



A de Montessus Type Convergence Study of a Least-Squares Vector-Valued Rational Interpolation Procedure II

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Abstract. We continue our study of convergence of IMPE, one of the vector-valued rational interpolation procedures proposed by the author in a recent paper, in the context of vector-valued meromorphic functions with simple poles. So far, this study has been carried out in the presence of corresponding residues that are mutually orthogonal. In the present work, we continue to study IMPE in the same context, but in the presence of corresponding residues that are not necessarily orthogonal. Choosing the interpolation points appropriately, we derive de Montessus type convergence results for the interpolants and König type results for the poles and residues.

Keywords. Vector-valued rational interpolation, Hermite interpolation, Newton interpolation formula, de Montessus Theorem, König Theorem.

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1. Introduction

In a recent work [15], we presented three different kinds of vector-valued rational interpolation procedures. These were modeled after some rational approximation procedures from the MacLaurin series of vector-valued functions developed in Sidi [13], which in turn had their origin in vector extrapolation methods. Vector extrapolation methods are used for accelerating the convergence of certain kinds of vector sequences, such as those produced by fixed-point iterative methods on linear and non-linear systems of algebraic equations. Some of the algebraic properties of these interpolants were already mentioned in [15], and their study was continued in another paper [16] by the author.

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All three methods produce two-dimensional arrays of rational functions

$$R_{p,k}(z) = \frac{U_{p,k}(z)}{V_{p,k}(z)},$$

where $U_{p,k}(z)$ is a vector-valued polynomial of degree at most $p-1$, while $V_{p,k}(z)$ is a scalar-valued polynomial of degree k . In all three methods, the $R_{p,k}(z)$ interpolate $F(z)$ at p points counting multiplicities. The methods differ only in the way their denominators $V_{p,k}(z)$ are determined. We can order the approximations such that the sequence $\{R_{p,k}(z)\}_{p=1}^{\infty}$ form the k th row in the table.

Two of these procedures, namely, those denoted IMMPE and IMPE, were studied in the context of meromorphic vector-valued functions by the author recently: IMMPE was studied in [17] for functions having simple poles. IMPE was studied in [18] for functions having simple poles and mutually orthogonal residues, and the orthogonality of the residues enabled the author to employ the techniques of [17] successfully with minor changes.

In the present work, we continue to study IMPE. As in [18], we assume that the functions being interpolated are meromorphic with simple poles, but we do not assume that the corresponding residues are mutually orthogonal. The techniques we use to tackle this more general situation are based on those we used in [17] and [18], but are considerably more involved.

As the definition, construction, and algebraic properties of IMPE have been reviewed in [18], we do not go into the details of these topics. We give a very brief summary of them in the next section, and refer the reader to [15, 16, 18] for details. We also use the next section to set part of the notation that we use throughout.

We first consider the application of IMPE to vector-valued rational functions $F(z)$. In Section 3, we derive a closed-form expression for the error when the function $F(z)$ being interpolated is rational with simple poles. The main results of this section are Theorems 3.2–3.4, which form the starting point of the convergence analysis in the subsequent sections. In Section 4, we present the choice of the points of interpolation and its consequences. Starting with the developments of Sections 3 and 4, in Section 5, we present a detailed convergence theory, concerning vector-valued rational functions $F(z)$ with simple poles, for sequences of interpolants $R_{p,k}(z)$ whose denominators are of a fixed degree that may be much smaller than the number of poles of $F(z)$, while the number of interpolation conditions (hence the degree of the numerators) tends to infinity. This theory provides us with a de Montessus [10] type theorem (Theorem 5.4) concerning the convergence of $R_{p,k}(z)$ as $p \rightarrow \infty$, and König [9] type theorems (Theorems 5.2 and 5.3) concerning the denominator polynomials $V_{p,k}(z)$ and their zeros as $p \rightarrow \infty$.¹ The results of Section 5 show that rational interpolation with

¹Actually, König's Theorem concerns only the $[n/1]$ Padé approximants as $n \rightarrow \infty$. The generalization of König's theorem concerning the convergence of the denominator polynomials

a small and fixed number of poles can help approximate a rational function $F(z)$ that has a large number of poles very accurately in the largest possible region depending on the location of the poles of $F(z)$ and the number of poles of the interpolants being considered.

Finally, Section 6 is concerned with the extension of the results of Section 5 to functions that are meromorphic in some domain of the complex plane but are not necessarily rational.

The following conclusions are drawn from the results of [17, 18] and the present work, as IMMPE and IMPE are being applied to meromorphic vector-valued functions $F(z)$:

- (i) IMPE and IMMPE provide the same rates of convergence for the interpolants.
- (ii) When the residues of $F(z)$ are not mutually orthogonal, IMPE and IMMPE provide the same rates of convergence for the denominator polynomials and poles of the interpolants as well.
- (iii) When the residues of $F(z)$ are mutually orthogonal, IMPE produces twice as fast convergence for the denominator polynomials and poles as IMMPE.
- (iv) The error formula for IMPE obtained in the present work is valid in the presence of both orthogonal and non-orthogonal residues, and it reduces precisely to that of [18] when the residues of $F(z)$ are mutually orthogonal. This is not true for the error formula pertaining to poles, however; poles related to orthogonal residues seem to have a special and more favorable convergence property.

Our results are in the spirit of those given by Saff [11] for the scalar rational interpolation problem and by Graves-Morris and Saff [4, 5, 6, 7] for vector-valued rational interpolants and vector-valued Padé approximants: the conditions imposed on the points of interpolation in our case are exactly those of [11, 4, 6], and, when expressed as a p th root asymptotic result, our de Montessus type convergence result for $R_{p,k}(z)$ as $p \rightarrow \infty$ is analogous to those of [11, 4, 6]. We are also aware of a König type result [concerning the denominator of $R_{p,k}(z)$] in [4, Eq. (2.10)]. Our method of interpolation (that is, IMPE) is different from those in [4, 6]. So are our proofs; they employ linear algebra techniques that are analogous to those developed in Sidi, Ford, and Smith [19] and used in Sidi [12] in the study of Padé approximants. In addition, the techniques we use here enable us to obtain optimally refined results in the form of asymptotic expansions and asymptotic equalities.

of the $[n/k]$ Padé approximants with arbitrary k is already contained in the arguments given in [10]. For different treatments concerning this generalization, see Golomb [2], Gragg and Householder [3], Householder [8], and Sidi [12], for example. In addition, rates of convergence of the poles of the $[n/k]$ Padé approximants as $n \rightarrow \infty$ are provided in [12]. For a summary of this subject, see Sidi [14, Cha. 17, Sect. 17.8].

2. Review of algebraic structure of IMPE

Let $F(z)$ be a vector-valued function such that $F: \mathbb{C} \rightarrow \mathbb{C}^N$. Assume that $F(z)$ is defined on a bounded open set $\Omega \subset \mathbb{C}$ and consider the problem of interpolating $F(z)$ at the points ξ_1, ξ_2, \dots , in this set. We do not assume that the ξ_i are necessarily distinct; thus we allow interpolation in the sense of Hermite. See [15] and [16].

First, we define the scalar polynomials $\psi_{m,n}(z)$ via

$$(2.1) \quad \psi_{m,n}(z) = \prod_{r=m}^n (z - \xi_r), \quad n \geq m \geq 1; \quad \psi_{m,m-1}(z) = 1, \quad m \geq 1.$$

Next, we define the vectors $D_{m,n}$ via

$$(2.2) \quad D_{m,n} = F[\xi_m, \xi_{m+1}, \dots, \xi_n], \quad n \geq m,$$

where $F[\xi_i] = F(\xi_i)$ and $F[\xi_r, \xi_{r+1}, \dots, \xi_{r+s}]$ is the divided difference of order s of $F(z)$ over the set of points $\{\xi_r, \xi_{r+1}, \dots, \xi_{r+s}\}$. With these, $G_{m,n}(z)$, the vector-valued polynomial (of degree at most $n - m$) that interpolates $F(z)$ at the points $\xi_m, \xi_{m+1}, \dots, \xi_n$ in the sense of Hermite, has the Newtonian form

$$(2.3) \quad G_{m,n}(z) = \sum_{i=m}^n D_{m,i} \psi_{m,i-1}(z).$$

For divided differences and the Newton interpolation formula, see, for example, Atkinson [1] and Stoer and Bulirsch [20].

The vector-valued rational interpolants to the function $F(z)$ developed in [15] are all of the general form

$$(2.4) \quad R_{p,k}(z) = \frac{U_{p,k}(z)}{V_{p,k}(z)} = \frac{\sum_{j=0}^k c_j \psi_{1,j}(z) G_{j+1,p}(z)}{\sum_{j=0}^k c_j \psi_{1,j}(z)},$$

where p and k are arbitrary positive integers, and c_0, c_1, \dots, c_k are complex scalars. For IMPE, the c_j are defined as the solution to the linear least-squares problem

$$(2.5) \quad \min_{c_0, c_1, \dots, c_{k-1}} \left\| \sum_{j=0}^k c_j D_{j+1,p+1} \right\|, \quad \text{subject to } c_k = 1.$$

Here $\|\cdot\|$ is a vector l_2 -norm that is induced by some inner product (\cdot, \cdot) . That is, for any vector $x \in \mathbb{C}^N$, we have $\|x\| = \sqrt{(x, x)}$. We also define this inner product such that, for arbitrary $x, y \in \mathbb{C}^N$ and $\alpha, \beta \in \mathbb{C}$, we have $(\alpha x, \beta y) = \bar{\alpha}\beta(x, y)$.

Note that the c_j are determined by the function values $F(\xi_i)$, $1 \leq i \leq p + 1$, while $R_{p,k}(\xi_i) = F(\xi_i)$, $1 \leq i \leq p$.

The denominator polynomial $V_{p,k}(z)$ of $R_{p,k}(z)$ is a symmetric function of ξ_i , $i = 1, \dots, p + 1$, and $R_{p,k}(z)$ itself is a symmetric function of ξ_i , $i = 1, \dots, p$, provided $V_{p,k}(\xi_i) \neq 0$, $i = 1, \dots, p$.

Provided a unique solution to these equations exists, $R_{p,k}(z)$ has a determinantal representation given as in

$$(2.6) \quad R_{p,k}(z) = \frac{P(z)}{Q(z)} = \frac{\begin{vmatrix} \psi_{1,0}(z) G_{1,p}(z) & \psi_{1,1}(z) G_{2,p}(z) & \cdots & \psi_{1,k}(z) G_{k+1,p}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}{\begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \cdots & \psi_{1,k}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}},$$

where

$$(2.7) \quad u_{i,j} = (D_{i,p+1}, D_{j+1,p+1}).$$

Here, the numerator determinant $P(z)$ is vector-valued and is defined by its expansion with respect to its first row. That is, if M_j is the cofactor of the term $\psi_{1,j}(z)$ in the denominator determinant $Q(z)$, then

$$(2.8) \quad R_{p,k}(z) = \frac{\sum_{j=0}^k M_j \psi_{1,j}(z) G_{j+1,p}(z)}{\sum_{j=0}^k M_j \psi_{1,j}(z)}.$$

A unique solution for the c_j exists provided $M_k \neq 0$. This also guarantees the uniqueness of $R_{p,k}(z)$ provided $V_{p,k}(\xi_i) \neq 0, i = 1, \dots, p$. For $M_k \neq 0$ to be true, it is necessary and sufficient that the vectors $D_{1,p+1}, D_{2,p+1}, \dots, D_{k,p+1}$ be linearly independent. It is shown in [16, Sec. 2 & 5] that this holds when $F(z)$ is a vector-valued rational function of the form

$$(2.9) \quad F(z) = u(z) + \sum_{s=1}^{\sigma} \sum_{j=1}^{r_s} \frac{v_{sj}}{(z - z_s)^j},$$

where $u(z)$ is an arbitrary vector-valued polynomial, the vectors $v_{sj} \in \mathbb{C}^N, 1 \leq j \leq r_s, 1 \leq s \leq \sigma$, are linearly independent, z_1, \dots, z_{σ} are distinct points in \mathbb{C} and $k \leq \sum_{s=1}^{\sigma} r_s \leq N$.

The denominator polynomial $V_{p,k}(z)$ of the IMPE interpolant $R_{p,k}(z)$ is a symmetric function of all the ξ_i used to construct it, namely, of $\xi_1, \xi_2, \dots, \xi_{p+1}$, while $R_{p,k}(z)$ itself is a symmetric function of the points of interpolation, namely, of $\xi_1, \xi_2, \dots, \xi_p$. That is, $R_{p,k}(z)$ is independent of the order of the interpolation points ξ_1, \dots, ξ_p . See [16, Lem. 3.4 and Thm. 3.5].

Let $F(z)$ be a vector-valued rational function of the form $F(z) = \tilde{U}(z)/\tilde{V}(z)$, where $\tilde{U}(z)$ is a vector-valued polynomial of degree at most $p - 1$ and $\tilde{V}(z)$ is a scalar polynomial of degree exactly k . Provided the cofactor M_k in (2.8) is

non-zero and $V_{p,k}(\xi_i) \neq 0, i = 1, \dots, p$, holds, IMPE reproduces $F(z)$, that is $R_{p,k}(z) \equiv F(z)$. See [16, Thm. 4.1].

Finally, the error in $R_{p,k}(z)$ has the determinantal representation

$$(2.10) \quad F(z) - R_{p,k}(z) = \frac{\Delta(z)}{Q(z)},$$

where $Q(z)$ is the denominator determinant of $R_{p,k}(z)$ in (2.6) and

$$(2.11) \quad \Delta(z) = \begin{vmatrix} \Delta_0(z) & \Delta_1(z) & \cdots & \Delta_k(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}, \quad \Delta_j(z) = \psi_{1,j}(z) [F(z) - G_{j+1,p}(z)],$$

3. IMPE error formula for $F(z)$ a vector-valued rational function

As in [17] and [18], we start our study of IMPE for the case in which the function $F(z)$ is a vector-valued rational function with simple poles, namely,

$$(3.1) \quad F(z) = u(z) + \sum_{s=1}^{\mu} \frac{v_s}{z - z_s},$$

where

- (i) $u(z)$ is an arbitrary vector-valued polynomial, z_1, \dots, z_{μ} are distinct complex numbers, and
- (ii) the corresponding residues, namely, v_1, \dots, v_{μ} , are constant vectors in \mathbb{C}^N , which we assume to be linearly independent, and
- (iii) the v_i are not assumed to be mutually orthogonal with respect to the inner product used in defining IMPE.

Clearly, $\mu \leq N$.

Example. Let A be an $N \times N$ diagonalizable matrix with eigenpairs $(\lambda_i, w_i), i = 1, \dots, N$, and let b be an N -vector, and consider the solution to the linear system of equations $(I - zA)x = b$. Because w_1, \dots, w_N span \mathbb{C}^N , we have $b = \sum_{i=1}^N \alpha_i w_i$ for some scalars α_i . Then, for $z \neq \lambda_i^{-1}, i = 1, \dots, N$, the solution to $(I - zA)x = b$ has the representation

$$x = F(z) = (I - zA)^{-1}b = \sum_{i=1}^N \frac{\alpha_i w_i}{1 - z\lambda_i}.$$

Thus, $F(z)$ is precisely of the form described in (3.1). In case A is singular, $u(z) \equiv v_0$, where v_0 is either an eigenvector of A corresponding to its zero

eigenvalue or $v_0 = 0$; therefore, $u(z)$ is a constant polynomial. If A is non-singular, $u(z) \equiv 0$. Whether A is singular or not, the z_s in (3.1) are the reciprocals of some or all of the distinct non-zero λ_i (hence $\mu \leq N$), and, for each s , v_s is a linear combination of the eigenvectors corresponding to the eigenvalue z_s^{-1} , hence is itself an eigenvector of A , that is, $Av_s = z_s^{-1}v_s$, $s = 1, \dots, \mu$.

We now recall some technical tools that were used in [17] and will be used throughout this work as well. The following lemma is the same as Lemma 3.4 in [17], with the exception of (3.4), which can be proved by invoking (3.1) in $(D_{m',n'}, D_{m,n})$. Both parts of this lemma can be proved with the help of the result

$$(3.2) \quad \omega_a(z) = (z - a)^{-1} \quad \Rightarrow \quad \omega_a[\xi_m, \dots, \xi_n] = -\frac{1}{\psi_{m,n}(a)} = -\frac{\psi_{1,m-1}(a)}{\psi_{1,n}(a)},$$

and by recalling that $g[x_0, x_1, \dots, x_q] = 0$ whenever $g(x)$ is a polynomial in x of degree less than q .

Lemma 3.1. *Let $F(z)$ be given as in (3.1). Let $n - m > \deg(u)$. Then, whether the ξ_i are distinct or not, the following are true:*

(i) $D_{m,n} = F[\xi_m, \dots, \xi_n]$ is given as in

$$(3.3) \quad D_{m,n} = -\sum_{s=1}^{\mu} \frac{v_s}{\psi_{m,n}(z_s)} = -\sum_{s=1}^{\mu} v_s \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)}.$$

Therefore, we also have

$$(3.4) \quad (D_{m',n'}, D_{m,n}) = \sum_{r=1}^{\mu} \sum_{s=1}^{\mu} \alpha_{r,s} \frac{\overline{\psi_{1,m'-1}(z_r)} \psi_{1,m-1}(z_s)}{\psi_{1,n'}(z_r) \psi_{1,n}(z_s)}, \quad \alpha_{r,s} = (v_r, v_s).$$

(ii) $F(z) - G_{m,n}(z) = F[z, \xi_m, \dots, \xi_n] \psi_{m,n}(z)$ is given as in

$$(3.5) \quad F(z) - G_{m,n}(z) = \psi_{m,n}(z) \sum_{s=1}^{\mu} e_s(z) \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)}, \quad e_s(z) = \frac{v_s}{z - z_s}.$$

We start with the analysis of $Q(z)$, the denominator determinant of $F(z) - R_{p,k}(z)$ and of $R_{p,k}(z)$ in equations (2.10) and (2.6), respectively. The following theorem gives a closed form expression for $Q(z)$ in simple terms, and is the analogue of [17, Thm. 3.6].

Theorem 3.2. *Let $F(z)$ be the vector-valued rational function in (3.1), and precisely as described in the first paragraph of this section, with the notation therein. Let also*

$$(3.6) \quad \Psi_p(z) = \psi_{1,p+1}(z).$$

Then, with $p > k + \deg(u)$, the denominator determinant $Q(z)$ in (2.6) and (2.10) has the expansion

$$(3.7) \quad Q(z) = \sum_{1 \leq r_1 < r_2 < \dots < r_k \leq \mu} \overline{V(z_{r_1}, z_{r_2}, \dots, z_{r_k})} \left[\prod_{i=1}^k \overline{\Psi_p(z_{r_i})} \right]^{-1} \\ \times \sum_{1 \leq s_1 < s_2 < \dots < s_k \leq \mu} T_{s_1, \dots, s_k}^{r_1, \dots, r_k} V(z, z_{s_1}, z_{s_2}, \dots, z_{s_k}) \left[\prod_{i=1}^k \overline{\Psi_p(z_{s_i})} \right]^{-1},$$

where

$$(3.8) \quad T_{s_1, \dots, s_k}^{r_1, \dots, r_k} = \begin{vmatrix} \alpha_{r_1, s_1} & \alpha_{r_1, s_2} & \dots & \alpha_{r_1, s_k} \\ \alpha_{r_2, s_1} & \alpha_{r_2, s_2} & \dots & \alpha_{r_2, s_k} \\ \vdots & \vdots & & \vdots \\ \alpha_{r_k, s_1} & \alpha_{r_k, s_2} & \dots & \alpha_{r_k, s_k} \end{vmatrix}, \quad \alpha_{r, s} = (v_r, v_s),$$

and $V(x_0, x_1, \dots, x_n)$ is the Vandermonde determinant defined by

$$(3.9) \quad V(x_0, x_1, \dots, x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} = \prod_{0 \leq i < j \leq n} (x_j - x_i).$$

Proof. Taking $p > k + \deg(u)$, and invoking (3.4) in the determinant representation of $Q(z)$ in (2.6), we obtain

$$Q(z) = \begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \dots & \psi_{1,k}(z) \\ \sum_{r_1, s_1} \alpha_{r_1, s_1} \frac{\overline{\psi_{1,0}(z_{r_1})}}{\overline{\Psi_p(z_{r_1})}} \frac{\psi_{1,0}(z_{s_1})}{\Psi_p(z_{s_1})} & \sum_{r_1, s_1} \alpha_{r_1, s_1} \frac{\overline{\psi_{1,0}(z_{r_1})}}{\overline{\Psi_p(z_{r_1})}} \frac{\psi_{1,1}(z_{s_1})}{\Psi_p(z_{s_1})} & \dots & \sum_{r_1, s_1} \alpha_{r_1, s_1} \frac{\overline{\psi_{1,0}(z_{r_1})}}{\overline{\Psi_p(z_{r_1})}} \frac{\psi_{1,k}(z_{s_1})}{\Psi_p(z_{s_1})} \\ \sum_{r_2, s_2} \alpha_{r_2, s_2} \frac{\overline{\psi_{1,1}(z_{r_2})}}{\overline{\Psi_p(z_{r_2})}} \frac{\psi_{1,0}(z_{s_2})}{\Psi_p(z_{s_2})} & \sum_{r_2, s_2} \alpha_{r_2, s_2} \frac{\overline{\psi_{1,1}(z_{r_2})}}{\overline{\Psi_p(z_{r_2})}} \frac{\psi_{1,1}(z_{s_2})}{\Psi_p(z_{s_2})} & \dots & \sum_{r_2, s_2} \alpha_{r_2, s_2} \frac{\overline{\psi_{1,1}(z_{r_2})}}{\overline{\Psi_p(z_{r_2})}} \frac{\psi_{1,k}(z_{s_2})}{\Psi_p(z_{s_2})} \\ \vdots & \vdots & & \vdots \\ \sum_{r_k, s_k} \alpha_{r_k, s_k} \frac{\overline{\psi_{1, k-1}(z_{r_k})}}{\overline{\Psi_p(z_{r_k})}} \frac{\psi_{1,0}(z_{s_k})}{\Psi_p(z_{s_k})} & \sum_{r_k, s_k} \alpha_{r_k, s_k} \frac{\overline{\psi_{1, k-1}(z_{r_k})}}{\overline{\Psi_p(z_{r_k})}} \frac{\psi_{1,1}(z_{s_k})}{\Psi_p(z_{s_k})} & \dots & \sum_{r_k, s_k} \alpha_{r_k, s_k} \frac{\overline{\psi_{1, k-1}(z_{r_k})}}{\overline{\Psi_p(z_{r_k})}} \frac{\psi_{1,k}(z_{s_k})}{\Psi_p(z_{s_k})} \end{vmatrix}.$$

Because determinants are multilinear in their rows (and columns), we can take the summations outside. Following that, we take out the common factors from each row of the remaining determinant. We obtain

$$(3.10) \quad Q(z) = \sum_{r_1, s_1} \sum_{r_2, s_2} \dots \sum_{r_k, s_k} \left(\prod_{i=1}^k \alpha_{r_i, s_i} \right) \left[\prod_{i=1}^k \frac{\overline{\psi_{1, i-1}(z_{r_i})}}{\overline{\Psi_p(z_{r_i})}} \right] \\ \times \left[\prod_{i=1}^k \overline{\Psi_p(z_{s_i})} \right]^{-1} X(z, z_{s_1}, z_{s_2}, \dots, z_{s_k}),$$

where

$$(3.11) \quad X(y_0, y_1, y_2, \dots, y_n) = \begin{vmatrix} \psi_{1,0}(y_0) & \psi_{1,1}(y_0) & \cdots & \psi_{1,k}(y_0) \\ \psi_{1,0}(y_1) & \psi_{1,1}(y_1) & \cdots & \psi_{1,k}(y_1) \\ \vdots & \vdots & & \vdots \\ \psi_{1,0}(y_k) & \psi_{1,1}(y_k) & \cdots & \psi_{1,k}(y_k) \end{vmatrix}.$$

Now, since $\psi_{1,r}(z)$ is a monic polynomial in z of degree r , [17, Lem. 3.2] applies, and we also have

$$(3.12) \quad X(y_0, y_1, \dots, y_n) = V(y_0, y_1, \dots, y_n) = \prod_{0 \leq i < j \leq n} (y_j - y_i)$$

is the Vandermonde determinant. Consequently,

$$(3.13) \quad X(z, z_{s_1}, z_{s_2}, \dots, z_{s_k}) = V(z, z_{s_1}, z_{s_2}, \dots, z_{s_k}).$$

Since, by (3.11), the product

$$\left[\prod_{i=1}^k \Psi_p(z_{s_i}) \right]^{-1} X(z, z_{s_1}, z_{s_2}, \dots, z_{s_k})$$

is odd under an interchange of any two of the indices s_1, \dots, s_k , [17, Lem. 3.1] (originally, given in [19]) applies to the summation $\sum_{s_1} \sum_{s_2} \cdots \sum_{s_k}$, and we obtain

$$(3.14) \quad Q(z) = \sum_{r_1} \sum_{r_2} \cdots \sum_{r_k} \left[\prod_{i=1}^k \frac{\psi_{1,i-1}(z_{r_i})}{\Psi_p(z_{r_i})} \right] \\ \times \sum_{1 \leq s_1 < s_2 < \cdots < s_k \leq \mu} T_{s_1, \dots, s_k}^{r_1, \dots, r_k} V(z, z_{s_1}, z_{s_2}, \dots, z_{s_k}) \left[\prod_{i=1}^k \Psi_p(z_{s_i}) \right]^{-1}.$$

Let us rewrite this in the form

$$(3.15) \quad Q(z) = \sum_{1 \leq s_1 < s_2 < \cdots < s_k \leq \mu} V(z, z_{s_1}, z_{s_2}, \dots, z_{s_k}) \left[\prod_{i=1}^k \Psi_p(z_{s_i}) \right]^{-1} \\ \times \sum_{r_1} \sum_{r_2} \cdots \sum_{r_k} T_{s_1, \dots, s_k}^{r_1, \dots, r_k} \left[\prod_{i=1}^k \Psi_p(z_{r_i}) \right]^{-1} \left[\prod_{i=1}^k \psi_{1,i-1}(z_{r_i}) \right].$$

Observing, by (3.8), that the product

$$T_{s_1, \dots, s_k}^{r_1, \dots, r_k} \left[\prod_{i=1}^k \Psi_p(z_{r_i}) \right]^{-1}$$

is odd under an interchange of any two of the indices r_1, \dots, r_k , this time, we apply [17, Lem. 3.1] to the summation $\sum_{r_1} \sum_{r_2} \cdots \sum_{r_k}$, to obtain

$$(3.16) \quad Q(z) = \sum_{1 \leq s_1 < s_2 < \cdots < s_k \leq \mu} V(z, z_{s_1}, z_{s_2}, \dots, z_{s_k}) \left[\prod_{i=1}^k \Psi_p(z_{s_i}) \right]^{-1} \\ \times \sum_{1 \leq r_1 < r_2 < \cdots < r_k \leq \mu} T_{s_1, \dots, s_k}^{r_1, \dots, r_k} \left[\prod_{i=1}^k \overline{\Psi_p(z_{r_i})} \right]^{-1} \overline{X(z_{r_1}, \dots, z_{r_k})}.$$

Invoking (3.12) in (3.16), we obtain the result in (3.7). ■

Note that, even though the functions $\psi_{m,n}(z)$ that define $X(y_0, y_1, \dots, y_n)$ in (3.11) depend on the ξ_i , $X(y_0, y_1, \dots, y_n)$ itself is *independent of* the ξ_i . As a result, as is clear from (3.7), $Q(z)$ depends on the ξ_i only via the products $\prod_{i=1}^k \Psi_p(z_{r_i})$ and $\prod_{i=1}^k \Psi_p(z_{s_i})$. This has important implications in the asymptotic behavior of $Q(z)$ and hence of $R_{p,k}(z)$ as $p \rightarrow \infty$, as we shall soon see.

We next turn to $\Delta(z)$, the numerator determinant of $F(z) - R_{p,k}(z)$ in (2.10).

Theorem 3.3. *Let $F(z)$ be the vector-valued rational function in (3.1), and precisely as described in the first paragraph of this section, with the notation therein. With $\alpha_{r,s}$, $e_s(z)$, and $\Psi_p(z)$ as in (3.4), (3.5), and (3.6), respectively, define*

$$(3.17) \quad \widehat{e}_s^{(p)}(z) = e_s(z)(z_s - \xi_{p+1})$$

and

$$(3.18) \quad \widehat{T}_{s_0, s_1, \dots, s_k}^{r_1, \dots, r_k}(z; p) = \begin{vmatrix} \widehat{e}_{s_0}^{(p)}(z) & \widehat{e}_{s_1}^{(p)}(z) & \cdots & \widehat{e}_{s_k}^{(p)}(z) \\ \alpha_{r_1, s_0} & \alpha_{r_1, s_1} & \cdots & \alpha_{r_1, s_k} \\ \alpha_{r_2, s_0} & \alpha_{r_2, s_1} & \cdots & \alpha_{r_2, s_k} \\ \vdots & \vdots & & \vdots \\ \alpha_{r_k, s_0} & \alpha_{r_k, s_1} & \cdots & \alpha_{r_k, s_k} \end{vmatrix}.$$

Then, with $p > k + \deg(u)$, we have

$$(3.19) \quad \frac{\Delta(z)}{\psi_{1,p}(z)} = \sum_{1 \leq r_1 < r_2 < \cdots < r_k \leq \mu} \overline{V(z_{r_1}, z_{r_2}, \dots, z_{r_k})} \left[\prod_{i=1}^k \overline{\Psi_p(z_{r_i})} \right]^{-1} \\ \times \sum_{1 \leq s_0 < s_1 < \cdots < s_k \leq \mu} \widehat{T}_{s_0, s_1, \dots, s_k}^{r_1, \dots, r_k}(z; p) V(z_{s_0}, z_{s_1}, \dots, z_{s_k}) \left[\prod_{i=0}^k \Psi_p(z_{s_i}) \right]^{-1}.$$

Proof. Taking $p > k + \deg(u)$, and invoking (3.5) in (2.11), we first have

$$\begin{aligned}
 (3.20) \quad \Delta_j(z) &= \psi_{1,j}(z) \left[\psi_{j+1,p}(z) \sum_{s=1}^{\mu} e_s(z) \frac{\psi_{1,j}(z_s)}{\psi_{1,p}(z_s)} \right] \\
 &= \psi_{1,p}(z) \sum_{s=1}^{\mu} \widehat{e}_s^{(p)}(z) \frac{\psi_{1,j}(z_s)}{\Psi_p(z_s)}.
 \end{aligned}$$

Substituting (3.20) and (3.4) in the determinant $\Delta(z)$ of (2.11), and factoring out $\psi_{1,p}(z)$ from the first row, we have

$$(3.21) \quad \Delta(z) = \psi_{1,p}(z)W(z),$$

where

$$W(z) = \begin{vmatrix} \sum_{s_0} \widehat{e}_{s_0}^{(p)}(z) \frac{\psi_{1,0}(z_{s_0})}{\Psi_p(z_{s_0})} & \sum_{s_0} \widehat{e}_{s_0}^{(p)}(z) \frac{\psi_{1,1}(z_{s_0})}{\Psi_p(z_{s_0})} & \cdots & \sum_{s_0} \widehat{e}_{s_0}^{(p)}(z) \frac{\psi_{1,k}(z_{s_0})}{\Psi_p(z_{s_0})} \\ \sum_{r_1,s_1} \alpha_{r_1,s_1} \frac{\overline{\psi_{1,0}(z_{r_1})}}{\overline{\Psi_p(z_{r_1})}} \frac{\psi_{1,0}(z_{s_1})}{\Psi_p(z_{s_1})} & \sum_{r_1,s_1} \alpha_{r_1,s_1} \frac{\overline{\psi_{1,0}(z_{r_1})}}{\overline{\Psi_p(z_{r_1})}} \frac{\psi_{1,1}(z_{s_1})}{\Psi_p(z_{s_1})} & \cdots & \sum_{r_1,s_1} \alpha_{r_1,s_1} \frac{\overline{\psi_{1,0}(z_{r_1})}}{\overline{\Psi_p(z_{r_1})}} \frac{\psi_{1,k}(z_{s_1})}{\Psi_p(z_{s_1})} \\ \sum_{r_2,s_2} \alpha_{r_2,s_2} \frac{\overline{\psi_{1,1}(z_{r_2})}}{\overline{\Psi_p(z_{r_2})}} \frac{\psi_{1,0}(z_{s_2})}{\Psi_p(z_{s_2})} & \sum_{r_2,s_2} \alpha_{r_2,s_2} \frac{\overline{\psi_{1,1}(z_{r_2})}}{\overline{\Psi_p(z_{r_2})}} \frac{\psi_{1,1}(z_{s_2})}{\Psi_p(z_{s_2})} & \cdots & \sum_{r_2,s_2} \alpha_{r_2,s_2} \frac{\overline{\psi_{1,1}(z_{r_2})}}{\overline{\Psi_p(z_{r_2})}} \frac{\psi_{1,k}(z_{s_2})}{\Psi_p(z_{s_2})} \\ \vdots & \vdots & & \vdots \\ \sum_{r_k,s_k} \alpha_{r_k,s_k} \frac{\overline{\psi_{1,k-1}(z_{r_k})}}{\overline{\Psi_p(z_{r_k})}} \frac{\psi_{1,0}(z_{s_k})}{\Psi_p(z_{s_k})} & \sum_{r_k,s_k} \alpha_{r_k,s_k} \frac{\overline{\psi_{1,k-1}(z_{r_k})}}{\overline{\Psi_p(z_{r_k})}} \frac{\psi_{1,1}(z_{s_k})}{\Psi_p(z_{s_k})} & \cdots & \sum_{r_k,s_k} \alpha_{r_k,s_k} \frac{\overline{\psi_{1,k-1}(z_{r_k})}}{\overline{\Psi_p(z_{r_k})}} \frac{\psi_{1,k}(z_{s_k})}{\Psi_p(z_{s_k})} \end{vmatrix}.$$

Proceeding as in the proof of Theorem 3.2, we first take the summations outside. Following that, we take out the common factors from each row of the remaining determinant. We obtain

$$\begin{aligned}
 (3.22) \quad W(z) &= \sum_{s_0} \sum_{r_1,s_1} \cdots \sum_{r_k,s_k} \widehat{e}_{s_0}^{(p)}(z) \left(\prod_{i=1}^k \alpha_{r_i,s_i} \right) \left[\prod_{i=1}^k \frac{\overline{\psi_{1,i-1}(z_{r_i})}}{\overline{\Psi_p(z_{r_i})}} \right] \\
 &\quad \times \left[\prod_{i=0}^k \Psi_p(z_{s_i}) \right]^{-1} X(z_{s_0}, z_{s_1}, \dots, z_{s_k}),
 \end{aligned}$$

with $X(y_0, y_1, y_2, \dots, y_n)$ as given in (3.11). Since the product

$$\left[\prod_{i=0}^k \Psi_p(z_{s_i}) \right]^{-1} X(z_{s_0}, z_{s_1}, \dots, z_{s_k})$$

is odd under an interchange of any two of the indices s_0, s_1, \dots, s_k , [17, Lem. 3.1] applies to the summation $\sum_{s_0} \sum_{s_1} \dots \sum_{s_k}$. Invoking also (3.12), we obtain

$$(3.23) \quad W(z) = \sum_{r_1} \sum_{r_2} \dots \sum_{r_k} \left[\prod_{i=1}^k \frac{\overline{\psi_{1,i-1}(z_{r_i})}}{\Psi_p(z_{r_i})} \right] \\ \times \sum_{1 \leq s_0 < s_1 < \dots < s_k \leq \mu} \widehat{T}_{s_0, s_1, \dots, s_k}^{r_1, \dots, r_k}(z; p) V(z_{s_0}, z_{s_1}, \dots, z_{s_k}) \left[\prod_{i=0}^k \Psi_p(z_{s_i}) \right]^{-1}.$$

Let us rewrite this in the form

$$(3.24) \quad W(z) = \sum_{1 \leq s_0 < s_1 < \dots < s_k \leq \mu} V(z_{s_0}, z_{s_1}, \dots, z_{s_k}) \left[\prod_{i=0}^k \Psi_p(z_{s_i}) \right]^{-1} \\ \times \sum_{r_1} \sum_{r_2} \dots \sum_{r_k} \widehat{T}_{s_0, s_1, \dots, s_k}^{r_1, \dots, r_k}(z; p) \left[\prod_{i=1}^k \overline{\Psi_p(z_{r_i})} \right]^{-1} \left[\prod_{i=1}^k \overline{\psi_{1,i-1}(z_{r_i})} \right].$$

Observing, by (3.18), that now the product

$$\widehat{T}_{s_0, s_1, \dots, s_k}^{r_1, \dots, r_k}(z; p) \left[\prod_{i=1}^k \overline{\Psi_p(z_{r_i})} \right]^{-1}$$

is odd under an interchange of any two of the indices r_1, \dots, r_k , we apply [17, Lem. 3.1] to the summation $\sum_{r_1} \sum_{r_2} \dots \sum_{r_k}$, to obtain

$$(3.25) \quad W(z) = \sum_{1 \leq s_0 < s_1 < \dots < s_k \leq \mu} V(z_{s_0}, z_{s_1}, \dots, z_{s_k}) \left[\prod_{i=0}^k \Psi_p(z_{s_i}) \right]^{-1} \\ \times \sum_{1 \leq r_1 < r_2 < \dots < r_k \leq \mu} \widehat{T}_{s_0, s_1, \dots, s_k}^{r_1, \dots, r_k}(z; p) \left[\prod_{i=1}^k \overline{\Psi_p(z_{r_i})} \right]^{-1} \overline{X(z_{r_1}, \dots, z_{r_k})}.$$

Invoking (3.12) in (3.25), we obtain the result in (3.19). ■

Finally, combining (3.7) and (3.19) in (2.10), we obtain a simple and elegant expression for $F(z) - R_{p,k}(z)$ when $F(z)$ is a vector-valued rational function with simple poles. This is the subject of the following theorem.

Theorem 3.4. *For the error in $R_{p,k}(z)$, with $p > k + \deg(u)$, we have the closed-form expression*

$$(3.26) \quad F(z) - R_{p,k}(z) = \psi_{1,p}(z) \\ \times \frac{\sum_{1 \leq r_1 < \dots < r_k \leq \mu} \frac{V(z_{r_1}, \dots, z_{r_k})}{\prod_{i=1}^k \overline{\Psi_p(z_{r_i})}} \sum_{1 \leq s_0 < s_1 < \dots < s_k \leq \mu} \widehat{T}_{s_0, s_1, \dots, s_k}^{r_1, \dots, r_k}(z; p) \frac{V(z_{s_0}, z_{s_1}, \dots, z_{s_k})}{\prod_{i=0}^k \Psi_p(z_{s_i})}}{\sum_{1 \leq r_1 < r_2 < \dots < r_k \leq \mu} \frac{V(z_{r_1}, \dots, z_{r_k})}{\prod_{i=1}^k \overline{\Psi_p(z_{r_i})}} \sum_{1 \leq s_1 < \dots < s_k \leq \mu} T_{s_1, \dots, s_k}^{r_1, \dots, r_k} \frac{V(z_{s_1}, \dots, z_{s_k})}{\prod_{i=1}^k \Psi_p(z_{s_i})}}.$$

Remarks.

- When $k = \mu$ in Theorem 3.4, the summation on s_0, s_1, \dots, s_k in the numerator on the right-hand side of (3.26) is empty, and this implies that $R_{p,k}(z) \equiv F(z)$. Thus, this theorem provides an independent proof of the reproducing property of IMPE.
- The error formula (3.26) for IMPE obtained in the present work is valid in the presence of both orthogonal and non-orthogonal residues, and it reduces precisely to that of [18] when the residues of $F(z)$ are mutually orthogonal, as it should.

4. Preliminaries to convergence theory

Let E be a closed and bounded set in the z -plane, whose complement K , including the point at infinity, is connected and has a classical Green’s function $g(z)$ with a pole at infinity, which is continuous on ∂E , the boundary of E , and is zero on ∂E . For each σ , let Γ_σ be the locus $g(z) = \log \sigma$, and let E_σ denote the interior of Γ_σ . Then, E_1 is the interior of E and, for $1 < \sigma < \sigma'$, we have $E \subset E_\sigma \subset E_{\sigma'}$.

For each $p \in \{1, 2, \dots\}$, let

$$(4.1) \quad \Xi_p = \left\{ \xi_1^{(p)}, \xi_2^{(p)}, \dots, \xi_{p+1}^{(p)} \right\}$$

be the set of interpolation points used in constructing the IMPE interpolant $R_{p,k}(z)$. Assume that the sets Ξ_p are such that $\xi_i^{(p)}$ have no limit points in K and

$$(4.2) \quad \lim_{p \rightarrow \infty} \left| \prod_{i=1}^{p+1} (z - \xi_i^{(p)}) \right|^{1/p} = \kappa \Phi(z), \quad \kappa = \text{cap}(E), \quad \Phi(z) = \exp[g(z)],$$

uniformly in z on every compact subset of K , where $\text{cap}(E)$ is the logarithmic capacity of E defined by

$$\text{cap}(E) = \lim_{n \rightarrow \infty} \left(\min_{r \in \mathcal{P}_n} \max_{z \in E} |r(z)| \right)^{1/n}, \quad \mathcal{P}_n = \{r(z) : r \in \Pi_n \text{ and monic}\}.$$

Such sequences $\{\xi_1^{(p)}, \xi_2^{(p)}, \dots, \xi_{p+1}^{(p)}\}$, $p = 1, 2, \dots$, exist, see Walsh [21, p. 74]. Note that, in terms of $\Phi(z)$, the locus Γ_σ is defined by $\Phi(z) = \sigma$ for $\sigma > 1$, while $\partial E = \Gamma_1$ is simply the locus $\Phi(z) = 1$.

Recalling that $\prod_{i=1}^{p+1} (z - \xi_i^{(p)}) = \Psi_p(z)$ (see (3.6)), we can write (4.2) also as in

$$(4.3) \quad \lim_{p \rightarrow \infty} |\Psi_p(z)|^{1/p} = \kappa \Phi(z),$$

uniformly in z on every compact subset of K .

It is clear that if $z' \in \Gamma_{\sigma'}$ and $z'' \in \Gamma_{\sigma''}$ and $1 < \sigma' < \sigma''$, then $\Phi(z') < \Phi(z'')$.

The following lemma that we use in our convergence study later gathers the results of [17, Lem. 4.1–4.3].

Lemma 4.1.

(i) *Let K' be some compact subset of K . Then, for every $\epsilon > 0$, there is an integer p_0 depending only on ϵ , such that*

$$(4.4) \quad [(1 - \epsilon)\kappa\Phi(z)]^p < |\Psi_p(z)| < [(1 + \epsilon)\kappa\Phi(z)]^p \quad \text{for all } z \in K' \text{ and } p > p_0.$$

(ii) *For every $\epsilon > 0$, there is an integer p_0 depending only on ϵ , such that*

$$(4.5) \quad |\Psi_p(z)| < [(1 + \epsilon)\kappa]^p \quad \text{for all } z \in E \text{ and } p > p_0.$$

As a result, we also have that

$$(4.6) \quad \limsup_{p \rightarrow \infty} |\Psi_p(z)|^{1/p} \leq \kappa \quad \text{for all } z \in E.$$

(iii) *Let (a) $z', z'' \in K$ and $\Phi(z') < \Phi(z'')$, or (b) $z' \in E$ and $z'' \in K$. Then*

$$(4.7) \quad \lim_{p \rightarrow \infty} \left| \frac{\Psi_p(z')}{\Psi_p(z'')} \right|^{1/p} = \frac{\Phi(z')}{\Phi(z'')} < 1 \quad \text{in case (a),}$$

$$(4.8) \quad \limsup_{p \rightarrow \infty} \left| \frac{\Psi_p(z')}{\Psi_p(z'')} \right|^{1/p} \leq \frac{1}{\Phi(z'')} < 1 \quad \text{in case (b).}$$

In both cases,

$$(4.9) \quad \lim_{p \rightarrow \infty} \frac{\Psi_p(z')}{\Psi_p(z'')} = 0.$$

The result of (4.4) in part (i) of Lemma 4.1 suggests that $\Psi_p(z)$ behaves practically like $[\kappa\Phi(z)]^p$ as $p \rightarrow \infty$.

5. Convergence theory for rational $F(z)$

In this section, we provide a convergence theory for the sequences $\{R_{p,k}(z)\}_{p=1}^{\infty}$ with $k < \mu$ and fixed, in case $F(z)$ is a vector-valued rational function with simple poles as in (3.1). The theorems that follow can be proved as those given in [17, Sec. 5]. Therefore, also to keep this work short, we only sketch some of the proofs. In what follows, we continue to use the notation of the preceding sections.

Note that, by the reproducing property mentioned at the end of Section 3, for $k = \mu$, $R_{p,k}(z) \equiv F(z)$ for all $p \geq p_0$, where $p_0 - 1$ is the degree of the numerator of $F(z)$, namely, $p_0 - 1 = \mu + \deg(u)$. Also, as we will let $p \rightarrow \infty$ in our analysis, the condition that $p > k + \deg(u)$ is satisfied for all large p . Recall that it is this condition that makes the results of Section 3 possible.

We now turn to $F(z)$ in (3.1). We assume that $F(z)$ is analytic in E . This implies that its poles z_1, \dots, z_μ are all in K . We order the poles of $F(z)$ such that

$$(5.1) \quad \Phi(z_1) \leq \Phi(z_2) \leq \dots \leq \Phi(z_\mu).$$

By (4.7) in part (i) of Lemma 4.1, if z' and z'' are two different poles of $F(z)$, and $\Phi(z') < \Phi(z'')$, then z' and z'' lie on two different loci $\Gamma_{\sigma'}$ and $\Gamma_{\sigma''}$. In addition, $\sigma' < \sigma''$, that is, the set $E_{\sigma'}$ is in the interior of $E_{\sigma''}$.

Lemma 5.1 below plays an important role in the proofs of the results that follow.

Lemma 5.1. *Under the condition that the vectors v_s in (3.1) are linearly independent, with $T_{s_1, \dots, s_k}^{r_1, \dots, r_k}$ as in (3.8), we have*

$$(5.2) \quad T_{1, \dots, k}^{1, \dots, k} > 0.$$

Proof. By (3.8),

$$(5.3) \quad T_{1, \dots, k}^{1, \dots, k} = \begin{vmatrix} (v_1, v_1) & (v_1, v_2) & \cdots & (v_1, v_k) \\ (v_2, v_1) & (v_2, v_2) & \cdots & (v_2, v_k) \\ \vdots & \vdots & \ddots & \vdots \\ (v_k, v_1) & (v_k, v_2) & \cdots & (v_k, v_k) \end{vmatrix}.$$

In words, $T_{1, \dots, k}^{1, \dots, k}$ is the Gram determinant of the vectors v_1, \dots, v_k , hence is positive by the linear independence of these vectors. ■

5.1. Convergence analysis for $V_{p,k}(z)$. We now state a König-type convergence theorem for $V_{p,k}(z)$, the denominator (monic) polynomial of $R_{p,k}(z)$ in (2.4) and another theorem concerning its zeros (equivalently, poles of $R_{p,k}(z)$), assuming that $\Phi(z_k) < \Phi(z_{k+1})$. These results are analogous to, and in the spirit of, the ones given in [12] for denominators of Padé approximants. They are also similar to the corresponding results pertaining to IMMPE given in [17].

Theorem 5.2. *Assume*

$$(5.4) \quad \Phi(z_k) < \Phi(z_{k+1}) = \dots = \Phi(z_{k+r}) < \Phi(z_{k+r+1}),$$

in addition to (5.1). In case $k + r = \mu$, we define $\Phi(z_{k+r+1}) = \infty$. Then,

$$(5.5) \quad Q(z) = (-1)^k T_{1, \dots, k}^{1, \dots, k} \left| \frac{V(z_1, \dots, z_k)}{\prod_{i=1}^k \Psi_p(z_i)} \right|^2 \\ \times \left[S(z) + \mathcal{O} \left(\frac{\Psi_p(z_k)}{\tilde{\Psi}_{p,k}} \right) \right] \quad \text{as } p \rightarrow \infty,$$

uniformly in every compact subset of $\mathbb{C} \setminus \{z_1, z_2, \dots, z_k\}$, where

$$(5.6) \quad \left| \tilde{\Psi}_{p,k} \right| = \min_{1 \leq j \leq r} |\Psi_p(z_{k+j})|,$$

and

$$(5.7) \quad S(z) = \prod_{i=1}^k (z - z_i).$$

Consequently,

$$(5.8) \quad V_{p,k}(z) - S(z) = \mathcal{O}\left(\frac{\Psi_p(z_k)}{\widetilde{\Psi}_{p,k}}\right) \quad \text{as } p \rightarrow \infty,$$

from which we also have

$$(5.9) \quad \limsup_{p \rightarrow \infty} |V_{p,k}(z) - S(z)|^{1/p} \leq \frac{\Phi(z_k)}{\Phi(z_{k+1})} < 1.$$

Proof. By (5.1), (5.4), part (i) of Lemma 4.1, and Lemma 5.1, the largest term in (3.7) is that with the indices

$$(r_1, \dots, r_k) = (s_1, \dots, s_k) = (1, \dots, k).$$

The next largest terms are those with

$$(r_1, \dots, r_k) = (1, \dots, k) \text{ and } (s_1, \dots, s_k) = (1, \dots, k - 1, k + j)$$

and with

$$(r_1, \dots, r_k) = (1, \dots, k - 1, k + j) \text{ and } (s_1, \dots, s_k) = (1, \dots, k),$$

$1 \leq j \leq r$. Obviously, we have

$$\lim_{p \rightarrow \infty} \frac{\Psi_p(z_k)}{\widetilde{\Psi}_{p,k}} = 0.$$

In addition,

$$(5.10) \quad V(z, z_1, \dots, z_k) = (-1)^k V(z_1, \dots, z_k) \prod_{i=1}^k (z - z_i).$$

This completes the proof of (5.5). The result in (5.8) follows from (5.5), and that in (5.9) follows from (5.8) and (4.3). ■

Theorem 5.2 implies that, for all large p , $V_{p,k}(z)$ has precisely k zeros that tend to those of $S(z)$. In the next theorem, we provide the rate of convergence of each of these zeros.

Theorem 5.3. *Under the conditions of Theorem 5.2, $V_{p,k}(z)$ is of degree exactly k . Let us denote its zeros $z_1^{(p)}, \dots, z_k^{(p)}$. Then $\lim_{p \rightarrow \infty} z_m^{(p)} = z_m$, $m = 1, \dots, k$. In addition, we have the refined result*

$$(5.11) \quad z_m^{(p)} - z_m \sim \sum_{j=1}^r C_j^{(m)} \frac{\Psi_p(z_m)}{\Psi_p(z_{k+j})} + \dots \quad \text{as } p \rightarrow \infty,$$

where $C_j^{(m)}$ are scalars independent of p given by

$$(5.12) \quad C_j^{(m)} = (-1)^{k-m} \frac{T_{1,\dots,m-1,m+1,\dots,k,k+j}^{1,\dots,k} S(z_{k+j})}{T_{1,\dots,k}^{1,\dots,k} S'(z_m)}, \quad j = 1, \dots, r,$$

from which, for $r \geq 2$,

$$(5.13) \quad z_m^{(p)} - z_m = \mathcal{O}\left(\frac{\Psi_p(z_m)}{\tilde{\Psi}_{p,k}}\right) \quad \text{as } p \rightarrow \infty,$$

with $\tilde{\Psi}_{p,k}$ as in (5.6). From this, it follows that

$$(5.14) \quad \limsup_{p \rightarrow \infty} |z_m^{(p)} - z_m|^{1/p} \leq \frac{\Phi(z_m)}{\Phi(z_{k+1})} < 1,$$

In case $r = 1$ in (5.4), and provided $C_1^{(m)} \neq 0$, we have the asymptotic equality

$$(5.15) \quad z_m^{(p)} - z_m \sim C_1^{(m)} \frac{\Psi_p(z_m)}{\Psi_p(z_{k+1})} \quad \text{as } p \rightarrow \infty,$$

hence

$$(5.16) \quad \lim_{p \rightarrow \infty} |z_m^{(p)} - z_m|^{1/p} = \frac{\Phi(z_m)}{\Phi(z_{k+1})} < 1.$$

Proof. We start with the following asymptotic equality that is given in [17]:

$$(5.17) \quad z_m^{(p)} - z_m \sim -\frac{V_{p,k}(z_m)}{V'_{p,k}(z_m)} = -\frac{Q(z_m)}{Q'(z_m)} \quad \text{as } p \rightarrow \infty.$$

First, it is not difficult to see that $Q'(z_m)$ satisfies the asymptotic equality

$$(5.18) \quad Q'(z_m) \sim (-1)^k T_{1,\dots,k}^{1,\dots,k} \left| \frac{V(z_1, \dots, z_k)}{\prod_{i=1}^k \Psi_p(z_i)} \right|^2 S'(z_m) \quad \text{as } p \rightarrow \infty,$$

that is obtained simply by differentiating that in (5.5) formally. The validity of this can be shown by actually differentiating the expansion of $Q(z)$ given in Theorem 3.2, letting $z = z_m$ in the resulting expansion, and noting that the dominant term in this expansion as $p \rightarrow \infty$ is that given on the right-hand side of (5.18) and is non-zero.

Next, setting $z = z_m$ in (3.7), and recalling that $V(y_0, y_1, \dots, y_k)$ vanishes when any two of the y_j are equal, we have that the summation on s_1, \dots, s_k there does not contain the terms for which $s_i = m, i = 1, \dots, k$. Given this fact, the largest terms in the expansion of $Q(z_m)$ are those with $(r_1, \dots, r_k) = (1, \dots, k)$

and $(s_1, \dots, s_k) = (1, \dots, m-1, m+1, \dots, k, k+j)$, $j = 1, \dots, r$. Consequently, as $p \rightarrow \infty$, $Q(z_m)$ is as in

$$(5.19) \quad Q(z_m) \sim \frac{\overline{V(z_1, \dots, z_k)}}{\prod_{i=1}^k \Psi_p(z_i)} \times \sum_{j=1}^r T_{1, \dots, m-1, m+1, \dots, k, k+j}^{1, \dots, k} \times \frac{V(z_m, z_1, \dots, z_{m-1}, z_{m+1}, \dots, z_k, z_{k+j})}{\prod_{i=1}^k \Psi_p(z_{s_i})} \times \frac{\Psi_p(z_m)}{\Psi_p(z_{k+j})} + \dots$$

Now, by (3.9) and (5.10),

$$(5.20) \quad \begin{aligned} V(z_m, z_1, \dots, z_{m-1}, z_{m+1}, \dots, z_k, z_{k+j}) &= (-1)^{m-1} V(z_1, \dots, z_k, z_{k+j}) \\ &= (-1)^{m-1} V(z_1, \dots, z_k) S(z_{k+j}). \end{aligned}$$

Combining (5.18), (5.19), and (5.20) in (5.17), we obtain (5.11) with (5.12). (5.13) and (5.15) follow directly from (5.11), while (5.14) follows from (5.13) and (4.3). \blacksquare

Remarks.

- In proving (5.11), we divided the right-hand side of (5.19) by the right-hand side of (5.18). We would like to emphasize that this operation is made possible strictly on account of the asymptotic equality for $Q'(z_m)$ given in (5.18).
- Being a limit result, (5.16) concerning $r = 1$ is stronger than the limsup result in (5.14) for $r \geq 2$. The result in (5.14) for $r \geq 2$ is the best that can be obtained unless we have more information about $\Psi_p(z)$, hence about the $\xi_i^{(p)}$, than that given in (4.3).
- The results of Theorems 5.2 and 5.3 are the best that can be obtained when the residues v_s are not mutually orthogonal. This shows that the corresponding results of [18] pertaining to orthogonal residues (that show twice as fast convergence as those of Theorems 5.2 and 5.3) are indeed quite special.

5.2. Convergence analysis for $R_{p,k}(z)$. We now continue to the analysis of $F(z) - R_{p,k}(z)$, as $p \rightarrow \infty$. Throughout the rest of this work, $\|Y\|$ denotes the vector norm of $Y \in \mathbb{C}^N$.

Theorem 5.4. *Under the conditions of Theorem 5.2, $R_{p,k}(z)$ exists and is unique for all large p and satisfies*

$$(5.21) \quad F(z) - R_{p,k}(z) \sim (-1)^k \sum_{j=1}^r \frac{\widehat{T}_{1,\dots,k,k+j}^{1,\dots,k}(z;p) S(z_{k+j})}{T_{1,\dots,k}^{1,\dots,k} S(z)} \times \frac{\psi_{1,p}(z)}{\Psi_p(z_{k+j})} + \dots \quad \text{as } p \rightarrow \infty,$$

and hence

$$(5.22) \quad F(z) - R_{p,k}(z) = \mathcal{O}\left(\frac{\Psi_p(z)}{\widetilde{\Psi}_{p,k}}\right) \quad \text{as } p \rightarrow \infty,$$

uniformly on every compact subset of $\mathbb{C} \setminus \{z_1, \dots, z_\mu\}$, with $\widetilde{\Psi}_{p,k}$ as defined in (5.6). From this, it also follows that

$$(5.23) \quad \limsup_{p \rightarrow \infty} \|F(z) - R_{p,k}(z)\|^{1/p} \leq \frac{\Phi(z)}{\Phi(z_{k+1})}, \quad z \in \widetilde{K} = K \setminus \{z_1, \dots, z_\mu\},$$

uniformly on each compact subset of \widetilde{K} , and

$$(5.24) \quad \limsup_{p \rightarrow \infty} \|F(z) - R_{p,k}(z)\|^{1/p} \leq \frac{1}{\Phi(z_{k+1})}, \quad z \in E,$$

uniformly on E . Thus, uniform convergence takes place for z in any compact subset of the set \widetilde{K}_k , where

$$\widetilde{K}_k = \{z : \Phi(z) < \Phi(z_{k+1})\} \setminus \{z_1, \dots, z_k\}.$$

Proof. We have already analyzed $Q(z)$ in Theorem 5.2 and obtained the result in (5.5), from which we also have the asymptotic equality

$$(5.25) \quad Q(z) \sim (-1)^k T_{1,\dots,k}^{1,\dots,k} \left| \frac{V(z_1, \dots, z_k)}{\prod_{i=1}^k \Psi_p(z_i)} \right|^2 S(z) \quad \text{as } p \rightarrow \infty,$$

that holds uniformly in every compact subset of $\mathbb{C} \setminus \{z_1, z_2, \dots, z_k\}$. This shows that, for all large p , $V_{p,k}(z)$ is such that $V_{p,k}(\xi_i^{(p)}) \neq 0$, for $i = 1, \dots, p$, since the $\xi_i^{(p)}$ have no limit points in the set K , the complement of E , whereas the zeros of $V_{p,k}(z)$ are all in the set K . In addition, M_k , the cofactor of $\psi_{1,k}(z)$ in the determinant $Q(z)$ of (2.6), is non-zero because

$$M_k = (-1)^k \begin{vmatrix} u_{1,0} & u_{1,1} & \cdots & u_{1,k-1} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k-1} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k-1} \end{vmatrix} = \frac{Q^{(k)}(0)}{k!},$$

and, by (5.25),

$$Q^{(k)}(0) \sim (-1)^k k! T_{1,\dots,k}^{1,\dots,k} \left| \frac{V(z_1, \dots, z_k)}{\prod_{i=1}^k \Psi_p(z_i)} \right|^2 \neq 0 \quad \text{as } p \rightarrow \infty.$$

Under these, $R_{p,k}(z)$ exists and is unique for all large p , as mentioned in Section 2, following (2.8).

To complete the proof, we need to analyze the asymptotic behavior of $\Delta(z)$ in (2.11). From (3.19) in Theorem 3.3, we realize that it is necessary to first analyze the asymptotic behavior of the $\widehat{T}_{s_0, s_1, \dots, s_k}^{1, \dots, k}(z; p)$ as $p \rightarrow \infty$. Expanding the determinant representation of $\widehat{T}_{s_0, s_1, \dots, s_k}^{1, \dots, k}(z; p)$ given in (3.18) with respect to its first row, we have

$$\widehat{T}_{s_0, s_1, \dots, s_k}^{1, \dots, k}(z; p) = \sum_{i=0}^k w_i \widehat{e}_{s_i}^{(p)}(z), \quad w_i = (-1)^i T_{s_0, \dots, s_{i-1}, s_{i+1}, \dots, s_k}^{1, \dots, k}.$$

By (3.8), the cofactors w_i are independent of z and p . By (3.17), and by the fact that the $\xi_i^{(p)}$, and hence $\widehat{e}_{s_i}^{(p)}(z)$, are all bounded in p , we get

$$\widehat{T}_{s_0, s_1, \dots, s_k}^{1, \dots, k}(z; p) = \mathcal{O}(1) \quad \text{as } p \rightarrow \infty.$$

Turning now to $\Delta(z)$, arguing as before, we have that, by (5.4), the dominant terms in the summation in (3.19) as $p \rightarrow \infty$ are those having indices

$$(r_1, \dots, r_k) = (1, \dots, k) \text{ and } (s_0, s_1, \dots, s_k) = (1, \dots, k, k + j), \quad 1 \leq j \leq r.$$

The rest of the terms are negligible by Lemma 4.1. Thus, uniformly in every compact subset of the set $\mathbb{C} \setminus \{z_1, \dots, z_\mu\}$,

$$(5.26) \quad \frac{\Delta(z)}{\psi_{1,p}(z)} \sim \frac{\overline{V(z_1, \dots, z_k)}}{\prod_{i=1}^k \overline{\Psi_p(z_i)}} \times \sum_{j=1}^r \widehat{T}_{1, \dots, k, k+j}^{1, \dots, k}(z; p) \frac{V(z_1, \dots, z_k, z_{k+j})}{\Psi_p(z_{k+j}) \prod_{i=1}^k \Psi_p(z_i)} + \dots \quad \text{as } p \rightarrow \infty,$$

which, by (5.10), becomes

$$(5.27) \quad \frac{\Delta(z)}{\psi_{1,p}(z)} \sim \left| \frac{V(z_1, \dots, z_k)}{\prod_{i=1}^k \Psi_p(z_i)} \right|^2 \times \sum_{j=1}^r \widehat{T}_{1, \dots, k, k+j}^{1, \dots, k}(z; p) \frac{S(z_{k+j})}{\Psi_p(z_{k+j})} + \dots \quad \text{as } p \rightarrow \infty.$$

Combining (5.25) and (5.27) in (2.10), we obtain (5.21). (5.22) follows directly from (5.21), while (5.23) and (5.24) follow from (5.22). This completes the proof. ■

5.3. Approximation of residues. With Theorems 5.2 and 5.4 available, we can prove that the residues of $R_{p,k}(z)$ converge to corresponding residues of $F(z)$, their rates of convergence being the same as those of the corresponding poles.

Theorem 5.5. *Assume that the conditions of Theorems 5.4 and 5.3 are fulfilled. For $m = 1, \dots, k$, let*

$$v_m^{(p)} = \text{Res } R_{p,k}(z) \Big|_{z=z_m^{(p)}}.$$

Then, $\lim_{p \rightarrow \infty} v_m^{(p)} = v_m$. In fact, we have

$$(5.28) \quad \limsup_{p \rightarrow \infty} \|v_m^{(p)} - v_m\|^{1/p} \leq \frac{\Phi(z_m)}{\Phi(z_{k+1})} < 1.$$

Another result that concerns the approximation of $H(z_m)$, where $H(z)$ is a scalar-valued or vector-valued function analytic at $z = z_m$, is given in the next theorem.

Theorem 5.6. *Let $H(z)$ be a scalar-valued or vector-valued function analytic at $z = z_m$, $m \in \{1, \dots, k\}$. Then $H(z_m)$ can be approximated by $H(z_m^{(p)})$ as follows:*

$$(5.29) \quad H(z_m^{(p)}) - H(z_m) \sim H'(z_m)(z_m^{(p)} - z_m) \quad \text{as } p \rightarrow \infty,$$

hence

$$(5.30) \quad \limsup_{p \rightarrow \infty} |H(z_m^{(p)}) - H(z_m)|^{1/p} \leq \frac{\Phi(z_m)}{\Phi(z_{k+1})}.$$

Here, $|T|$ stands for the modulus or the norm of T in case T is a scalar or a vector, respectively.

The proofs of both of these theorems are identical to those of Theorems 5.4 and 5.5 in [18].

6. Convergence theory for meromorphic $F(z)$ with simple poles

Let the sets of interpolation points $\{\xi_1^{(p)}, \dots, \xi_{p+1}^{(p)}\}$ be as in Sections 4 and 5. We now turn to the convergence analysis of $R_{p,k}(z)$ as $p \rightarrow \infty$, when the function $F(z)$ is analytic in E and meromorphic in $E_\rho = \text{int } \Gamma_\rho$, where Γ_ρ , as before, is the locus $\Phi(z) = \rho$ for some $\rho > 1$. Assume that $F(z)$ has μ distinct simple poles z_1, \dots, z_μ in E_ρ . Thus, $F(z)$ has the following form:

$$(6.1) \quad F(z) = \sum_{s=1}^{\mu} \frac{v_s}{z - z_s} + \Theta(z),$$

$\Theta(z)$ being analytic in E_ρ . We assume, as before, that the vectors v_1, \dots, v_μ are linearly independent.

The treatment of this case is based entirely on that of the preceding section, the differences being minor. Note that the polynomial $u(z)$ of (3.1) is now replaced

by $\Theta(z)$ in (6.1). Previously, we had $u[\xi_m, \dots, \xi_n] = 0$ for all large $n - m$, as a consequence of which, we had (3.3) for $D_{m,n}$ and (3.5) for $F(z) - G_{m,n}(z)$. Instead of these, we now have

$$(6.2) \quad D_{m,n} = - \sum_{s=1}^{\mu} v_s \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)} + \Theta[\xi_m, \dots, \xi_n]$$

and

$$(6.3) \quad F(z) - G_{m,n}(z) = \psi_{m,n}(z) \left(\sum_{s=1}^{\mu} e_s(z) \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)} + \Theta[z, \xi_m, \dots, \xi_n] \right),$$

with $e_s(z)$ as in (3.5).

It is clear that the treatment of the general meromorphic $F(z)$ will be the same as that of the rational $F(z)$ provided the contributions from $\Theta(z)$ to $u_{i,j}$ in (2.7) and $\Delta_j(z)$ in (2.11), as $p \rightarrow \infty$, are negligible compared to the relevant dominant and subdominant terms we encountered earlier. This is guaranteed by the following lemma, which is [17, Lem. 6.1].

Lemma 6.1. *With $F(z)$ as in the first paragraph, we have*

$$(6.4) \quad \limsup_{p \rightarrow \infty} \left\| \Theta[\xi_{j+1}^{(p)}, \dots, \xi_{p+1}^{(p)}] \right\|^{1/p} \leq \frac{1}{\kappa\rho}.$$

We also have

$$(6.5) \quad \limsup_{p \rightarrow \infty} \left\| \Theta[z, \xi_{j+1}^{(p)}, \dots, \xi_p^{(p)}] \right\|^{1/p} \leq \frac{1}{\kappa\rho},$$

uniformly in every compact subset of E_ρ .

With this information, we can now prove the following theorems for general meromorphic $F(z)$. Again, we order the poles z_1, \dots, z_μ of $F(z)$ such that

$$(6.6) \quad \Phi(z_1) \leq \Phi(z_2) \leq \dots \leq \Phi(z_\mu) < \rho.$$

We also adopt the notation of Theorems 5.2, 5.3, and 5.4.

Theorem 6.2.

(i) *When $k < \mu$, assume that*

$$(6.7) \quad \Phi(z_k) < \Phi(z_{k+1}) = \dots = \Phi(z_{k+r}) < \begin{cases} \Phi(z_{k+r+1}) & \text{if } k+r < \mu, \\ \rho & \text{if } k+r = \mu, \end{cases}$$

in addition to (6.6). Then, all the results of Theorem 5.2 hold.

(ii) *When $k = \mu$,*

$$(6.8) \quad \limsup_{p \rightarrow \infty} |V_{p,k}(z) - S(z)|^{1/p} \leq \frac{\Phi(z_k)}{\rho}$$

uniformly on every compact subset of $\mathbb{C} \setminus \{z_1, \dots, z_\mu\}$.

Theorem 6.2 implies that $V_{p,k}(z)$ has precisely k zeros that tend to those of $S(z)$. Let us denote the zeros of $V_{p,k}(z)$ by $z_m^{(p)}$, $m = 1, \dots, k$. Then $\lim_{p \rightarrow \infty} z_m^{(p)} = z_m$, $m = 1, \dots, k$. In the next theorem, we provide the rate of convergence of each of these zeros.

Theorem 6.3. *Assume the conditions of Theorem 5.3.*

- (i) *When $k < \mu$, all the results of Theorem 5.3 hold.*
- (ii) *When $k = \mu$,*

$$(6.9) \quad \limsup_{p \rightarrow \infty} |z_m^{(p)} - z_m|^{1/p} \leq \frac{\Phi(z_m)}{\rho}, \quad m = 1, \dots, k.$$

Theorem 6.4. *Assume the conditions of Theorem 5.4. Then $R_{p,k}(z)$ exists and is unique.*

- (i) *When $k < \mu$, all the results of Theorem 5.4 hold with $\tilde{K} = E_\rho \setminus \{z_1, \dots, z_\mu\}$.*
- (ii) *When $k = \mu$, we have*

$$(6.10) \quad \limsup_{p \rightarrow \infty} \|F(z) - R_{p,k}(z)\|^{1/p} \leq \frac{\Phi(z)}{\rho}, \quad z \in \tilde{K} = E_\rho \setminus \{z_1, \dots, z_\mu\},$$

uniformly on each compact subset of \tilde{K} , and

$$(6.11) \quad \limsup_{p \rightarrow \infty} \|F(z) - R_{p,k}(z)\|^{1/p} \leq \frac{1}{\rho}, \quad z \in E,$$

uniformly on E .

Theorem 6.5. *Assume that the conditions of Theorems 6.4 and 6.3 hold. For $m = 1, \dots, k$, let*

$$v_m^{(p)} = \text{Res } R_{p,k}(z) \Big|_{z=z_m^{(p)}}.$$

Then, $\lim_{p \rightarrow \infty} v_m^{(p)} = v_m$. In fact, we have the following:

- (i) *When $k < \mu$, the result of Theorem 5.5 is true.*
- (ii) *When $k = \mu$, we have*

$$(6.12) \quad \limsup_{p \rightarrow \infty} \|v_m^{(p)} - v_m\|^{1/p} \leq \frac{\Phi(z_m)}{\rho} < 1.$$

The proofs of Lemma 6.1 and Theorems 6.2–6.4 are the same as those of the corresponding results in [17, Sec. 6]. The proof of Theorem 6.5 is similar to that of Theorem 5.5 of the present work.

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