbf{k}$th summation on the right-hand side of (6.8) leaving only $|\mathbf{I_k}|$ of the "dominant" terms with indices $\mathbf{i} \in \mathbf{I_k}$, $\mathbf{k} \in \mathbf{Q_m}$, replace $I[f]$ and $\beta_{\mathbf{k},\mathbf{i}}$ by $A_{\mathbf{I},\mathbf{X}}$ and $-\bar{\beta}_{\mathbf{k},\mathbf{i}}$ respectively, and collocate at the $|\mathbf{X}|$ points $\mathbf{x} \in \mathbf{X}$, where $\mathbf{X}$ is a discrete set of points in $\mathbb{R}_0^s$, and

$$\mathbf{I} = \{\mathbf{I_k} \mid \mathbf{k} \in \mathbf{Q_m}\}, \quad |\mathbf{X}| = \sum_{\mathbf{k} \in \mathbf{Q_m}} |\mathbf{I_k}| + 1.$$

This results in the $|\mathbf{X}| \times |\mathbf{X}|$ linear system

$$A_{\mathbf{x}}[f] = A_{\mathbf{I},\mathbf{X}} + \sum_{\mathbf{k}\in\mathbf{Q_m}} \mathbf{x}^{\mathbf{k}}\mathbf{D}^{\mathbf{k}-\mathbf{1}}f(\mathbf{x}) \sum_{\mathbf{i}\in\mathbf{I_k}} \bar{\beta}_{\mathbf{k},\mathbf{i}}\mathbf{x}^{-\mathbf{i}}, \quad \mathbf{x} \in \mathbf{X}. \tag{6.9}$$

Here $A_{\mathbf{I},\mathbf{X}}$ is the required approximation to $I[f]$ and $\bar{\beta}_{\mathbf{k},\mathbf{i}}$ are the additional unknowns.

From (6.7) it is clear that $A_{\mathbf{x}}[f]$ involves the computation of 1-D, 2-D, ..., $(s-1)$-D integrals. For example, when $s = 2$, with $\mathbf{x} = (x, y)$,

$$A_{\mathbf{x}}[f] = \int_0^x \left[\int_0^\infty f(u,v)\,dv\right] du + \int_0^y \left[\int_0^\infty f(u,v)\,du\right] dv - \int_0^x \left[\int_0^y f(u,v)\,dv\right] du.$$

We assume that we already know how to compute such integrals.

What remains to be done is to actually derive the asymptotic expansion given in (6.8) for important classes of s-D infinite-range integrals $\int_{\mathbb{R}_0^s} f(\mathbf{t})\,d\mathbf{t}$.

6.3 MGREP for Multiple Series

The derivation of MGREP for s-D infinite series is similar to that for s-D infinite-range integrals. We begin with the simple case of $S(\{\mathbf{a_r}\}) = \sum_{\mathbf{r}\in\mathbb{Z}_+^s}^{\infty} a_{\mathbf{r}}$, where $a_{\mathbf{r}} = \prod_{j=1}^s a_{r_j}^{(j)}$, with $\{a_r^{(j)}\}_{r=1}^\infty \in \mathbf{b}^{(m_j)}$ for some integers m_j. Since $a_n^{(j)} = \sum_{k=1}^{m_j} p_{j,k}(n)\Delta^k a_n^{(j)}$ with $p_{j,k}(n) \in \mathbf{A}_0^{(k)}$, $k = 1, \ldots, m_j$, and the difference operator Δ operating only on the index n, we have that

$$a_{\mathbf{n}} = \sum_{\mathbf{k}\in\mathbf{Q_m}} p_{\mathbf{k}}(\mathbf{n})\mathbf{\Delta}^{\mathbf{k}} a_{\mathbf{n}}, \tag{6.10}$$

where $p_{\mathbf{k}}(\mathbf{n}) = \prod_{j=1}^s p_{j,k_j}(n_j)$ and $\mathbf{\Delta}^{\mathbf{k}} \equiv \Delta_1^{k_1}\Delta_2^{k_2}\cdots\Delta_s^{k_s}$, with $\mathbf{k} = (k_1, \ldots, k_s)$, $\mathbf{k} \in \mathbf{Q_m}$, and $\mathbf{Q_m}$ as defined above. Here Δ_j operates only on the index n_j.

As Theorem 4.3 applies to each one of the series $\sum_{r=1}^\infty a_r^{(j)}$, we can proceed precisely as in the previous subsection on integrals: We define

$$R_{\mathbf{n}}(\{\mathbf{a_r}\}) = \sum_{\mathbf{r}\in\mathbb{Z}_{\mathbf{n}}^s} a_{\mathbf{r}}, \quad A_{\mathbf{n}}(\{\mathbf{a_r}\}) = S(\{\mathbf{a_r}\}) - R_{\mathbf{n}}(\{\mathbf{a_r}\}). \tag{6.11}$$

Then $A_{\mathbf{n}}(\{\mathbf{a_r}\})$ has the asymptotic expansion

$$A_{\mathbf{n}}(\{\mathbf{a_r}\}) \sim S(\{\mathbf{a_r}\}) - \sum_{\mathbf{k}\in\mathbf{Q_m}} \mathbf{n}^{\mathbf{k}}\mathbf{\Delta}^{\mathbf{k}-\mathbf{1}}a_{\mathbf{n}} \sum_{\mathbf{i}\in\mathbb{Z}_0^s} \beta_{\mathbf{k},\mathbf{i}}\mathbf{n}^{-\mathbf{i}} \quad \text{as } \mathbf{n}\to\infty, \quad \mathbf{n}\in\mathbb{Z}_+^s. \tag{6.12}$$

Using this asymptotic expansion, we now define MGREP for $S(\{\mathbf{a_r}\})$ as follows: We truncate the $\mathbf{k}$th summation on the right-hand side of (6.12) leaving only $|\mathbf{I_k}|$ of the "dominant" terms with indices $\mathbf{i} \in \mathbf{I_k}$, $\mathbf{k} \in \mathbf{Q_m}$, replace $I[f]$ and $\beta_{\mathbf{k},\mathbf{i}}$ by $A_{\mathbf{I},\mathbf{N}}$ and $-\bar{\beta}_{\mathbf{k},\mathbf{i}}$ respectively, and collocate at the $|\mathbf{N}|$ points $\mathbf{n} \in \mathbf{N}$, where $\mathbf{N}$ is a discrete set of points in $\mathbb{Z}_+^s$, and

$$\mathbf{I} = \{\mathbf{I_k} \mid \mathbf{k} \in \mathbf{Q_m}\}, \quad |\mathbf{N}| = \sum_{\mathbf{k}\in\mathbf{Q_m}} |\mathbf{I_k}| + 1.$$

This results in the $|\mathbf{N}| \times |\mathbf{N}|$ linear system

$$A_{\mathbf{n}}(\{\mathbf{a_r}\}) = A_{\mathbf{I},\mathbf{N}} + \sum_{\mathbf{k}\in\mathbf{Q_m}} \mathbf{n}^{\mathbf{k}}\mathbf{\Delta}^{\mathbf{k}-\mathbf{1}}a_{\mathbf{n}} \sum_{\mathbf{i}\in\mathbf{I_k}} \bar{\beta}_{\mathbf{k},\mathbf{i}}\mathbf{n}^{-\mathbf{i}}, \quad \mathbf{n} \in \mathbf{N}. \tag{6.13}$$

Here $A_{\mathbf{I},\mathbf{N}}$ is the required approximation to $S(\{\mathbf{a_r}\})$ and $\bar{\beta}_{\mathbf{k},\mathbf{i}}$ are the additional unknowns.

From (6.11) we realize that the computation if $A_{\mathbf{n}}(\{\mathbf{a_r}\})$ involves the summation of 1-D, 2-D, ..., $(s-1)$-D infinite series. For example, when $s=3$, and $\mathbf{n}=(p,q,r)$, we have

$$A_{\mathbf{n}}(\{\mathbf{a_r}\}) = \left(\sum_{i=1}^{p}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} + \sum_{j=1}^{q}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} + \sum_{k=1}^{r}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \right.$$
$$\left. -\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{\infty} - \sum_{i=1}^{p}\sum_{k=1}^{r}\sum_{j=1}^{\infty} - \sum_{j=1}^{q}\sum_{k=1}^{r}\sum_{i=1}^{\infty} + \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r}\right) a_{i,j,k}.$$

Of course, we assume that we already know how to perform these summations efficiently.

What remains to be done is to actually derive the asymptotic expansion given in (6.12) for important classes of s-D infinite series $\sum_{\mathbf{r}\in\mathbb{Z}_+^s} a_{\mathbf{r}}$.

7 Multi-Dimensional Asymptotic Expansions

The asymptotic expansions presented here for the remainders of s-D infinite integrals and series are direct generalizations of the asymptotic expansions derived in [13] for the 1-D case and in [8] for the 2-D case.

7.1 Asymptotic Expansions for Integrals

Let us start with the case of s-D infinite integrals which is simpler to grasp. First, we need to generalize Definition 3.1:

Definition 7.1 A function $\alpha : \mathbb{R}_0^s \to \mathbb{R}$ belongs to the set $\mathbf{A}^{(\mathbf{c})}$, where $\mathbf{c}=(c_1,\ldots,c_s)$, if it is infinitely differentiable in $\mathbb{R}_0^s$ and has a Poincaré-type asymptotic expansion of the form

$$\alpha(\mathbf{x}) \sim \sum_{\mathbf{i}\in\mathbb{Z}_0^s} \alpha_{\mathbf{i}}\mathbf{x}^{\mathbf{c}-\mathbf{i}} \quad \text{as } \mathbf{x}\to\infty, \tag{7.1}$$

and its derivatives have Poincaré-type asymptotic expansions obtained by differentiating that in (7.1) formally term by term.

Definition 7.2 We say that a function $f(\mathbf{x})$ belongs to the set $\mathbf{B}^{(\mathbf{m})}$ for some $\mathbf{m}\in\mathbb{Z}_+^s$, if it satisfies a partial differential equation (PDE) of the form

$$f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbf{Q_m}} p_{\mathbf{k}}(\mathbf{x})\,\mathbf{D}^{\mathbf{k}} f(\mathbf{x}), \tag{7.2}$$

where $\mathbf{Q_m}=\{\mathbf{k}\mid \mathbf{1}\le\mathbf{k}\le\mathbf{m}\}$, $p_{\mathbf{k}}\in\mathbf{A}^{(\mathbf{k})}$, $\mathbf{k}\in\mathbf{Q_m}$, $p_{\mathbf{k}}\in\mathbf{A}^{(\mathbf{k})}$, and $\mathbf{D}^{\mathbf{k}}\equiv\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}\cdots\partial_{x_s}^{k_s}$.

For deriving the asymptotic expansions and for applying the related transformations one does not have to know the coefficients $p_{\mathbf{k}}(\mathbf{x})$ of the PDE in (7.2) explicitly. Mere knowledge of their existence is enough, except for the following technical condition that they should satisfy.

Assumption 7.3
A. Behavior at infinity: With $\mathbf{Q}_{\mathbf{m},0}=\{\mathbf{i}\mid \mathbf{0}\le\mathbf{i}\le\mathbf{m}\}$, there holds

$$\lim_{\mathbf{x}\to\infty}\mathbf{x}^{\mathbf{u_k}}\,\mathbf{D}^{\mathbf{k}} f(\mathbf{x}) = 0 \ \text{ for some } \mathbf{u_k}\in\mathbf{Q}_{\mathbf{m},0}, \ \text{ and all } \ \mathbf{k}\in\mathbf{Q}_{\mathbf{m},0}. \tag{7.3}$$

This assumption is analogous to that in (3.3) for 1-D integrals; the $\mathbf{u_k}$ depend on the $p_{\mathbf{k}}(\mathbf{x})$ and their exact values can be obtained by repeated integration by parts of $\int_{\mathbb{R}^s_{\mathbf{x}}} f(\mathbf{x})\,d\mathbf{x}$.
B. With $\overline{p}_{\mathbf{k}} \equiv \lim_{\mathbf{x}\to\infty} \mathbf{x}^{-\mathbf{k}} p_{\mathbf{k}}(\mathbf{x})$, and $\Pi_{\mathbf{j},\mathbf{k}} = [\mathbf{D^k x^j}]_{\mathbf{x}=\mathbf{1}}$, there holds

$$\alpha^{(\mathbf{j})} \equiv \sum_{\mathbf{k}\in\mathbf{Q_m}} \Pi_{\mathbf{j},\mathbf{k}} \overline{p}_{\mathbf{k}} \neq 1, \quad \text{for all } \mathbf{j} \in \mathbb{Z}^s_0. \tag{7.4}$$

Using the same steps as in [13] for the 1-D case and in [8] for the 2-D case, we derive the main theorem on the asymptotic expansion of the infinite multiple integrals:

Theorem 7.4 *Let f be integrable on $\mathbb{R}^s_0$ and satisfy a linear PDE of the form given in (7.2), with coefficients $p_{\mathbf{k}}(\mathbf{x})$ satisfying Assumption 7.3. Then, there exist functions $g_{\mathbf{k}}(\mathbf{x}) \in \mathbf{A}^{(\mathbf{0})}$, $\mathbf{k} \in \mathbf{Q_m}$, for which $T_{\mathbf{x}}[f] = \int_{\mathbb{R}^s_{\mathbf{x}}} f(\mathbf{t})\,d\mathbf{t}$ satisfies*

$$T_{\mathbf{x}}[f] = \sum_{\mathbf{k}\in\mathbf{Q_m}} \mathbf{x^k} \left[\mathbf{D^{k-1}} f(\mathbf{x})\right] g_{\mathbf{k}}(\mathbf{x}) \sim \sum_{\mathbf{k}\in\mathbf{Q_m}} \mathbf{x^k} \left[\mathbf{D^{k-1}} f(\mathbf{x})\right] \sum_{\mathbf{i}\in\mathbb{Z}^s_0} \beta_{\mathbf{k},\mathbf{i}} \mathbf{x}^{-\mathbf{i}} \quad \textit{as } \mathbf{x}\to\infty. \tag{7.5}$$

Remark. The conditions and assumptions used here for deriving the above results may be relaxed. In particular, the set of indices $\mathbf{Q_m}$ may be any finite set in $\mathbb{Z}^s_0$. The particular form used here is so chosen to simplify the presentation.

7.2 More General Asymptotic Expansions for Integrals

Multivariate asymptotic expansions are less obvious than the univariate ones since in many important cases the coefficients $p_{\mathbf{k}}(\mathbf{x})$ look simple and yet they fail to satisfy Assumption 7.3. For example, in 2-D, $p(x,y) = \frac{1.}{xy+x+y} \in \mathbf{A}^{(-1,-1)} \subset \mathbf{A}^{(0,0)}$, but $p(x,y) = \frac{1.}{x+y} \notin \mathbf{A}^{(0,0)}$.

In this section we suggest a framework for treating such cases. We present a generalization of Assumption 7.3, and the consequent generalization of the expansion of Theorem 7.4. The derivation of Theorem 7.4 is by repeated integration by parts, and we observe that the quantities involved are derivatives and products of derivatives of the coefficients $p_{\mathbf{k}}(\mathbf{x})$ of the PDE in (7.2). In case these coefficients satisfy Assumption 7.3, the derivatives $\mathbf{\Delta^j} p_{\mathbf{k}}(\mathbf{x})$ and also products of such derivatives have asymptotic expansions in negative powers of $\mathbf{x}$ as $\mathbf{x}\to\infty$. For the general case we introduce the following assumption:

Assumption 7.5 Consider the set G of all the functions which are products of derivatives $\mathbf{D^j} p_{\mathbf{k}}(\mathbf{x})$, $\mathbf{k} \in \mathbf{Q_m}$, $\mathbf{j} \in \mathbb{Z}^s_{\mathbf{0}}$. We assume there exists a set of functions $H = \{h_{\mathbf{j}}(\mathbf{x})\}_{\mathbf{j}\in\mathbb{Z}^r_0}$ (with $r < s$ or $r = s$ or $r > s$), such that
A. $h_{\mathbf{j}}(\mathbf{x}) = o\big(h_{\mathbf{i}}(\mathbf{x})\big)$ *as* $\mathbf{x}\to\infty$, *for* $\mathbf{i} < \mathbf{j}$, $\mathbf{i},\mathbf{j} \in \mathbb{Z}^r_{\mathbf{0}}$.
B. $G \subset \mathrm{span}\{h_{\mathbf{j}}(\mathbf{x}) \mid \mathbf{j} \in \mathbb{Z}^r_{\mathbf{0}}\}$.

For example, in case all the coefficients of the PDE in (7.2) are combinations of positive powers of $p(x,y) = \frac{1.}{x+y}$, the set H is simply $H = \{(x+y)^{-j} \mid j \in \mathbb{Z}_0\}$.

Based upon the above assumption we may rewrite Theorem 7.4 replacing the asymptotic expansions in negative powers of $\mathbf{x}$ by expansions in the functions $\{h_{\mathbf{i}}(\mathbf{x})\}_{\mathbf{i}\in\mathbb{Z}^r_{\mathbf{0}}}$.

7.3 Asymptotic Expansions for Series

Definition 7.6 A function $\alpha : \mathbb{R}^s_0 \to \mathbb{R}$ belongs to the set $\mathbf{A}^{(\mathbf{c})}_0$, where $\mathbf{c} = (c_1, \ldots, c_s)$, if it has a Poincaré-type asymptotic expansion of the form

$$\alpha(\mathbf{x}) \sim \sum_{\mathbf{i}\in\mathbb{Z}^s_0} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{c}-\mathbf{i}} \quad \text{as } \mathbf{x}\to\infty. \tag{7.6}$$

We consider a convergent s-D infinite series $S(\{a_{\mathbf{r}}\}) = \sum_{\mathbf{r}\in\mathbb{Z}_+^s} a_{\mathbf{r}}$, where $\{a_{\mathbf{k}}\}$ is known to satisfy a partial difference equation of the form

$$a_{\mathbf{n}} = \sum_{\mathbf{k}\in\mathbf{Q_m}} p_{\mathbf{k}}(\mathbf{n})\mathbf{\Delta}^{\mathbf{k}} a_{\mathbf{n}}, \tag{7.7}$$

with $p_{\mathbf{k}} \in \mathbf{A}^{(\mathbf{k})}$, $\mathbf{k} \in \mathbf{Q_m}$.

Assumption 7.7 A. Behavior at infinity: With $\mathbf{Q}_{\mathbf{m},0} = \{\mathbf{i} \mid \mathbf{0} \leq \mathbf{i} \leq \mathbf{m}\}$, there holds

$$\lim_{\mathbf{n}\to\infty} \mathbf{n}^{\mathbf{u_k}} \mathbf{\Delta}^{\mathbf{k}} a_{\mathbf{n}} = 0 \quad \text{for some } \mathbf{u_k} \in \mathbf{Q}_{\mathbf{m},0}, \quad \text{and all } \mathbf{k} \in \mathbf{Q}_{\mathbf{m},0}. \tag{7.8}$$

This assumption is analogous to that in (4.3) for 1-D series; the $\mathbf{u_k}$ depend on the $p_{\mathbf{k}}(\mathbf{n})$ and their exact values can be obtained by repeated summation by parts of $\sum_{\mathbf{r}\in\mathbb{Z}^s_{\mathbf{n+1}}} a_{\mathbf{r}}$.
B. Same as Assumption 7.3-B.

Using the same steps as in the development of Theorem 7.4, replacing the integration by parts by summation by parts, we derive the main theorem on the asymptotic expansion of the infinite multiple series :

Theorem 7.8 *Let $\sum_{\mathbf{r}\in\mathbb{Z}_+^s} a_{\mathbf{r}}$ be convergent and assume that $\{a_{\mathbf{k}}\}$ satisfies the linear difference equation of the form (7.7), with coefficients satisfying Assumption 7.7. Then, as $\mathbf{n} \to \infty$, $R_{\mathbf{n}}(\{a_{\mathbf{r}}\}) = \sum_{\mathbf{r}\in\mathbb{Z}^s_{\mathbf{n+1}}} a_{\mathbf{r}}$ satisfies*

$$R_{\mathbf{n}}(\{a_{\mathbf{r}}\}) = \sum_{\mathbf{k}\in\mathbf{Q_m}} q_{\mathbf{k}}(\mathbf{n})\mathbf{\Delta}^{\mathbf{k-1}} a_{\mathbf{n}}, \tag{7.9}$$

where $q_{\mathbf{k}} \in \mathbf{A}^{(\mathbf{k})}$, $\mathbf{k} \in \mathbf{Q_m}$.

The general framework presented in the previous subsection is also applicable for the case of multiple series. Here, one should consider the set G of all the functions which are products of differences $\mathbf{\Delta}^{\mathbf{j}} p_{\mathbf{k}}(\mathbf{n})$, $\mathbf{k} \in \mathbf{Q_m}$, $\mathbf{j} \in \mathbb{Z}_{\mathbf{0}}^s$. Assumption 7.5 may now be repeated for this set G, and the asymptotic expansion for the remainder may be written in terms of the the functions $\{h_{\mathbf{i}}(\mathbf{n})\}_{\mathbf{i}\in\mathbb{Z}_0^r}$.

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