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A de Montessus type convergence study of a least-squares vector-valued rational interpolation procedure

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Abstract

In a recent paper of the author [A. Sidi, A new approach to vector-valued rational interpolation, J. Approx. Theory 130 (2004) 177–187], three new interpolation procedures for vector-valued functions $F(z)$, where $F : \mathbb{C} \to \mathbb{C}^N$, were proposed, and some of their algebraic properties were studied. One of these procedures, denoted IMPE, was defined via the solution of a linear least-squares problem. In the present work, we concentrate on IMPE, and study its convergence properties when it is applied to meromorphic functions with simple poles and orthogonal vector residues. We prove de Montessus and Koenig type theorems when the points of interpolation are chosen appropriately.

Keywords: Vector-valued rational interpolation; Hermite interpolation; Newton interpolation formula; de Montessus theorem; Koenig theorem

1. Introduction

In a recent work, Sidi [10], we presented three different kinds of vector-valued rational interpolation procedures. These were modelled after some rational approximation procedures from the Maclaurin series of vector-valued functions developed in Sidi [8], which in turn had their origin in vector extrapolation methods. Vector extrapolation methods are used for accelerating

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the convergence of certain kinds of vector sequences, such as those produced by fixed-point iterative methods on linear and nonlinear systems of algebraic equations.

Some of the algebraic properties of these interpolants were already mentioned in [10], and their study was continued in another paper [11] by the author. In yet another recent work [12], we continued to study one of the three interpolation procedures that was denoted IMMPE in [10]. In particular, we studied the convergence properties of IMMPE as it is being applied to vector-valued meromorphic functions $F(z)$ that have simple poles. We provided (i) a de Montessus type convergence theory for the approximants and (ii) Koenig type theory pertaining to convergence of the poles of the approximants to the poles of $F(z)$.

In the present work, we turn to the interpolation procedure that was denoted IMPE in [10]. This procedure is defined via the solution to a linear least-squares problem and is technically more involved than IMMPE.

In the next section, we provide a brief description of IMPE. Following this, in Section 3, we derive a closed-form expression for the error when the function $F(z)$ being interpolated is rational with simple poles and orthogonal vector residues. The main results of this section are Theorems 3.6 and 3.8. In Section 4, we present the choice of the points of interpolation and its consequences.

Starting with the developments of Sections 3 and 4, in Section 5, we present a detailed convergence theory, concerning vector-valued rational functions $F(z)$ (with simple poles and orthogonal residues), for sequences of interpolants whose denominators are of a fixed degree that may be much smaller than the number of poles of $F(z)$, while the number of interpolation conditions (hence the degree of the numerators) tends to infinity. This theory provides us with de Montessus and Koenig type theorems for the sequence of interpolants being studied. The main results of this section are Theorems 5.1 and 5.2 that concern the convergence of the poles of the interpolants and Theorem 5.3 that concerns the convergence of the interpolants themselves. Theorem 5.4 concerns the convergence of the residues of the interpolants. The results of Section 5 show that rational interpolation with a small number of poles can help approximate a function $F(z)$ that has a large number of poles very accurately in a large set of the complex plane. Finally, Section 6 concerns the extension of the results of Section 5 to functions that are meromorphic in the whole complex plane and that reside in infinite dimensional inner product spaces.

Our theory is in the spirit of that given by Saff [6] for the scalar rational interpolation problem and by Graves-Morris and Saff [2–4] for vector-valued rational interpolants, in particular, simultaneous Padé approximants and generalized inverse vector-valued Padé approximants. Our proofs are completely different, however. They employ linear algebra techniques and are analogous to those developed in Sidi, Ford, and Smith [13] and used in Sidi [7] in the study of Padé approximants. In addition, the techniques we use here enable us to obtain optimally refined results in the form of asymptotic expansions and asymptotic equalities. In particular, they enable us to prove the surprising result that the convergence of the poles of the interpolants to the poles of the functions $F(z)$ considered in this work (namely, meromorphic with simple poles and orthogonal residues) with IMPE is *twice as fast* as that with IMMPE.

2. Definition and algebraic properties of IMPE

We start with a brief description of the developments in [10,11] that concern IMPE. By this, we shall also introduce some of the notation that we use in what follows.

Let z be a complex variable and let $F(z)$ be a vector-valued function such that $F : \mathbb{C} \to \mathbb{C}^N$. Assume that $F(z)$ is defined on a bounded open set $\Omega \subset \mathbb{C}$ and consider the problem of interpolating $F(z)$ at the points $\xi_1, \xi_2, \ldots$, in this set. We do not assume that the ξ_i are necessarily distinct; thus we allow interpolation in the sense of Hermite. See [10,11].

Let $n \geq m$ and $G_{m,n}(z)$ be the vector-valued polynomial (of degree at most $n - m$) that interpolates $F(z)$ at the points $\xi_m, \xi_{m+1}, \ldots, \xi_n$ in the sense of Hermite. Thus, in Newtonian form, this polynomial is given as in (see, e.g., Stoer and Bulirsch [14, Chapter 2] or Atkinson [1, Chapter 3])

$$G_{m,n}(z) = F[\xi_m] + F[\xi_m, \xi_{m+1}](z - \xi_m) + F[\xi_m, \xi_{m+1}, \xi_{m+2}](z - \xi_m)(z - \xi_{m+1}) \\ + \cdots + F[\xi_m, \xi_{m+1}, \ldots, \xi_n](z - \xi_m)(z - \xi_{m+1}) \cdots (z - \xi_{n-1}). \tag{2.1}$$

Here, $F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}]$ is the divided difference of order s of $F(z)$ over the set of points $\{\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}\}$. Obviously, $F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}]$ are all vectors in $\mathbb{C}^N$.

We define the scalar polynomials $\psi_{m,n}(z)$ via

$$\psi_{m,n}(z) = \prod_{r=m}^{n} (z - \xi_r), \quad n \geq m \geq 1; \quad \psi_{m,m-1}(z) = 1, \quad m \geq 1. \tag{2.2}$$

We also define the vectors $D_{m,n}$ via

$$D_{m,n} = F[\xi_m, \xi_{m+1}, \ldots, \xi_n], \quad n \geq m. \tag{2.3}$$

With this notation, we can rewrite (2.1) in the form

$$G_{m,n}(z) = \sum_{i=m}^{n} D_{m,i}\, \psi_{m,i-1}(z). \tag{2.4}$$

The vector-valued rational interpolants to the function $F(z)$ that we developed in [10] are all of the general form

$$R_{p,k}(z) = \frac{U_{p,k}(z)}{V_{p,k}(z)} = \frac{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)\, G_{j+1,p}(z)}{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)}, \tag{2.5}$$

where $c_0, c_1, \ldots, c_k$ are, for the time being, arbitrary complex scalars, and p is an arbitrary integer. Obviously, $U_{p,k}(z)$ is a vector-valued polynomial of degree at most $p - 1$ and $V_{p,k}(z)$ is a scalar polynomial of degree at most k. It is also clear from (2.5) that $k \leq p - 1$.

It turns out, whether the ξ_i are distinct or not, provided $V_{p,k}(\xi_i) \neq 0$, $i = 1, \ldots, p$, $R_{p,k}(z)$ interpolates $F(z)$ at the points $\xi_1, \ldots, \xi_p$ in the sense of Hermite. See [10, Lemma 2.1 and Lemma 2.3].

Of course, the quality of $R_{p,k}(z)$ as an approximation to $F(z)$ depends very strongly on the choice of the c_j. Naturally, the c_j must depend on $F(z)$ and on the ξ_i. Fixing the integers k and p such that $p \geq k + 1$, we define the c_j for IMPE to be the solution to the linear least-squares problem

$$\min_{c_0, c_1, \ldots, c_{k-1}} \left\| \sum_{j=0}^{k} c_j D_{j+1,p+1} \right\|; \quad \text{subject to } c_k = 1. \tag{2.6}$$

Here $\|\cdot\|$ is a vector l_2-norm that is induced by some inner product $(\cdot\,,\cdot)$. That is, for any vector $x \in \mathbb{C}^N$, we have $\|x\| = \sqrt{(x, x)}$. We also define this inner product such that, for arbitrary

$x, y \in \mathbb{C}^N$ and $\alpha, \beta \in \mathbb{C}$, there holds $(\alpha x, \beta y) = \overline{\alpha}\beta(x, y)$. The minimization problem in (2.6) gives rise to the following set of normal equations for the c_j:

$$\sum_{j=0}^{k-1} u_{i,j} c_j = -u_{i,k}, \quad i = 1, \ldots, k; \quad c_k = 1; \quad u_{i,j} = \left(D_{i,p+1}, D_{j+1,p+1}\right). \tag{2.7}$$

Note that the c_j are determined by the function values $F(\xi_i)$, $1 \le i \le p+1$, while $R_{p,k}(\xi_i) = F(\xi_i)$, $1 \le i \le p$.

It has been shown in [10] that, provided a unique solution to these equations exists, $R_{p,k}(z)$ has a determinantal representation given as in

$$R_{p,k}(z) = \frac{P(z)}{Q(z)} = \frac{\begin{vmatrix} \psi_{1,0}(z)\, G_{1,p}(z) & \psi_{1,1}(z)\, G_{2,p}(z) & \cdots & \psi_{1,k}(z)\, G_{k+1,p}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}{\begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \cdots & \psi_{1,k}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}. \tag{2.8}$$

Here, the numerator determinant $P(z)$ is vector-valued and is defined by its expansion with respect to its first row. That is, if M_j is the cofactor of the term $\psi_{1,j}(z)$ in the denominator determinant $Q(z)$, then

$$R_{p,k}(z) = \frac{\sum_{j=0}^{k} M_j\, \psi_{1,j}(z)\, G_{j+1,p}(z)}{\sum_{j=0}^{k} M_j\, \psi_{1,j}(z)}. \tag{2.9}$$

Note that this determinantal representation has been used throughout [11] extensively. It seems to offer a very effective tool for the study of $R_{p,k}(z)$, as we will see later in this work as well.

Here is a summary of the results of [11] that concern IMPE:

1. A sufficient condition for the equations in (2.7) to have a unique solution is that (see [11, Lemma 2.1 and Theorem 2.2]

$$\begin{vmatrix} u_{1,0} & u_{1,1} & \cdots & u_{1,k-1} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k-1} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k-1} \end{vmatrix} \neq 0; \qquad u_{i,j} = \left(D_{i,p+1}, D_{j+1,p+1}\right). \tag{2.10}$$

This also guarantees the uniqueness of $R_{p,k}(z)$ provided $V_{p,k}(\xi_i) \neq 0$, $i = 1, \ldots, p$. For (2.10) to be true, it is necessary and sufficient that the vectors $D_{1,p+1}, D_{2,p+1}, \ldots, D_{k,p+1}$ be linearly independent. It is shown in [11, Sections 2 and 5] that this holds when $F(z)$ is a vector-valued rational function of the form

$$F(z) = u(z) + \sum_{s=1}^{\sigma} \sum_{j=1}^{r_s} \frac{v_{sj}}{(z - z_s)^j}, \tag{2.11}$$

where $u(z)$ is an arbitrary vector-valued polynomial, the vectors $v_{sj} \in \mathbb{C}^N$, $1 \le j \le r_s$, $1 \le s \le \sigma$, are linearly independent, $z_1, \ldots, z_\sigma$ are distinct points in $\mathbb{C}$, and $k \le \sum_{s=1}^{\sigma} r_s \le N$.

2. The denominator polynomial $V_{p,k}(z)$ of the IMPE interpolant $R_{p,k}(z)$ is a symmetric function of all the ξ_i used to construct it, namely, of $\xi_1, \xi_2, \ldots, \xi_{p+1}$, while $R_{p,k}(z)$ itself is a symmetric function of the points of interpolation, namely, of $\xi_1, \xi_2, \ldots, \xi_p$. That is, $R_{p,k}(z)$ is independent of the order of the interpolation points $\xi_1, \ldots, \xi_p$. See [11, Lemma 3.4 and Theorem 3.5].
3. Let $F(z)$ be a vector-valued rational function of the form $F(z) = \tilde{U}(z)/\tilde{V}(z)$, where $\tilde{U}(z)$ is a vector-valued polynomial of degree at most $p - 1$ and $\tilde{V}(z)$ is a scalar polynomial of degree exactly k. Provided (2.10) is satisfied and $V_{p,k}(\xi_i) \neq 0$, $i = 1, \ldots, p$, holds, IMPE reproduces $F(z)$, that is, there holds $R_{p,k}(z) \equiv F(z)$. See [11, Theorem 4.1].

3. IMPE error formula for $F(z)$ a vector-valued rational function with orthogonal residues

As in [11], we start our study of IMPE for the case in which the function $F(z)$ is a vector-valued rational function with simple poles, namely,

$$F(z) = u(z) + \sum_{s=1}^{\mu} \frac{v_s}{z - z_s}, \tag{3.1}$$

where $u(z)$ is an arbitrary vector-valued polynomial, $z_1, \ldots, z_\mu$ are distinct nonzero complex numbers, and $v_1, \ldots, v_\mu$ are linearly independent constant vectors in $\mathbb{C}^N$. Clearly, $\mu \le N$. In addition, we assume in this work that the residues of $F(z)$ at its poles, namely, the vectors v_i, form an orthogonal set with respect to the inner product used in defining IMPE. Thus,

$$(v_i, v_j) = 0 \quad \text{if } i \neq j. \tag{3.2}$$

Example. Let A be an $N \times N$ diagonalizable matrix with eigenpairs (λ_i, w_i), $i = 1, \ldots, N$, and let b be an N-vector, and consider the solution to the linear system of equations $(I - zA)x = b$. Since $w_1, \ldots, w_N$ span $\mathbb{C}^N$, there holds $b = \sum_{i=1}^{N} \alpha_i w_i$ for some scalars α_i. Then, for $z \neq \lambda_i^{-1}$, $i = 1, \ldots, N$, the solution to $(I - zA)x = b$ has the representation

$$x = F(z) = (I - zA)^{-1} b = \sum_{i=1}^{N} \frac{\alpha_i w_i}{1 - z\lambda_i}.$$

Thus, $F(z)$ is precisely of the form described in (3.1). In case A is singular, $u(z) \equiv v_0$, where v_0 is either an eigenvector of A corresponding to its zero eigenvalue or $v_0 = 0$; therefore, $u(z)$ is a constant polynomial. If A is nonsingular, $u(z) \equiv 0$. Whether A is singular or not, the z_s in (3.1) are the reciprocals of some or all of the nonzero distinct λ_i (hence $\mu \le N$), and, for each s, v_s is a linear combination of the eigenvectors corresponding to the eigenvalue z_s^{-1}, hence is itself an eigenvector of A, that is, $Av_s = z_s^{-1} v_s$, $s = 1, \ldots, \mu$. If A is a normal matrix, that is, if $A^*A = AA^*$, then A is diagonalizable and its eigenvectors are orthogonal in the sense that $w_i^* w_j = 0$ for $i \neq j$. By this, we also have that the v_s satisfy (3.2) with $(x, y) = x^* y$.

We now recall some technical tools that were used in [12], and will be used throughout this work as well. The following lemma was stated and proved as Lemma A.1 in [13].

Lemma 3.1. *Let $i_0, i_1, \ldots, i_k$ be positive integers, and assume that the scalars $v_{i_0,i_1,\ldots,i_k}$ are odd under an interchange of any two of the indices $i_0, i_1, \ldots, i_k$. Let $t_{i,j}$, $i, j \geq 1$, be scalars and let σ_i, $i \geq 1$ be all scalars or vectors. Define*

$$I_{k,N} = \sum_{i_0=1}^{N} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \sigma_{i_0} \left(\prod_{p=1}^{k} t_{i_p,p} \right) v_{i_0,i_1,\ldots,i_k}$$

and

$$J_{k,N} = \sum_{1 \leq i_0 < i_1 < \cdots < i_k \leq N} \begin{vmatrix} \sigma_{i_0} & \sigma_{i_1} & \cdots & \sigma_{i_k} \\ t_{i_1,1} & t_{i_2,1} & \cdots & t_{i_k,1} \\ t_{i_1,2} & t_{i_2,2} & \cdots & t_{i_k,2} \\ \vdots & \vdots & & \vdots \\ t_{i_1,k} & t_{i_2,k} & \cdots & t_{i_k,k} \end{vmatrix} v_{i_0 i_1,\ldots,i_k}.$$

Then

$$I_{k,N} = J_{k,N}.$$

We note that Lemma 3.1 is used in conjunction with the multilinearity property of determinants. The next lemma is Lemma 1.2 in [9].

Lemma 3.2. *Let $Q_i(x) = \sum_{j=0}^{i} a_{ij} x^j$, with $a_{ii} \neq 0$, $i = 0, 1 \ldots, n$, and let x_i, $i = 0, 1, \ldots, n$, be arbitrary complex numbers. Then*

$$\begin{vmatrix} Q_0(x_0) & Q_0(x_1) & \cdots & Q_0(x_n) \\ Q_1(x_0) & Q_1(x_1) & \cdots & Q_1(x_n) \\ \vdots & \vdots & & \vdots \\ Q_n(x_0) & Q_n(x_1) & \cdots & Q_n(x_n) \end{vmatrix} = \left(\prod_{i=0}^{n} a_{ii} \right) V(x_0, x_1, \ldots, x_n), \tag{3.3}$$

where $V(x_0, x_1, \ldots, x_n) = \prod_{0 \leq i < j \leq n} (x_j - x_i)$ is a Vandermonde determinant.

The next lemma is Lemma 3.3 in [12].

Lemma 3.3. *Let $\omega_a(z) = (z - a)^{-1}$. Then, whether the ξ_i are distinct or not, $\omega_a[\xi_m, \ldots, \xi_n]$, the divided difference of $\omega_a(z)$ over the set of points $\{\xi_m, \ldots, \xi_n\}$, is given by*

$$\omega_a[\xi_m, \ldots, \xi_n] = -\frac{1}{\psi_{m,n}(a)} = -\frac{\psi_{1,m-1}(a)}{\psi_{1,n}(a)}. \tag{3.4}$$

The following lemma is the same as Lemma 3.4 in [12], with the exception of (3.6), which can be proved by invoking (3.2) and (3.5) in $(D_{i,n}, D_{m,n})$.

Lemma 3.4. *Let $F(z)$ be given as in (3.1). Let $n - m > \deg(u)$. Then, whether the ξ_i are distinct or not, the following are true:*

(i) $D_{m,n} = F[\xi_m, \ldots, \xi_n]$ *is given as in*

$$D_{m,n} = -\sum_{s=1}^{\mu} \frac{v_s}{\psi_{m,n}(z_s)} = -\sum_{s=1}^{\mu} v_s \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)}. \tag{3.5}$$

Therefore, if (3.2) is satisfied, we also have

$$\big(D_{i,n}, D_{m,n}\big) = \sum_{s=1}^{\mu} \alpha_{i,s} \frac{\psi_{1,m-1}(z_s)}{|\psi_{1,n}(z_s)|^2}, \qquad \alpha_{i,s} = \|v_s\|^2\, \overline{\psi_{1,i-1}(z_s)}. \tag{3.6}$$

(ii) $F(z) - G_{m,n}(z) = \psi_{m,n}(z) F[z, \xi_m, \ldots, \xi_n]$ *is given as in*

$$F(z) - G_{m,n}(z) = \psi_{m,n}(z) \sum_{s=1}^{\mu} e_s(z) \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)}; \quad e_s(z) = \frac{v_s}{z - z_s}. \tag{3.7}$$

The next lemma, which is Lemma 3.5 in [12], gives the determinant representation of the error function $F(z) - R_{p,k}(z)$, and we will be analyzing it in what follows.

Lemma 3.5. *Let*

$$\Delta_j(z) = \psi_{1,j}(z) \big[F(z) - G_{j+1,p}(z)\big], \quad j = 0, 1, \ldots. \tag{3.8}$$

Then the error in $R_{p,k}(z)$ has the determinantal representation

$$F(z) - R_{p,k}(z) = \frac{\Delta(z)}{Q(z)}, \tag{3.9}$$

where $Q(z)$ is the denominator determinant of $R_{p,k}(z)$ in (2.8) and

$$\Delta(z) = \begin{vmatrix} \Delta_0(z) & \Delta_1(z) & \cdots & \Delta_k(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}. \tag{3.10}$$

We start with the analysis of $Q(z)$, the denominator determinant of $R_{p,k}(z)$ in (2.8). The following theorem gives a closed-form expression for $Q(z)$ in simple terms, and is the analogue of Theorem 3.6 in [12].

Theorem 3.6. *Let $F(z)$ be the vector-valued rational function in (3.1), and precisely as described in the first paragraph of this section, with the notation therein. Let also*

$$\Psi_p(z) = \psi_{1,p+1}(z). \tag{3.11}$$

Then, with $p > k + \deg(u)$,

$$Q(z) = \sum_{1 \le s_1 < s_2 < \cdots < s_k \le \mu} T_{s_1,\ldots,s_k} V(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \left| \prod_{i=1}^{k} \Psi_p(z_{s_i}) \right|^{-2}, \tag{3.12}$$

where,

$$T_{s_1,\ldots,s_k} = \begin{vmatrix} \alpha_{1,s_1} & \alpha_{1,s_2} & \cdots & \alpha_{1,s_k} \\ \alpha_{2,s_1} & \alpha_{2,s_2} & \cdots & \alpha_{2,s_k} \\ \vdots & \vdots & & \vdots \\ \alpha_{k,s_1} & \alpha_{k,s_2} & \cdots & \alpha_{k,s_k} \end{vmatrix} = \left(\prod_{i=1}^{k} \|v_{s_i}\|^2 \right) \overline{V(z_{s_1}, z_{s_2}, \ldots, z_{s_k})}, \tag{3.13}$$

and $V(x_0, x_1, \ldots, x_n)$ is the Vandermonde determinant defined in Lemma 3.2.

Proof. Taking $p > k + \deg(u)$, and invoking (3.6) in (2.7), we first have

$$u_{i,j} = \big(D_{i,p+1}, D_{j+1,p+1}\big) = \sum_{s=1}^{\mu} \alpha_{i,s} \frac{\psi_{1,j}(z_s)}{|\Psi_p(z_s)|^2}, \tag{3.14}$$

where $\alpha_{i,s}$ are as in (3.6). Thus, the determinant representation of $Q(z)$ in (2.8) becomes

$$Q(z) = \begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \cdots & \psi_{1,k}(z) \\ \sum_{s_1} \alpha_{1,s_1} \frac{\psi_{1,0}(z_{s_1})}{|\Psi_p(z_{s_1})|^2} & \sum_{s_1} \alpha_{1,s_1} \frac{\psi_{1,1}(z_{s_1})}{|\Psi_p(z_{s_1})|^2} & \cdots & \sum_{s_1} \alpha_{1,s_1} \frac{\psi_{1,k}(z_{s_1})}{|\Psi_p(z_{s_1})|^2} \\ \sum_{s_2} \alpha_{2,s_2} \frac{\psi_{1,0}(z_{s_2})}{|\Psi_p(z_{s_2})|^2} & \sum_{s_2} \alpha_{2,s_2} \frac{\psi_{1,1}(z_{s_2})}{|\Psi_p(z_{s_2})|^2} & \cdots & \sum_{s_2} \alpha_{2,s_2} \frac{\psi_{1,k}(z_{s_2})}{|\Psi_p(z_{s_2})|^2} \\ \vdots & \vdots & & \vdots \\ \sum_{s_k} \alpha_{k,s_k} \frac{\psi_{1,0}(z_{s_k})}{|\Psi_p(z_{s_k})|^2} & \sum_{s_k} \alpha_{k,s_k} \frac{\psi_{1,1}(z_{s_k})}{|\Psi_p(z_{s_k})|^2} & \cdots & \sum_{s_k} \alpha_{k,s_k} \frac{\psi_{1,k}(z_{s_k})}{|\Psi_p(z_{s_k})|^2} \end{vmatrix}.$$

Since determinants are multilinear in their rows (and columns), we can take the summations outside. Following that, we take out the common factors from each row of the remaining determinant. We obtain

$$Q(z) = \sum_{s_1} \sum_{s_2} \cdots \sum_{s_k} \left(\prod_{i=1}^{k} \alpha_{i,s_i} \right) \left| \prod_{i=1}^{k} \Psi_p(z_{s_i}) \right|^{-2} X(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k}),$$

where

$$X(y_0, y_1, y_2, \ldots, y_n) = \begin{vmatrix} \psi_{1,0}(y_0) & \psi_{1,1}(y_0) & \cdots & \psi_{1,k}(y_0) \\ \psi_{1,0}(y_1) & \psi_{1,1}(y_1) & \cdots & \psi_{1,k}(y_1) \\ \vdots & \vdots & & \vdots \\ \psi_{1,0}(y_k) & \psi_{1,1}(y_k) & \cdots & \psi_{1,k}(y_k) \end{vmatrix}. \tag{3.15}$$

Now, since $\psi_{1,r}(z)$ is a monic polynomial in z of degree r, Lemma 3.2 applies, and we also have

$$X(y_0, y_1, \ldots, y_n) = V(y_0, y_1, \ldots, y_n) = \prod_{0 \le i < j \le n} (y_j - y_i), \tag{3.16}$$

is the Vandermonde determinant. Since the product

$$\left| \prod_{i=1}^{k} \Psi_p(z_{s_i}) \right|^{-2} X(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k})$$

is odd under an interchange of any two of the indices $s_1, \ldots, s_k$, Lemma 3.1 applies, and we obtain the result in (3.12). ■

It is worth noting that, even though the functions $\psi_{m,n}(z)$ that define $X(y_0, y_1, \ldots, y_n)$ in (3.15) depend on the ξ_i, $X(y_0, y_1, \ldots, y_n)$ itself is *independent of* the ξ_i. As a result, $Q(z)$ depends on the ξ_i only via the products $\prod_{i=1}^{k} \Psi_p(z_{s_i})$. This has important implications in the asymptotic behavior of $Q(z)$ and hence of $R_{p,k}(z)$ as $p \to \infty$, as we shall see later in this work.

We next turn to $\Delta(z)$, the numerator determinant of $F(z) - R_{p,k}(z)$ in Lemma 3.5.

Theorem 3.7. *Let $F(z)$ be the vector-valued rational function in (3.1), and precisely as described in the first paragraph of this section, with the notation therein. With $\alpha_{i,s}$, $e_s(z)$, and $\Psi_p(z)$ as in (3.6), (3.7) and (3.11), respectively, define*

$$\widehat{e}_s^{(p)}(z) = e_s(z)\overline{\Psi_p(z_s)}(z_s - \xi_{p+1}) = \frac{(z_s - \xi_{p+1})\,\overline{\Psi_p(z_s)}}{z - z_s}\, v_s \tag{3.17}$$

and

$$\widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z) = \begin{vmatrix} \widehat{e}_{s_0}^{(p)}(z) & \widehat{e}_{s_1}^{(p)}(z) & \cdots & \widehat{e}_{s_k}^{(p)}(z) \\ \alpha_{1,s_0} & \alpha_{1,s_1} & \cdots & \alpha_{1,s_k} \\ \alpha_{2,s_0} & \alpha_{2,s_1} & \cdots & \alpha_{2,s_k} \\ \vdots & \vdots & & \vdots \\ \alpha_{k,s_0} & \alpha_{k,s_1} & \cdots & \alpha_{k,s_k} \end{vmatrix}. \tag{3.18}$$

Then, with $p > k + \deg(u)$, we have

$$\Delta(z) = \psi_{1,p}(z) \sum_{1\le s_0<s_1<\cdots<s_k\le\mu} \widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z)\, V(z_{s_0}, z_{s_1}, \ldots, z_{s_k}) \left| \prod_{i=0}^{k} \Psi_p(z_{s_i}) \right|^{-2}. \tag{3.19}$$

Proof. Taking $p > k + \deg(u)$, and invoking (3.7) of Lemma 3.4 in (3.8), we first have

$$\begin{aligned} \Delta_j(z) &= \psi_{1,p}(z) F[z, \xi_{j+1}, \ldots, \xi_p] \\ &= \psi_{1,p}(z) \sum_{s=1}^{\mu} e_s(z) \frac{\psi_{1,j}(z_s)}{\psi_{1,p}(z_s)} \\ &= \psi_{1,p}(z) \sum_{s=1}^{\mu} \widehat{e}_s^{(p)}(z) \frac{\psi_{1,j}(z_s)}{\left|\Psi_p(z_s)\right|^2}. \end{aligned} \tag{3.20}$$

Substituting (3.20) and (3.14) in (3.10), and factoring out $\psi_{1,p}(z)$ from the first row, we have

$$\Delta(z) = \psi_{1,p}(z) W(z), \tag{3.21}$$

where

$$W(z) = \begin{vmatrix} \sum_{s_0} \widehat{e}_{s_0}^{(p)}(z) \frac{\psi_{1,0}(z_{s_0})}{|\Psi_p(z_{s_0})|^2} & \sum_{s_0} \widehat{e}_{s_0}^{(p)}(z) \frac{\psi_{1,1}(z_{s_0})}{|\Psi_p(z_{s_0})|^2} & \cdots & \sum_{s_0} \widehat{e}_{s_0}^{(p)}(z) \frac{\psi_{1,k}(z_{s_0})}{|\Psi_p(z_{s_0})|^2} \\ \sum_{s_1} \alpha_{1,s_1} \frac{\psi_{1,0}(z_{s_1})}{|\Psi_p(z_{s_1})|^2} & \sum_{s_1} \alpha_{1,s_1} \frac{\psi_{1,1}(z_{s_1})}{|\Psi_p(z_{s_1})|^2} & \cdots & \sum_{s_1} \alpha_{1,s_1} \frac{\psi_{1,k}(z_{s_1})}{|\Psi_p(z_{s_1})|^2} \\ \sum_{s_2} \alpha_{2,s_2} \frac{\psi_{1,0}(z_{s_2})}{|\Psi_p(z_{s_2})|^2} & \sum_{s_2} \alpha_{2,s_2} \frac{\psi_{1,1}(z_{s_2})}{|\Psi_p(z_{s_2})|^2} & \cdots & \sum_{s_2} \alpha_{2,s_2} \frac{\psi_{1,k}(z_{s_2})}{|\Psi_p(z_{s_2})|^2} \\ \vdots & \vdots & & \vdots \\ \sum_{s_k} \alpha_{k,s_k} \frac{\psi_{1,0}(z_{s_k})}{|\Psi_p(z_{s_k})|^2} & \sum_{s_k} \alpha_{k,s_k} \frac{\psi_{1,1}(z_{s_k})}{|\Psi_p(z_{s_k})|^2} & \cdots & \sum_{s_k} \alpha_{k,s_k} \frac{\psi_{1,k}(z_{s_k})}{|\Psi_p(z_{s_k})|^2} \end{vmatrix}. \tag{3.22}$$

Proceeding as in the proof of Theorem 3.6, we first take the summations outside. Following that, we take out the common factors from each row of the remaining determinant. We obtain

$$W(z) = \sum_{s_0}\sum_{s_1}\cdots\sum_{s_k} \widehat{e}_{s_0}^{(p)}(z) \left(\prod_{i=1}^{k} \alpha_{i,s_i} \right) \left| \prod_{i=0}^{k} \Psi_p(z_{s_i}) \right|^{-2} X(z_{s_0}, z_{s_1}, \ldots, z_{s_k}),$$

with $X(y_0, y_1, y_2, \ldots, y_n)$ as given in (3.15). Since the product

$$\left|\prod_{i=0}^{k} \Psi_p(z_{s_i})\right|^{-2} X(z_{s_0}, z_{s_1}, \ldots, z_{s_k})$$

is odd under an interchange of any two of the indices $s_0, s_1, \ldots, s_k$, Lemma 3.1 applies. Invoking also (3.16), we obtain the result in (3.19). ■

Finally, combining (3.12) and (3.19) in (3.9), we obtain a simple and elegant expression for $F(z) - R_{p,k}(z)$ when $F(z)$ is a vector-valued rational function. This is the subject of the following theorem.

Theorem 3.8. *For the error in $R_{p,k}(z)$, with $p > k + \deg(u)$, we have the closed-form expression*

$$F(z) - R_{p,k}(z) = \psi_{1,p}(z) \times \frac{\displaystyle\sum_{1\le s_0 < s_1 < \cdots < s_k \le \mu} \widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z)\, V(z_{s_0}, z_{s_1}, \ldots, z_{s_k}) \left|\prod_{i=0}^{k} \Psi_p(z_{s_i})\right|^{-2}}{\displaystyle\sum_{1\le s_1 < s_2 < \cdots < s_k \le \mu} T_{s_1,s_2,\ldots,s_k} V(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \left|\prod_{i=1}^{k} \Psi_p(z_{s_i})\right|^{-2}}. \tag{3.23}$$

Remark. When $k = \mu$ in Theorem 3.8, the summation in the numerator on the right-hand side of (3.23) is empty. Thus, this theorem provides an independent proof of the reproducing property of IMPE.

4. Preliminaries to convergence theory

Let E be a closed and bounded set in the z-plane, whose complement K, including the point at infinity, is connected and has a classical Green function $g(z)$ with a pole at infinity, which is continuous on ∂E, the boundary of E, and is zero on ∂E. For each σ, let Γ_σ be the locus $g(z) = \log \sigma$, and let E_σ denote the interior of Γ_σ. Then, E_1 is the interior of E and, for $1 < \sigma < \sigma'$, there holds $E \subset E_\sigma \subset E_{\sigma'}$.

For each $p \in \{1, 2, \ldots\}$, let

$$\Xi_p = \left\{\xi_1^{(p)}, \xi_2^{(p)}, \ldots, \xi_{p+1}^{(p)}\right\} \tag{4.1}$$

be the set of interpolation points used in constructing the IMPE interpolant $R_{p,k}(z)$. Assume that the sets Ξ_p are such that $\xi_i^{(p)}$ have no limit points in K and

$$\lim_{p\to\infty} \left|\prod_{i=1}^{p+1} \left(z - \xi_i^{(p)}\right)\right|^{1/p} = \kappa\, \Phi(z); \qquad \kappa = \operatorname{cap}(E), \qquad \Phi(z) = \exp[g(z)], \tag{4.2}$$

uniformly in z on every compact subset of K, where $\operatorname{cap}(E)$ is the logarithmic capacity of E defined by

$$\operatorname{cap}(E) = \lim_{n\to\infty} \left(\min_{r\in\mathcal{P}_n} \max_{z\in E} |r(z)|\right)^{1/n}; \qquad \mathcal{P}_n = \{r(z) :\ r \in \Pi_n \text{ and monic}\}.$$

Such sequences $\left\{\xi_1^{(p)}, \xi_2^{(p)}, \ldots, \xi_{p+1}^{(p)}\right\}$, $p = 1, 2, \ldots$, exist, see Walsh [15, p. 74]. Note that, in terms of $\Phi(z)$, the locus Γ_σ is defined by $\Phi(z) = \sigma$ for $\sigma > 1$, while $\partial E = \Gamma_1$ is simply the locus $\Phi(z) = 1$.

Recalling that $\prod_{i=1}^{p+1}\left(z - \xi_i^{(p)}\right) = \Psi_p(z)$ [see (3.11)], we can write (4.2) also as in

$$\lim_{p\to\infty} \left|\Psi_p(z)\right|^{1/p} = \kappa\,\Phi(z), \tag{4.3}$$

uniformly in z on every compact subset of K.

It is clear that if $z' \in \Gamma_{\sigma'}$ and $z'' \in \Gamma_{\sigma''}$ and $1 < \sigma' < \sigma''$, then $\Phi(z') < \Phi(z'')$.

The following lemmas that we use in our convergence study later were proved in [12].

Lemma 4.1. *Let K' be some compact subset of K. Then, for every $\epsilon > 0$, there is an integer p_0 depending only on ϵ, such that*

$$[(1-\epsilon)\kappa\,\Phi(z)]^p < \left|\Psi_p(z)\right| < [(1+\epsilon)\kappa\,\Phi(z)]^p, \quad \textit{for all } z \in K' \textit{ and for all } p > p_0. \tag{4.4}$$

Lemma 4.2. *For every $\epsilon > 0$, there is an integer p_0 depending only on ϵ, such that*

$$\left|\Psi_p(z)\right| < [(1+\epsilon)\kappa]^p, \quad \textit{for all } z \in E \textit{ and for all } p > p_0. \tag{4.5}$$

As a result, we also have that

$$\limsup_{p\to\infty} \left|\Psi_p(z)\right|^{1/p} \le \kappa \quad \textit{for all } z \in E. \tag{4.6}$$

Lemma 4.3. *Let* (i) *$z', z'' \in K$ and $\Phi(z') < \Phi(z'')$, or* (ii) *$z' \in E$ and $z'' \in K$. Then*

$$\lim_{p\to\infty} \left|\frac{\Psi_p(z')}{\Psi_p(z'')}\right|^{1/p} = \frac{\Phi_p(z')}{\Phi_p(z'')} < 1, \quad \textit{case } (\mathrm{i}). \tag{4.7}$$

$$\limsup_{p\to\infty} \left|\frac{\Psi_p(z')}{\Psi_p(z'')}\right|^{1/p} \le \frac{1}{\Phi_p(z'')} < 1, \quad \textit{case } (\mathrm{ii}). \tag{4.8}$$

In both cases,

$$\lim_{p\to\infty} \frac{\Psi_p(z')}{\Psi_p(z'')} = 0. \tag{4.9}$$

The result of Lemma 4.1 suggests that $\Psi_p(z)$ behaves practically like $[\kappa\,\Phi(z)]^p$ as $p \to \infty$.

5. Convergence theory for rational $F(z)$

In this section, we provide a convergence theory for the sequences $\{R_{p,k}(z)\}_{p=1}^{\infty}$ with $k < \mu$ and fixed, in case $F(z)$ is a vector-valued rational function with simple poles as in (3.1) and with orthogonal residues as in (3.2). The theorems that we state in what follows can be proved as those given in [12, Section 5]. Therefore, also to keep this work short, we only sketch some of the proofs. In what follows, we continue to use the notation of the preceding sections.

Note that, by the reproducing property mentioned in Section 1, for $k = \mu$, $R_{p,k}(z) \equiv F(z)$ for all $p \geq p_0$, where $p_0 - 1$ is the degree of the numerator of $F(z)$. Also, as we will let $p \to \infty$ in our analysis, the condition that $p > k + \deg(u)$ is satisfied for all large p. Recall that it is this condition that makes the results of Section 3 possible.

We now turn to $F(z)$ in (3.1). We assume that $F(z)$ is analytic in E. This implies that its poles $z_1, \ldots, z_\mu$ are all in K. We order the poles of $F(z)$ such that

$$\Phi(z_1) \leq \Phi(z_2) \leq \cdots \leq \Phi(z_\mu). \tag{5.1}$$

By Lemma 4.3, if z' and z'' are two different poles of $F(z)$, and $\Phi(z') < \Phi(z'')$, then z' and z'' lie on two different loci $\Gamma_{\sigma'}$ and $\Gamma_{\sigma''}$. In addition, $\sigma' < \sigma''$, that is, the set $E_{\sigma'}$ is in the interior of $E_{\sigma''}$.

5.1. *Convergence analysis for $V_{p,k}(z)$*

We now state a Koenig type convergence theorem for $V_{p,k}(z)$ and another theorem concerning its zeros [equivalently, poles of $R_{p,k}(z)$], assuming that $\Phi(z_k) < \Phi(z_{k+1})$. These results are analogous to, and in the spirit of, the ones given in Sidi [7] for denominators of Padé approximants. They are also similar to the corresponding results pertaining to IMMPE given in [12], but show *twice* as good performance for IMPE as that for IMMPE. We will remark on this further at the end of this subsection.

Theorem 5.1. *Assume*

$$\Phi(z_k) < \Phi(z_{k+1}) = \cdots = \Phi(z_{k+r}) < \Phi(z_{k+r+1}), \tag{5.2}$$

in addition to (5.1). In case $k + r = \mu$, we define $\Phi(z_{k+r+1}) = \infty$. Then, there holds

$$Q(z) = T_{1,\ldots,k} V(z, z_1, \ldots, z_k) \left| \prod_{i=1}^{k} \Psi_p(z_i) \right|^{-2} \left[1 + O\left(\left| \frac{\Psi_p(z_k)}{\widetilde{\Psi}_{p,k}} \right|^2 \right) \right] \quad \text{as } p \to \infty, \tag{5.3}$$

uniformly in every compact subset of $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_k\}$, where

$$\left| \widetilde{\Psi}_{p,k} \right| = \min_{1 \leq j \leq r} \left| \Psi_p(z_{k+j}) \right|. \tag{5.4}$$

Thus, with the normalization that $c_k = 1$, and letting

$$S(z) = \prod_{i=1}^{k} (z - z_i), \tag{5.5}$$

there holds

$$V_{p,k}(z) - S(z) = O\left(\left| \frac{\Psi_p(z_k)}{\widetilde{\Psi}_{p,k}} \right|^2 \right) \quad \text{as } p \to \infty, \tag{5.6}$$

from which we also have

$$\limsup_{p \to \infty} \left| V_{p,k}(z) - S(z) \right|^{1/p} \leq \left[\frac{\Phi(z_k)}{\Phi(z_{k+1})} \right]^2 < 1. \tag{5.7}$$

Proof. By (5.1), (5.2), Lemma 4.3, and the fact that

$$T_{1,\ldots,k} = \left(\prod_{i=1}^{k} \|v_i\|^2\right) \overline{V(z_1,\ldots,z_k)} \neq 0, \tag{5.8}$$

which follows from (3.13), asymptotically as $p \to \infty$, the largest term in (3.12) is that with the indices $(s_1,\ldots,s_k) = (1,\ldots,k)$. The next largest terms are those with $(s_1,\ldots,s_k) = (1,\ldots,k-1,k+j)$, $1 \le j \le r$. Obviously, we have $\lim_{p\to\infty}\left[\Psi_p(z_k)/\widetilde{\Psi}_{p,k}\right] = 0$. This completes the proof of (5.3). The proof of (5.6) can be achieved by noting that

$$V(z,z_1,\ldots,z_k) = (-1)^k V(z_1,\ldots,z_k)\prod_{i=1}^{k}(z-z_i). \tag{5.9}$$

The proof of (5.7) follows from (5.6) and (4.3). ■

Theorem 5.1 implies that, for all large p, $V_{p,k}(z)$ has precisely k zeros that tend to those of $S(z)$. In the next theorem, we provide the rate of convergence of each of these zeros.

Theorem 5.2. *Under the conditions of Theorem 5.1, $V_{p,k}(z)$ is of degree exactly k. Let us denote its zeros $z_1^{(p)},\ldots,z_k^{(p)}$. Then $\lim_{p\to\infty} z_m^{(p)} = z_m$, $m = 1,\ldots,k$. In addition, we have the refined result*

$$z_m^{(p)} - z_m \sim \sum_{j=1}^{r} C_j^{(m)} \left|\frac{\Psi_p(z_m)}{\Psi_p(z_{k+j})}\right|^2 + \cdots \quad \text{as } p \to \infty, \tag{5.10}$$

where $C_j^{(m)}$ are scalars independent of p given by

$$C_j^{(m)} = \frac{\|v_{k+j}\|^2}{\|v_m\|^2}\left|\prod_{\substack{i=1\\ i\neq m}}^{k}\frac{z_{k+j}-z_i}{z_m-z_i}\right|^2 (z_{k+j}-z_m), \quad j = 1,\ldots,r, \tag{5.11}$$

from which

$$z_m^{(p)} - z_m = O\left(\left|\frac{\Psi_p(z_m)}{\widetilde{\Psi}_{p,k}}\right|^2\right) \quad \text{as } p \to \infty, \tag{5.12}$$

with $\widetilde{\Psi}_{p,k}$ as in (5.4). From this, it follows that

$$\limsup_{p\to\infty}\left|z_m^{(p)} - z_m\right|^{1/p} \le \left[\frac{\Phi(z_m)}{\Phi(z_{k+1})}\right]^2 < 1. \tag{5.13}$$

Remark. The summation in (5.10) is the first term in the asymptotic expansion of $z_m^{(p)} - z_m$, and "…" stands for the rest of the terms in this expansion that are of higher order.

Proof. We start with the following asymptotic equality that is given in [12]:

$$z_m^{(p)} - z_m \sim -\frac{V_{p,k}(z_m)}{V'_{p,k}(z_m)} \quad \text{as } p \to \infty. \tag{5.14}$$

Since $Q(z)$ in (2.8) is a constant multiple of $V_{p,k}(z)$, this asymptotic equality can be rewritten as in

$$z_m^{(p)} - z_m \sim -\frac{Q(z_m)}{Q'(z_m)} \quad \text{as } p \to \infty. \tag{5.15}$$

Differentiating both sides of (3.12), and letting $z = z_m$, we have

$$Q'(z_m) = \sum_{1 \le s_1 < s_2 < \cdots < s_k \le \mu} T_{s_1,\ldots,s_k} a_{s_1,\ldots,s_k}^{(m)} \left| \prod_{i=1}^{k} \Psi_p(z_{s_i}) \right|^{-2}, \tag{5.16}$$

where

$$\begin{aligned} a_{s_1,s_2,\ldots,s_k}^{(m)} &= \frac{\mathrm{d}}{\mathrm{d}z} V(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \Big|_{z=z_m} \\ &= (-1)^k V(z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \left[\frac{\mathrm{d}}{\mathrm{d}z} \prod_{i=1}^{k} (z - z_{s_i}) \right] \Bigg|_{z=z_m}. \end{aligned} \tag{5.17}$$

Proceeding as in the proof of Theorem 5.1, we see that, because

$$a_{1,\ldots,k}^{(m)} = -V(z_1, \ldots, z_k) \prod_{\substack{i=1 \\ i \ne m}}^{k} (z_i - z_m) \ne 0,$$

the dominant term as $p \to \infty$ in the summation of (5.16) is that with $(s_1, \ldots, s_k) = (1, \ldots, k)$, the rest of the terms being negligible. Therefore, $Q'(z_m)$ satisfies the asymptotic equality

$$Q'(z_m) \sim T_{1,\ldots,k} a_{1,\ldots,k}^{(m)} \left| \prod_{i=1}^{k} \Psi_p(z_i) \right|^{-2} \quad \text{as } p \to \infty. \tag{5.18}$$

Setting $z = z_m$ in (3.12), and recalling that $V(y_0, y_1, \ldots, y_k)$ vanishes when any two of the y_j are equal, we have

$$Q(z_m) = \sum_{\substack{1 \le s_1 < \cdots < s_k \le \mu \\ s_1, \ldots, s_k \ne m}} T_{s_1,\ldots,s_k} V(z_m, z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \left| \prod_{i=1}^{k} \Psi_p(z_{s_i}) \right|^{-2}. \tag{5.19}$$

The dominant terms in this summation are those with

$$(s_1, \ldots, s_k) = (1, \ldots, m-1, m+1, \ldots, k, k+j), \quad 1 \le j \le r,$$

the rest of the terms being negligible. Thus,

$$Q(z_m) \sim \sum_{j=1}^{r} A_j^{(m)} \left| \prod_{i=1}^{k} \Psi_p(z_i) \right|^{-2} \left| \frac{\Psi_p(z_m)}{\Psi_p(z_{k+j})} \right|^2 + \cdots \quad \text{as } p \to \infty, \tag{5.20}$$

where

$$\begin{aligned} A_j^{(m)} &= T_{1,\ldots,m-1,m+1,\ldots,k,k+j} V(z_m, z_1, \ldots, z_{m-1}, z_{m+1}, \ldots, z_k, z_{k+j}), \\ &\quad j = 1, \ldots, r. \end{aligned} \tag{5.21}$$

Combining (5.18) and (5.20) in (5.15), we obtain

$$z_m^{(p)} - z_m \sim -\sum_{j=1}^{r} \frac{A_j^{(m)}}{T_{1,\ldots,k} a_{1,\ldots,k}^{(m)}} \left| \frac{\Psi_p(z_m)}{\Psi_p(z_{k+j})} \right|^2 + \cdots \quad \text{as } p \to \infty, \tag{5.22}$$

which is legitimate because (5.18) is an asymptotic equality. After some tedious manipulations, it can be shown that

$$-\frac{A_j^{(m)}}{T_{1,\ldots,k} a_{1,\ldots,k}^{(m)}} = C_j^{(m)}.$$

From this and from (5.22), the result in (5.13) now follows. (5.12) follows directly from (5.10), while (5.13) follows from (5.12) and (4.3). ■

Remark. If we replace IMPE by IMMPE, from [12, Theorems 5.1 and 5.2], we have that

$$\limsup_{p\to\infty} \left| V_{p,k}(z) - S(z) \right|^{1/p} \le \frac{\Phi(z_k)}{\Phi(z_{k+1})}$$

and

$$\limsup_{p\to\infty} \left| z_m^{(p)} - z_m \right|^{1/p} \le \frac{\Phi(z_m)}{\Phi(z_{k+1})}, \quad m = 1, \ldots, k.$$

Comparing these results for IMMPE with (5.7) in Theorem 5.1 and with (5.13) in Theorem 5.2 of the present work, we realize that, in the presence of orthogonal residues, $V_{p,k}(z)$ and $z_m^{(p)}$, $m = 1, \ldots, k$, converge with IMPE *twice as fast* as they do with IMMPE.

5.2. *Convergence analysis for $R_{p,k}(z)$*

We now continue with the analysis of $F(z) - R_{p,k}(z)$, as $p \to \infty$. Throughout the rest of this work, $\|Y\|$ denotes the vector norm of $Y \in \mathbb{C}^N$.

Theorem 5.3. *Under the conditions of Theorem 5.1, $R_{p,k}(z)$ exists and is unique for all large p and satisfies*

$$F(z) - R_{p,k}(z) \sim \sum_{j=1}^{r} \left(\prod_{i=1}^{k} \frac{z_{k+j} - z_i}{z - z_i} \right) \frac{v_{k+j}}{z - z_{k+j}} \frac{\psi_{1,p}(z)}{\psi_{1,p}(z_{k+j})} + \cdots \quad \textit{as } p \to \infty, \tag{5.23}$$

and hence

$$F(z) - R_{p,k}(z) = O\left(\frac{\Psi_p(z)}{\widetilde{\Psi}_{p,k}} \right) \quad \textit{as } p \to \infty, \tag{5.24}$$

uniformly on every compact subset of $\mathbb{C} \setminus \{z_1, \ldots, z_\mu\}$, with $\widetilde{\Psi}_{p,k}$ as defined in (5.4). From this, it also follows that

$$\limsup_{p\to\infty} \left\| F(z) - R_{p,k}(z) \right\|^{1/p} \le \frac{\Phi(z)}{\Phi(z_{k+1})}, \quad z \in \widetilde{K} = K \setminus \{z_1, \ldots, z_\mu\}, \tag{5.25}$$

uniformly on each compact subset of $\widetilde{K}$, *and*

$$\limsup_{p\to\infty} \left\| F(z) - R_{p,k}(z) \right\|^{1/p} \leq \frac{1}{\Phi(z_{k+1})}, \quad z \in E, \tag{5.26}$$

uniformly on E. *Thus, uniform convergence takes place for* z *in any compact subset of the set* $\widetilde{K}_k$, *where*

$$\widetilde{K}_k = \{z : \Phi(z) < \Phi(z_{k+1})\} \setminus \{z_1, \ldots, z_k\}.$$

Proof. We have already analyzed $Q(z)$ in Theorem 5.1 and obtained the result in (5.3), from which we also have the asymptotic equality

$$Q(z) \sim T_{1,\ldots,k} V(z, z_1, \ldots, z_k) \left| \prod_{i=1}^{k} \Psi_p(z_i) \right|^{-2} \quad \text{as } p \to \infty, \tag{5.27}$$

that holds uniformly in every compact subset of $\mathbb{C} \setminus \{z_1, z_2, \ldots, z_k\}$. This shows that, for all large p, $V_{p,k}(z)$ is such that $V_{p,k}(\xi_i^{(p)}) \neq 0$, for $i = 1, \ldots, p$, and large p, and that the condition in (2.10) is satisfied because

$$\begin{vmatrix} u_{1,0} & u_{1,1} & \cdots & u_{1,k-1} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k-1} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k-1} \end{vmatrix} = (-1)^k Q^{(k)}(0)/k!,$$

and that, by (5.27),

$$Q^{(k)}(0) \sim (-1)^k k! T_{1,\ldots,k} V(z_1, \ldots, z_k) \left| \prod_{i=1}^{k} \Psi_p(z_i) \right|^{-2} \neq 0 \quad \text{as } p \to \infty.$$

Under these, $R_{p,k}(z)$ exists and is unique for all large p, as mentioned in Section 2.

To complete the proof, we need to analyze the asymptotic behavior of $\Delta(z)$. From (3.19) in Theorem 3.7, we realize that it is necessary to analyze the asymptotic behavior of the $\widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z)$ as $p \to \infty$ first. Expanding the determinant representation of $\widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z)$ given in (3.18) with respect to its first row, we have

$$\widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z) = \sum_{i=0}^{k} w_i \, \widehat{e}^{(p)}_{s_i}(z); \quad w_i = (-1)^i T_{s_0,\ldots,s_{i-1},s_{i+1},\ldots,s_k}.$$

By (3.13), the cofactors w_i are independent of z and p. By (3.17), since $s_0 < s_1 < \cdots < s_k$ and due to Lemma 4.3, and (5.1), there holds

$$\widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z) = O\left(\Psi_p(z_{s_k})\right) \quad \text{as } p \to \infty.$$

In case $\Phi(z_{s_k}) > \Phi(z_{s_{k-1}})$, again by Lemma 4.3, we actually have the asymptotic equality

$$\begin{aligned} \widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z) &\sim w_k \, \widehat{e}^{(p)}_{s_k}(z) \quad \text{as } p \to \infty \\ &\sim (-1)^k T_{s_0,s_1,\ldots,s_{k-1}} \frac{(z_{s_k} - \xi^{(p)}_{p+1}) \overline{\Psi_p(z_{s_k})}}{z - z_{s_k}} v_{s_k} \quad \text{as } p \to \infty. \end{aligned} \tag{5.28}$$

Turning now to $\Delta(z)$, arguing as before, we have that, by (5.2), the dominant terms in the summation in (3.19) as $p \to \infty$ are those having indices

$$(s_0, s_1, \ldots, s_k) = (1, \ldots, k, k+j), \quad 1 \le j \le r.$$

The rest of the terms are negligible by Lemma 4.3. Thus, uniformly in every compact subset of the set $\mathbb{C} \setminus \{z_1, \ldots, z_\mu\}$,

$$\frac{\Delta(z)}{\psi_{1,p}(z)} \sim \sum_{j=1}^{r} \widehat{T}^{(p)}_{1,\ldots,k,k+j}(z) \frac{V(z_1, \ldots, z_k, z_{k+j})}{\left| \Psi_p(z_{k+j}) \prod_{i=1}^{k} \Psi_p(z_i) \right|^2} + \cdots \quad \text{as } p \to \infty, \tag{5.29}$$

which, by (5.28) and (3.11), becomes

$$\frac{\Delta(z)}{\psi_{1,p}(z)} \sim (-1)^k \sum_{j=1}^{r} T_{1,\ldots,k} \frac{V(z_1, \ldots, z_k) \prod_{i=1}^{k} (z_{k+j} - z_i)}{\psi_{1,p}(z_{k+j}) \left| \prod_{i=1}^{k} \Psi_p(z_i) \right|^2} \frac{v_{k+j}}{z - z_{k+j}} + \cdots$$
$$\text{as } p \to \infty. \tag{5.30}$$

Combining (5.27) and (5.30) in (3.9), and invoking (5.9), we obtain (5.23). (5.24) follows directly from (5.23), while (5.25) follows from (5.24). This completes the proof. ■

5.3. Approximation of residues

With Theorems 5.1 and 5.3 available, we now show that the residues of $R_{p,k}(z)$ converge to corresponding residues of $F(z)$.

Theorem 5.4. *Assume the conditions of Theorems 5.3 and 5.2. For $m = 1, \ldots, k$, let*

$$v_m^{(p)} = Res\, R_{p,k}(z)\big|_{z=z_m^{(p)}}.$$

Then, $\lim_{p\to\infty} v_m^{(p)} = v_m$. In fact, we have

$$\limsup_{p\to\infty} \|v_m^{(p)} - v_m\|^{1/p} \le \frac{\Phi(z_m)}{\Phi(z_{k+1})} < 1. \tag{5.31}$$

Proof. Let $\epsilon > 0$ be such that the set $D_m(\epsilon) = \{z : |z - z_m| \le \epsilon\}$ does not contain any of the poles z_i, $i \ne m$. By Theorem 5.2, for all large p, $D_m(\epsilon)$ contains $z_m^{(p)}$ but not $z_i^{(p)}$, $i \ne m$. Then, by Cauchy's theorem,

$$v_m^{(p)} - v_m = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D_m(\epsilon)} \left[R_{p,k}(z) - F(z) \right] \mathrm{d}z.$$

Here, the path $\partial D_m(\epsilon)$ is traversed in the counterclockwise direction. By the fact that (see, Ortega [5, pp. 142–143])

$$\left\| \oint_{\partial D_m(\epsilon)} H(z)\, \mathrm{d}z \right\| \le \oint_{\partial D_m(\epsilon)} \|H(z)\|\, |\mathrm{d}z|,$$

we have

$$\|v_m^{(p)} - v_m\| \le \frac{1}{2\pi} \oint_{\partial D_m(\epsilon)} \|R_{p,k}(z) - F(z)\|\,|\mathrm{d}z| \le \epsilon \max_{z\in\partial D_m(\epsilon)} \|R_{p,k}(z) - F(z)\|.$$

Thus,

$$\limsup_{p\to\infty} \|v_m^{(p)} - v_m\|^{1/p} \le \limsup_{p\to\infty} \left(\max_{z\in\partial D_m(\epsilon)} \|R_{p,k}(z) - F(z)\| \right)^{1/p} \le \frac{\Phi(z_m + \delta(\epsilon))}{\Phi(z_{k+1})},$$

for some $\delta(\epsilon)$, $|\delta(\epsilon)| \le \epsilon$. Clearly, $\Phi(z_m + \delta(\epsilon)) > \Phi(z_m)$. By Theorem 5.3 and the fact that ϵ can be taken to be arbitrarily close to zero, the result in (5.31) follows. ■

Another result that concerns the approximation of $H(z_m)$, where $H(z)$ is a scalar-valued or vector-valued function analytic at $z = z_m$, is given in the next theorem.

Theorem 5.5. *Let $H(z)$ be a scalar-valued or vector-valued function analytic at $z = z_m$, $m \in \{1, \ldots, k\}$. Then $H(z_m)$ can be approximated by $H(z_m^{(p)})$ as follows:*

$$H(z_m^{(p)}) - H(z_m) \sim H'(z_m)(z_m^{(p)} - z_m) \quad \text{as } p \to \infty, \tag{5.32}$$

hence

$$\limsup_{p\to\infty} |H(z_m^{(p)}) - H(z_m)|^{1/p} \le \left[\frac{\Phi(z_m)}{\Phi(z_{k+1})} \right]^2. \tag{5.33}$$

Proof. The assertion in (5.32) follows from

$$H(z_m^{(p)}) = H(z_m) + H'(z_m)(z_m^{(p)} - z_m) + O(|z_m^{(p)} - z_m|^2) \quad \text{as } p \to \infty,$$

and from the fact that $\lim_{p\to\infty} z_m^{(p)} = z_m$. The assertion in (5.33) follows from (5.32) and from Theorem 5.2. ■

6. Extension to infinite dimensional spaces

In this section, we extend the results of the previous sections to functions $F(z)$ that are meromorphic in the whole complex plane and that belong to an infinite dimensional inner product space X. Thus, we are interested in functions $F : \mathbb{C} \to X$ that are of the form

$$F(z) = \sum_{i=0}^{\nu} u_i z^i + \sum_{s=1}^{\infty} \frac{v_s}{z - z_s}, \tag{6.1}$$

where u_i and v_s are vectors in X, u_i being arbitrary while v_s satisfy

$$(v_i, v_j) = 0 \quad \text{if } i \ne j. \tag{6.2}$$

Here, $(\cdot\,,\cdot)$ is the inner product on X. The scalars z_s are distinct and satisfy $\lim_{s\to\infty} |z_s| = \infty$. Consequently, there can be only a finite number of them having the same modulus. Of course, the infinite series in (6.1) converges in the complex plane with the poles z_s excluded.

Such functions arise, for example, when X is a Hilbert space, and $F(z)$ is the solution to the operator equation $(I - zA)x = b$, where A is a compact self-adjoint operator on X. It is known that A has eigenpairs (λ_i, w_i) (i.e., $Aw_i = \lambda_i w_i$), $i = 1, 2, \ldots$, such that $\lambda_i \neq 0$, $\lim_{i\to\infty} \lambda_i = 0$, $(w_i, w_j) = \delta_{ij}$, and $\{w_1, w_2, \ldots\}$ is a basis for X. Then, letting $b = \sum_{i=1}^{\infty} \alpha_i w_i$, we obtain,

$$F(z) = (I - zA)^{-1} b = \sum_{i=1}^{\infty} \frac{\alpha_i}{1 - z\lambda_i} w_i.$$

It is clear that this $F(z)$ is of the form given in (6.1) and (6.2), with the z_s being some or all of the λ_i^{-1}; cf. the example in Section 3.

An important example of such $F(z)$ arises in the Hilbert–Schmidt theory of Fredholm integral equations of the second kind, namely,

$$u(x) - z \int_a^b K(x, t) u(t)\, dt = f(x), \quad a \le x \le b, \tag{6.3}$$

where $K(x, t)$ is real and continuous for $(x, t) \in [a, b] \times [a, b]$ and satisfies $K(t, x) = K(x, t)$. The space X in this case is $L_2[a, b]$, the space of square-integrable functions on $[a, b]$, with the inner product $(g, h) = \int_a^b g(x)h(x)\, dx$. Also, there exist functions $u_i(x) \in L_2[a, b]$, and real scalars z_i, $i = 1, 2, \ldots$, of finite multiplicity, such that $z_i \int_a^b K(x, t)u_i(t)\, dt = u_i(x)$, $\lim_{i\to\infty} z_i = \infty$, and $\int_a^b u_i(x)u_j(x)\, dx = \delta_{i,j}$. In addition, these functions span $L_2[a, b]$. Thus, $u(x)$, the solution to (6.3), is given by

$$u(x) = \sum_{i=1}^{\infty} \frac{(u_i, f)}{1 - z/z_i} u_i(x) = \sum_{i=1}^{\infty} \frac{z_i (u_i, f)}{z_i - z} u_i(x) \equiv F(z). \tag{6.4}$$

The analysis of the previous sections concerning rational $F(z)$ carries over to the cases of this section without any changes once we replace the integer μ in the previous sections by ∞. In what follows, we sketch the justification of this claim.

As before, we assume that the points of interpolation are as in Section 4 and that the poles z_s of $F(z)$ are ordered such that

$$\Phi(z_1) \le \Phi(z_2) \le \cdots. \tag{6.5}$$

By the fact that $\lim_{s\to\infty} |z_s| = \infty$, we have that $\lim_{s\to\infty} \Phi(z_s) = \infty$ as well.

We first choose integers k and μ such that

$$k < \mu, \quad \Phi(z_k) < \Phi(z_{k+1}), \quad \Phi(z_\mu) < \Phi(z_{\mu+1}) \equiv \rho. \tag{6.6}$$

[That there are infinitely many integers k and μ for which (6.5) holds follows from the fact that there can be only a finite number of the z_s having the same modulus.] Next, we rewrite (6.1) as in

$$F(z) = F_0(z) + \Theta(z), \tag{6.7}$$

where

$$F_0(z) = \sum_{i=0}^{\nu} u_i z^i + \sum_{s=1}^{\mu} \frac{v_s}{z - z_s} \quad \text{and} \quad \Theta(z) = \sum_{s=\mu+1}^{\infty} \frac{v_s}{z - z_s}. \tag{6.8}$$

Clearly, $F_0(z)$ is a rational function with simple poles $z_1, \ldots, z_\mu$ all in the set E_ρ and $\Theta(z)$ is an analytic function in E_ρ, where $E_\rho = \{z : \Phi(z) < \rho = \Phi(z_{\mu+1})\}$. Concerning the function

$\Theta(z)$, we know that (see [12, Lemma 6.1])

$$\limsup_{p\to\infty} \left\| \Theta[\xi_{j+1}^{(p)}, \ldots, \xi_{p+1}^{(p)}] \right\|^{1/p} \leq \frac{1}{\kappa\,\Phi(z_{\mu+1})} = \frac{1}{\kappa\rho}, \tag{6.9}$$

and

$$\limsup_{p\to\infty} \left\| \Theta[z, \xi_{j+1}^{(p)}, \ldots, \xi_{p}^{(p)}] \right\|^{1/p} \leq \frac{1}{\kappa\,\Phi(z_{\mu+1})} = \frac{1}{\kappa\rho},$$
$$\text{uniformly in every compact subset of } E_\rho. \tag{6.10}$$

Now,

$$D_{m,n} = F[\xi_m, \ldots, \xi_n] = F_0[\xi_m, \ldots, \xi_n] + \Theta[\xi_m, \ldots, \xi_n], \tag{6.11}$$

where, by (3.5),

$$F_0[\xi_m, \ldots, \xi_n] = -\sum_{s=1}^{\mu} v_s \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)}, \quad n-m > \nu, \tag{6.12}$$

$$\Theta[\xi_m, \ldots, \xi_n] = -\sum_{s=\mu+1}^{\infty} v_s \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)}, \tag{6.13}$$

and the infinite series in (6.13) converges because that in (6.1) does. Consequently, by (6.2),

$$(F_0[\xi_m, \ldots, \xi_n],\, \Theta[\xi_{m'}, \ldots, \xi_{n'}]) = 0, \quad n-m > \nu. \tag{6.14}$$

As a result, with $n-m > \nu$ and $n'-m' > \nu$,

$$\begin{aligned}\left(D_{m,n}, D_{m',n'}\right) &= (F_0[\xi_m, \ldots, \xi_n],\, F_0[\xi_{m'}, \ldots, \xi_{n'}]) \\ &\quad + (\Theta[\xi_m, \ldots, \xi_n],\, \Theta[\xi_{m'}, \ldots, \xi_{n'}]).\end{aligned} \tag{6.15}$$

With (6.12) and (3.6), and recalling (2.7), $u_{i,j} = (D_{i,p+1}, D_{j+1,p+1})$ becomes

$$u_{i,j} = \sum_{s=1}^{\mu} \alpha_{i,s} \frac{\psi_{1,j}(z_s)}{|\Psi_p(z_s)|^2} + \left(\Theta[\xi_i, \ldots, \xi_{p+1}],\, \Theta[\xi_{j+1}, \ldots, \xi_{p+1}]\right),$$
$$\text{for all large } p, \tag{6.16}$$

where $\alpha_{i,s}$ are as in (3.6). Now, by the Cauchy–Schwarz inequality,

$$\left|\left(\Theta[\xi_i, \ldots, \xi_{p+1}],\, \Theta[\xi_{j+1}, \ldots, \xi_{p+1}]\right)\right| \leq \left\|\Theta[\xi_i, \ldots, \xi_{p+1}]\right\| \, \left\|\Theta[\xi_{j+1}, \ldots, \xi_{p+1}]\right\|.$$

From this and (6.9), we therefore have

$$\limsup_{p\to\infty} \left|\left(\Theta[\xi_i^{(p)}, \ldots, \xi_{p+1}^{(p)}],\, \Theta[\xi_{j+1}^{(p)}, \ldots, \xi_{p+1}^{(p)}]\right)\right|^{1/p} \leq \frac{1}{[\kappa\,\Phi(z_{\mu+1})]^2} = \frac{1}{(\kappa\rho)^2}.$$

In other words, the term contributed to $u_{i,j}$ by $\Theta(z)$ is asymptotically of the order of $1/|\Psi_p(z_{\mu+1})|^2$ and hence is dominated by the μth term of the summation in (6.16). Substituting (6.16) in the determinant $Q(z)$ in (2.8) and (3.9), and going through the steps of the proofs of Theorems 3.6 and 5.1, we can show that the dominant term in the expansion of $Q(z)$ is that given in Theorem 5.1, despite the presence of $\Theta(z)$ as part of $F(z)$.

Similarly, $\Delta_j(z)$ in (3.8) becomes, for all large p,

$$\begin{aligned}\Delta_j(z) &= \psi_{1,p}(z)\left(F_0[z,\xi_{j+1},\ldots,\xi_p] + \Theta[z,\xi_{j+1},\ldots,\xi_{p+1}]\right)\\ &= \psi_{1,p}(z)\left(\sum_{s=1}^{\mu} e_s(z)\frac{\psi_{1,j}(z_s)}{\psi_{1,p}(z_s)} + \Theta[z,\xi_{j+1},\ldots,\xi_{p+1}]\right)\\ &= \psi_{1,p}(z)\left(\sum_{s=1}^{\mu} \widehat{e}_s^{(p)}(z)\frac{\psi_{1,j}(z_s)}{|\Psi_p(z_s)|^2} + \Theta[z,\xi_{j+1},\ldots,\xi_{p+1}]\right),\end{aligned} \tag{6.17}$$

with $\widehat{e}_s^{(p)}(z)$ as in (3.17). From (6.10), it is clear that, with $\xi_i = \xi_i^{(p)}$, the term $\Theta[z,\xi_{j+1}^{(p)},\ldots,\xi_{p+1}^{(p)}]$ in (6.17) is asymptotically of the order of $1/\Psi_p(z_{\mu+1})$ uniformly in every compact subset of E_ρ and hence is dominated by the μth term of the summation in (6.17). Substituting (6.17) and (6.16)in the determinant $\Delta(z)$ in (3.9) and (3.10), and going through the steps of the proofs of Theorems 3.7 and 5.3, we can show that the dominant term in the expansion of $\Delta(z)$ is that given in the proof of Theorem 5.3, despite the presence of $\Theta(z)$ as part of $F(z)$.

The rest of the results now follow easily.

Finally, we would like to remark that if the polynomial $\sum_{i=0}^{\nu} u_i z^i$ in (6.1) is replaced by a vector-valued *entire* function $H(z)$, so that

$$F(z) = H(z) + \sum_{s=1}^{\infty}\frac{v_s}{z-z_s},$$

the results of Section 5 continue to remain unchanged. The reason for this is that the divided differences of $H(z)$ now satisfy, *for every* $\rho > 1$,

$$\limsup_{p\to\infty}\left\|H[\xi_i^{(p)},\ldots,\xi_{p+1}^{(p)}]\right\|^{1/p} \le \frac{1}{\kappa\rho}$$

and

$$\limsup_{p\to\infty}\left\|H[z,\xi_i^{(p)},\ldots,\xi_{p+1}^{(p)}]\right\|^{1/p} \le \frac{1}{\kappa\rho},\quad \text{uniformly in every compact subset of } \mathbb{C}.$$

This is so by [12, Lemma 6.1], because $H(z)$ is analytic in every set E_ρ, where ρ can be arbitrarily large. In other words, even though none of the divided differences of $H(z)$ of high order vanishes, $H[z,\xi_i^{(p)},\ldots,\xi_{p+1}^{(p)}]$ tends to zero as $p\to\infty$, uniformly in every compact subset of $\mathbb{C}$, faster than $1/\Psi_p(z_s)$ for *every* s; thus, $H[z,\xi_i^{(p)},\ldots,\xi_{p+1}^{(p)}] = O(\mathrm{e}^{-\gamma p})$ as $p\to\infty$, for every $\gamma > 0$. As a result, the contribution of $H(z)$ to $u_{i,j}$ and $\Delta_j(z)$ can be safely ignored in the asymptotic analyses of $Q(z)$ and $\Delta(z)$. We can now continue as above with

$$F(z) = F_0(z) + \Theta(z) + H(z),$$

where

$$F_0(z) = \sum_{s=1}^{\mu}\frac{v_s}{z-z_s} \quad\text{and}\quad \Theta(z) = \sum_{s=\mu+1}^{\infty}\frac{v_s}{z-z_s},$$

but

$$F[\xi_i^{(p)},\ldots,\xi_{p+1}^{(p)}] \sim F_0[\xi_i^{(p)},\ldots,\xi_{p+1}^{(p)}] + \Theta[\xi_i^{(p)},\ldots,\xi_{p+1}^{(p)}] \quad \text{as } p\to\infty,$$

that is, as if $F(z)$ were given as in (6.1), and again produce the results of Section 5 without any changes.

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