



Algebraic properties of some new vector-valued rational interpolants

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Abstract

In a recent paper of the author [A. Sidi, A new approach to vector-valued rational interpolation, *J. Approx. Theory*, 130 (2004) 177–187], three new interpolation procedures for vector-valued functions $F(z)$, where $F : \mathbb{C} \rightarrow \mathbb{C}^N$, were proposed, and some of their properties were studied. In this work, after modifying their definition slightly, we continue the study of these interpolation procedures. We show that the interpolants produced via these procedures are unique in some sense and that they are symmetric functions of the points of interpolation. We also show that, under the conditions that guarantee uniqueness, they also reproduce $F(z)$ in case $F(z)$ is a rational function.

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1. Introduction

In a recent work, Sidi [6], we presented three different kinds of vector-valued rational interpolation procedures, denoted IMPE, IMMPE, and ITEA there. These were modelled after the rational approximation procedures from Maclaurin series of vector-valued functions developed in Sidi [3], which, in turn had their origin in the vector extrapolation methods MPE (the minimal polynomial extrapolation), MMPE (the modified minimal polynomial extrapolation), and TEA

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(the topological epsilon algorithm). The methods MPE, RRE, and TEA are used for accelerating the convergence of certain kinds of vector sequences, such as those produced by fixed-point iterative methods on linear and nonlinear systems of algebraic equations.

Some of the properties of IMPE, IMMPE, and ITEA interpolants have already been mentioned in [6]. In this paper, we continue to study these interpolants by concentrating on some of their algebraic properties. To set the stage for later developments, and to fix the notation as well, we start with a summary of the developments in [6]. In this summary, we modify the definitions of the interpolants slightly.

Let z be a complex variable and let $F(z)$ be a vector-valued function such that $F : \mathbb{C} \rightarrow \mathbb{C}^N$. Assume that $F(z)$ is defined on a bounded open set $\Omega \in \mathbb{C}$ and consider the problem of interpolating $F(z)$ at some of the points ζ_1, ζ_2, \dots , in this set. We do not assume that the ζ_i are necessarily distinct. The general picture is described in the next paragraph:

Let a_1, a_2, \dots , be distinct complex numbers, and let

$$\begin{aligned} \zeta_1 &= \zeta_2 = \dots = \zeta_{r_1} = a_1, \\ \zeta_{r_1+1} &= \zeta_{r_1+2} = \dots = \zeta_{r_1+r_2} = a_2, \\ \zeta_{r_1+r_2+1} &= \zeta_{r_1+r_2+2} = \dots = \zeta_{r_1+r_2+r_3} = a_3 \\ &\text{and so on.} \end{aligned} \tag{1.1}$$

Let $G_{m,n}(z)$ be the vector-valued polynomial (of degree at most $n - m$) that interpolates $F(z)$ at the points $\zeta_m, \zeta_{m+1}, \dots, \zeta_n$ in the generalized Hermite sense. Thus, in Newtonian form, this polynomial is given as in (see, e.g., Stoer and Bulirsch [7, Chapter 2] or Atkinson [1, Chapter 3])

$$\begin{aligned} G_{m,n}(z) &= F[\zeta_m] + F[\zeta_m, \zeta_{m+1}](z - \zeta_m) + F[\zeta_m, \zeta_{m+1}, \zeta_{m+2}](z - \zeta_m)(z - \zeta_{m+1}) \\ &\quad + \dots + F[\zeta_m, \zeta_{m+1}, \dots, \zeta_n](z - \zeta_m)(z - \zeta_{m+1}) \dots (z - \zeta_{n-1}). \end{aligned} \tag{1.2}$$

Here, $F[\zeta_r, \zeta_{r+1}, \dots, \zeta_{r+s}]$ is the divided difference of order s of $F(z)$ over the set of points $\{\zeta_r, \zeta_{r+1}, \dots, \zeta_{r+s}\}$. The $F[\zeta_r, \zeta_{r+1}, \dots, \zeta_{r+s}]$ are defined, as in the scalar case, by the recursion relations

$$\begin{aligned} F[\zeta_r, \zeta_{r+1}, \dots, \zeta_{r+s}] &= \frac{F[\zeta_r, \zeta_{r+1}, \dots, \zeta_{r+s-1}] - F[\zeta_{r+1}, \zeta_{r+2}, \dots, \zeta_{r+s}]}{\zeta_r - \zeta_{r+s}}, \\ r &= 1, 2, \dots, \quad s = 1, 2, \dots, \end{aligned} \tag{1.3}$$

with the initial conditions

$$F[\zeta_r] = F(\zeta_r), \quad r = 1, 2, \dots \tag{1.4}$$

Note that, in case $\zeta_r = \zeta_{r+1} = \dots = \zeta_{r+s}$, the right-hand side of (1.3) is defined via a limiting process, with the result

$$F[\zeta_r, \zeta_{r+1}, \dots, \zeta_{r+s}] = \frac{F^{(s)}(\zeta_r)}{s!}. \tag{1.5}$$

Obviously, $F[\zeta_r, \zeta_{r+1}, \dots, \zeta_{r+s}]$ are all vectors in \mathbb{C}^N .

For simplicity of notation, we define the scalar polynomials $\psi_{m,n}(z)$ via

$$\psi_{m,n}(z) = \prod_{r=m}^n (z - \zeta_r), \quad n \geq m \geq 1; \quad \psi_{m,m-1}(z) = 1, \quad m \geq 1. \tag{1.6}$$

We also define the vectors $D_{m,n}$ via

$$D_{m,n} = F[\xi_m, \xi_{m+1}, \dots, \xi_n], \quad n \geq m. \tag{1.7}$$

With this notation, we can rewrite (1.2) in the form

$$G_{m,n}(z) = \sum_{i=m}^n D_{m,i} \psi_{m,i-1}(z). \tag{1.8}$$

The vector-valued rational interpolants to the function $F(z)$ we developed in [6] are all of the general form

$$R(z) = \frac{U(z)}{V(z)} = \frac{\sum_{j=0}^k c_j \psi_{1,j}(z) G_{j+1,p}(z)}{\sum_{j=0}^k c_j \psi_{1,j}(z)}, \tag{1.9}$$

where c_0, c_1, \dots, c_k are, for the time being, arbitrary complex scalars, and p is an arbitrary integer. Obviously, $U(z)$ is a vector-valued polynomial of degree at most $p - 1$ and $V(z)$ is a scalar polynomial of degree at most k . It is also clear from (1.9) that $k \leq p - 1$.

The following theorem says that, whether the ξ_i are distinct or not, $R(z)$ interpolates $F(z)$. See [6, Lemmas 2.1 and 2.3].

Theorem 1.1. *Let the vector-valued rational function $R(z)$ be as in (1.9), and assume that $V(\xi_i) \neq 0, i = 1, 2, \dots, p$.*

- (i) *When the ξ_i are distinct, $R(z)$ interpolates $F(z)$ at the points $\xi_1, \xi_2, \dots, \xi_p$ in the ordinary sense:*

$$R(\xi_i) = F(\xi_i), \quad i = 1, \dots, p. \tag{1.10}$$

- (ii) *When the ξ_i are not necessarily distinct and are ordered as in (1.1), $R(z)$ interpolates $F(z)$ in the generalized Hermite sense as follows: let t and ρ be the unique integers satisfying $t \geq 0$ and $0 \leq \rho < r_{t+1}$ for which $p = \sum_{i=1}^t r_i + \rho$. Then,*

$$R^{(s)}(a_i) = F^{(s)}(a_i) \quad \text{for } s = 0, 1, \dots, r_i - 1 \text{ when } i = 1, \dots, t, \\ \text{and for } s = 0, 1, \dots, \rho - 1 \text{ when } i = t + 1. \tag{1.11}$$

(Of course, when $\rho = 0$, there is no interpolation at a_{t+1} .)

Remark. It must be noted that the condition $V(\xi_i) \neq 0, i = 1, \dots, p$, features throughout this work. Because $k < p$ and because p can be arbitrarily large, this condition might look too restrictive at first. This is not the case, however. Indeed, the condition $V(\xi_i) \neq 0, i = 1, \dots, p$, is natural for the following reason: normally, we take the points of interpolation ξ_i in a set Ω on which the function $F(z)$ is regular. If $R_{p,k}(z)$ is to approximate $F(z)$, it should also be a regular function over Ω and hence free of singularities there. Since the singularities of $R_{p,k}(z)$ are the zeroes of $V(z)$, this implies that $V(z)$ should not vanish on Ω . [We expect the singularities of $R_{p,k}(z)$ —the zeroes of $V(z)$ —to be close to the singularities of $F(z)$, which are outside the set Ω .]

So far, the c_j in (1.9) are arbitrary. Of course, the quality of $R(z)$ as an approximation to $F(z)$ depends very strongly on the choice of the c_j . Naturally, the c_j must depend on $F(z)$ and on

the ξ_j . Fixing the integers k and p such that $p \geq k + 1$, we determine the c_j as follows:

1. With the normalization $c_k = 1$, we determine c_0, c_1, \dots, c_{k-1} as the solution to the problem

$$\min_{c_0, c_1, \dots, c_{k-1}} \left\| \sum_{j=0}^k c_j D_{j+1, p+1} \right\| \quad \text{subject to } c_k = 1, \tag{1.12}$$

where $\| \cdot \|$ stands for an arbitrary vector norm in \mathbb{C}^N . With the l_1 - and l_∞ -norms, the optimization problem can be solved by using linear programming. With the l_2 -norm, it becomes a least-squares problem, which can be solved numerically via standard techniques. Of course, the inner product (\cdot, \cdot) that defines the l_2 -norm [that is, $\|u\| = \sqrt{(u, u)}$] is not restricted to the standard inner product $(u, v) = u^*v$; it can be given by $(u, v) = u^*Mv$, where M is a hermitian positive definite matrix. We let $\| \cdot \|$ in (1.12) be the l_2 -norm.

We denote the resulting rational interpolation procedure IMPE and the interpolant in (1.9) $R_{p,k}^{\text{IMPE}}(z)$.

2. Again, with the normalization $c_k = 1$, we determine c_0, c_1, \dots, c_{k-1} via the solution of the linear system

$$\left(q_i, \sum_{j=0}^k c_j D_{j+1, p+1} \right) = 0, \quad i = 1, \dots, k; \quad c_k = 1, \tag{1.13}$$

where q_1, \dots, q_k are linearly independent vectors in \mathbb{C}^N . Note that we can choose the vectors q_1, \dots, q_k to be independent of p or to depend on p .

We denote the resulting rational interpolation procedure IMMPE and the interpolant in (1.9) $R_{p,k}^{\text{IMMPE}}(z)$.

3. Again, with the normalization $c_k = 1$, we determine c_0, c_1, \dots, c_{k-1} via the solution of the linear system

$$\left(q, \sum_{j=0}^k c_j D_{j+1, p+s} \right) = 0, \quad s = 1, 2, \dots, k; \quad c_k = 1, \tag{1.14}$$

where q is a nonzero vector in \mathbb{C}^N .

We denote the resulting rational interpolation procedure ITEA and the interpolant in (1.9) $R_{p,k}^{\text{ITEA}}(z)$.

Remarks.

1. The way we determine the c_j here differs from the one given in [6] in that the normalization of $V(z)$ in [6] is $c_0 = 1$, whereas we have chosen $c_k = 1$ here.
2. Under the present normalization $c_k = 1$, the denominator polynomials $V(z)$ for $R_{p,k}^{\text{IMMPE}}(z)$ and for $R_{p,k}^{\text{ITEA}}(z)$ turn out to be the same as those given in [6], up to a constant multiplicative factor. The denominator polynomial $V(z)$ for $R_{p,k}^{\text{IMPE}}(z)$ is different from the corresponding one given in [6].
3. $V(z)$ for $R_{p,k}^{\text{IMPE}}(z)$ in [6] is a symmetric function of the points ξ_2, \dots, ξ_{p+1} , but not of ξ_1, \dots, ξ_{p+1} , all the points used in its construction. Under the present normalization $c_k = 1$, it does become symmetric in ξ_1, \dots, ξ_{p+1} , however. This was the motivation for switching to

- $c_k = 1$. $V(z)$ for $R_{p,k}^{IMMPE}(z)$ and for $R_{p,k}^{ITEA}(z)$ are symmetric functions of all the points ξ_i used in their construction, namely, of ξ_1, \dots, ξ_{p+1} for IMMPE and of ξ_1, \dots, ξ_{p+k} for ITEA.¹
4. Because $R_{p,k}(z)$ in the present paper achieves interpolation at the points $\xi_1, \xi_2, \dots, \xi_p$, and because its denominator $V(z)$ is symmetric in $\xi_1, \xi_2, \dots, \xi_p$, $R_{p,k}(z)$ also achieves symmetry in $\xi_1, \xi_2, \dots, \xi_p$.

The assertions of Remarks 1 and 2 can be verified by comparing the determinantal representations in the next theorem with those given in [6, Theorem 4.1]. The proof of this theorem is exactly the same as that of Theorem 4.1 in [6]. The proofs of the assertions in Remarks 3 and 4 are given in Section 3.

Theorem 1.2. *Let the vector-valued rational interpolant $R_{p,k}(z)$ to $F(z)$ be given by*

$$R_{p,k}(z) = \frac{U(z)}{V(z)} = \frac{\sum_{j=0}^k c_j \psi_{1,j}(z) G_{j+1,p}(z)}{\sum_{j=0}^k c_j \psi_{1,j}(z)}, \tag{1.15}$$

such that $R_{p,k}(\xi_i) = F(\xi_i)$, $i = 1, \dots, p$, and the scalars c_j are defined by (1.12) for IMPE, by (1.13) for IMMPE, and by (1.14) for ITEA. Then $R_{p,k}(z)$ has a determinant representation of the form

$$R_{p,k}(z) = \frac{P(z)}{Q(z)} = \frac{\begin{vmatrix} \psi_{1,0}(z) G_{1,p}(z) & \psi_{1,1}(z) G_{2,p}(z) & \cdots & \psi_{1,k}(z) G_{k+1,p}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}{\begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \cdots & \psi_{1,k}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}, \tag{1.16}$$

where

$$u_{i,j} = \begin{cases} (D_{i,p+1}, D_{j+1,p+1}) & \text{for IMPE,} \\ (q_i, D_{j+1,p+1}) & \text{for IMMPE,} \\ (q, D_{j+1,p+i}) & \text{for ITEA.} \end{cases} \tag{1.17}$$

Here, the numerator determinant $P(z)$ is vector-valued and is defined by its expansion with respect to its first row. That is, if M_j is the cofactor of the term $\psi_{1,j}(z)$ in the denominator determinant $Q(z)$, then

$$R_{p,k}(z) = \frac{\sum_{j=0}^k M_j \psi_{1,j}(z) G_{j+1,p}(z)}{\sum_{j=0}^k M_j \psi_{1,j}(z)}. \tag{1.18}$$

¹ A function $f(x_1, \dots, x_m)$ is symmetric in x_1, \dots, x_m if $f(x_{i_1}, \dots, x_{i_m}) = f(x_1, \dots, x_m)$ for every permutation $(x_{i_1}, \dots, x_{i_m})$ of (x_1, \dots, x_m) .

Now, in order to be acceptable as interpolants, the functions $R_{p,k}(z)$ must satisfy the following criteria:

1. They must be unique in some sense.
2. They must be symmetric in the points of interpolation. In other words, $R_{p,k}(z)$ must be the same rational function whatever the ordering of $\xi_1, \xi_2, \dots, \xi_p$.
3. If $F(z)$, the function being interpolated, is of the form $F(z) = \tilde{U}(z)/\tilde{V}(z)$, with $\tilde{U}(z)$ a vector-valued polynomial of degree at most $p - 1$ and $\tilde{V}(z)$ a scalar polynomial of degree exactly k , the rational interpolants $R_{p,k}(z)$ must reproduce $F(z)$ in the sense that $R_{p,k}(z) \equiv F(z)$, under appropriate conditions.

We treat the question of uniqueness in the next section. Even though the denominators $V(z)$ are defined in different ways, this treatment can be unified.

In Section 3, we discuss the symmetry of $R_{p,k}(z)$ in the interpolation points. This discussion is not straightforward because these interpolants are defined with the points of interpolation ordered as ξ_1, ξ_2, \dots . We are nevertheless able to show that $R_{p,k}(z)$ are symmetric functions of the underlying points of interpolation. In this study, the determinantal representations given in Theorem 1.2 prove to be very useful.

In Section 4, we turn to the reproducing property of the $R_{p,k}(z)$.

In Section 5, we provide an example function $F(z)$ for which the main condition for uniqueness and the reproducing property is satisfied.

Finally, as already mentioned in [6], the methods we have proposed for determining the c_j can be extended to the case in which $F(z)$ is such that $F : \mathbb{C} \rightarrow \mathbb{B}$, where \mathbb{B} is a general linear space, exactly as is shown in [3, Section 6]. This amounts to the introduction of the norm defined in \mathbb{B} when the latter is a normed space (for IMPE), and to the introduction of some bounded linear functionals (for IMMPE and ITEA). With these, the determinant representations of Theorem 1.2 remain unchanged as well. We refer the reader to [3] for the details.

2. Uniqueness of $R_{p,k}(z)$

As emphasized in [6], what differentiates between the various interpolants $R_{p,k}(z)$ is how their corresponding c_j are determined. With this in mind, the following lemma is the first step towards the answer to the question of uniqueness in some sense.

Lemma 2.1. *Let $V(z)$ be a fixed scalar polynomial of degree k , such that $V(\xi_i) \neq 0$, $i = 1, \dots, p$. Define $R(z)$ to be a vector-valued rational function of the form $R(z) = U(z)/V(z)$, where $U(z)$ is a vector-valued polynomial of degree at most $p - 1$, and $R(\xi_i) = F(\xi_i)$, $i = 1, 2, \dots, p$. Then, $R(z)$ is unique. In particular, if we express $V(z)$ in the form $V(z) = \sum_{j=0}^k c_j \psi_{1,j}(z)$, which is possible, then $R(z)$ is as given in (1.9).*

Proof. Let $\tilde{R}(z) = \tilde{U}(z)/V(z)$ be another vector-valued rational function, where $\tilde{U}(z)$ is a vector-valued polynomial of degree at most $p - 1$, such that $\tilde{R}(\xi_i) = F(\xi_i)$, $i = 1, \dots, p$. Then $\tilde{R}(\xi_i) = R(\xi_i)$, $i = 1, \dots, p$. Because $V(\xi_i) \neq 0$, $i = 1, \dots, p$, this implies that $\tilde{U}(\xi_i) - U(\xi_i) = 0$, $i = 1, \dots, p$. Since $\tilde{U}(z) - U(z)$ is a (vector-valued) polynomial of degree at most $p - 1$, this is possible only if $\tilde{U}(z) \equiv U(z)$. Thus, $R(z)$ is unique. The rest of the proof is immediate. \square

From Lemma 2.1, it is clear that the uniqueness of $R_{p,k}(z) = U(z)/V(z)$ for IMPE, IMMPE, and ITEA depends on the uniqueness of the denominator polynomial $V(z)$. The uniqueness of

$V(z)$, in turn, hinges on the uniqueness of the coefficients c_j . When the c_j are determined as in (1.12) or (1.13) or (1.14), we have the following result:

Theorem 2.2. *Let*

$$R_{p,k}(z) = \frac{U(z)}{V(z)} = \frac{\sum_{j=0}^k c_j \psi_{1,j}(z) G_{j+1,p}(z)}{\sum_{j=0}^k c_j \psi_{1,j}(z)},$$

with the c_j defined via (1.12) or (1.13) or (1.14). Then $R_{p,k}(z)$ is unique provided

$$\begin{vmatrix} u_{1,0} & u_{1,1} & \cdots & u_{1,k-1} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k-1} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k-1} \end{vmatrix} \neq 0, \tag{2.1}$$

where $u_{i,j}$ are as defined in (1.17), and $V(\zeta_i) \neq 0, i = 1, \dots, p$.

Proof. We first note that the equations in (1.12) or (1.13) or (1.14) that define the c_j can be rewritten as in

$$\sum_{j=0}^{k-1} u_{i,j} c_j = -u_{i,k}, \quad i = 1, \dots, k. \tag{2.2}$$

Thus, the condition in (2.1) guarantees the existence and uniqueness of the c_j . The proof now follows by invoking Lemma 2.1. \square

Note that the condition in (2.1) is equivalent to the conditions we state next:

1. The vectors $D_{i,p+1}, i = 1, \dots, k$, are linearly independent in case of $R_{p,k}^{\text{IMPE}}(z)$. This also means that $k \leq N$.
2. The vectors $D_{i,p+1}, i = 1, \dots, k$, are linearly independent, and the $k \times k$ matrix Q^*D , where

$$Q = [q_1 | q_2 | \dots | q_k] \in \mathbb{C}^{N \times k} \quad \text{and} \quad D = [D_{1,p+1} | D_{2,p+1} | \dots | D_{k,p+1}] \in \mathbb{C}^{N \times k}$$

has full rank in case of $R_{p,k}^{\text{IMMPE}}(z)$. This also means that $k \leq N$.

3. $\begin{vmatrix} (q, D_{1,p+1}) & (q, D_{2,p+1}) & \cdots & (q, D_{k,p+1}) \\ (q, D_{1,p+2}) & (q, D_{2,p+2}) & \cdots & (q, D_{k,p+2}) \\ \vdots & \vdots & & \vdots \\ (q, D_{1,p+k}) & (q, D_{2,p+k}) & \cdots & (q, D_{k,p+k}) \end{vmatrix} \neq 0$ in case of $R_{p,k}^{\text{ITEA}}(z)$.

As we have seen, in order for the conditions stated in (2.1) that pertain to the uniqueness of $R_{p,k}(z)$ for IMPE and IMMPE to be satisfied, the vectors $D_{i,p+1}, i = 1, \dots, k$, must be linearly independent. In Section 5, we will see that this is the case when the function $F(z)$ is of the form $F(z) = \tilde{U}(z)/\tilde{V}(z)$, where $\tilde{U}(z)$ is a vector-valued polynomial and $\tilde{V}(z)$ is a scalar polynomial, subject to certain conditions on the Laurent expansion of $F(z)$: (i) when the poles of $F(z)$ are all simple, that is, when

$$F(z) = \sum_{i=0}^v u_i z^i + \sum_{s=1}^{\mu} \frac{v_s}{z - z_s},$$

where u_i are arbitrary vectors, $\mu \leq N$, and z_1, \dots, z_μ are distinct points in \mathbb{C} , the vectors v_1, \dots, v_μ must be linearly independent. (ii) When some or all of the poles of $F(z)$ are multiple, that is, when

$$F(z) = \sum_{i=0}^v u_i z^i + \sum_{s=1}^{\sigma} \sum_{j=1}^{r_s} \frac{v_{sj}}{(z - z_s)^j},$$

where u_i are arbitrary vectors, $\mu = \sum_{s=1}^{\sigma} r_s \leq N$, and z_1, \dots, z_σ are distinct points in \mathbb{C} , the vectors v_{sj} , $1 \leq j \leq r_s$, $1 \leq s \leq \sigma$, must be linearly independent.

3. Symmetry of $R_{p,k}(z)$

3.1. Preliminaries

In this section, we show that, in case the points of interpolation ξ_i are distinct, $R_{p,k}(z)$ (either for IMPE or for IMMPE or for ITEA) does not depend on the order in which the ξ_i are introduced into the interpolation process, that is, $R_{p,k}(z)$ is a symmetric function of the points ξ_1, \dots, ξ_p . We start with the following lemma:

Lemma 3.1. *Define $R(z)$ to be a vector-valued rational function of the form $R(z) = U(z)/V(z)$, where $U(z)$ is a vector-valued polynomial of degree at most $p - 1$ and $V(z)$ is a scalar polynomial of degree k . Assume that $V(\xi_i) \neq 0$, $i = 1, \dots, p$, and that $R(\xi_i) = F(\xi_i)$, $i = 1, 2, \dots, p$. Then, $R(z)$ is a symmetric function of ξ_1, \dots, ξ_p provided $V(z)$ is too.*

Proof. Because $V(z)$ is a symmetric function of ξ_1, \dots, ξ_p , $R(z)$ will also be a symmetric function of ξ_1, \dots, ξ_p provided $U(z)$ is too. Now, $U(z) = V(z)R(z)$. Therefore,

$$U(\xi_i) = V(\xi_i)R(\xi_i) = V(\xi_i)F(\xi_i), \quad i = 1, \dots, p, \tag{3.1}$$

that is, $U(z)$ interpolates $V(z)F(z)$ at the p points ξ_1, \dots, ξ_p . Being a (vector-valued) polynomial of degree at most $p - 1$, $U(z)$ is the unique polynomial of interpolation to $V(z)F(z)$ at ξ_1, \dots, ξ_p . Hence $U(z)$ is a symmetric function of ξ_1, \dots, ξ_p . Consequently, so is $R(z) = U(z)/V(z)$. \square

In view of Lemma 3.1, in order to establish that $R_{p,k}(z) = U(z)/V(z)$, for the interpolation procedures considered in this work, is a symmetric function of ξ_1, \dots, ξ_p , it is sufficient to show that $V(z)$ is a symmetric function of ξ_1, \dots, ξ_p . We do this separately for $R_{p,k}^{\text{IMPE}}(z)$, $R_{p,k}^{\text{IMMPE}}(z)$, and $R_{p,k}^{\text{ITEA}}(z)$. We actually show that the polynomials $V(z)$ are symmetric functions of *all* the ξ_i used in their construction.

The next lemma (see, e.g. Bourbaki [2, Chapter 1, 5.7, p. 63, Proposition 9]) too will be of use in the sequel.

Lemma 3.2. *Let (i_1, i_2, \dots, i_s) denote the permutation $\begin{pmatrix} 1 & 2 & \dots & s \\ i_1 & i_2 & \dots & i_s \end{pmatrix}$. Then (i_1, i_2, \dots, i_s) is a product of transpositions of the form $(j, j + 1)$, $j \in \{1, 2, \dots, s - 1\}$.*

We illustrate this lemma via an example that indicates the way to the general proof. Let $s = 5$, and consider the permutation $(3, 5, 2, 1, 4)$. This permutation can be obtained from $(1, 2, 3, 4, 5)$

via the following sequence of transpositions:

$$(12345) \mapsto (13245) \mapsto (31245) \mapsto (31254) \mapsto (31524) \mapsto (35124) \mapsto (35214).$$

Thus, as a product of transpositions of the form $(j, j + 1)$, we have

$$(35214) = (34)(23)(34)(45)(12)(23),$$

the transpositions being performed from right to left.

The following lemma helps to unify the treatments of the different rational interpolation procedures.

Lemma 3.3. *Define*

$$g(\xi) = \frac{1}{z - \xi} \quad (\xi : \text{variable}, z : \text{fixed parameter}), \tag{3.2}$$

and denote $g[\xi_m, \xi_{m+1}, \dots, \xi_{m+q}]$, the divided difference of order q of $g(\xi)$ on the set of points $\{\xi_m, \xi_{m+1}, \dots, \xi_{m+q}\}$, by $g_{m,m+1,\dots,m+q}$. Then, the denominator determinant $Q(z)$ of $R_{p,k}(z)$ in (1.16), namely,

$$Q(z) = \begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \cdots & \psi_{1,k}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}, \tag{3.3}$$

can be rewritten in the form

$$Q(z) = \psi_{1,n}(z) W(\xi_1, \xi_2, \dots, \xi_n; z), \tag{3.4}$$

where n is an integer greater than k and

$$W(\xi_1, \xi_2, \dots, \xi_n; z) = \begin{vmatrix} g_{1,\dots,n} & g_{2,\dots,n} & \cdots & g_{k+1,\dots,n} \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}. \tag{3.5}$$

Remark. In the sequel, we take n to be the number of the ξ_i used to construct $V(z)$. Thus, $n = p + 1$ for $R_{p,k}^{\text{IMPE}}(z)$ and $R_{p,k}^{\text{IMMPE}}(z)$, while $n = p + k$ for $R_{p,k}^{\text{ITEA}}(z)$.

Proof. By (1.6),

$$\psi_{1,r}(z) = \prod_{i=1}^r (z - \xi_i) = \frac{\psi_{1,n}(z)}{\psi_{r+1,n}(z)}, \quad 0 \leq r \leq n - 1. \tag{3.6}$$

Furthermore, with the function $g(\xi)$ as defined in (3.2), using the recursion relation in (1.3), it can be shown by induction that $g[\xi_m, \xi_{m+1}, \dots, \xi_s]$, is given by

$$g[\xi_m, \xi_{m+1}, \dots, \xi_s] = \frac{1}{\psi_{m,s}(z)}. \tag{3.7}$$

Consequently,

$$\psi_{1,r}(z) = \psi_{1,n}(z) g[\zeta_{r+1}, \zeta_{r+2}, \dots, \zeta_n] = \psi_{1,n}(z) g_{r+1,r+2,\dots,n}, \quad 0 \leq r \leq n - 1. \quad (3.8)$$

Substituting (3.8) in (3.3), and factoring out $\psi_{1,n}(z)$ from the first row, the result follows. \square

Now, the factor $\psi_{1,n}(z)$ in (3.4) is a symmetric function of $\zeta_1, \zeta_2, \dots, \zeta_n$. We therefore need to analyze only the determinant $W(\zeta_1, \zeta_2, \dots, \zeta_n; z)$.

What we want to show now is that, for any permutation $(\zeta_{i_1}, \zeta_{i_2}, \dots, \zeta_{i_n})$ of $(\zeta_1, \zeta_2, \dots, \zeta_n)$, where (i_1, i_2, \dots, i_n) is a permutation of $(1, 2, \dots, n)$, there holds

$$W(\zeta_{i_1}, \zeta_{i_2}, \dots, \zeta_{i_n}; z) \equiv W(\zeta_1, \zeta_2, \dots, \zeta_n; z).$$

By Lemma 3.2, it is enough to show that this holds when, for any $i \in \{1, 2, \dots, n - 1\}$, ζ_i and ζ_{i+1} are interchanged in $W(\zeta_1, \zeta_2, \dots, \zeta_n; z)$. That is, it is enough to show that

$$\widehat{W}_i(z) \equiv W(z), \quad (3.9)$$

where we have denoted

$$W(z) = W(\zeta_1, \zeta_2, \dots, \zeta_n; z) \quad (3.10)$$

and

$$\widehat{W}_i(z) = W(\zeta_1, \zeta_2, \dots, \zeta_{i-1}, \zeta_{i+1}, \zeta_i, \zeta_{i+2}, \dots, \zeta_n; z), \quad (3.11)$$

for short. We now turn to this subject. In the remainder of this section, we use the notation introduced above freely.

Note that, in the analysis below, we also make use of the facts

$$\left(a, \sum_{r=1}^s \beta_r b_r \right) = \sum_{r=1}^s \beta_r (a, b_r), \quad \left(\sum_{r=1}^s \alpha_r a_r, b \right) = \sum_{r=1}^s \bar{\alpha}_r (a_r, b).$$

Here a, b, a_r, b_r are vectors and α_r, β_r are scalars, and $\bar{\alpha}_r$ stands for the complex conjugate of α_r .

3.2. Treatment of $R_{p,k}^{\text{IMMPE}}(z)$

Lemma 3.4. *The denominator polynomial $V(z)$ of $R_{p,k}^{\text{IMMPE}}(z)$ is a symmetric function of $\zeta_1, \zeta_2, \dots, \zeta_{p+1}$ used to construct $V(z)$.*

Proof. With the notation

$$n = p + 1 \quad \text{and} \quad w_{m,\dots,n}^{(i)} = (q_i, D_{m,n}), \quad \Rightarrow \quad u_{i,j} = w_{j+1,\dots,n}^{(i)} \quad (3.12)$$

(3.5) becomes

$$W(\zeta_1, \zeta_2, \dots, \zeta_n; z) = \begin{vmatrix} g_{1,\dots,n} & g_{2,\dots,n} & \cdots & g_{k+1,\dots,n} \\ w_{1,\dots,n}^{(1)} & w_{2,\dots,n}^{(1)} & \cdots & w_{k+1,\dots,n}^{(1)} \\ w_{1,\dots,n}^{(2)} & w_{2,\dots,n}^{(2)} & \cdots & w_{k+1,\dots,n}^{(2)} \\ \vdots & \vdots & & \vdots \\ w_{1,\dots,n}^{(k)} & w_{2,\dots,n}^{(k)} & \cdots & w_{k+1,\dots,n}^{(k)} \end{vmatrix}. \quad (3.13)$$

From (3.12) and (3.13), it is clear that the elements in each column of the determinant expression for $W(\xi_1, \xi_2, \dots, \xi_n; z)$ are divided differences of the same order and over the same set of points, hence satisfy the same recursion relations. Specifically, the elements in the r th column are divided differences of order $n - r = p - r + 1$ over the set of points $\{\xi_r, \xi_{r+1}, \dots, \xi_n\}$. This allows us to perform on the determinant elementary column transformations easily.

What we want to show now is that, for any $i \in \{1, 2, \dots, n - 1\}$, (3.9) holds. There are two cases to consider: (i) $i \geq k + 1$, and (ii) $1 \leq i \leq k$. In the sequel, we make use of the fact that a divided difference on the set of points $\{\xi_m, \xi_{m+1}, \dots, \xi_n\}$ is a symmetric function of $\xi_m, \xi_{m+1}, \dots, \xi_n$.

By (3.13), by the fact that $k \leq p$, and by the symmetry property of divided differences, it follows that $\widehat{W}_i(z)$ has exactly the same columns as $W(z)$ when $i \geq k + 1$, hence (3.9) holds trivially.

When $1 \leq i \leq k$, $\widehat{W}_i(z)$ differs from $W(z)$ columnwise. However, due to the symmetry property of divided differences, $\widehat{W}_i(z)$ differs from $W(z)$ only in its $(i + 1)$ st column, this column being

$$[g_{i,i+2,\dots,n}, w_{i,i+2,\dots,n}^{(1)}, w_{i,i+2,\dots,n}^{(2)}, \dots, w_{i,i+2,\dots,n}^{(k)}]^T.$$

Now, by (1.3), there holds

$$g_{i,\dots,n} = \frac{g_{i,i+2,\dots,n} - g_{i+1,i+2,\dots,n}}{\xi_i - \xi_{i+1}}, \tag{3.14}$$

from which

$$g_{i+1,i+2,\dots,n} = g_{i,i+2,\dots,n} + (\xi_{i+1} - \xi_i)g_{i,\dots,n}. \tag{3.15}$$

The same holds with $g_{m,\dots,n}$ replaced by $w_{m,\dots,n}^{(s)}$, that is,

$$w_{i+1,i+2,\dots,n}^{(s)} = w_{i,i+2,\dots,n}^{(s)} + (\xi_{i+1} - \xi_i)w_{i,\dots,n}^{(s)}.$$

Thus, if we multiply the i th column in $\widehat{W}_i(z)$ by $(\xi_{i+1} - \xi_i)$ and add to the $(i + 1)$ st column, the $(i + 1)$ st column becomes the same as that in $W(z)$, without changing the value of the determinant $\widehat{W}_i(z)$, of course. This proves the validity of (3.9). \square

Combining Lemmas 3.1 and 3.4, we have the following main result:

Theorem 3.5. *Let $V(z)$ in $R_{p,k}^{\text{IMPE}}(z)$ be such that $V(\xi_i) \neq 0, i = 1, 2, \dots, p$. Then $R_{p,k}^{\text{IMPE}}(z)$ is a symmetric function of $\xi_1, \xi_2, \dots, \xi_p$.*

3.3. Treatment of $R_{p,k}^{\text{IMPE}}(z)$

Due to the complicated nature of the matrix elements $u_{i,j}$ of $R_{p,k}^{\text{IMPE}}(z)$ in Theorem 1.2, the treatment of this interpolant is more involved than that of $R_{p,k}^{\text{IMPE}}(z)$.

Lemma 3.6. *The denominator polynomial $V(z)$ of $R_{p,k}^{\text{IMPE}}(z)$ is a symmetric function of $\xi_1, \xi_2, \dots, \xi_{p+1}$ used to construct $V(z)$.*

Proof. With the notation

$$n = p + 1 \quad \text{and} \quad w_{m,\dots,n}^{(i)} = (D_{i,n}, D_{m,n}) \Rightarrow u_{i,j} = w_{j+1,\dots,n}^{(i)}, \tag{3.16}$$

(3.5) becomes

$$W(\xi_1, \xi_2, \dots, \xi_n; z) = \begin{vmatrix} g_{1,\dots,n} & g_{2,\dots,n} & \cdots & g_{k+1,\dots,n} \\ w_{1,\dots,n}^{(1)} & w_{2,\dots,n}^{(1)} & \cdots & w_{k+1,\dots,n}^{(1)} \\ w_{1,\dots,n}^{(2)} & w_{2,\dots,n}^{(2)} & \cdots & w_{k+1,\dots,n}^{(2)} \\ \vdots & \vdots & & \vdots \\ w_{1,\dots,n}^{(k)} & w_{2,\dots,n}^{(k)} & \cdots & w_{k+1,\dots,n}^{(k)} \end{vmatrix}. \tag{3.17}$$

What we want to show now is that, for any $i \in \{1, 2, \dots, n - 1\}$, (3.9) holds. There are two cases to consider: (i) $i \geq k + 1$ and (ii) $1 \leq i \leq k$.

By (3.16) and (3.17), by the fact that $k \leq p$, and by the symmetry property of divided differences, it follows that $\widehat{W}_i(z)$ has exactly the same rows and columns as $W(z)$ when $i \geq k + 1$, hence (3.9) holds trivially.

When $1 \leq i \leq k$, however, $\widehat{W}_i(z)$ differs from $W(z)$ in a way that is more complicated than what we had in Lemma 3.4 for IMMPE. In this case, it is best to do the proof for a special case that can be generalized easily.

Let us consider the case $k = 3$ and $p = 5$, hence $n = 6$. Then

$$W(z) = \begin{vmatrix} g_{123456} & g_{23456} & g_{3456} & g_{456} \\ (D_{123456}, D_{123456}) & (D_{123456}, D_{23456}) & (D_{123456}, D_{3456}) & (D_{123456}, D_{456}) \\ (D_{23456}, D_{123456}) & (D_{23456}, D_{23456}) & (D_{23456}, D_{3456}) & (D_{23456}, D_{456}) \\ (D_{3456}, D_{123456}) & (D_{3456}, D_{23456}) & (D_{3456}, D_{3456}) & (D_{3456}, D_{456}) \end{vmatrix}.$$

We now employ Lemma 3.2 and show that this determinant remains the same under an interchange of ξ_i and ξ_{i+1} in $\{\xi_1, \dots, \xi_6\}$. Let us take $i = 1$. Then

$$\widehat{W}_1(z) = \begin{vmatrix} g_{213456} & g_{13456} & g_{3456} & g_{456} \\ (D_{213456}, D_{213456}) & (D_{213456}, D_{13456}) & (D_{213456}, D_{3456}) & (D_{213456}, D_{456}) \\ (D_{13456}, D_{213456}) & (D_{13456}, D_{13456}) & (D_{13456}, D_{3456}) & (D_{13456}, D_{456}) \\ (D_{3456}, D_{213456}) & (D_{3456}, D_{13456}) & (D_{3456}, D_{3456}) & (D_{3456}, D_{456}) \end{vmatrix}.$$

By the symmetry property of divided differences, we have

$$\widehat{W}_1(z) = \begin{vmatrix} g_{123456} & g_{13456} & g_{3456} & g_{456} \\ (D_{123456}, D_{123456}) & (D_{123456}, D_{13456}) & (D_{123456}, D_{3456}) & (D_{123456}, D_{456}) \\ (D_{13456}, D_{123456}) & (D_{13456}, D_{13456}) & (D_{13456}, D_{3456}) & (D_{13456}, D_{456}) \\ (D_{3456}, D_{123456}) & (D_{3456}, D_{13456}) & (D_{3456}, D_{3456}) & (D_{3456}, D_{456}) \end{vmatrix}.$$

Now, in the first row of $\widehat{W}_1(z)$,

$$g_{123456} = \frac{g_{13456} - g_{23456}}{\xi_1 - \xi_2},$$

from which

$$g_{23456} = g_{13456} + (\xi_2 - \xi_1)g_{123456}.$$

We have analogous relations for the remaining rows of $\widehat{W}_1(z)$. Thus, if we multiply the first ($i = 1$) column in $\widehat{W}_1(z)$ by $(\xi_2 - \xi_1)$ and add to the second ($i + 1 = 2$) column, we obtain

$$\widehat{W}'_1(z) = \begin{vmatrix} g_{123456} & g_{23456} & g_{3456} & g_{456} \\ (D_{123456}, D_{123456}) & (D_{123456}, D_{23456}) & (D_{123456}, D_{3456}) & (D_{123456}, D_{456}) \\ (D_{13456}, D_{123456}) & (D_{13456}, D_{23456}) & (D_{13456}, D_{3456}) & (D_{13456}, D_{456}) \\ (D_{3456}, D_{123456}) & (D_{3456}, D_{23456}) & (D_{3456}, D_{3456}) & (D_{3456}, D_{456}) \end{vmatrix}.$$

If we now multiply the second ($i + 1 = 2$) row in $\widehat{W}'_1(z)$ by $(\xi_2 - \xi_1)$ and add to the second ($i + 2 = 3$) row, the resulting determinant $\widehat{W}''_1(z)$ is precisely $W(z)$, and this is what we needed to prove. \square

Note. As can be seen from the proof of Lemma 3.6, if we would stick with the normalization $c_0 = 1$ in the definition of $V(z)$, this polynomial would be a symmetric function of ξ_2, \dots, ξ_n , but not of $\xi_1, \xi_2, \dots, \xi_n$. Precisely this was the reason for the normalization $c_k = 1$.

Combining Lemmas 3.1 and 3.6, we have the following main result:

Theorem 3.7. *Let $V(z)$ in $R_{p,k}^{IMPE}(z)$ be such that $V(\xi_i) \neq 0, i = 1, 2, \dots, p$. Then $R_{p,k}^{IMPE}(z)$ is a symmetric function of $\xi_1, \xi_2, \dots, \xi_p$.*

3.4. Treatment of $R_{p,k}^{ITEA}(z)$

Lemma 3.8. *The denominator polynomial $V(z)$ of $R_{p,k}^{ITEA}(z)$ is a symmetric function of $\xi_1, \xi_2, \dots, \xi_{p+k}$ used to construct $V(z)$.*

Proof. With the notation

$$n = p + k \quad \text{and} \quad w_{r,\dots,s} = (q, D_{r,s}) \quad \Rightarrow \quad u_{i,j} = w_{j+1,\dots,p+i}, \tag{3.18}$$

(3.5) becomes

$$W(\xi_1, \xi_2, \dots, \xi_n; z) = \begin{vmatrix} g_{1,\dots,n} & g_{2,\dots,n} & \cdots & g_{k+1,\dots,n} \\ w_{1,\dots,p+1} & w_{2,\dots,p+1} & \cdots & w_{k+1,\dots,p+1} \\ w_{1,\dots,p+2} & w_{2,\dots,p+2} & \cdots & w_{k+1,\dots,p+2} \\ \vdots & \vdots & & \vdots \\ w_{1,\dots,p+k} & w_{2,\dots,p+k} & \cdots & w_{k+1,\dots,p+k} \end{vmatrix}. \tag{3.19}$$

Obviously, being a divided difference, $w_{r,\dots,s}$ is symmetric in the points $\xi_r, \xi_{r+1}, \dots, \xi_s$, hence, equivalently, in its indices $r, r + 1, \dots, s$.

What we want to show now is that, for any $i \in \{1, 2, \dots, n - 1\}$, (3.9) holds. There are two cases to consider: (i) $i \geq k + 1$ and (ii) $1 \leq i \leq k$.

By (3.18) and (3.19), by the fact that $k \leq p$, and by the symmetry property of divided differences, it follows that $\widehat{W}_i(z)$ has exactly the same columns as $W(z)$ when $i \geq k + 1$, hence (3.9) holds trivially.

When $1 \leq i \leq k$, $\widehat{W}_i(z)$ differs from $W(z)$ columnwise. However, due to the symmetry property of divided differences, $\widehat{W}_i(z)$ differs from $W(z)$ only in its $(i + 1)$ st column, this column being

$$[g_{i,i+2,\dots,n}, w_{i,i+2,\dots,p+1}, w_{i,i+2,\dots,p+2}, \dots, w_{i,i+2,\dots,p+k}]^T.$$

Again, $g_{m,\dots,n}$ satisfy (3.14) and (3.15). The same holds with $g_{m,\dots,n}$ replaced by $w_{m,\dots,p+s}$, $s = 1, \dots, k$, even though these divided differences are not of the same order. That is,

$$w_{i+1,i+2,\dots,p+s} = w_{i,i+2,\dots,p+s} + (\xi_{i+1} - \xi_i)w_{i,\dots,p+s}.$$

Thus, if we multiply the i th column in $\widehat{W}_i(z)$ by $(\xi_{i+1} - \xi_i)$ and add to the $(i + 1)$ st column, the $(i + 1)$ st column becomes the same as that in $W(z)$, without changing the value of the determinant $\widehat{W}_i(z)$. This proves the validity of (3.9). \square

Combining Lemmas 3.1 and 3.8, we have the following main result:

Theorem 3.9. *Let $V(z)$ in $R_{p,k}^{\text{ITEA}}(z)$ be such that $V(\xi_i) \neq 0, i = 1, 2, \dots, p$. Then $R_{p,k}^{\text{ITEA}}(z)$ is a symmetric function of $\xi_1, \xi_2, \dots, \xi_p$.*

4. Reproducing property of $R_{p,k}(z)$

In the next theorem, we show that, provided the conditions pertaining to the uniqueness of the denominator polynomial $V(z)$ are satisfied, the interpolant $R_{p,k}(z)$ reproduces $F(z)$ when the latter is itself a vector-valued rational function.

Theorem 4.1. *Let $F(z)$ be of the form $F(z) = \widetilde{U}(z)/\widetilde{V}(z)$, with $\widetilde{U}(z)$ a vector-valued polynomial of degree at most $p - 1$ and $\widetilde{V}(z)$ a scalar polynomial of degree exactly k . Then, all three rational interpolants $R_{p,k}(z)$ reproduce $F(z)$ in the sense that $R_{p,k}(z) \equiv F(z)$, provided the condition in (2.1) of Theorem 2.2 holds.*

Proof. By the fact that $\widetilde{U}(z)$ is a polynomial of degree at most $p - 1$, we first have that all divided differences of $\widetilde{U}(z)$ of order p or more vanish, that is,

$$\widetilde{U}[\xi_1, \dots, \xi_p, \xi_{p+1}, \dots, \xi_{p+s}] = 0, \quad s = 1, 2, \dots \tag{4.1}$$

Now, since $\widetilde{U}(z) = \widetilde{V}(z)F(z)$, by the Leibnitz rule for divided differences, we have

$$\widetilde{U}[\xi_1, \dots, \xi_m] = \sum_{i=1}^m \widetilde{V}[\xi_1, \dots, \xi_i] F[\xi_i, \dots, \xi_m]. \tag{4.2}$$

But, because $\widetilde{V}(z)$ is a polynomial of degree k , there holds

$$\widetilde{V}[\xi_1, \dots, \xi_i] = 0, \quad i \geq k + 2.$$

Furthermore, writing $\widetilde{V}(z)$ in the form

$$\widetilde{V}(z) = \sum_{j=0}^k \tilde{c}_j \psi_{1,j}(z),$$

which is legitimate, and comparing with the Newtonian form

$$\widetilde{V}(z) = \sum_{i=1}^{k+1} \widetilde{V}[\xi_1, \dots, \xi_i] \psi_{1,i-1}(z),$$

we realize that

$$\tilde{c}_j = \tilde{V}[\xi_1, \dots, \xi_{j+1}], \quad j = 0, 1, \dots, k.$$

Substituting this in (4.2) and letting $m = p + s$ there, switching to the notation $D_{i,m} = F[\xi_i, \xi_{i+1}, \dots, \xi_m]$ [recall (1.7)], and invoking (4.1), we see that \tilde{c}_j satisfy the equations

$$\sum_{j=0}^k \tilde{c}_j D_{j+1,p+s} = 0, \quad s = 1, 2, \dots \tag{4.3}$$

Therefore, they also satisfy (1.12)–(1.14). It is now easy to see that, when (2.1) holds, we have $c_j = \tilde{c}_j, j = 0, 1, \dots, k$. This completes the proof. \square

Note that Theorem 4.1 and its proof can also serve to *define* the rational interpolation procedures. That is, these interpolation procedures can be obtained by demanding that $R_{p,k}(z) \equiv F(z)$ when $F(z)$ is a vector-valued rational function, as described in Theorem 4.1.

Finally, the vector-valued rational functions $F(z)$ described in the next section (also described in the last paragraph of Section 2) satisfy the conditions of Theorem 4.1 in case of IMPE and IMMPE.

5. Rational $F(z)$ and the conditions (2.1)

As we have seen, in order for the conditions stated in (2.1) that pertain to the uniqueness of $R_{p,k}(z)$ for IMPE and IMMPE to be satisfied, the vectors $D_{i,p+1}, i = 1, \dots, k$, must be linearly independent. We will now see that this is the case when the function $F(z)$ is of the form $F(z) = \tilde{U}(z)/\tilde{V}(z)$, where $\tilde{U}(z)$ is a vector-valued polynomial of degree $\nu + \mu$ and $\tilde{V}(z)$ is a scalar polynomial of degree exactly $\mu, \mu \geq k$, provided certain conditions are satisfied by $F(z)$. The poles of $F(z)$ may be simple or multiple. Below, we first treat the case in which all the poles of $F(z)$ are simple. Following that, we allow some or all of the poles of $F(z)$ to be multiple.

5.1. $F(z)$ has simple poles

Let us assume that the poles of $F(z)$ are all simple and its corresponding residues are linearly independent vectors in \mathbb{C}^N . In this case, $F(z)$ is of the form

$$F(z) = \sum_{i=0}^{\nu} u_i z^i + \sum_{s=1}^{\mu} \frac{v_s}{z - z_s},$$

where u_i are arbitrary vectors in $\mathbb{C}^N, \mu \leq N, z_1, \dots, z_\mu$ are distinct points in \mathbb{C} , and v_1, \dots, v_μ are linearly independent constant vectors in \mathbb{C}^N . For example, with $A \in \mathbb{C}^{N \times N}$ a diagonalizable matrix and $b \in \mathbb{C}^N$ a nonzero constant vector, $F(z) = (zI - A)^{-1}b$ is such a function; in this case, $u_0 = \dots = u_\nu = 0, z_1, \dots, z_\mu$ are some or all of the distinct eigenvalues of A , and v_1, \dots, v_μ are corresponding eigenvectors (i.e., $Av_i = z_i v_i, i = 1, \dots, \mu$), and $\mu \leq N$ necessarily. See Sidi [4, Section 2].

Now, with $m - i \geq v + 1$, the divided difference of the vector-valued polynomial $\sum_{i=0}^v u_i z^i$ over the set of points $\{\xi_i, \xi_{i+1}, \dots, \xi_m\}$ vanishes; consequently, the vector $D_{i,m}$ is given by

$$D_{i,m} = F[\xi_i, \xi_{i+1}, \dots, \xi_m] = - \sum_{s=1}^{\mu} \frac{v_s}{\psi_{i,m}(z_s)}, \quad m \geq v + i + 1,$$

where we have used the fact that

$$\phi(z) = \frac{1}{z - z_s} \Rightarrow \phi[\xi_i, \xi_{i+1}, \dots, \xi_m] = - \frac{1}{\psi_{i,m}(z_s)}.$$

[This can be proved via (1.3) and by induction on m .]

Let

$$D = [D_{1,m} | D_{2,m} | \dots | D_{k,m}] \in \mathbb{C}^{N \times k}; \quad m \geq v + k + 2.$$

Then, D can be factorized as in

$$D = -XM,$$

where

$$X = [v_1 | v_2 | \dots | v_{\mu}] \in \mathbb{C}^{N \times \mu}$$

and

$$M = \begin{bmatrix} 1/\psi_{1,m}(z_1) & 1/\psi_{2,m}(z_1) & \cdots & 1/\psi_{k,m}(z_1) \\ 1/\psi_{1,m}(z_2) & 1/\psi_{2,m}(z_2) & \cdots & 1/\psi_{k,m}(z_2) \\ \vdots & \vdots & & \vdots \\ 1/\psi_{1,m}(z_{\mu}) & 1/\psi_{2,m}(z_{\mu}) & \cdots & 1/\psi_{k,m}(z_{\mu}) \end{bmatrix} \in \mathbb{C}^{\mu \times k}.$$

We wish to show that $\text{rank}(D) = k$.

Obviously, $\text{rank}(X) = \mu$ because the vectors v_1, \dots, v_{μ} are linearly independent and $\mu \leq N$. We now want to establish that $\text{rank}(M) = k$. We start by observing that

$$M = EM',$$

where

$$E = \text{diag}(1/\psi_{1,m}(z_1), 1/\psi_{1,m}(z_2), \dots, 1/\psi_{1,m}(z_{\mu})) \in \mathbb{C}^{\mu \times \mu}$$

and

$$M' = \begin{bmatrix} 1 & \psi_{1,1}(z_1) & \psi_{1,2}(z_1) & \cdots & \psi_{1,k-1}(z_1) \\ 1 & \psi_{1,1}(z_2) & \psi_{1,2}(z_2) & \cdots & \psi_{1,k-1}(z_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \psi_{1,1}(z_{\mu}) & \psi_{1,2}(z_{\mu}) & \cdots & \psi_{1,k-1}(z_{\mu}) \end{bmatrix} \in \mathbb{C}^{\mu \times k}.$$

Next, we have (see Sidi [5, Chapter 6, Lemma 6.8.1])

$$\begin{vmatrix} 1 & \psi_{1,1}(z_1) & \psi_{1,2}(z_1) & \cdots & \psi_{1,k-1}(z_1) \\ 1 & \psi_{1,1}(z_2) & \psi_{1,2}(z_2) & \cdots & \psi_{1,k-1}(z_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \psi_{1,1}(z_k) & \psi_{1,2}(z_k) & \cdots & \psi_{1,k-1}(z_k) \end{vmatrix} = V(z_1, z_2, \dots, z_k),$$

where $V(z_1, z_2, \dots, z_k)$ is the Vandermonde determinant defined by

$$V(z_1, z_2, \dots, z_k) = \begin{vmatrix} 1 & z_1 & z_1^2 & \dots & z_1^{k-1} \\ 1 & z_2 & z_2^2 & \dots & z_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_k & z_k^2 & \dots & z_k^{k-1} \end{vmatrix} = \prod_{1 \leq i < j \leq k} (z_j - z_i).$$

Since the z_i are distinct, it is clear that $V(z_1, z_2, \dots, z_k) \neq 0$, and this implies that $\text{rank}(M') = k$. This and the fact that E is a nonsingular square matrix imply that $\text{rank}(M) = k$.

As a result, the matrix D has rank k , that is, its columns $D_{1,m}, D_{2,m}, \dots, D_{k,m}$ are linearly independent. This holds, in particular, for $m = p + 1$.

5.2. $F(z)$ has multiple poles

Let us assume that the poles of $F(z)$ may be simple or multiple, that is, $F(z)$ is of the form

$$F(z) = \sum_{i=0}^v u_i z^i + \sum_{s=1}^{\sigma} \sum_{j=1}^{r_s} \frac{v_{sj}}{(z - z_s)^j},$$

where u_i are arbitrary vectors in \mathbb{C}^N , $\mu = \sum_{s=1}^{\sigma} r_s \leq N$, and that v_{sj} , $1 \leq j \leq r_s$, $1 \leq s \leq \sigma$, are linearly independent vectors in \mathbb{C}^N . For example, with $A \in \mathbb{C}^{N \times N}$ a nondiagonalizable matrix and $b \in \mathbb{C}^N$ a nonzero constant vector, $F(z) = (zI - A)^{-1}b$ is such a function; in this case, $u_0 = \dots = u_v = 0$, z_1, \dots, z_{σ} are some or all of the distinct eigenvalues of A , and, for each s , v_{sr_s} is an eigenvector of A corresponding to the eigenvalue z_s , while v_{sj} , $j < r_s$ are linear combinations of eigenvectors and principal vectors corresponding to the eigenvalue z_s . All these vectors, μ in number, are linearly independent. For details, see Sidi [4, Section 2].

Let us define

$$\phi_j(z; \alpha) = \frac{1}{(z - \alpha)^j}.$$

Then, again, for $m - i \geq v + 1$, we have

$$D_{i,m} = F[\xi_i, \xi_{i+1}, \dots, \xi_m] = \sum_{s=1}^{\sigma} \sum_{j=1}^{r_s} v_{sj} \phi_j[\xi_i, \xi_{i+1}, \dots, \xi_m; z_s].$$

Here, $\phi_j[\xi_i, \xi_{i+1}, \dots, \xi_m; \alpha]$ is the divided difference of $\phi_j(z; \alpha)$ over the set of points $\{\xi_i, \xi_{i+1}, \dots, \xi_m\}$, as a function of z (α being viewed as a fixed parameter).

Because

$$\phi_j(z; \alpha) = \frac{1}{(j - 1)!} \frac{\partial^{j-1}}{\partial \alpha^{j-1}} \phi_1(z; \alpha), \quad j = 1, 2, \dots,$$

and because z and α vary independently, we have

$$\phi_j[\xi_i, \xi_{i+1}, \dots, \xi_m; \alpha] = \frac{1}{(j - 1)!} \frac{\partial^{j-1}}{\partial \alpha^{j-1}} \phi_1[\xi_i, \xi_{i+1}, \dots, \xi_m; \alpha], \quad j = 1, 2, \dots$$

Noting again that

$$\phi_1[\xi_i, \xi_{i+1}, \dots, \xi_m; \alpha] = -\frac{1}{\psi_{i,m}(\alpha)},$$

and denoting

$$\tilde{v}_{sj} = \frac{v_{sj}}{(j-1)!}, \quad \rho_i(z) = \frac{1}{\psi_{i,m}(z)},$$

we can rewrite $D_{i,m}$ in the form

$$D_{i,m} = -\sum_{s=1}^{\sigma} \sum_{j=1}^{r_s} \tilde{v}_{sj} \rho_s^{(j-1)}(z_s).$$

We now turn to the matrix

$$D = [D_{1,m} | D_{2,m} | \dots | D_{k,m}] \in \mathbb{C}^{N \times k}; \quad m \geq v + k + 2.$$

This matrix can be factorized as in

$$D = -XM,$$

where

$$X = [\tilde{v}_{11} | \tilde{v}_{12} | \dots | \tilde{v}_{1r_1} | \dots \dots | \tilde{v}_{\sigma 1} | \tilde{v}_{\sigma 2} | \dots | \tilde{v}_{\sigma r_\sigma}] \in \mathbb{C}^{N \times \mu}$$

and

$$M = \begin{bmatrix} \rho_1(z_1) & \rho_2(z_1) & \dots & \dots & \rho_k(z_1) \\ \rho'_1(z_1) & \rho'_2(z_1) & \dots & \dots & \rho'_k(z_1) \\ \vdots & \vdots & & & \vdots \\ \rho_1^{(\omega_1)}(z_1) & \rho_2^{(\omega_1)}(z_1) & \dots & \dots & \rho_k^{(\omega_1)}(z_1) \\ \hline \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \hline \rho_1(z_\sigma) & \rho_2(z_\sigma) & \dots & \dots & \rho_k(z_\sigma) \\ \rho'_1(z_\sigma) & \rho'_2(z_\sigma) & \dots & \dots & \rho'_k(z_\sigma) \\ \vdots & \vdots & & & \vdots \\ \rho_1^{(\omega_\sigma)}(z_\sigma) & \rho_2^{(\omega_\sigma)}(z_\sigma) & \dots & \dots & \rho_k^{(\omega_\sigma)}(z_\sigma) \end{bmatrix} \in \mathbb{C}^{\mu \times k},$$

with $\omega_s = r_s - 1, s = 1, \dots, \sigma$.

We wish to show that $\text{rank}(D) = k$. Obviously, $\text{rank}(X) = \mu$ since the vectors v_{sj} are linearly independent. If we show that $\text{rank}(M) = k$, we will be done. The analysis of the matrix M , however, turns out to be more involved than before. As before, we look at the determinant of the $k \times k$ matrix M_1 formed by the first k rows of M . It is easy to see that we can consider $k = \mu = \sum_{s=1}^{\sigma} r_s$ without loss of generality. This way we also avoid the need for introducing additional notation. In addition, $M_1 = M$ now.

We start by noting that

$$\det M = \left[\left(\prod_{s=1}^{\sigma} \prod_{j=0}^{\omega_s} \frac{\partial^j}{\partial z_{sj}^j} \right) \det \tilde{M}(z_{10}, z_{11}, \dots, z_{1\omega_1}; \dots; z_{\sigma 0}, z_{\sigma 1}, \dots, z_{\sigma\omega_\sigma}) \right] \Big|_{z_{sj}=z_s},$$

where, suppressing the arguments z_{sj} in $\tilde{M}(\dots)$,

$$\tilde{M} = \begin{bmatrix} \rho_1(z_{10}) & \rho_2(z_{10}) & \cdots & \cdots & \rho_k(z_{10}) \\ \rho_1(z_{11}) & \rho_2(z_{11}) & \cdots & \cdots & \rho_k(z_{11}) \\ \vdots & \vdots & & & \vdots \\ \rho_1(z_{1\omega_1}) & \rho_2(z_{1\omega_1}) & \cdots & \cdots & \rho_k(z_{1\omega_1}) \\ \hline \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \hline \rho_1(z_{\sigma 0}) & \rho_2(z_{\sigma 0}) & \cdots & \cdots & \rho_k(z_{\sigma 0}) \\ \rho_1(z_{\sigma 1}) & \rho_2(z_{\sigma 1}) & \cdots & \cdots & \rho_k(z_{\sigma 1}) \\ \vdots & \vdots & & & \vdots \\ \rho_1(z_{\sigma\omega_\sigma}) & \rho_2(z_{\sigma\omega_\sigma}) & \cdots & \cdots & \rho_k(z_{\sigma\omega_\sigma}) \end{bmatrix}.$$

Letting $(z_{10}, z_{11}, \dots, z_{1\omega_1}; \dots; z_{\sigma 0}, z_{\sigma 1}, \dots, z_{\sigma\omega_\sigma}) = (\eta_1, \eta_2, \dots, \eta_k)$ for short, by the preceding subsection, we have

$$\tilde{M} = \tilde{E} \tilde{M}',$$

where

$$\tilde{E} = \text{diag}(1/\psi_{1,m}(\eta_1), 1/\psi_{1,m}(\eta_2), \dots, 1/\psi_{1,m}(\eta_k)),$$

and

$$\tilde{M}' = \begin{bmatrix} 1 & \psi_{1,1}(\eta_1) & \psi_{1,2}(\eta_1) & \cdots & \psi_{1,k-1}(\eta_1) \\ 1 & \psi_{1,1}(\eta_2) & \psi_{1,2}(\eta_2) & \cdots & \psi_{1,k-1}(\eta_2) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \psi_{1,1}(\eta_k) & \psi_{1,2}(\eta_k) & \cdots & \psi_{1,k-1}(\eta_k) \end{bmatrix}.$$

But, by the preceding subsection, there holds

$$\det \tilde{M} = \det \tilde{E} \cdot \det \tilde{M}' = \frac{V(\eta_1, \eta_2, \dots, \eta_k)}{\prod_{i=1}^k \psi_{1,m}(\eta_i)}.$$

Consequently,

$$\det M = \left[\left(\prod_{s=1}^{\sigma} \prod_{j=0}^{\omega_s} \frac{\partial^j}{\partial z_{sj}^j} \right) \frac{V(\eta_1, \eta_2, \dots, \eta_k)}{\prod_{i=1}^k \psi_{1,m}(\eta_i)} \right] \Big|_{z_{sj}=z_s}.$$

Since $V(\eta_1, \eta_2, \dots, \eta_k) = \prod_{1 \leq i < j \leq k} (\eta_j - \eta_i)$, all of the terms obtained upon differentiating the quotient $V(\eta_1, \eta_2, \dots, \eta_k) / \prod_{i=1}^k \psi_{1,m}(\eta_i)$ vanish except one, and we obtain

$$\det M = \left[\frac{1}{\prod_{i=1}^k \psi_{1,m}(\eta_i)} \left(\prod_{s=1}^{\sigma} \prod_{j=0}^{\omega_s} \frac{\partial^j}{\partial z_{sj}^j} \right) V(\eta_1, \eta_2, \dots, \eta_k) \right] \Big|_{z_{sj} = z_s}.$$

But

$$\begin{aligned} \left[\left(\prod_{s=1}^{\sigma} \prod_{j=0}^{\omega_s} \frac{1}{j!} \frac{\partial^j}{\partial z_{sj}^j} \right) V(\eta_1, \eta_2, \dots, \eta_k) \right] \Big|_{z_{sj} = z_s} &= V(z_1, r_1; z_2, r_2; \dots; z_{\sigma}, r_{\sigma}) \\ &= \prod_{1 \leq i < j \leq \sigma} (z_j - z_i)^{r_i r_j} \end{aligned}$$

is the confluent Vandermonde determinant. Since the z_i are distinct, this determinant is nonzero. Combining everything, we have

$$\det M = \left(\prod_{s=1}^{\sigma} \prod_{j=0}^{\omega_s} j! \right) \frac{\prod_{1 \leq i < j \leq \sigma} (z_j - z_i)^{r_i r_j}}{\prod_{s=1}^{\sigma} [\psi_{1,m}(z_s)]^{r_s}} \neq 0.$$

This completes the proof of the assertion $\text{rank}(M) = k$.

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