Asymptotic Analysis 65 (2009) 175–190
DOI 10.3233/ASY-2009-0950
IOS Press

Asymptotic expansions of Legendre series coefficients for functions with endpoint singularities

Avram Sidi
Computer Science Department, Technion – Israel Institute of Technology, Haifa 32000, Israel
E-mail: asidi@cs.technion.ac.il; URL: http://www.cs.technion.ac.il/~asidi/

Abstract. Let $\sum_{n=0}^{\infty} e_n[f]P_n(x)$ be the Legendre expansion of a function $f(x)$ on $(-1,1)$. In this work, we derive an asymptotic expansion as $n\to\infty$ for $e_n[f]$, assuming that $f\in C^{\infty}(-1,1)$, but may have arbitrary algebraic-logarithmic singularities at one or both endpoints $x=\pm1$. Specifically, we assume that $f(x)$ has asymptotic expansions of the forms

$$f(x)\sim\sum_{s=0}^{\infty}U_s\big(\log(1-x)\big)(1-x)^{\alpha_s}\quad\text{as } x\to 1-,$$

$$f(x)\sim\sum_{s=0}^{\infty}V_s\big(\log(1+x)\big)(1+x)^{\beta_s}\quad\text{as } x\to -1+,$$

where $U_s(y)$ and $V_s(y)$ are some polynomials in y. Here, α_s and β_s are in general complex and $\Re\alpha_s,\Re\beta_s>-1$. An important special case is that in which $U_s(y)$ and $V_s(y)$ are constant polynomials; for this case, the asymptotic expansion of $e_n[f]$ assumes the form

$$e_n[f]\sim\sum_{\substack{s=0\\ \alpha_s\notin\mathbb{Z}^+}}^{\infty}\sum_{i=0}^{\infty}a_{si}h^{\alpha_s+i+1/2}+(-1)^n\sum_{\substack{s=0\\ \beta_s\notin\mathbb{Z}^+}}^{\infty}\sum_{i=0}^{\infty}b_{si}h^{\beta_s+i+1/2}\quad\text{as } n\to\infty,$$

where $h=(n+1/2)^{-2}$, $\mathbb{Z}^+=\{0,1,2,\ldots\}$, and a_{si} and b_{si} are constants independent of n.
Keywords: Legendre expansion, endpoint singularities, asymptotic expansions

1. Introduction

Let $\sum_{n=0}^{\infty} e_n[f]P_n(x)$ be the Legendre series of a function $f(x)$ on $(-1,1)$. Here $P_n(x)$ is the nth Legendre polynomial standardized such that $P_n(1)=1$, so that

$$\int_{-1}^{1}P_m(x)P_n(x)\,\mathrm{d}x=\frac{1}{n+1/2}\delta_{m,n},\quad m,n=0,1,\ldots,\tag{1.1}$$

and hence

$$e_n[f]=(n+1/2)\int_{-1}^{1}f(x)P_n(x)\,\mathrm{d}x,\quad n=0,1,\ldots.\tag{1.2}$$

It is known that when $f(x)$ and $|f(x)|^2$ are integrable on $(-1,1)$, from Parseval's theorem (see, e.g., Szegő [9] or Freud [4]), we have

$$e_n[f] = \mathrm{o}\bigl(\sqrt{n}\bigr) \quad \text{as } n \to \infty. \tag{1.3}$$

When $f \in C^r[-1,1]$ for some integer $r \geqslant 0$, then, by (1.2) and by the fact that

$$\int_{-1}^{1} P_n(x) q(x)\,\mathrm{d}x = 0 \quad \text{for every } q \in \Pi_{n-1},$$

we first have

$$e_n[f] = (n+1/2)\int_{-1}^{1} \bigl[f(x) - q(x)\bigr] P_n(x)\,\mathrm{d}x \quad \text{for every } q \in \Pi_{n-1}.$$

Here Π_m is the set of all polynomials of degree at most m. Thus, by the Cauchy–Schwartz inequality (see, e.g., Apostol [2]) and by (1.1),

$$\bigl|e_n[f]\bigr|^2 \leqslant (n+1/2)\int_{-1}^{1} \bigl|f(x) - q(x)\bigr|^2\,\mathrm{d}x \quad \text{for every } q \in \Pi_{n-1}.$$

Next, by one of Jackson's theorems, we have

$$\min_{q \in \Pi_{n-1}} \max_{-1 \leqslant x \leqslant 1} \bigl|f(x) - q(x)\bigr| = \mathrm{O}\bigl(n^{-r}\omega_{f^{(r)}}(2/n)\bigr) \quad \text{as } n \to \infty, \tag{1.4}$$

where $\omega_g(\delta)$ stands for the modulus of continuity of $g(x)$ on $(-1,1)$. (For moduli of continuity, see Davis [3], pp. 7–8, or Lorentz [5], pp. 43–46, for example. For the relevant theorem of Jackson, see [5], p. 66, Theorem 2, for example.) Therefore,

$$e_n[f] = \mathrm{O}\bigl(n^{-r+1/2}\omega_{f^{(r)}}(2/n)\bigr) \quad \text{as } n \to \infty. \tag{1.5}$$

Clearly, when $f(x)$ is continuously differentiable only r times on $[-1,1]$, the best we can say about $e_n[f]$ is (1.5), and that the smaller r is, the slower the convergence of $e_n[f]$ to zero becomes.

From (1.5), it is easy to see that when $f \in C^\infty[-1,1]$, $e_n[f]$ tends to zero as $n \to \infty$ faster than all negative powers of n, that is,

$$e_n[f] = \mathrm{o}\bigl(n^{-\mu}\bigr) \quad \text{as } n \to \infty, \quad \text{for every } \mu > 0. \tag{1.6}$$

In particular, when $f(z)$ is analytic in an open set of the z-plane that contains the interval $[-1,1]$ in its interior, there holds

$$e_n[f] = \mathrm{O}\bigl(\mathrm{e}^{-\sigma n}\bigr) \quad \text{as } n \to \infty, \quad \text{for some } \sigma > 0. \tag{1.7}$$

Let us consider the case when $f(x) = (1+x)^{r+\nu}$, $0 < \nu < 1$. Since $f^{(r)}(x) = (r+\nu)(r+\nu-1)\cdots$

$(\nu+1)(1+x)^\nu$, $f \in C^r[-1,1]$ but $f \notin C^{r+1}[-1,1]$, and we have $\omega_{f^{(r)}}(\delta) = M\delta^\nu$ for some constant $M > 0$; see [3], p. 8, Example 1. Therefore, (1.5) gives $e_n[f] = \mathrm{O}(n^{-r-\nu+1/2})$ as $n \to \infty$. However, as we will show later, there holds exactly $e_n[f] \sim Kn^{-2(r+\nu)-1}$ as $n \to \infty$ for some constant K, which is a much better result. The important thing to note here is that this $f(x)$ has an algebraic *endpoint* singularity. Thus, it seems that when singularities of $f(x)$ occur only at the endpoints $x = \pm 1$ but $f \in C^\infty(-1,1)$, the result in (1.5) can be improved substantially, at least in some cases, and this is the subject of the present work.

In this paper, we first study the case in which $f \in C^\infty(-1,1)$ and has *arbitrary* integrable algebraic-logarithmic singularities at one or both endpoints $x = \pm 1$. We derive a full asymptotic expansion for $e_n[f]$ as $n \to \infty$. One special case of such functions is $f(x) = [(1-x)^\alpha \log(1-x)^p] \times [(1+x)^\beta \log(1+x)^q]g(x)$, where $g \in C^\infty[-1,1]$ and p and q are nonnegetive integers. The main results concerning such $f(x)$ are presented in the next section, and their proofs are provided in Section 3. An interesting feature of the asymptotic expansions derived here is that they can be written down easily by looking only at the asymptotic expansions of $f(x)$ as $x \to \pm 1$, nothing else being needed.

In Section 4, we relax the assumption that $f \in C^\infty(-1,1)$, and extend the results of Section 2 to the cases in which $f \in C^r(-1,1)$ for some nonnegative integer r. Assuming further that $f \in C^\infty(-1, -1+\eta)$ and $f \in C^\infty(1-\eta, 1)$ for some $\eta > 0$, and that $f(x)$ again has *arbitrary* integrable algebraic-logarithmic singularities at one or both endpoints $x = \pm 1$, we derive an asymptotic expansion for $e_n[f]$ as $n \to \infty$ for this case as well.

To the best of our knowledge, these expansions have not been given before.

The results of this work, in addition to being of interest by themselves, can have applications in asymptotic analyses involving Legendre expansions, such as integral equations, numerical quadrature, and in series of spherical harmonics. They can be used to obtain the form of the asymptotic expansions of partial sums of Legendre series, which can then be used in accelerating the convergence of these series. For this topic, see Sidi [7], Chapter 6, for example.

Before we go on, we would like to mention that the proofs of Section 3 are almost identical to those in the recent paper [8] by the author concerning the asymptotic expansions of Gauss–Legendre quadrature rules when the integrand is as described in Section 2. This is made possible by the special form of the asymptotic expansion of $e_n[u]$, $u(x) = (1-x)^\omega$, that is derived in Theorem 2.1. We have chosen to provide the proofs in complete form to make this paper self-contained.

2. Main results

Throughout this section, we assume that the function $f(x)$ in (1.2) has the following properties:

1. $f \in C^\infty(-1,1)$ and has the asymptotic expansions

$$
\begin{aligned}
f(x) &\sim \sum_{s=0}^{\infty} U_s\big(\log(1-x)\big)(1-x)^{\alpha_s} \quad \text{as } x \to 1-, \\
f(x) &\sim \sum_{s=0}^{\infty} V_s\big(\log(1+x)\big)(1+x)^{\beta_s} \quad \text{as } x \to -1+,
\end{aligned} \tag{2.1}
$$

where $U_s(y)$ and $V_s(y)$ are some polynomials in y, and α_s and β_s are in general complex and satisfy

$$\begin{aligned} &-1 < \Re\alpha_0 \leqslant \Re\alpha_1 \leqslant \Re\alpha_2 \leqslant \cdots; \qquad \lim_{s\to\infty} \Re\alpha_s = +\infty, \\ &-1 < \Re\beta_0 \leqslant \Re\beta_1 \leqslant \Re\beta_2 \leqslant \cdots; \qquad \lim_{s\to\infty} \Re\beta_s = +\infty. \end{aligned} \tag{2.2}$$

Here, $\Re z$ stands for the real part of z.

2. If we let $u_s = \deg(U_s)$ and $v_s = \deg(V_s)$, then the α_s and β_s are ordered such that

$$u_s \geqslant u_{s+1} \quad \text{if } \Re\alpha_{s+1} = \Re\alpha_s; \qquad v_s \geqslant v_{s+1} \quad \text{if } \Re\beta_{s+1} = \Re\beta_s. \tag{2.3}$$

3. By (2.1), we mean that, for each $r = 1, 2, \ldots,$

$$\begin{aligned} &f(x) - \sum_{s=0}^{r-1} U_s\big(\log(1-x)\big)(1-x)^{\alpha_s} = \mathrm{O}\big(U_r\big(\log(1-x)\big)(1-x)^{\alpha_r}\big) \quad \text{as } x \to 1-, \\ &f(x) - \sum_{s=0}^{r-1} V_s\big(\log(1+x)\big)(1+x)^{\beta_s} = \mathrm{O}\big(V_r\big(\log(1+x)\big)(1+x)^{\beta_r}\big) \quad \text{as } x \to -1+. \end{aligned} \tag{2.4}$$

4. For each $k = 1, 2, \ldots,$ the kth derivative of $f(x)$ also has asymptotic expansions as $x \to \pm 1$ that are obtained by differentiating those in (2.1) term by term.
   The following are consequences of (2.2) and (2.3):
   (i) There are only a finite number of α_s that have the same real parts. Similarly, there are only a finite number of β_s that have the same real parts. Consequently, $\Re\alpha_s < \Re\alpha_{s+1}$ and $\Re\beta_{s'} < \Re\beta_{s'+1}$ for infinitely many values of the indices s and s'.
   (ii) The sequences $\{U_s(\log(1-x))(1-x)^{\alpha_s}\}_{s=0}^{\infty}$ and $\{V_s(\log(1+x))(1+x)^{\beta_s}\}_{s=0}^{\infty}$ are asymptotic scales, respectively, as $x \to 1-$ and $x \to -1+$, in the following sense: For $\{U_s(\log(1-x)) \times (1-x)^{\alpha_s}\}_{s=0}^{\infty}$, we have

$$\lim_{x\to 1-} \left| \frac{U_{s+1}(\log(1-x))(1-x)^{\alpha_{s+1}}}{U_s(\log(1-x))(1-x)^{\alpha_s}} \right| = \begin{cases} 0 & \text{if } \Re\alpha_s < \Re\alpha_{s+1}, \text{ or if } \Re\alpha_s = \Re\alpha_{s+1} \text{ and } u_s > u_{s+1}, \\ C_s & \text{if } \Re\alpha_s = \Re\alpha_{s+1} \text{ and } u_s = u_{s+1}. \end{cases}$$

   Here C_s are some positive constants. This limit is zero an infinite number of times since $\Re\alpha_s < \Re\alpha_{s+1}$ an infinite number of times. In the same sense, the sequence $\{V_s(\log(1+x))(1+x)^{\beta_s}\}_{s=0}^{\infty}$ is an asymptotic scale.
   (For a discussion of asymptotic scales, see Olver [6], p. 25, for example.)

   In view of (2.2)–(2.4), the expansions in (2.1) are thus genuine asymptotic expansions.

As for the fourth assumption on the termwise differentiability of the asymptotic expansions in (2.1), we mention that this assumption is crucial. It is automatically satisfied in many cases of practical interest. One such example is that for which $f(x) = [(1-x)^{\alpha} \log(1-x)^p][(1+x)^{\beta} \log(1+x)^q]g(x)$, where

$g \in C^\infty[-1,1]$ and p and q are nonnegative integers. For simplicity, let us consider the case $p = q = 0$ and $\beta = 0$. For this case, $f(x) = (1-x)^\alpha g(x)$, and we have

$$f(x) \sim \sum_{s=0}^{\infty} (-1)^s \frac{g^{(s)}(1)}{s!}(1-x)^{\alpha+s} \quad \text{as } x \to 1-.$$

Note that this expansion is nothing but the product of $(1-x)^\alpha$ and the Taylor series of $g(x)$ at $x = 1$. By the fact that $f^{(k)} \in C^\infty(-1,1)$ for all $k \geqslant 0$, it is easy to show that, just like $f(x)$, $f^{(k)}(x)$, $k = 1,2,\ldots,$ has an asymptotic expansion as $x \to 1-$, and that termwise differentiation k times of the asymptotic expansion of $f(x)$ results in the same expansion.

We now proceed to the main results of this work. Let us define

$$f_\omega^\pm(x) = (1 \pm x)^\omega. \tag{2.5}$$

Then, by (1.2) and by the fact that $P_n(-x) = (-1)^n P_n(x)$,

$$e_n[f_\omega^+] = (-1)^n e_n[f_\omega^-]. \tag{2.6}$$

By invoking Rodrigues' formula for $P_n(x)$ in $\int_{-1}^{1}(1-x)^\omega P_n(x)\,\mathrm{d}x$, and by repeated integration by parts, we also have (see Davis [3], p. 327, Exercise 16)

$$e_n[f_\omega^-] = 2^\omega(2n+1)\frac{(-\omega)_n}{(1+\omega)_{n+1}}, \quad n = 0,1,\ldots, \tag{2.7}$$

where $(a)_k$ is the Pochhammer symbol defined by $(a)_0 = 1$ and $(a)_k = \prod_{i=1}^{k}(a+i-1)$, $k = 1,2,\ldots.$ Note that $e_n[f_\omega^\pm]$ are analytic for $\Re\omega > -1$. We now state and prove a result concerning the asymptotic expansion of $e_n[f_\omega^\pm]$ as $n \to \infty$ that we will use in the rest of this work.

Theorem 2.1. *Let $f_\omega^\pm(x)$ be as in* (2.5)*, with $\Re\omega > -1$ but $\omega \notin \mathbb{Z}^+$, where $\mathbb{Z}^+ = \{0,1,2,\ldots\}$. Then, with $h = (n+1/2)^{-2}$, we have the asymptotic expansion*

$$(-1)^n e_n[f_\omega^+] = e_n[f_\omega^-] \sim \sum_{k=0}^{\infty} c_k(\omega) h^{\omega+k+1/2} \quad \text{as } n \to \infty,$$

$$c_k(\omega) = 2^{\omega+1}\frac{\Gamma(1+\omega)}{\Gamma(-\omega)}\frac{B_{2k}^{(\sigma)}(\sigma/2)}{(2k)!}\frac{\Gamma(2k+2\omega+2)}{\Gamma(2\omega+2)}, \quad k = 0,1,\ldots,$$

$$\sigma = -2\omega - 1, \tag{2.8}$$

that is valid uniformly in every strip $-1 < d_1 \leqslant \Re\omega \leqslant d_2 < \infty$ of the ω-plane. The $c_k(\omega)$ are analytic functions of ω for $\Re\omega > -1$. $B_s^{(\sigma)}(u)$ is the sth generalized Bernoulli polynomial.[1] *When $\omega \in \mathbb{Z}^+$, there holds $c_k(\omega) = 0$ for each $k = 0,1,\ldots$; in this case, we also have $e_n[f_\omega^\pm] = 0$ for all $n > \omega$.*

[1]The generalized Bernoulli polynomials $B_s^{(\sigma)}(u)$ are defined via (see Andrews, Askey and Roy [1], p. 615, for example)

$$\left(\frac{t}{\mathrm{e}^t - 1}\right)^\sigma \mathrm{e}^{ut} = \sum_{s=0}^{\infty} B_s^{(\sigma)}(u)\frac{t^s}{s!}, \quad |t| < 2\pi.$$

They satisfy $B_s^{(\sigma)}(\sigma - u) = (-1)^s B_s^{(\sigma)}(u)$, hence $B_s^{(\sigma)}(\sigma/2) = 0$ for $s = 1,3,5,\ldots.$

Proof. We start by rewriting (2.7) in the form

$$e_n[f_\omega^-] = 2^\omega(2n+1)\frac{\Gamma(1+\omega)}{\Gamma(-\omega)}\frac{\Gamma(n-\omega)}{\Gamma(n+\omega+2)}, \quad n = 0, 1, \ldots. \tag{2.9}$$

Applying to $\Gamma(n-\omega)/\Gamma(n+\omega+2)$ a result concerning the ratio of two Gamma functions, namely,

$$\begin{aligned} &\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim \sum_{k=0}^{\infty} \frac{B_{2k}^{(\sigma)}(\sigma/2)}{(2k)!}\frac{\Gamma(2k+b-a)}{\Gamma(b-a)}\frac{1}{(x+a-\sigma/2)^{2k+b-a}} \quad \text{as } x \to \infty, \\ &\sigma = a+1-b, \qquad \Re(b-a) > 0, \end{aligned} \tag{2.10}$$

that is given in Andrews, Askey and Roy [1], p. 216, for example, we obtain (2.8).

We leave the details to the reader. □

Remark.

1. By using a different asymptotic expansion for $\Gamma(x+a)/\Gamma(x+b)$ that is simpler and less sophisticated than that in (2.10), we obtain an asymptotic expansion for $e_n[f_\omega^\pm]$ that contains the powers $n^{-2\omega-k-1}$, $k = 0, 1, \ldots$. It is easy to see that this expansion has *twice as many* terms as that given in (2.8), however. See [1], pp. 215–216.
2. From Theorem 2.1 it follows that, with α_s and β_s as in (2.2), the sequences $\{e_n[f_{\alpha_s}^-]\}_{s=0}^\infty$ and $\{e_n[f_{\beta_s}^+]\}_{s=0}^\infty$ are both asymptotic scales as $n \to \infty$.

We now state the main results of this work. We start with the following special case of pure algebraic (nonlogarithmic) endpoint singularities that is important and of interest in itself:

Theorem 2.2. *Let $f(x)$ be exactly as described in the first paragraph of this section with the same notation, $U_s(y) = A_s \neq 0$ and $V_s(y) = B_s \neq 0$ being constant polynomials for all s. Then, with $h = (n+1/2)^{-2}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, there holds*

$$e_n[f] \sim \sum_{\substack{s=0 \\ \alpha_s \notin \mathbb{Z}^+}}^{\infty} A_s e_n[f_{\alpha_s}^-] + \sum_{\substack{s=0 \\ \beta_s \notin \mathbb{Z}^+}}^{\infty} B_s e_n[f_{\beta_s}^+] \quad \text{as } n \to \infty. \tag{2.11}$$

Consequently,

$$\begin{aligned} e_n[f] \sim &\sum_{\substack{s=0 \\ \alpha_s \notin \mathbb{Z}^+}}^{\infty} A_s \sum_{k=0}^{\infty} c_k(\alpha_s) h^{\alpha_s+k+1/2} \\ &+ (-1)^n \sum_{\substack{s=0 \\ \beta_s \notin \mathbb{Z}^+}}^{\infty} B_s \sum_{k=0}^{\infty} c_k(\beta_s) h^{\beta_s+k+1/2} \quad \text{as } n \to \infty. \end{aligned} \tag{2.12}$$

Here, $c_k(\omega)$ are precisely as given in Theorem 2.1.

Remark.

1. By (2.2), the sequences $\{h^{\alpha_s+k+1/2}\}_{s=0}^{\infty}$ and $\{h^{\beta_s+k+1/2}\}_{s=0}^{\infty}$ are asymptotic scales as $n\to\infty$, and the expansion in (2.12) is a genuine asymptotic expansion when its terms are reordered according to their size.
2. Note that, when $U_s(y)$ and $V_s(y)$ are constants, the nonnegative integer powers $(1-x)^s$ and $(1+x)^s$, if present in the asymptotic expansions of (2.1), do not contribute to the expansion of $e_n[f]$ as $n\to\infty$.
3. In case α_s,β_s are all nonnegative integers in Theorem 2.2, of course, $f\in C^\infty[-1,1]$, and the asymptotic expansion in (2.12) is empty (zero). This does not necessarily mean that $e_n[f]=0$, however. It only means that $e_n[f]$ tends to zero as $n\to\infty$ faster than all negative powers of n, which is consistent with the known result we mentioned in Section 1. Of course, when $f(x)$ is a polynomial, $e_n[f]=0$ for all $n>\deg f$.
4. If $\alpha_s=\alpha+s$ and $\beta_s=s$ for all $s=0,1,\dots,$ in Theorem 2.2, then $f(x)$ is of the form $f(x)=(1-x)^\alpha g(x)$ with $g\in C^\infty[-1,1]$ and $A_s=(-1)^s g^{(s)}(1)/s!$, $s=0,1,\dots.$ In this case, the second double sum in (2.12) disappears and the first double sum can be rearranged so that

$$e_n[f]\sim\sum_{k=0}^{\infty}a_k h^{\alpha+k+1/2}\quad\text{as } n\to\infty, \tag{2.13}$$

where a_k are functions of α given by

$$a_k=\sum_{s=0}^{k}A_s c_{k-s}(\alpha+s),\quad k=0,1,\dots, \tag{2.14}$$

and are analytic in every strip $-1<d_1\leqslant\Re\alpha\leqslant d_2<\infty$ of the α-plane.
5. If $\alpha_s=\alpha+s$ and $\beta_s=\beta+s$ for all $s=0,1,\dots,$ in Theorem 2.2, then $f(x)$ is of the form $f(x)=(1-x)^\alpha(1+x)^\beta g(x)$ with $g\in C^\infty[-1,1]$, and A_s and B_s are given by

$$\begin{aligned}A_s&=\frac{(-1)^s}{s!}\frac{\mathrm{d}^s}{\mathrm{d}x^s}\big[(1+x)^\beta g(x)\big]\Big|_{x=1}=(-1)^s\sum_{i=0}^{s}\binom{\beta}{i}\frac{g^{(s-i)}(1)}{(s-i)!}2^{\beta-i},\\ B_s&=\frac{1}{s!}\frac{\mathrm{d}^s}{\mathrm{d}x^s}\big[(1-x)^\alpha g(x)\big]\Big|_{x=-1}=\sum_{i=0}^{s}(-1)^i\binom{\alpha}{i}\frac{g^{(s-i)}(-1)}{(s-i)!}2^{\alpha-i}.\end{aligned} \tag{2.15}$$

Note that A_s are functions of β only, while B_s are functions of α only. In addition, they are entire, because $\binom{z}{i}2^z$ is an entire function of the complex variable z. In this case, by rearranging both of the double sums in (2.12), we have

$$e_n[f]\sim\sum_{k=0}^{\infty}a_k h^{\alpha+k+1/2}+(-1)^n\sum_{k=0}^{\infty}b_k h^{\beta+k+1/2}\quad\text{as } n\to\infty. \tag{2.16}$$

Here, a_k and b_k are functions of both α and β given by

$$a_k=\sum_{s=0}^{k}A_s c_{k-s}(\alpha+s),\qquad b_k=\sum_{s=0}^{k}B_s c_{k-s}(\beta+s),\quad k=0,1,\dots, \tag{2.17}$$

and are analytic when α and β are such that $-1 < d_1 \leqslant \Re\alpha \leqslant d_2 < \infty$ and $-1 < d_1' \leqslant \Re\beta \leqslant d_2' < \infty$, respectively.

The next theorem deals with the general case, in which algebraic-logarithmic singularities may occur at the endpoints.

Theorem 2.3. *Let $f(x)$ be exactly as described in the first paragraph of this section with the same notation, and let $U_s(y) = \sum_{i=0}^{u_s} \sigma_{si} y^i$ and $V_s(y) = \sum_{i=0}^{v_s} \tau_{si} y^i$. Denote $\frac{\mathrm{d}}{\mathrm{d}\omega}$ by D_ω. For an arbitrary polynomial $W(y) = \sum_{i=0}^{k} \epsilon_i y^i$ and an arbitrary function g that depends on ω, define also*

$$W(D_\omega)g := \sum_{i=0}^{k} \epsilon_i \big[D_\omega^i g\big] = \sum_{i=0}^{k} \epsilon_i \frac{\mathrm{d}^i g}{\mathrm{d}\omega^i}.$$

Then, with $h = (n+1/2)^{-2}$, there holds

$$e_n[f] \sim \sum_{s=0}^{\infty} U_s(D_{\alpha_s}) e_n\big[f_{\alpha_s}^-\big] + \sum_{s=0}^{\infty} V_s(D_{\beta_s}) e_n\big[f_{\beta_s}^+\big] \quad \text{as } n \to \infty. \tag{2.18}$$

Consequently,

$$\begin{aligned} e_n[f] \sim{}& \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} U_s(D_{\alpha_s})\big[c_k(\alpha_s) h^{\alpha_s+k+1/2}\big] \\ &+ (-1)^n \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} V_s(D_{\beta_s})\big[c_k(\beta_s) h^{\beta_s+k+1/2}\big] \quad \text{as } n \to \infty. \end{aligned} \tag{2.19}$$

Here, $c_k(\omega)$ are precisely as given in Theorem 2.1.

Remark.

1. To see the explicit form of the expansion in Theorem 2.3, we also need

$$D_\omega^i\big[c_k(\omega) h^{\omega+k+1/2}\big] = h^{\omega+k+1/2} \sum_{j=0}^{i} \binom{i}{j} c_k^{(i-j)}(\omega)(\log h)^j,$$

where $c_k^{(r)}(\omega)$ stands for the rth derivative of $c_k(\omega)$. Using this, it can be seen that, for example,

$$U_s(D_{\alpha_s})\big[c_k(\alpha_s) h^{\alpha_s+k+1/2}\big] = h^{\alpha_s+k+1/2} \sum_{j=0}^{u_s} e_{sj} (\log h)^j,$$

where

$$e_{sj} = \sum_{i=j}^{u_s} \binom{i}{j} \sigma_{si} c_k^{(i-j)}(\alpha_s), \quad j = 0, 1, \ldots, u_s.$$

Note that $e_{su_s} = \sigma_{su_s} c_k(\alpha_s)$. By Theorem 2.1, this implies that $e_{su_s} = 0$ when $\alpha_s \in \mathbb{Z}^+$. Thus, (2.19) assumes the following explicit form:

$$e_n[f] \sim \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \widehat{U}_{sk}(\log h) h^{\alpha_s+k+1/2} + (-1)^n \sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \widehat{V}_{sk}(\log h) h^{\beta_s+k+1/2} \quad \text{as } n \to \infty, \tag{2.20}$$

where $\widehat{U}_{sk}(y)$ and $\widehat{V}_{sk}(y)$ are polynomials in y with $\deg(\widehat{U}_{sk}) \leqslant u_s$ and $\deg(V_s) \leqslant v_s$. If $\alpha_s \in \mathbb{Z}^+$, then $\deg(\widehat{U}_{sk}) \leqslant u_s - 1$; otherwise, $\deg(\widehat{U}_{sk}) = u_s$. Similarly, if $\beta_s \in \mathbb{Z}^+$, then $\deg(\widehat{V}_{sk}) \leqslant v_s - 1$; otherwise, $\deg(\widehat{V}_{sk}) = v_s$.

2. Invoking now (2.2) and (2.3), we conclude that the sequences

$$\{U_s(D_{\alpha_s})[c_k(\alpha_s)h^{\alpha_s+k+1/2}]\}_{s=0}^{\infty} \quad \text{and} \quad \{V_s(D_{\beta_s})[c_k(\beta_s)h^{\beta_s+k+1/2}]\}_{s=0}^{\infty}$$

are asymptotic scales as $n \to \infty$, and that the expansion in (2.19) is a genuine asymptotic expansion.

3. When $\alpha_s = \alpha + s$ and $\beta_s = \beta + s$, for all $s = 0, 1, \ldots$, and $u_0 = u_1 = \cdots = p$ and $v_0 = v_1 = \cdots = q$, we can rearrange the double sums in (2.20), and obtain

$$e_n[f] \sim \sum_{k=0}^{\infty} \check{U}_k(\log h) h^{\alpha+k+1/2} + (-1)^n \sum_{k=0}^{\infty} \check{V}_k(\log h) h^{\beta+k+1/2} \quad \text{as } n \to \infty, \tag{2.21}$$

where $\check{U}_k(y) = \sum_{s=0}^{k} \widehat{U}_{s,k-s}(y)$ and $\check{V}_k(y) = \sum_{s=0}^{k} \widehat{V}_{s,k-s}(y)$ are polynomials in y of degree at most p and q, respectively. If $\alpha \in \mathbb{Z}^+$, then $\deg(\check{U}_k) \leqslant p - 1$. Similarly, if $\beta \in \mathbb{Z}^+$, then $\deg(\check{V}_k) \leqslant q - 1$.

4. The case in the preceding remark arises, for example, when

$$f(x) = (1-x)^{\alpha}(1+x)^{\beta}[\log(1-x)]^p[\log(1+x)]^q g(x)$$

with $g \in C^{\infty}[-1, 1]$. In this case, the asymptotic expansion of $e_n[f]$ can be obtained by differentiating the asymptotic expansion of $e_n[\widetilde{f}]$, where $\widetilde{f}(x) = (1-x)^{\alpha}(1+x)^{\beta} g(x)$, p times with respect to α and q times with respect to β. Note that $\widetilde{f}(x)$ here is precisely the function $f(x)$ given in Remark 5 following Theorem 2.2, and the asymptotic expansion of $e_n[\widetilde{f}]$ is as given in (2.16) and (2.17). Recall that the a_k and b_k there are analytic functions of both α and β. Thus, applying $\partial^{p+q}/\partial\alpha^p\,\partial\beta^q$ to (2.16), we obtain the expansion in (2.21).

A simpler special case is one in which $\beta = 0$ and $q = 0$. For this case, we have $f(x) = (1-x)^{\alpha}[\log(1-x)]^p g(x)$ with p a positive integer and $g \in C^{\infty}[-1, 1]$. The asymptotic expansion of $e_n[f]$ is now of the form

$$e_n[f] \sim \sum_{k=0}^{\infty} \check{U}_k(\log h) h^{\alpha+k+1/2} \quad \text{as } n \to \infty, \tag{2.22}$$

where $\check{U}_k(y)$, as before, are polynomials in y of degree at most p, and this can be obtained by differentiating the asymptotic expansion of $e_n[\widetilde{f}]$, where $\widetilde{f}(x) = (1-x)^\alpha g(x)$, p times with respect to α.

3. Proofs of main results

3.1. *Proof of Theorem* 2.2

With $U_s(y) = A_s$ and $V_s(y) = B_s$, and an arbitrary positive integer m, let

$$p(x) = \sum_{s=0}^{m-1} A_s(1-x)^{\alpha_s} + \sum_{s=0}^{m-1} B_s(1+x)^{\beta_s} = \sum_{s=0}^{m-1} A_s f^-_{\alpha_s}(x) + \sum_{s=0}^{m-1} B_s f^+_{\beta_s}(x). \tag{3.1}$$

Here, $f^\pm_\omega(x)$ are as defined in (2.5). Then,

$$f(x) = p(x) + \phi(x); \qquad \phi(x) := f(x) - p(x). \tag{3.2}$$

Thus,

$$e_n[f] = e_n[p] + e_n[\phi]. \tag{3.3}$$

By Theorem 2.1,

$$\begin{aligned} e_n[p] &= \sum_{s=0}^{m-1} A_s e_n\left[f^-_{\alpha_s}\right] + \sum_{s=0}^{m-1} B_s e_n\left[f^+_{\beta_s}\right] \\ &\sim \sum_{s=0}^{m-1} A_s \sum_{k=0}^{\infty} c_k(\alpha_s) h^{\alpha_s+k+1/2} + (-1)^n \sum_{s=0}^{m-1} B_s \sum_{k=0}^{\infty} c_k(\beta_s) h^{\beta_s+k+1/2} \qquad \text{as } n \to \infty. \end{aligned} \tag{3.4}$$

We now have to analyze $e_n[\phi]$. For this, we need to know the differentiability properties of $\phi(x)$ on $[-1,1]$. First, $\phi \in C^\infty(-1,1)$. At $x = \pm 1$, $\phi(x)$ has the asymptotic expansions

$$\begin{aligned} \phi(x) &\sim w^+_m(x) + \sum_{s=m}^{\infty} A_s(1-x)^{\alpha_s} \quad \text{as } x \to 1-; \qquad w^+_m(x) = -\sum_{s=0}^{m-1} B_s(1+x)^{\beta_s}, \\ \phi(x) &\sim w^-_m(x) + \sum_{s=m}^{\infty} B_s(1+x)^{\beta_s} \quad \text{as } x \to -1+; \quad w^-_m(x) = -\sum_{s=0}^{m-1} A_s(1-x)^{\alpha_s}. \end{aligned} \tag{3.5}$$

Note that $w^+_m(x)$ is infinitely differentiable at $x = 1$ while $w^-_m(x)$ is infinitely differentiable at $x = -1$. Thus, what determines the differentiability properties on $[-1,1]$ of $\phi(x)$ are the infinite sums in (3.5). By the fourth of the properties of $f(x)$ mentioned in the beginning of Section 2, the asymptotic expansions of $\phi(x)$ in (3.5) can be differentiated termwise as many times as we wish. Then, for every positive integer j, there holds

$$\begin{aligned}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\phi(x) &\sim \frac{\mathrm{d}^j}{\mathrm{d}x^j}w_m^+(x) + \sum_{s=m}^{\infty} A_s\alpha_s(\alpha_s-1)\cdots(\alpha_s-j+1)(1-x)^{\alpha_s-j} \quad \text{as } x\to 1-,\\ \frac{\mathrm{d}^j}{\mathrm{d}x^j}\phi(x) &\sim \frac{\mathrm{d}^j}{\mathrm{d}x^j}w_m^-(x) + \sum_{s=m}^{\infty} B_s\beta_s(\beta_s-1)\cdots(\beta_s-j+1)(1+x)^{\beta_s-j} \quad \text{as } x\to -1+.\end{aligned} \tag{3.6}$$

Clearly,

$$\begin{aligned}\lim_{x\to 1-}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\phi(x) &= -\frac{\mathrm{d}^j w_m^+}{\mathrm{d}x^j}\bigg|_{x=1}, \qquad j=0,1,\ldots,\lceil\Re\alpha_m-1\rceil,\\ \lim_{x\to -1+}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\phi(x) &= -\frac{\mathrm{d}^j w_m^-}{\mathrm{d}x^j}\bigg|_{x=-1}, \quad j=0,1,\ldots,\lceil\Re\beta_m-1\rceil,\end{aligned} \tag{3.7}$$

which also means that $\phi(x)$ has $\lceil\Re\alpha_m-1\rceil$ continuous derivatives at $x=1$ and $\lceil\Re\beta_m-1\rceil$ continuous derivatives at $x=-1$, in addition to being in $C^\infty(-1,1)$. Consequently, $\phi\in C^{\kappa_m}[-1,1]$, where $\kappa_m=\min\{\lceil\Re\alpha_m-1\rceil,\lceil\Re\beta_m-1\rceil\}$.

Next, by (1.5) in the second paragraph of Section 1,

$$e_n[\phi]=\mathrm{O}\big(n^{-\kappa_m+1}\big)=\mathrm{O}\big(h^{\kappa_m/2-1/2}\big) \quad \text{as } n\to\infty. \tag{3.8}$$

Combining (3.4) and (3.8) in (3.3), and considering only those terms with $\Re\alpha_s+k<\Re\alpha_m$ and $\Re\beta_s+k<\Re\beta_m$ in, respectively, the first and second double summations in (3.4), we have

$$\begin{aligned}e_n[f] = &\sum_{\substack{0\leqslant s\leqslant m-1\\ 0\leqslant k<\Re(\alpha_m-\alpha_s)}} A_s c_k(\alpha_s)h^{\alpha_s+k+1/2} + \mathrm{O}\big(h^{\alpha_m+1/2}\big)\\ &+(-1)^n\sum_{\substack{0\leqslant s\leqslant m-1\\ 0\leqslant k<\Re(\beta_m-\beta_s)}} B_s c_k(\beta_s)h^{\beta_s+k+1/2} + \mathrm{O}\big(h^{\beta_m+1/2}\big)\\ &+\mathrm{O}\big(h^{\kappa_m/2-1/2}\big) \quad \text{as } n\to\infty.\end{aligned} \tag{3.9}$$

Now, $\lim_{m\to\infty}\kappa_m=\infty$ and $\lim_{m\to\infty}\Re\alpha_m=\infty$ and $\lim_{m\to\infty}\Re\beta_m=\infty$ simultaneously, by (2.2). From this and from (3.9), we conclude that $e_n[f]$ has the true asymptotic expansion

$$e_n[f]\sim\sum_{s=0}^{\infty}A_s\sum_{k=0}^{\infty}c_k(\alpha_s)h^{\alpha_s+k+1/2}+(-1)^n\sum_{s=0}^{\infty}B_s\sum_{k=0}^{\infty}c_k(\beta_s)h^{\beta_s+k+1/2} \quad \text{as } n\to\infty. \tag{3.10}$$

Finally, the result in (2.12) follows by invoking the fact that $c_k(\omega)=0$ when $\omega\in\mathbb{Z}^+$.

3.2. *Proof of Theorem* 2.3

We first observe that, with $f^{\pm}_{\omega}(x)$ as defined in (2.5),

$$f^{\pm}_{\omega,i}(x) := \big[\log(1 \pm x)\big]^i (1 \pm x)^{\omega} = \frac{\mathrm{d}^i}{\mathrm{d}\omega^i} f^{\pm}_{\omega}(x). \tag{3.11}$$

Consequently, we also have

$$e_n\big[f^{\pm}_{\omega,i}\big] = \frac{\mathrm{d}^i}{\mathrm{d}\omega^i} e_n\big[f^{\pm}_{\omega}\big] \tag{3.12}$$

and

$$e_n\big[f^{+}_{\omega,i}\big] = (-1)^n e_n\big[f^{-}_{\omega,i}\big]. \tag{3.13}$$

The following theorem, which we employ in our proof, can be proved as Theorem 1.1 in [6], p. 323.

Theorem 3.1. *Let $\Re\omega > -1$. Then, with $h = (n+1/2)^{-2}$, for each $i = 1, 2, \ldots,$ we have the asymptotic expansion*

$$(-1)^n e_n\big[f^{+}_{\omega,i}\big] = e_n\big[f^{-}_{\omega,i}\big] \sim \sum_{k=0}^{\infty} \frac{\mathrm{d}^i}{\mathrm{d}\omega^i}\big[c_k(\omega) h^{\omega+k+1/2}\big] \quad \text{as } n \to \infty, \tag{3.14}$$

that is valid uniformly in every strip $-1 < d_1 \leqslant \Re\omega \leqslant d_2 < \infty$ of the ω-plane.

Remark. In other words, the asymptotic expansion of $e_n[f^{\pm}_{\omega,i}]$ is obtained by differentiating that of $e_n[f^{\pm}_{\omega}]$ term by term i times. Recall that the functions $e_n[f^{\pm}_{\omega}]$ are analytic for $\Re\omega > -1$ and so are the $c_k(\omega)$. Note, however, that even though $c_k(\omega)$ vanish when $\omega \in \mathbb{Z}^+$, $c_k^{(i)}(\omega)$, $i \geqslant 1$, do not have to.

For an arbitrary positive integer m, let

$$\begin{aligned} p(x) &= \sum_{s=0}^{m-1} U_s\big(\log(1-x)\big)(1-x)^{\alpha_s} + \sum_{s=0}^{m-1} V_s\big(\log(1+x)\big)(1+x)^{\beta_s} \\ &= \sum_{s=0}^{m-1}\sum_{i=0}^{u_s} \sigma_{si} f^{-}_{\alpha_s,i}(x) + \sum_{s=0}^{m-1}\sum_{i=0}^{v_s} \tau_{si} f^{+}_{\beta_s,i}(x) \end{aligned} \tag{3.15}$$

and write, as before,

$$f(x) = p(x) + \phi(x); \qquad \phi(x) := f(x) - p(x) \tag{3.16}$$

and

$$e_n[f] = e_n[p] + e_n[\phi]. \tag{3.17}$$

However, this time,

$$e_n[p] = \sum_{s=0}^{m-1} \sum_{i=0}^{u_s} \sigma_{si} e_n\big[f^-_{\alpha_s,i}\big] + \sum_{s=0}^{m-1} \sum_{i=0}^{v_s} \tau_{si} e_n\big[f^+_{\beta_s,i}\big]. \tag{3.18}$$

By Theorem 3.1, this gives

$$\begin{aligned} e_n[p] \sim{}& \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} U_s(D_{\alpha_s})\big[c_k(\alpha_s) h^{\alpha_s+k+1/2}\big] \\ &+ (-1)^n \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} V_s(D_{\beta_s})\big[c_k(\beta_s) h^{\beta_s+k+1/2}\big] \quad \text{as } n \to \infty. \end{aligned} \tag{3.19}$$

To analyze $e_n[\phi]$, we again need to study the differentiability properties of $\phi(x)$ on $[-1,1]$. Clearly, $\phi \in C^\infty(-1,1)$. At $x = \pm 1$, $\phi(x)$ has the asymptotic expansions

$$\begin{aligned} \phi(x) &\sim w^+_m(x) + \sum_{s=m}^{\infty} \sum_{i=0}^{u_s} \sigma_{si} \big[\log(1-x)\big]^i (1-x)^{\alpha_s} \quad \text{as } x \to 1-, \\ \phi(x) &\sim w^-_m(x) + \sum_{s=m}^{\infty} \sum_{i=0}^{v_s} \tau_{si} \big[\log(1+x)\big]^i (1+x)^{\beta_s} \quad \text{as } x \to -1+, \end{aligned} \tag{3.20}$$

with

$$\begin{aligned} w^+_m(x) &= -\sum_{s=0}^{m-1} V_s\big(\log(1+x)\big)(1+x)^{\beta_s}, \\ w^-_m(x) &= -\sum_{s=0}^{m-1} U_s\big(\log(1-x)\big)(1-x)^{\alpha_s}. \end{aligned} \tag{3.21}$$

As was the case in the proof of Theorem 2.2, again $w^+_m(x)$ is infinitely differentiable at $x = 1$ while $w^-_m(x)$ is infinitely differentiable at $x = -1$. By the fourth of the properties of $f(x)$ mentioned in the beginning of Section 2, the asymptotic expansions of $\phi(x)$ in (3.20) can be differentiated termwise as many times as we wish. Then, for every positive integer j, there holds

$$\begin{aligned} \frac{\mathrm{d}^j}{\mathrm{d}x^j}\phi(x) &\sim \frac{\mathrm{d}^j}{\mathrm{d}x^j} w^+_m(x) + \sum_{s=m}^{\infty} \widetilde{U}_s\big(\log(1-x)\big)(1-x)^{\alpha_s - j} \quad \text{as } x \to 1-, \\ \frac{\mathrm{d}^j}{\mathrm{d}x^j}\phi(x) &\sim \frac{\mathrm{d}^j}{\mathrm{d}x^j} w^-_m(x) + \sum_{s=m}^{\infty} \widetilde{V}_s\big(\log(1+x)\big)(1+x)^{\beta_s - j} \quad \text{as } x \to -1+, \end{aligned} \tag{3.22}$$

where $\widetilde{U}_s(y)$ and $\widetilde{V}_s(y)$ are polynomials in y of degree u_s and v_s, respectively. It is easy to see that, in this case too, we have

$$\lim_{x\to 1-}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\phi(x) = -\frac{\mathrm{d}^j w_m^+}{\mathrm{d}x^j}\bigg|_{x=1}, \qquad j=0,1,\ldots,\lceil\Re\alpha_m - 1\rceil,$$
$$\lim_{x\to -1+}\frac{\mathrm{d}^j}{\mathrm{d}x^j}\phi(x) = -\frac{\mathrm{d}^j w_m^-}{\mathrm{d}x^j}\bigg|_{x=-1}, \qquad j=0,1,\ldots,\lceil\Re\beta_m - 1\rceil, \tag{3.23}$$

which also means that $\phi(x)$ has $\lceil\Re\alpha_m - 1\rceil$ continuous derivatives at $x = 1$ and $\lceil\Re\beta_m - 1\rceil$ continuous derivatives at $x = -1$, in addition to being in $C^\infty(-1,1)$. Consequently, $\phi \in C^{\kappa_m}[-1,1]$, where $\kappa_m = \min\{\lceil\Re\alpha_m - 1\rceil, \lceil\Re\beta_m - 1\rceil\}$.

The proof of Theorem 2.3 can now be completed as that of Theorem 2.2. We leave the details to the reader.

4. Further developments

In the preceding sections, we assumed that the function $f(x)$ is infinitely differentiable on $(-1,1)$. However, the proofs of Theorems 2.2 and 2.3 suggest that these theorems can be extended to the case in which the function $f(x)$ is not necessarily in $C^\infty(-1,1)$.

Theorems 4.1 and 4.2 below are extensions of Theorems 2.2 and 2.3, respectively, precisely to this case. In these theorems, we assume that $f(x)$ is exactly as in the first paragraph of Section 2, except that it ceases to be infinitely differentiable at a finite number of points in $(-1,1)$, and that it is in $C^r(-1,1)$ for some nonnegative integer r. Of course, $f(x)$ continues to be infinitely differentiable in the open intervals $(-1,-1+\eta)$ and $(1-\eta,1)$, where η is sufficiently small and, in addition, as $x \to \pm 1$, $f(x)$ has the asymptotic expansions given in (2.1), with (2.2)–(2.4). Below, we adopt the notation of Sections 2 and 3.

Theorem 4.1. *Let $f(x)$ be as in the second paragraph of this section with the same notation, $U_s(y) = A_s \neq 0$ and $V_s(y) = B_s \neq 0$ being constant polynomials for all s. Let m_- and m_+ be the smallest integers for which*

$$r < \Re\alpha_{m_-} \quad \textit{and} \quad r < \Re\beta_{m_+}. \tag{4.1}$$

Then, with $h = (n+1/2)^{-2}$ and $\mathbb{Z}^+ = \{0,1,2,\ldots\}$, there holds

$$e_n[f] = \sum_{\substack{s=0\\ \alpha_s\notin\mathbb{Z}^+}}^{m_- -1}\ \sum_{k=0}^{\lceil r/2-\Re\alpha_s-1\rceil} A_s c_k(\alpha_s) h^{\alpha_s+k+1/2}$$
$$+ (-1)^n \sum_{\substack{s=0\\ \beta_s\notin\mathbb{Z}^+}}^{m_+ -1}\ \sum_{k=0}^{\lceil r/2-\Re\beta_s-1\rceil} B_s c_k(\beta_s) h^{\beta_s+k+1/2} + \mathrm{O}\big(h^{r/2-1/2}\big) \quad \textit{as } n\to\infty. \tag{4.2}$$

Theorem 4.2. *Let $f(x)$ be as in the second paragraph of this section with the same notation, $U_s(y)$ and $V_s(y)$ being polynomials in y of degree u_s and v_s, respectively. Let m_- and m_+ be the smallest integers for which*

$$r < \Re\alpha_{m_-} \quad \textit{and} \quad r < \Re\beta_{m_+}. \tag{4.3}$$

Then, with $h = (n+1/2)^{-2}$, there holds

$$\begin{aligned} e_n[f] = {} & \sum_{s=0}^{m_- -1} \sum_{k=0}^{\lceil r/2-\Re\alpha_s-1\rceil} U_s(D_{\alpha_s})\bigl[c_k(\alpha_s)h^{\alpha_s+k+1/2}\bigr] \\ & + \sum_{s=0}^{m_+ -1} \sum_{k=0}^{\lceil r/2-\Re\beta_s-1\rceil} V_s(D_{\beta_s})\bigl[c_k(\beta_s)h^{\beta_s+k+1/2}\bigr] + \mathrm{O}\bigl(h^{r/2-1/2}\bigr) \quad \textit{as } n\to\infty. \end{aligned} \tag{4.4}$$

The proof of Theorem 4.1 is achieved precisely as that of Theorem 2.2 by modifying $p(x)$ in (3.1) as in

$$p(x) = \sum_{s=0}^{m_- -1} A_s(1-x)^{\alpha_s} + \sum_{s=0}^{m_+ -1} B_s(1+x)^{\beta_s}. \tag{4.5}$$

Similarly, the proof of Theorem 4.2 is achieved precisely as that of Theorem 2.3 by modifying $p(x)$ in (3.15) as in

$$p(x) = \sum_{s=0}^{m_- -1} U_s\bigl(\log(1-x)\bigr)(1-x)^{\alpha_s} + \sum_{s=0}^{m_+ -1} V_s\bigl(\log(1+x)\bigr)(1+x)^{\beta_s}. \tag{4.6}$$

In both cases, the functions $\phi(x) := f(x) - p(x)$ are in $C^r[-1,1]$ so that $e_n[\phi] = \mathrm{O}(h^{r/2-1/2})$ as $n\to\infty$. We leave the details to the reader.

Note that the summations over the α_s (the β_s) in (4.2) and (4.4) are empty in case $\Re\alpha_0 \geqslant r/2-1$ ($\Re\beta_0 \geqslant r/2-1$).

Acknowledgement

This research was supported in part by the United States–Israel Binational Science Foundation Grant No. 2004353.

References

[1] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
[2] T.M. Apostol, *Mathematical Analysis*, Addison–Wesley, London, 1957.
[3] P.J. Davis, *Interpolation and Approximation*, Dover, New York, 1975.
[4] G. Freud, *Orthogonal Polynomials*, Pergamon Press, New York, 1971.
[5] G.G. Lorentz, *Approximation of Functions*, Chelsea, New York, 1986.

[6] F.W.J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.
[7] A. Sidi, *Practical Extrapolation Methods: Theory and Applications*, Cambridge Monographs on Applied and Computational Mathematics, Vol. 10, Cambridge University Press, Cambridge, 2003.
[8] A. Sidi, Asymptotic expansions of Gauss–Legendre quadrature rules for integrals with endpoint singularities, *Math. Comp.* **78** (2009), 241–253.
[9] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. 23, Amer. Math. Soc., Providence, RI, 1939.