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A simple approach to asymptotic expansions for Fourier integrals of singular functions

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ABSTRACT

In this work, we are concerned with the derivation of full asymptotic expansions for Fourier integrals $\int_a^b f(x)e^{\pm isx}\,dx$ as $s\to\infty$, where s is real positive, $[a,b]$ is a finite interval, and the functions $f(x)$ may have different types of algebraic and logarithmic singularities at $x=a$ and $x=b$. This problem has been treated in the literature by techniques involving neutralizers and Mellin transforms. Here, we derive the relevant asymptotic expansions by a method that employs simpler and less sophisticated tools.

1. Introduction

In this work, we are concerned with the derivation of full asymptotic expansions, as $s\to\infty$, for the Fourier integrals

$$F[f;\pm s]=\int_a^b f(x)e^{\pm isx}\,dx,\quad [a,b]\ \text{finite},\quad s>0, \tag{1.1}$$

when the function $f(x)$ has arbitrary algebraic and logarithmic singularities at the endpoints $x=a$ and/or $x=b$. Specifically, we assume that $f(x)$ has the following properties:

1. $f\in C^\infty(a,b)$ and has the asymptotic expansions

$$\begin{aligned} f(x)&\sim\sum_{j=0}^{\infty}U_j(\log(x-a))(x-a)^{\gamma_j}\quad \text{as } x\to a+,\\ f(x)&\sim\sum_{j=0}^{\infty}V_j(\log(b-x))(b-x)^{\delta_j}\quad \text{as } x\to b-, \end{aligned} \tag{1.2}$$

where $U_j(y)$ and $V_j(y)$ are some polynomials in y, and γ_j and δ_j are in general complex and satisfy

$$\begin{aligned} &\gamma_j\neq -1,-2,\ldots;\quad \Re\gamma_0\leqslant\Re\gamma_1\leqslant\Re\gamma_2\leqslant\cdots;\quad \lim_{j\to\infty}\Re\gamma_j=+\infty,\\ &\delta_j\neq -1,-2,\ldots;\quad \Re\delta_0\leqslant\Re\delta_1\leqslant\Re\delta_2\leqslant\cdots;\quad \lim_{j\to\infty}\Re\delta_j=+\infty. \end{aligned} \tag{1.3}$$

Here, $\Re z$ stands for the real part of z.

2. If we let $p_j=\deg(U_j)$ and $q_j=\deg(V_j)$ for each j, then γ_j and δ_j are ordered such that

$$p_j\geqslant p_{j+1}\quad \text{if } \Re\gamma_{j+1}=\Re\gamma_j;\quad q_j\geqslant q_{j+1}\quad \text{if } \Re\delta_{j+1}=\Re\delta_j. \tag{1.4}$$

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3. By (1.2), we mean that, for each $m = 1,2,\ldots$,

$$\begin{aligned} f(x) &\sim \sum_{j=0}^{m-1} U_j(\log(x-a))(x-a)^{\gamma_j} = O(U_m(\log(x-a))(x-a)^{\gamma_m}) \quad \text{as } x \to a+, \\ f(x) &\sim \sum_{s=0}^{m-1} V_j(\log(b-x))(b-x)^{\delta_j} = O(V_m(\log(b-x))(b-x)^{\delta_m}) \quad \text{as } x \to b-. \end{aligned} \tag{1.5}$$

   This is consistent with (1.3).
4. For each $k = 1,2,\ldots$, the kth derivative of $f(x)$ $[f^{(k)}(x)]$ also has asymptotic expansions as $x \to a+$ and $x \to b-$ that are obtained by differentiating those in (1.2) term by term.

Remark. Note that if

$$f(x) = (x-a)^{\gamma}[\log(x-a)]^{p} g_a(x) = (b-x)^{\delta}[\log(b-x)]^{q} g_b(x),$$

where p and q are nonnegative integers, g_a, $g_b \in C^{\infty}[a,b]$, and $g_a(x)$ and $g_b(x)$ have full Taylor series about $x = a$ and $x = b$, respectively, then $f(x)$ is as in (1.2)–(1.5) with

$$(\gamma_j = \gamma + j,\ p_j = p) \quad \text{and} \quad (\delta_j = \delta + j,\ q_j = q), \quad j = 0,1,\ldots.$$

The following are consequences of (1.3):

(i) There are only a finite number of γ_j and only a finite number of δ_j having the same real parts; consequently, $\Re\gamma_j < \Re\gamma_{j+1}$ and $\Re\delta_{j'} < \Re\delta_{j'+1}$ for infinitely many values of the indices j and j'.

(ii) The sequences $\{(x-a)^{\gamma_j}\}_{j=0}^{\infty}$ and $\{(b-x)^{\delta_j}\}_{j=0}^{\infty}$ are asymptotic scales as $x \to a+$ and $x \to b-$, respectively, in the following sense: For each $s = 0,1,\ldots$,

$$\lim_{x\to a+}\left|\frac{(x-a)^{\gamma_{j+1}}}{(x-a)^{\gamma_j}}\right| = \begin{cases} 1 & \text{if } \Re\gamma_j = \Re\gamma_{j+1}, \\ 0 & \text{if } \Re\gamma_j < \Re\gamma_{j+1}, \end{cases}$$

$$\lim_{x\to b-}\left|\frac{(b-x)^{\delta_{j+1}}}{(b-x)^{\delta_j}}\right| = \begin{cases} 1 & \text{if } \Re\delta_j = \Re\delta_{j+1}, \\ 0 & \text{if } \Re\delta_j < \Re\delta_{j+1}. \end{cases}$$

(iii) The integral $\int_a^b f(x)e^{\pm isx}\,dx$ exists in the ordinary sense provided $\Re\gamma_0 > -1$ and $\Re\delta_0 > -1$. Otherwise, it exists as an Hadamard finite part integral.

Asymptotic expansions for the integrals $F[f;\pm s] = \int_a^b f(x)e^{\pm isx}\,dx$ and others that have general oscillatory kernels have been derived in Bleistein and Handelsman [2, Sections 3.4, 6.3, and 6.4] by using a technique that involves neutralizers and Mellin transforms. In this work, we derive these expansions by using a method that employs simpler and less sophisticated tools; in fact, what is needed most is basic knowledge of asymptotic expansions. In the next section, we state the main results that contain the full asymptotic expansions for the integral $F[f;\pm s]$ as $s \to \infty$, when $f(x)$ is as described above. In Sections 3 and 4, we provide the proofs of these results. The asymptotic expansions of $F[f;\pm s]$ are expressed in simplest terms based only on the asymptotic expansions in (1.2). In Section 5, we provide some examples.

The leading terms in the asymptotic expansions derived here can also be obtained by using the method of stationary phase. For this method, see Olver [3] and Bender and Orszag [1], for example.

Before we end this section, we would like to comment very briefly on the proof technique that we use in this work. We split the integral $\int_a^b$ into two: $\int_a^r$ and $\int_r^b$, for some $r \in (a,b)$. Of course, under the conditions imposed in the first paragraph of this section, $f(x)$ is infinitely smooth in a neighborhood of $x = r$. We obtain the asymptotic expansions for both of these integrals and show that the first contains contributions from $x = a$ and $x = r$, while the second contains contributions from $x = r$ and $x = b$. It turns out that the two contributions from $x = r$ cancel each other out completely, the end result being that the asymptotic expansion of $F[f;\pm s]$ has contributions from $x = a$ and $x = b$ only.

2. Main results

We now state theorems on asymptotic expansion for the integral $\int_a^b f(t)e^{\pm isx}\,dx$ as $s \to \infty$, whether this integral converges and exists in the ordinary sense or diverges and is defined in the sense of Hadamard finite part. Theorem 2.1 concerns the special case of (1.2) in which $U_j(y)$ and $V_j(y)$ are constant polynomials. This case is of importance by itself. Theorem 2.2 covers the general case in which $U_j(y)$ and $V_j(y)$ are arbitrary polynomials.

Theorem 2.1. *Let $f(x)$ be as in the first paragraph of* Section 1, *such that $U_j(y)$ and $V_j(y)$ are constant polynomials. That is, $f(x)$ has asymptotic expansions of the form*

$$f(x)\sim\sum_{j=0}^{\infty}c_j(x-a)^{\gamma_j}\quad\text{as } x\to a+,$$
$$f(x)\sim\sum_{j=0}^{\infty}d_j(b-x)^{\delta_j}\quad\text{as } x\to b-, \tag{2.1}$$

where c_j and d_j are some nonzero constants. Let $w=\pm is$ with $s>0$. Then

$$\int_a^b f(x)e^{wx}\,dx\sim e^{wa}\sum_{j=0}^{\infty}c_j\frac{\Gamma(\gamma_j+1)}{(-w)^{\gamma_j+1}}+e^{wb}\sum_{j=0}^{\infty}d_j\frac{\Gamma(\delta_j+1)}{w^{\delta_j+1}}\quad\text{as } s\to\infty. \tag{2.2}$$

Theorem 2.2. *Let $f(x)$ be as in the first paragraph of Section 1, with $U_j(y)=\sum_{i=0}^{p_j}u_{ji}y^i$ and $V_j(y)=\sum_{i=0}^{q_j}v_{ji}y^i$, where u_{ji} and v_{ji} are constants, and let $w=\pm is$ with $s>0$. Denote $\frac{d}{d\omega}$ by D_ω. For an arbitrary polynomial $S(y)=\sum_{i=0}^{k}e_iy^i$ and an arbitrary function g that depends on ω, define also*

$$S(D_\omega)g=\sum_{i=0}^{k}e_i[D_\omega^i g]=\sum_{i=0}^{k}e_i\frac{d^ig}{d\omega^i}.$$

Then

$$\int_a^b f(x)e^{wx}\,dx\sim e^{wa}\sum_{j=0}^{\infty}U_j(D_{\gamma_j})\left[\frac{\Gamma(\gamma_j+1)}{(-w)^{\gamma_j+1}}\right]+e^{wb}\sum_{j=0}^{\infty}V_j(D_{\delta_j})\left[\frac{\Gamma(\delta_j+1)}{w^{\delta_j+1}}\right]\quad\text{as } s\to\infty. \tag{2.3}$$

Remarks

1. In case $f\in C^\infty[a,b]$, we have $\gamma_j=\delta_j=j$ and $c_j=f^{(j)}(a)/j!$, $d_j=(-1)^jf^{(j)}(b)/j!$, $j=0,1,\ldots$, in (2.2). This result can also be obtained by repeated integration by parts of $\int_a^b f(x)e^{wx}\,dx$. See, Wong [4, Chapter 1], for example.
2. For (2.2) to be a genuine asymptotic expansion, it is necessary (but not sufficient) that the sequences $\{w^{-\gamma_j-1}\}_{j=0}^{\infty}$ and $\{w^{-\delta_j-1}\}_{j=0}^{\infty}$ be asymptotic scales, and this is indeed the case.
3. For (2.3) to be a genuine asymptotic expansion, it is necessary (but not sufficient) that the sequences $\{U_j(D_{\gamma_j})[\Gamma(\gamma_j+1)/(-w)^{\gamma_j+1}]\}_{j=0}^{\infty}$ and $\{V_j(D_{\delta_j})[\Gamma(\delta_j+1)/w^{\delta_j+1}]\}_{j=0}^{\infty}$ be asymptotic scales, and this is also the case. To see that this is true, it is enough to observe that now

$$U_j(D_{\gamma_j})\left[\frac{\Gamma(\gamma_j+1)}{(-w)^{\gamma_j+1}}\right]=\frac{1}{(-w)^{\gamma_j+1}}\times(\text{a polynomial in}\log w\text{ of degree }p_j),$$

and

$$V_j(D_{\delta_j})\left[\frac{\Gamma(\delta_j+1)}{w^{\delta_j+1}}\right]=\frac{1}{w^{\delta_j+1}}\times(\text{a polynomial in}\log w\text{ of degree }q_j).$$

4. It is understood that, in both theorems, z^u is defined as follows: with $z=|z|e^{i\theta}$, $|\theta|<\pi$, and $u=\mu+i\nu$, we have $z^u=(|z|^\mu e^{-\nu\theta})e^{i(\mu\theta+\nu\log|z|)}$.
5. Of course, Theorems 2.1 and 2.2 can also be applied to the integrals $\int_A^B f(x)e^{wx}\,dx$, where $f(x)$ and/or its derivatives have singularities at one or more points in (A,B). For this, we subdivide the interval $[A,B]$ into several subintervals appropriately to ensure that $f(x)$ is infinitely differentiable in each of these (open) subintervals and may be singular at the endpoints only, and apply the theorems in each of these subintervals.

3. Proof of Theorem 2.1

3.1. Preliminaries

The following lemma will be used in the proof of Theorem 2.1.

Lemma 3.1. *Let α be a complex number different from $-1,-2,\ldots$, and let $\xi>0$ and $s>0$. Then, with $w=\pm is$, we have*

$$\int_0^\xi t^\alpha e^{wt}\,dt=\frac{\Gamma(\alpha+1)}{(-w)^{\alpha+1}}+e^{w\xi}\sum_{k=0}^{m-1}(-1)^k\frac{[\alpha]_k\xi^{\alpha-k}}{w^{k+1}}-(-1)^m\frac{[\alpha]_m}{w^m}\int_\xi^\infty t^{\alpha-m}e^{wt}\,dt,\quad m>\Re\alpha, \tag{3.1}$$

and the asymptotic expansion

$$\int_0^\xi t^\alpha e^{wt}\,dt\sim\frac{\Gamma(\alpha+1)}{(-w)^{\alpha+1}}+e^{w\xi}\sum_{k=0}^{\infty}(-1)^k\frac{[\alpha]_k\xi^{\alpha-k}}{w^{k+1}}\quad\text{as } s\to\infty, \tag{3.2}$$

where $[\alpha]_0=1$ and $[\alpha]_k=\prod_{i=0}^{k-1}(\alpha-i)$ for $k=1,2,\ldots$. In case of divergence, that is, in case $\Re\alpha\leqslant-1$, the integral $\int_0^\xi t^\alpha e^{wt}\,dt$ is defined in the sense of Hadamard finite part.

Proof. To begin, we assume that $-1 < \Re\alpha < 0$, so that the integral $\int_0^\xi t^\alpha e^{wt}\,dt$ exists in the ordinary sense for all w and is an analytic function of α. In addition, we can also write

$$\int_0^\xi t^\alpha e^{wt}\,dt = \int_0^\infty t^\alpha e^{wt}\,dt - \int_\xi^\infty t^\alpha e^{wt}\,dt, \tag{3.3}$$

because both $\int_0^\infty t^\alpha e^{wt}\,dt$ and $\int_\xi^\infty t^\alpha e^{wt}\,dt$ exist in the ordinary sense when $-1 < \Re\alpha < 0$. We compute the first of these integrals by rotating the contour of integration (that is the positive real axis in the t-plane) by 90° when $w = \mathrm{i}s$ and by −90° when $w = -\mathrm{i}s$. We obtain

$$\int_0^\infty t^\alpha e^{wt}\,dt = (\pm\mathrm{i})^{\alpha+1}\int_0^\infty \tau^\alpha e^{-s\tau}\,d\tau = (\pm\,\mathrm{i})^{\alpha+1}\frac{\Gamma(\alpha+1)}{s^{\alpha+1}} = \frac{\Gamma(\alpha+1)}{(-w)^{\alpha+1}}. \tag{3.4}$$

By repeated integration by parts, the second integral becomes

$$\int_\xi^\infty t^\alpha e^{wt}\,dt = -e^{w\xi}\sum_{k=0}^{m-1}(-1)^k\frac{[\alpha]_k\xi^{\alpha-k}}{w^{k+1}} + (-1)^m\frac{[\alpha]_m}{w^m}\int_\xi^\infty t^{\alpha-m}e^{wt}\,dt. \tag{3.5}$$

Thus, combining (3.4) and (3.5) in (3.3), we obtain (3.1).

The right-hand side of (3.1) is analytic for $\alpha \in S_1 = \{\alpha : \Re\alpha < m\}$ and has simple poles [those of $\Gamma(\alpha+1)$] at $\alpha = -1, -2, \ldots$, while the left-hand side is analytic for $\alpha \in S_2 = \{\alpha : \Re\alpha > -1\}$. Because $S_1 \cap S_2 \neq \emptyset$, and because m can be chosen arbitrarily large, the right-hand side of (3.1) is the analytic continuation of the left-hand side, as a function of α, to the whole α-plane. In addition,

$$\left|\int_\xi^\infty t^{\alpha-m}e^{wt}\,dt\right| \leqslant \frac{\xi^{\Re\alpha-m+1}}{m-\Re\alpha-1}, \quad \text{if } m > \Re\alpha+1, \tag{3.6}$$

independently of w. From (3.1) and (3.6), we thus have that the last term on the right-hand side of (3.1) is $O(w^{-m})$ as $s \to \infty$. From this and from the fact that m is an arbitrary integer, the result in (3.2) follows. It is also easy to see that, in case $\Re\alpha \leqslant -1$, the right-hand side of (3.1) is indeed the Hadamard finite part of $\int_0^\xi t^\alpha e^{wt}\,dt$. □

Remark. That the integral $\int_0^\xi t^\alpha e^{wt}\,dt$ can be continued to a meromorphic function of α can also be shown as follows: Expanding e^{wt} in powers of t, and integrating term by term, we obtain

$$\int_0^\xi t^\alpha e^{wt}\,dt = \sum_{k=0}^\infty \frac{w^k}{k!}\frac{\xi^{\alpha+k+1}}{\alpha+k+1}. \tag{3.7}$$

Because it converges absolutely and uniformly in α, the infinite series in (3.7) represents a function that is meromorphic in the α-plane with simple poles at $\alpha = -1, -2, \ldots$. Thus, the right-hand side of (3.7) is the analytic continuation of $\int_0^\xi t^\alpha e^{wt}\,dt$ to the whole α-plane, with $\alpha = -1, -2, \ldots$, removed; it is also the Hadamard finite part of it when $\Re\alpha \leqslant -1$ but $\alpha \neq -1, -2, \ldots$.

3.2. The proof

Let us write the integral $\int_a^b f(x)e^{wx}\,dx$ as a sum of two integrals as in

$$\int_a^b f(x)e^{wx}\,dx = I_{[a,r]} + I_{[r,b]}, \tag{3.8}$$

where

$$I_{[a,r]} = \int_a^r f(x)e^{wx}\,dx, \quad I_{[r,b]} = \int_r^b f(x)e^{wx}\,dx, \quad a < r < b. \tag{3.9}$$

Next, choose a positive integer m such that $\Re\gamma_m > 0$ and $\Re\delta_m > 0$, and set

$$\kappa_m = \min\{\lceil\Re\gamma_m - 1\rceil, \lceil\Re\delta_m - 1\rceil\}. \tag{3.10}$$

Such an integer m exists because $\lim_{j\to\infty}\Re\gamma_j = \infty$ and $\lim_{j\to\infty}\Re\delta_j = \infty$ by (1.3).

We now proceed to the proof of Theorem 2.1. We shall give all the details of this proof.

We first treat $I_{[a,r]}$. Making the substitution $t = x - a$ and $\xi = r - a$, we have

$$I_{[a,r]} = e^{wa}\int_0^\xi f(a+t)e^{wt}\,dt. \tag{3.11}$$

Invoking (2.1), let us define

$$P_m(t) = \sum_{j=0}^{m-1} c_j t^{\gamma_j}, \quad E_m(t) = f(a+t) - P_m(t). \tag{3.12}$$

Hence

$$I_{[a,r]} = e^{wa}\left[\int_0^{\xi} P_m(t)e^{wt}\,dt + \int_0^{\xi} E_m(t)e^{wt}\,dt\right]. \tag{3.13}$$

By (3.12) and (2.1) and our assumptions on $f(x)$,

$$E_m^{(k)}(t) \sim \sum_{j=m}^{\infty} c_j[\gamma_j]_k t^{\gamma_j-k} \quad \text{as } t \to 0+, \quad k = 0, 1, \ldots. \tag{3.14}$$

By (3.14) and (3.10), we have

$$E_m^{(k)}(0) = 0, \quad k = 0, 1, \ldots, \kappa_m. \tag{3.15}$$

Therefore, $E_m(t)$ is in $C^{\kappa_m}[0,\xi]$. Consequently, by integration by parts, we obtain

$$\int_0^{\xi} E_m(t)e^{wt}\,dt = e^{w\xi}\sum_{k=0}^{\kappa_m-1}(-1)^k\frac{E_m^{(k)}(\xi)}{w^{k+1}} + \frac{(-1)^{\kappa_m}}{w^{\kappa_m}}\int_0^{\xi} E_m^{(\kappa_m)}(t)e^{wt}\,dt, \tag{3.16}$$

and, because $E_m^{(\kappa_m)}(t)$ is continuous on $[0,\xi]$,

$$\int_0^{\xi} E_m^{(\kappa_m)}(t)e^{wt}\,dt = o(1) \quad \text{as } s \to \infty, \tag{3.17}$$

by the Riemann–Lebesgue lemma. Therefore,

$$e^{wa}\int_0^{\xi} E_m(t)e^{wt}\,dt = e^{wr}\sum_{k=0}^{\kappa_m-1}(-1)^k\frac{E_m^{(k)}(\xi)}{w^{k+1}} + o(s^{-\kappa_m}) \quad \text{as } s \to \infty. \tag{3.18}$$

By Lemma 3.1,

$$\begin{aligned} e^{wa}\int_0^{\xi} P_m(t)e^{wt}\,dt &= e^{wa}\sum_{j=0}^{m-1} c_j\int_0^{\xi} t^{\gamma_j}e^{wt}\,dt = e^{wa}\sum_{j=0}^{m-1} c_j\frac{\Gamma(\gamma_j+1)}{(-w)^{\gamma_j+1}} + e^{wr}\sum_{j=0}^{m-1} c_j\left[\sum_{k=0}^{\kappa_m-1}(-1)^k\frac{[\gamma_j]_k\xi^{\gamma_j-k}}{w^{k+1}} + O(s^{-\kappa_m})\right] \\ &= e^{wa}\sum_{j=0}^{m-1} c_j\frac{\Gamma(\gamma_j+1)}{(-w)^{\gamma_j+1}} + e^{wr}\sum_{k=0}^{\kappa_m-1}(-1)^k\frac{P_m^{(k)}(\xi)}{w^{k+1}} + O(s^{-\kappa_m}) \quad \text{as } s \to \infty. \end{aligned} \tag{3.19}$$

Here we have made use of the fact that

$$\sum_{j=0}^{m-1} c_j[\gamma_j]_k\xi^{\gamma_j-k} = \sum_{j=0}^{m-1} c_j\frac{d^k}{d\xi^k}\xi^{\gamma_j} = \frac{d^k}{d\xi^k}\sum_{j=0}^{m-1} c_j\xi^{\gamma_j} = P_m^{(k)}(\xi).$$

Substituting now (3.18) and (3.19) in (3.13), and noting that $E_m^{(k)}(\xi) + P_m^{(k)}(\xi) = f^{(k)}(r),\ k = 0, 1, \ldots$, by (3.12), we obtain

$$I_{[a,r]} = e^{wa}\sum_{j=0}^{m-1} c_j\frac{\Gamma(\gamma_j+1)}{(-w)^{\gamma_j+1}} + e^{wr}\sum_{k=0}^{\kappa_m-1}(-1)^k\frac{f^{(k)}(r)}{w^{k+1}} + O(s^{-\kappa_m}) \quad \text{as } s \to \infty. \tag{3.20}$$

The treatment of $I_{[r,b]}$ follows directly from that of $I_{[a,r]}$. We first rewrite $I_{[r,b]}$ as in

$$I_{[r,b]} = \int_{-b}^{-r} f(-x)e^{-wx}\,dx. \tag{3.21}$$

Next, we note that, by (2.1),

$$f(-x) \sim \sum_{j=0}^{\infty} d_j(x+b)^{\delta_j} \quad \text{as } x \to -b. \tag{3.22}$$

We can now apply the result in (3.20) with the substitutions $a \leftarrow -b$, $r \leftarrow -r$, $f(x) \leftarrow f(-x)$, $c_j \leftarrow d_j$, $\gamma_j \leftarrow \delta_j$, and $w \leftarrow -w$. We obtain

$$I_{[r,b]} = e^{wb}\sum_{j=0}^{m-1} d_j\frac{\Gamma(\delta_j+1)}{w^{\delta_j+1}} + e^{wr}\sum_{k=0}^{\kappa_m-1}\frac{f^{(k)}(r)}{(-w)^{k+1}} + O(s^{-\kappa_m}) \quad \text{as } s \to \infty. \tag{3.23}$$

Substituting (3.20) and (3.23) in (3.8), and observing that the two summations involving the $f^{(k)}(r)$ cancel each other out completely, we obtain

$$\int_a^b f(x)e^{wx}\,dx = e^{wa}\sum_{j=0}^{m-1} c_j\frac{\Gamma(\gamma_j+1)}{(-w)^{\gamma_j+1}} + e^{wb}\sum_{j=0}^{m-1} d_j\frac{\Gamma(\delta_j+1)}{w^{\delta_j+1}} + O(s^{-\kappa_m}) \quad \text{as } s \to \infty. \tag{3.24}$$

The result in (2.2) follows by realizing that $\lim_{m\to\infty}\kappa_m = \infty$. This completes the proof of Theorem 2.1. □

4. Proof of Theorem 2.2

4.1. Preliminaries

The following lemma will be used in the proof of Theorem 2.2.

Lemma 4.1. *Let α be a complex number different from $-1,-2,\ldots$, and let $\xi > 0$ and $s > 0$. Then, with $w = \pm is$ and $p = 1,2,\ldots$, we have*

$$\int_0^\xi (\log t)^p t^\alpha e^{wt}\,dt = \frac{d^p}{d\alpha^p}\left[\frac{\Gamma(\alpha+1)}{(-w)^{\alpha+1}} + e^{w\xi}\sum_{k=0}^{m-1}(-1)^k\frac{[\alpha]_k\xi^{\alpha-k}}{w^{k+1}} - (-1)^m\frac{[\alpha]_m}{w^m}\int_\xi^\infty t^{\alpha-m}e^{wt}\,dt\right], \quad m > \Re\alpha, \tag{4.1}$$

and the asymptotic expansion

$$\int_0^\xi (\log t)^p t^\alpha e^{wt}\,dt \sim \frac{d^p}{d\alpha^p}\left[\frac{\Gamma(\alpha+1)}{(-w)^{\alpha+1}}\right] + e^{w\xi}\sum_{k=0}^{\infty}(-1)^k\frac{\frac{\partial^k}{\partial\xi^k}\left[(\log\xi)^p\xi^\alpha\right]}{w^{k+1}} \quad \text{as } s\to\infty, \tag{4.2}$$

where $[\alpha]_0 = 1$ and $[\alpha]_k = \prod_{i=0}^{k-1}(\alpha - i)$ for $k = 1,2,\ldots$. In case of divergence, that is, in case $\Re\alpha \leqslant -1$, the integral $\int_0^\xi (\log t)^p t^\alpha e^{wt}\,dt$ is defined in the sense of Hadamard finite part.

Proof. To begin, we assume that $-1 < \Re\alpha < 0$, so that the integral $\int_0^\xi (\log t)^p t^\alpha e^{wt}\,dt$ exists in the ordinary sense for all w and is an analytic function of α. In addition,

$$\int_0^\xi (\log t)^p t^\alpha e^{wt}\,dt = \int_0^\xi \left(\frac{d^p}{d\alpha^p}t^\alpha\right)e^{wt}\,dt = \frac{d^p}{d\alpha^p}\int_0^\xi t^\alpha e^{wt}\,dt.$$

From this, from Lemma 3.1, and from the fact that the right-hand side of (3.1) is the analytic continuation in α of $\int_0^\xi t^\alpha e^{wt}\,dt$ to $\Re\alpha < m$, the result in (4.1) follows. The result in (4.2) can be shown to be true by observing that, in (4.1),

$$\frac{d^p}{d\alpha^p}\left([\alpha]_k\xi^{\alpha-k}\right) = \frac{\partial^p}{\partial\alpha^p}\left(\frac{\partial^k}{\partial\xi^k}\xi^\alpha\right) = \frac{\partial^k}{\partial\xi^k}\left(\frac{\partial^p}{\partial\alpha^p}\xi^\alpha\right) = \frac{\partial^k}{\partial\xi^k}\left[(\log\xi)^p\xi^\alpha\right]$$

and that the last term there is $O(s^{-m})$ as $s\to\infty$ since

$$\begin{aligned}\frac{d^p}{d\alpha^p}\left([\alpha]_m\int_\xi^\infty t^{\alpha-m}e^{wt}\,dt\right) &= \sum_{k=0}^{p}\binom{p}{k}\left(\frac{d^{p-k}}{d\alpha^{p-k}}[\alpha]_m\right)\left(\frac{d^k}{d\alpha^k}\int_\xi^\infty t^{\alpha-m}e^{wt}dt\right)\\ &= \sum_{k=0}^{p}\binom{p}{k}\left(\frac{d^{p-k}}{d\alpha^{p-k}}[\alpha]_m\right)\left(\int_\xi^\infty (\log t)^k t^{\alpha-m}e^{wt}dt\right)\\ &= O(1) \quad \text{as } s\to\infty, \quad \text{if } m > \Re\alpha + 1.\end{aligned}$$

This completes the proof of the lemma. □

4.2. The proof

The proof of Theorem 2.2 is achieved in exactly the same way as that of Theorem 2.1, with appropriate changes. We start by defining $I_{[a,r]}$ and $I_{[r,b]}$ as in (3.8) and (3.9), and κ_m as in (3.10).

In the treatment of $I_{[a,r]}$, as before, we make the substitution $t = x - a$ and $\xi = r - a$, and define

$$P_m(t) = \sum_{j=0}^{m-1}U_j(\log t)t^{\gamma_j}, \quad E_m(t) = f(a+t) - P_m(t). \tag{4.3}$$

After proceeding as in the preceding section, and using Lemma 4.1 this time, we obtain

$$I_{[a,r]} = e^{wa}\sum_{j=0}^{m-1}U_j(D_{\gamma_j})\left[\frac{\Gamma(\gamma_j+1)}{(-w)^{\gamma_j+1}}\right] + e^{wr}\sum_{k=0}^{\kappa_m-1}(-1)^k\frac{f^{(k)}(r)}{w^{k+1}} + O(s^{-\kappa_m}) \quad \text{as } s\to\infty. \tag{4.4}$$

Applying (4.4) to $I_{[r,b]}$, with appropriate substitutions, as in the preceding section, we obtain

$$I_{[r,b]} = e^{wb}\sum_{j=0}^{m-1}V_j(D_{\delta_j})\left[\frac{\Gamma(\delta_j+1)}{w^{\delta_j+1}}\right] + e^{wr}\sum_{k=0}^{\kappa_m-1}\frac{f^{(k)}(r)}{(-w)^{k+1}} + O(s^{-\kappa_m}) \quad \text{as } s\to\infty. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$\int_a^b f(x)\,dx = e^{wa}\sum_{j=0}^{m-1}U_j(D_{\gamma_j})\left[\frac{\Gamma(\gamma_j+1)}{(-w)^{\gamma_j+1}}\right] + e^{wb}\sum_{j=0}^{m-1}V_j(D_{\delta_j})\left[\frac{\Gamma(\delta_j+1)}{w^{\delta_j+1}}\right] + O(s^{-\kappa_m}) \quad \text{as } s\to\infty. \tag{4.6}$$

The result in (2.3) now follows from the fact that $\lim_{m\to\infty}\kappa_m = \infty$.

5. Applications

Example 1. In case $f(x)$ has no singularities on $[a,b]$, we have $\gamma_j = \delta_j = j$ and $c_j = f^{(j)}(a)/j!$, $d_j = (-1)^j f^{(j)}(b)/j!, j = 0,1,\ldots,$. Hence Theorem 2.1 gives the following known asymptotic expansion that can also be obtained by repeated integration by parts, as already mentioned in Section 2:

$$\int_a^b f(x)e^{wx}\,dx \sim \sum_{j=0}^{\infty}(-1)^j \frac{e^{bw}f^{(j)}(b) - e^{aw}f^{(j)}(a)}{w^{j+1}} \quad \text{as } s \to \infty, \quad \text{for } w = \pm is. \tag{5.1}$$

Example 2. In case $f(x) = g(x)|x-r|^{\sigma}$, where $g \in C^{\infty}[a,b]$, $a < r < b$, and σ is real and $\sigma \neq -1,-2,\ldots$, we can apply Theorem 2.1 to the integrals $\int_a^r f(x)e^{wx}\,dx = \int_a^r g(x)(r-a)^{\sigma}e^{wx}\,dx$ and $\int_r^b f(x)e^{wx}\,dx = \int_r^b g(x)(x-r)^{\sigma}e^{wx}\,dx$. After some manipulation, we obtain

$$\begin{aligned}\int_a^b f(x)e^{wx}\,dx &\sim \sum_{j=0}^{\infty}(-1)^j \frac{e^{bw}f^{(j)}(b) - e^{aw}f^{(j)}(a)}{w^{j+1}} \mp 2ie^{\pm\, i\sigma\pi/2}\sin(\sigma\pi/2)e^{rw}\\ &\times \sum_{j=0}^{\infty}(-1)^j \frac{g^{(j)}(r)}{j!}\,\frac{\Gamma(\sigma+j+1)}{w^{\sigma+j+1}} \quad \text{as } s \to \infty, \quad \text{for } w = \pm is.\end{aligned} \tag{5.2}$$

Example 3. In case $f(x) = g(x)\log|x-r|$, where $g \in C^{\infty}[a,b]$ and $a < r < b$, we can apply Theorem 2.2 to the integrals $\int_a^r f(x)e^{wx}\,dx$ and $\int_r^b f(x)e^{wx}\,dx$. This amounts to differentiating in (5.2) the term involving e^{wr} with respect to σ, and setting $\sigma = 0$ following that. This gives

$$\int_a^b f(x)e^{wx}\,dx \sim \sum_{j=0}^{\infty}(-1)^j \frac{e^{bw}f^{(j)}(b) - e^{aw}f^{(j)}(a)}{w^{j+1}} \mp i\pi e^{rw}\sum_{j=0}^{\infty}(-1)^j \frac{g^{(j)}(r)}{w^{j+1}} \text{ as } s \to \infty, \quad \text{for } w = \pm is. \tag{5.3}$$

Example 4. Let $f(x)$ be 2π-periodic on $(-\infty,\infty)$ and infinitely differentiable there, except at the points $x = r \pm 2k\pi$, $k = 0,1,2,\ldots$, where it has logarithmic singularities. Specifically, with $0 < r < 2\pi$, assume that $f(x)$ can be written as in

$$f(x) = h_1(x) + h_2(x), \quad 0 \leqslant x \leqslant 2\pi; \quad h_1(x) = g(x)\log|x-r|, \quad g, h_2 \in C^{\infty}[0,2\pi]. \tag{5.4}$$

[Note that $h_1(x)$ and $h_2(x)$ are not periodic by themselves, but their sum is.] Assume that we are interested in the nth Fourier coefficient of $f(x)$, namely,

$$e_n = \int_0^{2\pi} f(x)e^{inx}\,dx, \quad n = 0, \pm1, \pm2, \ldots, .$$

Applying (5.3) of the preceding example to the integral $\int_0^{2\pi} h_1(x)e^{wx}\,dx$, $w = \pm in$, we have

$$\int_0^{2\pi} h_1(x)e^{wx}\,dx \sim \sum_{j=0}^{\infty}(-1)^j \frac{h_1^{(j)}(2\pi) - h_1^{(j)}(0)}{w^{j+1}} \mp i\pi e^{rw}\sum_{j=0}^{\infty}(-1)^j \frac{g^{(j)}(r)}{w^{j+1}} \quad \text{as } n \to \infty. \tag{5.5}$$

Applying (5.1) in Example 1 to the integral $\int_0^{2\pi} h_2(x)e^{wx}\,dx$, $w = \pm in$, we have

$$\int_0^{2\pi} h_2(x)e^{wx}\,dx \sim \sum_{j=0}^{\infty}(-1)^j \frac{h_2^{(j)}(2\pi) - h_2^{(j)}(0)}{w^{j+1}} \quad \text{as } n \to \infty. \tag{5.6}$$

Adding (5.6) to (5.5), and recalling that $h_1(x) + h_2(x) = f(x)$ and that, being 2π-periodic, $f(x)$ satisfies $f^{(k)}(0) = f^{(k)}(2\pi)$, $k = 0,1,\ldots$, we obtain

$$e_{\pm n} = \int_0^{2\pi} f(x)e^{\pm inx}\,dx \sim \mp i\pi e^{\pm inr}\sum_{j=0}^{\infty}(-1)^j \frac{g^{(j)}(r)}{(\pm in)^{j+1}} \quad \text{as } n \to \infty. \tag{5.7}$$

That is, due to 2π-periodicity of $f(x)$ and its being infinitely differentiable at the points $x = 0$ and $x = 2\pi$, there are no contributions from these points to the asymptotic expansions of $e_{\pm n}$ as $n \to \pm\infty$. The only contribution comes from the point of singularity.

An example of functions considered here is $f(x) = u(x)\log\left(c|\sin\frac{1}{2}(x-r)|\right)$, where $0 < r < 2\pi$, $u \in C^{\infty}(-\infty,\infty)$ and is 2π-periodic, and c is some positive scalar. For this function,

$$h_1(x) = u(x)\log|x-r|, \quad h_2(x) = u(x)\log\left(c\left|\frac{\sin\frac{1}{2}(x-r)}{x-r}\right|\right).$$

Hence $g(x) = u(x)$. Such functions appear as kernels of some weakly singular Fredholm integral equations that arise from two-dimensional boundary value problems.

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