Euler–Maclaurin expansions for integrals with endpoint singularities: a new perspective

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Summary. In this note, we provide a new perspective on Euler–Maclaurin expansions of (offset) trapezoidal rule approximations of the finite-range integrals $I[f] = \int_a^b f(x)\,dx$, where $f \in C^\infty(a, b)$ but can have general algebraic-logarithmic singularities at one or both endpoints. These integrals may exist either as ordinary integrals or as Hadamard finite part integrals. We assume that $f(x)$ has asymptotic expansions of the general forms

$$f(x) \sim \sum_{s=0}^{\infty} P_s\big(\log(x - a)\big)(x - a)^{\gamma_s} \quad \text{as } x \to a+,$$
$$f(x) \sim \sum_{s=0}^{\infty} Q_s\big(\log(b - x)\big)(b - x)^{\delta_s} \quad \text{as } x \to b-,$$

where $P_s(y)$ and $Q_s(y)$ are some polynomials in y. Here the γ_s and δ_s are complex in general and different from $-1, -2, \dots$. The results we obtain in this work generalize, and include as special cases, those pertaining to the known special cases in which $f(x) = (x - a)^\gamma [\log(x - a)]^p g_a(x) = (b - x)^\delta [\log(b - x)]^q g_b(x)$, where p and q are nonnegative integers and $g_a \in C^\infty[a, b]$ and $g_b \in C^\infty[a, b]$. In addition, they have the pleasant feature that they are expressed in very simple terms based only on the asymptotic expansions of $f(x)$ as $x \to a+$ and $x \to b-$. With $h = (b - a)/n$, where n is a positive integer, and with $D_\omega = \frac{d}{d\omega}$, one of these results reads, as $h \to 0$,

$$h \sum_{i=1}^{n-1} f(a + ih) \sim I[f] + \sum_{s=0}^{\infty} P_s(D_{\gamma_s})\big[\zeta(-\gamma_s)\, h^{\gamma_s+1}\big]$$
$$+ \sum_{s=0}^{\infty} Q_s(D_{\delta_s})\big[\zeta(-\delta_s)\, h^{\delta_s+1}\big],$$

where $\zeta(z)$ is the Riemann Zeta function.

1 Introduction

Euler–Maclaurin expansions for trapezoidal rule approximations of finite-range integrals $\int_a^b f(x)\,dx$, and their various generalizations in the presence of possible endpoint singularities, have played an important role in the design of methods for the efficient numerical evaluation of these integrals. The known results concern either the case (i) $f \in C^\infty[a, b]$, or the case (ii) $f(x) = (x - a)^\gamma [\log(x - a)]^p g_a(x) = (b - x)^\delta [\log(b - x)]^q g_b(x)$, where $\Re\gamma > -1$ and $\Re\delta > -1$, p and q are nonnegative integers, and $g_a \in C^\infty[a, b)$ and $g_b \in C^\infty(a, b]$. The case (i) is treated in many books on numerical analysis; see, for example, Steffensen [10] or Davis and Rabinowitz [2]. The case (ii) was first treated in the papers by Navot [6], [7], and later, using a different method involving generalized functions, by Lyness and Ninham [4]. For a brief survey of the relevant results, see also Sidi [9, Appendix D]. Subsequently, in a paper by Ninham [8], Navot's expansions were shown to hold also for the case in which $\Re\gamma \leq -1$ or $\Re\delta \leq -1$ or both, such that γ and δ are different from $-1, -2, \dots$; in this case, $\int_a^b f(x)\,dx$ is defined as a *Hadamard finite part* integral.[1] Finally, the remaining case in which γ or δ or both are negative integers has recently been dealt with by Lyness [3] and by Monegato and Lyness [5]. The technique used in [5] unifies the treatments of the various expansions; it is based on an approach introduced by Verlinden [11] that employs the Mellin transform. For a summary of properties of Hadamard finite part integrals, we refer the reader to [2].

In this work, we present a new perspective to the subject that also allows us to extend these known results to functions $f(x)$ that may have a very general behavior as $x \to a+$ or as $x \to b-$. Specifically, we assume that $f(x)$ has the following properties:

1. $f \in C^\infty(a, b)$ and has the asymptotic expansions

(1.1)
$$f(x) \sim \sum_{s=0}^\infty P_s\big(\log(x - a)\big)\,(x - a)^{\gamma_s} \quad \text{as } x \to a+,$$
$$f(x) \sim \sum_{s=0}^\infty Q_s\big(\log(b - x)\big)\,(b - x)^{\delta_s} \quad \text{as } x \to b-,$$

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[1] The usual notation for Hadamard finite part integrals is $⨏_a^b f(x)\,dx$. For simplicity, in this work, we use $\int_a^b f(x)\,dx$ to denote both ordinary and Hadamard finite part integrals.

where $P_s(y)$ and $Q_s(y)$ are some polynomials in y, and γ_s and δ_s are in general complex and satisfy

(1.2)
$$\gamma_s \neq -1, -2, \ldots \; ; \; \Re\gamma_0 \leq \Re\gamma_1 \leq \Re\gamma_2 \leq \cdots \; ; \; \lim_{s\to\infty} \Re\gamma_s = +\infty,$$
$$\delta_s \neq -1, -2, \ldots \; ; \; \Re\delta_0 \leq \Re\delta_1 \leq \Re\delta_2 \leq \cdots \; ; \; \lim_{s\to\infty} \Re\delta_s = +\infty.$$

Here, $\Re z$ stands for the real part of z.

Note that, in case $f(x) = (x-a)^\gamma [\log(x-a)]^p g_a(x) = (b-x)^\delta [\log(b-x)]^q g_b(x)$ as in the first paragraph of this section, and $g_a(x)$ and $g_b(x)$ have full Taylor series about $x = a$ and $x = b$, respectively, we have $\gamma_s = \gamma + s$, $\delta_s = \delta + s$, $s = 0, 1, \ldots$.

2. If we let $p_s = \deg(P_s)$ and $q_s = \deg(Q_s)$ for each s, then the γ_s and δ_s are ordered such that

(1.3) $p_s \geq p_{s+1}$ if $\Re\gamma_{s+1} = \Re\gamma_s$; $q_s \geq q_{s+1}$ if $\Re\delta_{s+1} = \Re\delta_s$.

3. By (1.1), we mean that, for each $r = 1, 2, \ldots$,

(1.4)
$$f(x) - \sum_{s=0}^{r-1} P_s\big(\log(x-a)\big)(x-a)^{\gamma_s}$$
$$= O\big(P_r(\log(x-a))(x-a)^{\gamma_r}\big) \quad \text{as } x \to a+,$$
$$f(x) - \sum_{s=0}^{r-1} Q_s\big(\log(b-x)\big)(b-x)^{\delta_s}$$
$$= O\big(Q_r(\log(b-x))(b-x)^{\delta_r}\big) \quad \text{as } x \to b - .$$

This is consistent with (1.2) and (1.3).

4. For each $k = 1, 2, \ldots$, the kth derivative of $f(x)$ also has asymptotic expansions as $x \to a+$ and $x \to b-$ that are obtained by differentiating those in (1.1) term by term.

The following are consequences of (1.2):

(i) There are only a finite number of γ_s and only a finite number of δ_s having the same real parts; consequently, $\Re\gamma_s < \Re\gamma_{s+1}$ and $\Re\delta_{s'} < \Re\delta_{s'+1}$ for infinitely many values of the indices s and s'.

(ii) The sequences $\{(x-a)^{\gamma_s}\}_{s=0}^\infty$ and $\{(b-x)^{\delta_s}\}_{s=0}^\infty$ are asymptotic scales as $x \to a+$ and $x \to b-$, respectively, in the following sense: For each $s = 0, 1, \ldots$,

$$\lim_{x\to a+} \left| \frac{(x-a)^{\gamma_{s+1}}}{(x-a)^{\gamma_s}} \right| = \begin{cases} 1 & \text{if } \Re\gamma_s = \Re\gamma_{s+1}, \\ 0 & \text{if } \Re\gamma_s < \Re\gamma_{s+1}, \end{cases}$$

$$\lim_{x\to b-} \left| \frac{(b-x)^{\delta_{s+1}}}{(b-x)^{\delta_s}} \right| = \begin{cases} 1 & \text{if } \Re\delta_s = \Re\delta_{s+1}, \\ 0 & \text{if } \Re\delta_s < \Re\delta_{s+1}. \end{cases}$$

(iii) The integral $\int_a^b f(x)\,dx$ exists in the ordinary sense provided $\Re\gamma_0 > -1$ and $\Re\delta_0 > -1$. Otherwise, it exists as a Hadamard finite part integral. The latter is defined as follows: Let the integers μ and ν be such that

$$\Re\gamma_{\mu-1} \le -1 < \Re\gamma_\mu, \quad \Re\delta_{\nu-1} \le -1 < \Re\delta_\nu.$$

Define also

$$\phi_\mu(x) := f(x) - \sum_{s=0}^{\mu-1} P_s\big(\log(x-a)\big)\,(x-a)^{\gamma_s},$$

$$\psi_\nu(x) := f(x) - \sum_{s=0}^{\nu-1} Q_s\big(\log(b-x)\big)\,(b-x)^{\delta_s}.$$

Let also $P_s(y) = \sum_{i=0}^{p_s} c_{si}\,y^i$ and $Q_s(y) = \sum_{i=0}^{q_s} d_{si}\,y^i$ for some constants c_{si} and d_{si}. Then, for arbitrary $t \in (a,b)$,

$$\int_a^b f(x)\,dx = \sum_{s=0}^{\mu-1}\sum_{i=0}^{p_s} c_{si}\frac{d^i}{d\gamma_s^i}\frac{(t-a)^{\gamma_s+1}}{\gamma_s+1} + \int_a^t \phi_\mu(x)\,dx$$

$$+ \sum_{s=0}^{\nu-1}\sum_{i=0}^{q_s} d_{si}\frac{d^i}{d\delta_s^i}\frac{(b-t)^{\delta_s+1}}{\delta_s+1} + \int_t^b \psi_\nu(x)\,dx,$$

Here the integrals of $\phi_\mu(x)$ and $\psi_\nu(x)$ exist in the ordinary sense, as is clear from the way we have chosen μ and ν.

From the assumptions we have made above, it is obvious that the functions $f(x)$ treated in the literature so far and mentioned in the first paragraph are special cases of the ones we treat here. Now, the reader may be wondering as to whether nontrivial functions $f(x)$ that have the properties described here, but are different from the special cases mentioned, can be constructed in a reasonable way. At the end of this section, we give a simple procedure by which such functions can be constructed.

In the next section, we state the main results of this work, and in Section 3, we provide the proofs of these results. Our results have the pleasant feature that they are expressed in extremely simple terms based only on the asymptotic expansions in (1.1).

We would like to emphasize that our results do not follow from the known results on extensions of Euler–Maclaurin expansions in the presence of endpoint singularities. Actually, they contain the known results as special cases.

Before closing this section, we note that we have assumed that $f \in C^\infty(a,b)$ only for the sake of simplifying the presentation. We can assume that $f \in C^k(a,b)$ for some finite k just as well. The method of proof applies to this case without any changes.

1.1 Construction of functions satisfying (1.1)–(1.4)

We now describe a procedure by which one can construct functions $f(x)$ with the properties described here. Let us pick two integers $\tau_a \leqslant 0$ and $\tau_b \leqslant 0$, and choose the γ_s and δ_s such that

$$s + \tau_a \leq \Re\gamma_s < s + \tau_a + 1, \quad s + \tau_b \leq \Re\delta_s < s + \tau_b + 1, \quad s = 0, 1, \dots,$$

and (1.2) is satisfied. Otherwise, the γ_s and δ_s are arbitrary. Next, let us choose the polynomials $P_s(y)$ and $Q_s(y)$ such that their coefficients c_{si} and d_{si} satisfy

$$c_{si} = O\big((s!)^{-1}\big) \quad \text{and} \quad d_{si} = O\big((s!)^{-1}\big) \quad \text{as} \quad s \to \infty,$$

and their degrees p_s and q_s increase at most polynomially in s. Denote

$$R_m(x) := \sum_{s=m}^{\infty} P_s\big(\log(x-a)\big)(x-a)^{\gamma_s},$$

$$S_m(x) := \sum_{s=m}^{\infty} Q_s\big(\log(b-x)\big)(b-x)^{\delta_s},$$

and

$$R_m^{(k)}(x) := \sum_{s=m}^{\infty} \frac{d^k}{dx^k}\big[P_s\big(\log(x-a)\big)(x-a)^{\gamma_s}\big],$$

$$S_m^{(k)}(x) := \sum_{s=m}^{\infty} \frac{d^k}{dx^k}\big[Q_s\big(\log(b-x)\big)(b-x)^{\delta_s}\big].$$

Of course, for some integers $\mu \geq 0$ and $\nu \geq 0$, we have $\Re\gamma_\mu \geq 0$ and $\Re\delta_\nu \geq 0$. Then the series $R_\mu(x)$ and $S_\nu(x)$ converge uniformly on $[a, b]$, and thus represent functions that are continuous on $[a, b]$. This implies that the function $f_a(x)$ defined as the sum of the series $R_0(x)$ is in $C(a, b]$. Similarly, the function $f_b(x)$ defined as the sum of the series $S_0(x)$ is in $C[a, b)$. Furthermore, it can be shown from first principles that $R_0(x)$ represents $f_a(x)$ asymptotically as $x \to a+$, and $S_0(x)$ represents $f_b(x)$ asymptotically as $x \to b-$. In addition, the series $R_{\mu+k}^{(k)}(x)$ and $S_{\nu+k}^{(k)}(x)$ converge uniformly on $[a, b]$, and this implies that they are the kth derivatives of the functions defined by the sums of the series $R_{\mu+k}(x)$ and $S_{\nu+k}(x)$, respectively. The conclusion from all this discussion is that, for each $k = 1, 2, \dots$, the derivatives $f_a^{(k)}(x)$ and $f_b^{(k)}(x)$ exist and are given as the sums of the series $R_0^{(k)}(x)$ and $S_0^{(k)}(x)$, respectively. See, e.g., Apostol [1, p. 403, Theorem 13-14].

Thus, $f_a \in C^\infty(a, b]$ and $f_b \in C^\infty[a, b)$, $f_a(x)$ is the sum of the series $R_0(x)$ and is represented by the latter asymptotically as $x \to a+$, while

$f_b(x)$ is the sum of $S_0(x)$ and is represented by the latter asymptotically as $x \to b-$. Furthermore, $f_a^{(k)}(x)$ and $f_b^{(k)}(x)$ are the sums of $R_0^{(k)}(x)$ and $S_0^{(k)}(x)$, respectively, and represented by the latter asymptotically as $x \to a+$ and $x \to b-$, respectively.

Next, we construct functions $U_a(x)$ and $U_b(x)$ in $C^\infty[a, b]$ with the properties

$$U_a(a) = 1, \quad U_a(b) = 0, \quad U_a^{(k)}(a) = U_a^{(k)}(b) = 0, \quad k = 1, 2, \dots,$$
$$U_b(a) = 0, \quad U_b(b) = 1, \quad U_b^{(k)}(a) = U_b^{(k)}(b) = 0, \quad k = 1, 2, \dots.$$

For example,

$$U_a(x) = \frac{H(x)}{H(a)}, \quad U_b(x) = 1 - U_a(x);$$

$$H(x) = \int_x^b \exp\left[-\frac{1}{(t-a)(b-t)}\right] dt.$$

Finally, we set

$$f(x) = U_a(x) f_a(x) + U_b(x) f_b(x).$$

It is now easy to verify that $f(x)$ has all the properties mentioned in the second paragraph of this section.

2 Main results

Theorem 2.1 and Corollary 2.2 below concern the special case of (1.1) in which $P_s(y)$ and $Q_s(y)$ are constant polynomials. This case is of importance by itself. Following these, Theorem 2.3 covers the general case. As mentioned earlier, all these results reduce to the known results pertaining to the cases (i) and (ii) that were mentioned in the first paragraph of the preceding section.

Throughout the remainder of the paper, we use the notation

$$(2.1) \qquad\qquad I[f] := \int_a^b f(x)\, dx,$$

whether $\int_a^b f(x)\, dx$ exists as an ordinary integral or as a Hadamard finite part integral, and

$$(2.2)\quad \widetilde{T}_n[f; \theta] := h \sum_{i=0}^{n-1} f(a + ih + \theta h); \quad h = \frac{b-a}{n}, \quad n = 1, 2, \dots.$$

Here $\widetilde{T}_n[f; \theta]$ is the *offset trapezoidal rule* approximation to $I[f]$, and $\theta \in (0, 1)$. Because $f \in C^\infty(a, b)$, $\widetilde{T}_n[f; \theta]$ with $\theta \in (0, 1)$ is well-defined. Note

that $\tilde{T}_n[f; \frac{1}{2}]$ is simply the midpoint rule approximation to $I[f]$. For $\theta = 1$, and provided that $f(a)$ and/or $f(b)$ exist, we also use the notation

$$(2.3) \; \check{T}_n[f] := h \sum_{i=1}^{n-1} f(a + ih), \quad T_n[f] := \check{T}_n[f] + \frac{h}{2}[f(a) + f(b)],$$

$$T'_n[f] := \check{T}_n[f] + \frac{h}{2} f(b), \quad T^*_n[f] := \check{T}_n[f] + \frac{h}{2} f(a).$$

By the fact that $f \in C^\infty(a, b)$, $\check{T}_n[f]$ is always well-defined just as $\tilde{T}_n[f; \theta]$ with $0 < \theta < 1$. Note that $\check{T}_n[f]$ is analogous to (but not the same as) $\tilde{T}_n[f; 1]$. In addition, provided $f(a)$ and $f(b)$ exist, which is the case, for example, when $f \in C[a, b]$, $T_n[f]$ is the ordinary trapezoidal rule approximation to $I[f]$.

In our results below, $\zeta(z, \theta)$ denotes the *generalized Zeta function*, which is defined by the convergent Dirichlet series $\sum_{k=0}^\infty 1/(k+\theta)^z$ for $\Re z > 1$ and continued analytically to the whole complex z-plane, with the exception of $z = 1$, where it has a simple pole with residue 1. For $\theta = 1$, $\zeta(z, 1)$ is simply $\zeta(z)$, the *Riemann Zeta function*. At this point, we only note the following relations among the two Zeta functions and the Bernoulli polynomials $B_j(\theta)$ and the Bernoulli numbers B_j:

$$(2.4) \quad \zeta(-j, \theta) = -\frac{B_{j+1}(\theta)}{j+1}, \quad j = 0, 1, \dots,$$

$$B_j(0) = B_j, \quad j \geq 0; \quad B_1(1) = -B_1; \quad B_j(1) = B_j, \quad j \geq 0, \quad j \neq 1,$$

$$B_1 = -\frac{1}{2}; \quad B_{2j+1} = 0, \quad B_{2j} \neq 0, \quad j = 1, 2, \dots,$$

$$\zeta(0) = -\frac{1}{2}; \quad \zeta(-2j) = 0, \quad \zeta(1 - 2j) = -\frac{B_{2j}}{2j} \neq 0, \quad j = 1, 2, \dots,$$

$$B_{2j+1}(\tfrac{1}{2}) = 0, \quad \zeta(-2j, \tfrac{1}{2}) = 0, \quad j = 0, 1, \dots.$$

Theorem 2.1 *Let $f(x)$ be as in (1.1)–(1.4), with $P_s(y) = c_s$ and $Q_s(y) = d_s$ constants in (1.1). Then,*
(a) *for $0 < \theta < 1$,*

$$(2.5) \quad \tilde{T}_n[f; \theta] \sim I[f] + \sum_{s=0}^\infty c_s \, \zeta(-\gamma_s, \theta) \, h^{\gamma_s+1}$$

$$+ \sum_{s=0}^\infty d_s \, \zeta(-\delta_s, 1 - \theta) \, h^{\delta_s+1} \quad as \; h \to 0,$$

(b) *for $\theta = 1$,*
(2.6)

$$\check{T}_n[f] \sim I[f] + \sum_{s=0}^\infty c_s \, \zeta(-\gamma_s) \, h^{\gamma_s+1} + \sum_{s=0}^\infty d_s \, \zeta(-\delta_s) \, h^{\delta_s+1} \quad as \; h \to 0.$$

From (1.2), it is obvious that the expansions in (2.5) and (2.6) are genuine asymptotic expansions.

The following corollary is obtained by invoking in Theorem 2.1 the relations given in (2.4).

Corollary 2.2 *The result in (2.5) when $\theta = 1/2$ can be re-expressed as in*

$$(2.7) \quad \widetilde{T}_n[f; \tfrac{1}{2}] \sim I[f] + \sum_{\substack{s=0 \\ \gamma_s \notin \{0,2,4,\dots\}}}^{\infty} c_s \, \zeta(-\gamma_s, \tfrac{1}{2}) \, h^{\gamma_s+1}$$

$$+ \sum_{\substack{s=0 \\ \delta_s \notin \{0,2,4,\dots\}}}^{\infty} d_s \, \zeta(-\delta_s, \tfrac{1}{2}) \, h^{\delta_s+1} \quad as \ h \to 0,$$

while that in (2.6) can be re-expressed also as in

$$(2.8) \qquad \check{T}_n[f] \sim I[f] + \sum_{\substack{s=0 \\ \gamma_s \notin \{2,4,\dots\}}}^{\infty} c_s \, \zeta(-\gamma_s) \, h^{\gamma_s+1}$$

$$+ \sum_{\substack{s=0 \\ \delta_s \notin \{2,4,\dots\}}}^{\infty} d_s \, \zeta(-\delta_s) \, h^{\delta_s+1} \quad as \ h \to 0.$$

Remark In words, when $P_s(y) = c_s$ and $Q_s(y) = d_s$ are constants, the powers $(x-a)^{2s}$ and $(b-x)^{2s}$, if present in the asymptotic expansions of (1.1), do not contribute to the Euler–Maclaurin expansion of $\widetilde{T}_n[f; \tfrac{1}{2}]$ in (2.5) when $s \in \{0, 1, \dots\}$, and they do not contribute to the Euler–Maclaurin expansion of $\check{T}_n[f]$ in (2.6) when $s \in \{1, 2, \dots\}$.

Theorem 2.3 *Let $f(x)$ be as in (1.1)–(1.4), with $P_s(y) = \sum_{i=0}^{p_s} c_{si} y^i$ and $Q_s(y) = \sum_{i=0}^{q_s} d_{si} y^i$ in (1.1), where p_s and q_s are some nonnegative integers and c_{si} and d_{si} are constants. Denote $D_\omega = \frac{d}{d\omega}$. For an arbitrary polynomial $W(y) = \sum_{i=0}^{k} e_i y^i$ and an arbitrary function u that depends on the parameter ω, define also*

$$W(D_\omega)u := \sum_{i=0}^{k} e_i \, [D_\omega^i u] = \sum_{i=0}^{k} e_i \frac{d^i u}{d\omega^i}.$$

Then,
(a) for $0 < \theta < 1$,

$$(2.9) \qquad \widetilde{T}_n[f; \theta] \sim I[f] + \sum_{s=0}^{\infty} P_s(D_{\gamma_s}) \big[\zeta(-\gamma_s, \theta) \, h^{\gamma_s+1} \big]$$

$$+ \sum_{s=0}^{\infty} Q_s(D_{\delta_s}) \big[\zeta(-\delta_s, 1-\theta) \, h^{\delta_s+1} \big] \quad as \ h \to 0,$$

(b) *for $\theta = 1$,*

$$(2.10) \quad \check{T}_n[f] \sim I[f] + \sum_{s=0}^{\infty} P_s(D_{\gamma_s}) \big[\zeta(-\gamma_s) h^{\gamma_s+1} \big]$$

$$+ \sum_{s=0}^{\infty} Q_s(D_{\delta_s}) \big[\zeta(-\delta_s) h^{\delta_s+1} \big] \quad as\ h \to 0.$$

To see the explicit form of the expansions in Theorem 2.3, we also need

$$D_\omega^i \big[\zeta(-\omega, \theta) h^{\omega+1} \big] = h^{\omega+1} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \zeta^{(i-j)}(-\omega, \theta)(\log h)^j,$$

where $\zeta^{(k)}(z, \theta)$ is the kth derivative of $\zeta(z, \theta)$ with respect to z. Using this, it can be seen that, for example,

$$P_s(D_{\gamma_s}) \big[\zeta(-\gamma_s, \theta) h^{\gamma_s+1} \big] = h^{\gamma_s+1} \sum_{j=0}^{p_s} w_{sj} (\log h)^j,$$

where

$$w_{sj} = \sum_{i=j}^{p_s} (-1)^{i-j} \binom{i}{j} c_{si} \, \zeta^{(i-j)}(-\gamma_s, \theta), \quad i = 0, 1, \ldots, p_s.$$

From this and from (1.2), we see that the expansions in (2.9) and (2.10) too are genuine asymptotic expansions.

3 Proofs

3.1 Proof of Theorem 2.1

We begin by stating the classical result on the Euler–Maclaurin expansion for the trapezoidal rule. For a proof of this result, we refer the reader to Steffensen [10].

Theorem 3.1 *Let $g \in C^m[a, b]$. Then, for all $\theta \in [0, 1]$,*

$$\widetilde{T}_n[g; \theta] = I[g] + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} \big[g^{(k-1)}(b) - g^{(k-1)}(a) \big] h^k + U_m(h; \theta),$$

where the remainder term $U_m(h; \theta)$ is given by

$$U_m(h; \theta) = -h^m \int_a^b g^{(m)}(x) \frac{\bar{B}_m(\theta - n \frac{x-a}{b-a})}{m!} dx = O(h^m) \quad as\ h \to 0.$$

Here $\bar{B}_k(x)$ *is the* periodic Bernoullian function *that is the* 1-*periodic exten-sion of the Bernoulli polynomial* $B_k(x)$. *For the case* $\theta = 1$, *this result can be rewritten as*

$$T_n[f] = I[g] + \sum_{\substack{k=2 \\ k \text{ even}}}^{m} \frac{B_k}{k!} \left[g^{(k-1)}(b) - g^{(k-1)}(a) \right] h^k + U_m(h; 1).$$

We next state two results on the Euler–Maclaurin expansions for the inte-grals of the functions

(3.1) $u_\omega(x) = (x - a)^\omega$ and $v_\omega(x) = (b - x)^\omega.$

Note that $\int_a^b u_\omega(x)\, dx$ and $\int_a^b v_\omega(x)\, dx$ exist as ordinary integrals when $\Re\omega > -1$. In case $\Re\omega \le -1$ but $\omega \ne -1, -2, \ldots$, they exist as Hadamard finite part integrals. In any case,

$$I[u_\omega] = \frac{(b-a)^{\omega+1}}{(\omega+1)} = I[v_\omega].$$

These results follow from that of Navot [6].

Theorem 3.2 *Let* $\omega \ne -1, -2, \ldots$, *and let* m *be a nonnegative integer such that* $m > \Re\omega + 1$. *Then, for all* $\theta \in (0, 1]$,

$$\widetilde{T}_n[u_\omega; \theta] = I[u_\omega] + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} u_\omega^{(k-1)}(b)\, h^k + \zeta(-\omega, \theta)\, h^{\omega+1} + R_m(h; \theta),$$

where

$$R_m(h; \theta) = h^m \int_b^\infty u_\omega^{(m)}(x) \frac{\bar{B}_m(\theta - n\frac{x-a}{b-a})}{m!}\, dx = O(h^m) \quad \text{as } h \to 0.$$

For the special case $\theta = 1$, *this result can be rewritten as*

$$T_n'[u_\omega] = I[u_\omega] + \sum_{\substack{k=2 \\ k \text{ even}}}^{m} \frac{B_k}{k!} u_\omega^{(k-1)}(b)\, h^k + \zeta(-\omega)\, h^{\omega+1} + R_m(h; 1).$$

Theorem 3.3 *Let* $\omega \ne -1, -2, \ldots$, *and let* m *be a nonnegative integer such that* $m > \Re\omega + 1$. *Then, for all* $\theta \in [0, 1)$,

$$\widetilde{T}_n[v_\omega; \theta] = I[v_\omega] - \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} v_\omega^{(k-1)}(a)\, h^k + \zeta(-\omega, 1-\theta)\, h^{\omega+1} + S_m(h; \theta),$$

where

$$S_m(h; \theta) = h^m \int_b^\infty u_\omega^{(m)}(x) \frac{\bar{B}_m(1 - \theta - n\frac{x-a}{b-a})}{m!} \, dx = O(h^m) \quad \text{as } h \to 0.$$

For the special case $\theta = 1$, this result can be rewritten as

$$T_n^*[v_\omega] = I[v_\omega] - \sum_{\substack{k=2 \\ k \text{ even}}}^m \frac{B_k}{k!} v_\omega^{(k-1)}(a) \, h^k + \zeta(-\omega) h^{\omega+1} + S_m(h; 1).$$

Note that Theorem 3.3 follows from Theorem 3.2 once we realize that

$$\widetilde{T}_n[u_\omega; \theta] = h \sum_{i=0}^{n-1} (ih + \theta h)^\omega \quad \text{and} \quad \widetilde{T}_n[v_\omega; \theta] = h \sum_{i=0}^{n-1} (ih + (1 - \theta)h)^\omega,$$

so that

$$\widetilde{T}_n[v_\omega; \theta] = \widetilde{T}_n[u_\omega; 1 - \theta],$$

and $I[v_\omega] = I[u_\omega]$, $u_\omega^{(p)}(b) = (-1)^p v_\omega^{(p)}(a)$, and recall that $B_k(1 - \theta) = (-1)^k B_k(\theta)$ for all k.

We now turn to the proof of Theorem 2.1. We carry out the proof of the case $0 < \theta < 1$ only. That of the case $\theta = 1$ is almost identical. The proof of Corollary 2.2 can be carried out by using the relations given in (2.4), as mentioned already.

For $\mu \geq 0$ and $\nu \geq 0$ arbitrary integers, we can write, with $\phi_\mu(x)$ and $\psi_\nu(x)$ defined as in Section 1,

$$(3.2) \quad f(x) = \sum_{s=0}^{\mu-1} c_s u_{\gamma_s}(x) + \phi_\mu(x), \quad f(x) = \sum_{s=0}^{\nu-1} d_s v_{\delta_s}(x) + \psi_\nu(x),$$

where

$$(3.3) \quad \begin{aligned} \phi_\mu &\in C^\infty(a, b) \text{ and } \phi_\mu(x) = O\big((x-a)^{\gamma_\mu}\big) \text{ as } x \to a+, \\ \psi_\nu &\in C^\infty(a, b) \text{ and } \psi_\nu(x) = O\big((b-x)^{\delta_\nu}\big) \text{ as } x \to b-. \end{aligned}$$

Let m be an arbitrary large positive integer, and let μ and ν be the smallest integers for which $\gamma_{\mu-1} < m - 1 \leq \gamma_\mu$ and $\delta_{\nu-1} < m - 1 \leq \delta_\nu$. Because $\lim_{s\to\infty} \Re\gamma_s = +\infty$ and $\lim_{s\to\infty} \Re\delta_s = +\infty$, such μ and ν exist and are unique. Then, there hold

$$(3.4) \quad \begin{aligned} \Re\gamma_{\mu-1} &< \Re\gamma_\mu, \quad \Re\gamma_{\mu-1} + 1 < m < \Re\gamma_\mu + 2, \\ \Re\delta_{\nu-1} &< \Re\delta_\nu, \quad \Re\delta_{\nu-1} + 1 < m < \Re\delta_\nu + 2. \end{aligned}$$

Thus, by our assumption on the asymptotic expansions of $f^{(k)}(x)$ for each $k = 0, 1, \dots$, there hold

(3.5)
$$\phi_\mu^{(k)}(x) = O\big((x-a)^{\gamma_\mu - k}\big) \quad \text{as } x \to a+,$$
$$\psi_\nu^{(k)}(x) = O\big((b-x)^{\delta_\nu - k}\big) \quad \text{as } x \to b-,$$

from which we conclude that, for every $t \in (a, b)$,

(3.6)
$$\phi_\mu \in C^{m-2}[a, t], \ \phi_\mu^{(k)}(a) = 0, \ k = 0, 1, \dots, m-2,$$
$$\psi_\nu \in C^{m-2}[t, b], \ \psi_\nu^{(k)}(b) = 0, \ k = 0, 1, \dots, m-2.$$

Finally, we split the integral $I[f] = \int_a^b f(x)\,dx$ as in

(3.7)
$$I[f] = I_a^t[f] + I_t^b[f],$$

where

(3.8)
$$I_a^t[f] := \int_a^t f(x)\,dx, \quad I_t^b[f] := \int_t^b f(x)\,dx; \quad t = rh, \ r = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

We also split the trapezoidal rule $\widetilde{T}_n[f; \theta]$ as in

(3.9)
$$\widetilde{T}_n[f; \theta] = \widetilde{T}_n^{(a)}[f; \theta] + \widetilde{T}_n^{(b)}[f; \theta],$$

where

(3.10)
$$\widetilde{T}_n^{(a)}[f; \theta] := h \sum_{i=0}^{r-1} f(a + ih + \theta h),$$
$$\widetilde{T}_n^{(b)}[f; \theta] := h \sum_{i=r}^{n-1} f(a + ih + \theta h).$$

Thus, $\widetilde{T}_n^{(a)}[f; \theta]$ and $\widetilde{T}_n^{(b)}[f; \theta]$ are, respectively, the offset trapezoidal rule approximations for the integrals $I_a^t[f]$ and $I_t^b[f]$, with stepsize h. Note also that $t \sim (a+b)/2$ as $h \to 0$, so that the intervals $[a, t]$ and $[t, b]$ are both asymptotically of length $(b-a)/2$ as $h \to 0$.

In view of the above, for $\widetilde{T}_n^{(a)}[f; \theta]$, we have

(3.11)
$$\widetilde{T}_n^{(a)}[f; \theta] = \sum_{s=0}^{\mu-1} c_s \widetilde{T}_n^{(a)}[u_{\gamma_s}; \theta] + \widetilde{T}_n^{(a)}[\phi_\mu; \theta]$$

$$= \sum_{s=0}^{\mu-1} c_s \left\{ I_a^t[u_{\gamma_s}] + \zeta(-\gamma_s, \theta)\, h^{\gamma_s + 1} \right.$$

$$+ \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} u_{\gamma_s}^{(k-1)}(t) \, h^k + O(h^m) \bigg\}$$

$$+ \bigg\{ I_a^t[\phi_\mu] + \sum_{k=1}^{m-2} \frac{B_k(\theta)}{k!} \big[\phi_\mu^{(k-1)}(t) - \phi_\mu^{(k-1)}(a) \big] h^k$$

$$+ O(h^{m-2}) \bigg\} \quad \text{as } h \to 0$$

$$= I_a^t \bigg[\sum_{s=0}^{\mu-1} c_s u_{\gamma_s} + \phi_\mu \bigg] + \sum_{s=0}^{\mu-1} c_s \, \zeta(-\gamma_s, \theta) \, h^{\gamma_s+1}$$

$$+ \sum_{k=1}^{m-2} \frac{B_k(\theta)}{k!} \bigg[\sum_{s=0}^{\mu-1} c_s u_{\gamma_s}^{(k-1)}(t) + \phi_\mu^{(k-1)}(t) \bigg] h^k$$

$$+ O(h^{m-2}) \quad \text{as } h \to 0.$$

The first equality is an immediate consequence of (3.2); the second follows by invoking (3.6) and applying Theorem 3.1 to $\phi_\mu(x)$ and by applying Theorem 3.2 to the $u_{\gamma_s}(x)$; the third is obtained by employing (3.6). Invoking (3.2) again, we finally obtain

$$(3.12) \qquad \tilde{T}_n^{(a)}[f; \theta] = I_a^t[f] + \sum_{s=0}^{\mu-1} c_s \, \zeta(-\gamma_s, \theta) \, h^{\gamma_s+1}$$

$$+ \sum_{k=1}^{m-2} \frac{B_k(\theta)}{k!} f^{(k-1)}(t) \, h^k + O(h^{m-2}) \quad \text{as } h \to 0.$$

Similarly, for $\tilde{T}_n^{(b)}[f; \theta]$, we have

$$(3.13) \qquad \tilde{T}_n^{(b)}[f; \theta] = \sum_{s=0}^{\nu-1} d_s \tilde{T}_n^{(b)}[v_{\delta_s}; \theta] + \tilde{T}_n^{(b)}[\psi_\nu; \theta]$$

$$= \sum_{s=0}^{\nu-1} d_s \bigg\{ I_t^b[v_{\delta_s}] + \zeta(-\delta_s, 1 - \theta) \, h^{\delta_s+1}$$

$$- \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} v_{\delta_s}^{(k-1)}(t) \, h^k + O(h^m) \bigg\}$$

$$+ \bigg\{ I_t^b[\psi_\nu] + \sum_{k=1}^{m-2} \frac{B_k(\theta)}{k!} \big[\psi_\nu^{(k-1)}(b) - \psi_\nu^{(k-1)}(t) \big] h^k$$

$$+ O(h^{m-2}) \bigg\} \quad \text{as } h \to 0$$

$$= I_t^b \left[\sum_{s=0}^{\nu-1} d_s v_{\delta_s} + \psi_\nu \right] + \sum_{s=0}^{\nu-1} d_s \, \zeta(-\delta_s, 1 - \theta) \, h^{\delta_s + 1}$$

$$- \sum_{k=1}^{m-2} \frac{B_k(\theta)}{k!} \left[\sum_{s=0}^{\nu-1} d_s v_{\delta_s}^{(k-1)}(t) + \psi_\nu^{(k-1)}(t) \right] h^k$$

$$+ O(h^{m-2}) \quad \text{as } h \to 0.$$

The first equality is an immediate consequence of (3.2); the second follows by invoking (3.6) and applying Theorem 3.1 to $\psi_\nu(x)$ and by applying Theorem 3.3 to the $v_{\delta_s}(x)$; the third is obtained by employing (3.6). Invoking (3.2) again, we finally obtain

$$(3.14) \qquad \widetilde{T}_n^{(b)}[f; \theta] = I_t^b[f] + \sum_{s=0}^{\nu-1} d_s \, \zeta(-\delta_s, 1 - \theta) \, h^{\delta_s + 1}$$

$$- \sum_{k=1}^{m-2} \frac{B_k(\theta)}{k!} f^{(k-1)}(t) \, h^k + O(h^{m-2}) \quad \text{as } h \to 0.$$

Adding (3.12) to (3.14), and invoking (3.7) and (3.9), we get

$$(3.15) \qquad \widetilde{T}_n[f; \theta] = I[f] + \sum_{s=0}^{\mu-1} c_s \, \zeta(-\gamma_s, \theta) \, h^{\gamma_s + 1}$$

$$+ \sum_{s=0}^{\nu-1} d_s \, \zeta(-\delta_s, 1 - \theta) \, h^{\delta_s + 1} + O(h^{m-2}) \quad \text{as } h \to 0.$$

The result now follows by recalling that m is an arbitrary integer and that $\mu, \nu \to \infty$ as $m \to \infty$.

Before closing, we would like to note that the $O(h^m)$ terms in the first curly brackets of (3.12) and (3.14) appear as a result of the intervals $[a, t]$ and $[t, b]$ being of *nonzero* length as $h \to 0$, as can be verified by analyzing the term $R_m(h; \theta)$ in Theorem 3.2 and the term $S_m(h; \theta)$ in Theorem 3.3. This explains our choice of t.

3.2 Proof of Theorem 2.3

To prove Theorem 2.3, we need the Euler–Maclaurin expansions of the functions $u_{\omega,i}(x) = u_\omega(x)[\log(x - a)]^i$ and $v_{\omega,i}(x) = v_\omega(x)[\log(b - x)]^i$. Following Navot [7], we first observe that

$$u_{\omega,i}(x) = \frac{d^i}{d\omega^i} u_\omega(x) \quad \text{and} \quad v_{\omega,i}(x) = \frac{d^i}{d\omega^i} v_\omega(x).$$

Consequently,

$$I[u_{\omega,i}] = \frac{d^i}{d\omega^i} I[u_\omega], \qquad \widetilde{T}_n[u_{\omega,i};\theta] = \frac{d^i}{d\omega^i} \widetilde{T}_n[u_\omega;\theta],$$

$$\check{T}_n[u_{\omega,i}] = \frac{d^i}{d\omega^i} \check{T}_n[u_\omega],$$

$$I[v_{\omega,i}] = \frac{d^i}{d\omega^i} I[v_\omega], \qquad \widetilde{T}_n[v_{\omega,i};\theta] = \frac{d^i}{d\omega^i} \widetilde{T}_n[v_\omega;\theta],$$

$$\check{T}_n[v_{\omega,i}] = \frac{d^i}{d\omega^i} \check{T}_n[v_\omega].$$

Here, as can easily be shown, $I[u_{\omega,i}]$ and $I[v_{\omega,i}]$ are indeed Hadamard finite part integrals when, respectively, $I[u_\omega]$ and $I[v_\omega]$ are. In any case, they are given by

$$I[u_{\omega,i}] = \frac{d^i}{d\omega^i} \frac{(b-a)^{\omega+1}}{\omega+1} = I[v_{\omega,i}].$$

It is now easy to verify that the Euler–Maclaurin expansions we are interested in can be obtained by differentiating the ones given in Theorems 3.2 and 3.3 i times with respect to ω.

Applying the operator $D_\omega^i = \frac{d^i}{d\omega^i}$ to the result in Theorem 3.2, we then have

$$\widetilde{T}_n[u_{\omega,i};\theta] = I[u_{\omega,i}] + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} u_{\omega,i}^{(k-1)}(b)\, h^k$$

$$+ D_\omega^i \big[\zeta(-\omega,\theta)\, h^{\omega+1}\big] + D_\omega^i R_m(h;\theta),$$

where

$$D_\omega^i R_m(h;\theta) = h^m \int_b^\infty u_{\omega,i}^{(m)}(x) \frac{\bar{B}_m(\theta - n\frac{x-a}{b-a})}{m!}\, dx = O(h^m) \quad \text{as } h \to 0.$$

Applying $D_\omega^i = \frac{d^i}{d\omega^i}$ to the result in Theorem 3.3, we similarly have

$$\widetilde{T}_n[v_{\omega,i};\theta] = I[v_{\omega,i}] - \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} v_{\omega,i}^{(k-1)}(b)\, h^k$$

$$+ D_\omega^i \big[\zeta(-\omega, 1-\theta)\, h^{\omega+1}\big] + D_\omega^i S_m(h;\theta),$$

where

$$D_\omega^i S_m(h;\theta) = h^m \int_b^\infty u_{\omega,i}^{(m)}(x) \frac{\bar{B}_m(1-\theta - n\frac{x-a}{b-a})}{m!}\, dx = O(h^m) \quad \text{as } h \to 0.$$

Again, with $\mu \geq 0$ and $\nu \geq 0$ arbitrary integers, we can split the function $f(x)$ as in

$$f(x) = \sum_{s=0}^{\mu-1} \sum_{i=0}^{p_s} c_{si}\, u_{\gamma_s,i}(x) + \widehat{\phi}_\mu(x), \quad f(x) = \sum_{s=0}^{\nu-1} \sum_{i=0}^{q_s} d_{si}\, v_{\delta_s,i}(x) + \widehat{\psi}_\nu(x),$$

where

$$\widehat{\phi}_\mu \in C^\infty(a,b) \text{ and } \widehat{\phi}_\mu(x) = O\big((x-a)^{\gamma_\mu}[\log(x-a)]^{p_\mu}\big) \text{ as } x \to a+,$$

$$\widehat{\psi}_\nu \in C^\infty(a,b) \text{ and } \widehat{\psi}_\nu(x) = O\big((b-x)^{\delta_\nu}[\log(b-x)]^{q_\nu}\big) \text{ as } x \to b-\,.$$

Again, for m an arbitrary large positive integer, we can choose μ and ν such that (3.4) is satisfied.

From these, we have

$$\widehat{\phi}_\mu^{(k)}(x) = O\big((x-a)^{\gamma_\mu-k}[\log(x-a)]^{p_\mu}\big) \text{ as } x \to a+,$$

$$\widehat{\psi}_\nu^{(k)}(x) = O\big((b-x)^{\delta_\nu-k}[\log(b-x)]^{q_\nu}\big) \text{ as } x \to b-,$$

hence, for every $t \in (a,b)$,

$$\widehat{\phi}_\mu \in C^{m-2}[a,t], \ \widehat{\phi}_\mu^{(k)}(a) = 0, \ k = 0, 1, \dots, m-2,$$

$$\widehat{\psi}_\nu \in C^{m-2}[t,b], \ \widehat{\psi}_\nu^{(k)}(b) = 0, \ k = 0, 1, \dots, m-2.$$

We now proceed precisely as in the proof of Theorem 2.1. We leave the details to the reader.

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