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ASYMPTOTIC EXPANSION OF MELLIN TRANSFORMS IN THE COMPLEX PLANE

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Abstract: In an earlier paper by the author [A. Sidi, *SIAM J. Math. Anal.*, 16 (1985), pp. 896–906], asymptotic expansions for Mellin transforms $\widehat{f}(z) = \int_0^\infty t^{z-1}f(t)\,dt$ as $z \to \infty$, with z real and positive, were derived. In particular, it was shown there that, for certain classes of functions $u_k(t)$, $k = 0, 1, \ldots$, that form asymptotic scales as $t \to \infty$, if $f(t) \sim \sum_{k=0}^{\infty} A_k u_k(t)$ as $t \to \infty$, then $\widehat{f}(z) \sim \sum_{k=0}^{\infty} A_k \widehat{u}_k(z)$ as $z \to \infty$. In this note, we show that, for two of the cases considered there, $\widehat{f}(z) \sim \sum_{k=0}^{\infty} A_k \widehat{u}_k(z)$ as $z \to \infty$, also when z is complex and $|\Im z| \le \eta(\Re z)^c$, for some $c \in (0, 1)$ and some fixed, but otherwise arbitrary, $\eta > 0$.

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1. Introduction

Let $f(t)$ be locally integrable for $0 < t < \infty$, such that for some real constant σ, $t^\sigma f(t)$ is absolutely integrable in every interval of the form $[0, a]$, and that the Mellin transform of $f(t)$, namely,

$$\widehat{f}(z) = \int_0^\infty t^{z-1} f(t)\,dt, \tag{1.1}$$

exists for all large z, with $\Re z > 0$.

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Let the sequence of functions $\{u_k(t)\}_{k=0}^{\infty}$ be an asymptotic scale as $t \to \infty$, t being real, and let the sequence $\{\widehat{u}_k(z)\}_{k=0}^{\infty}$ of the Mellin transforms of the $u_k(t)$ be also an asymptotic scale as $z \to \infty$ in some fashion.[1] In an earlier paper Sidi [6], we showed that, for certain such sequences $\{u_k(t)\}_{k=0}^{\infty}$, if

$$f(t) \sim \sum_{k=0}^{\infty} A_k u_k(t) \quad \text{as } t \to \infty, \tag{1.2}$$

then $\widehat{f}(z)$ has the asymptotic expansion

$$\widehat{f}(z) \sim \sum_{k=0}^{\infty} A_k \widehat{u}_k(z) \quad \text{as } z \to \infty, \tag{1.3}$$

for z real and positive. Clearly, this result is analogous to Watson's lemma for Laplace transforms. For Watson's lemma, see Olver [4], for example.

In a subsequent paper, Frenzen [1] showed that some of the asymptotic expansions in (1.3) are valid in the complex z-plane provided z is restricted in a suitable manner.

In the present work, we follow [1] partly, and improve the proofs of some of the results of [6] and show that (1.3) is valid for complex z as well, provided z is restricted to a certain part of the complex plane.

Throughout this paper, we let

$$z = x + \mathrm{i}y, \quad x, y \ \text{ real}. \tag{1.4}$$

Two special and commonly occurring cases are considered in [6]. We reconsider these in the present work. Descriptions of these cases follow:

1. The first case is that with

$$u_k(t) = \exp(-\alpha_k t^{\beta}), \quad k = 0, 1, \ldots, \tag{1.5}$$

   where α_k are in general complex and β is real and positive, and

$$0 < \Re\alpha_0 < \Re\alpha_1 < \cdots, \quad \lim_{k\to\infty} \Re\alpha_k = \infty; \quad |\alpha_0| < |\alpha_1| < \cdots . \tag{1.6}$$

[1] A sequence of functions $\{h_k(t)\}_{k=0}^{\infty}$ is an asymptotic scale as $t \to t_0$ if

$$\lim_{t\to t_0} \frac{h_{k+1}(t)}{h_k(t)} = 0, \quad k = 0, 1, \ldots .$$

See Olver [4], for example.

The sequence $\{u_k(t)\}_{k=0}^{\infty}$ is an asymptotic scale as $t \to +\infty$, since for each $k = 0, 1, \ldots,$

$$\frac{u_{k+1}(t)}{u_k(t)} = \exp[-(\alpha_{k+1} - \alpha_k)t^{\beta}]. \tag{1.7}$$

For this case,

$$\widehat{u}_k(z) = \int_0^{\infty} t^{z-1} u_k(t)\, dt = \frac{1}{\beta}\, \alpha_k^{-z/\beta}\, \Gamma\Big(\frac{z}{\beta}\Big). \tag{1.8}$$

Clearly, the integral in (1.8) exists for all complex z with $x > 0$. That $\{\widehat{u}_k(z)\}_{k=0}^{\infty}$ is an asymptotic scale as $z \to \infty$ along any path in the z-plane, such that $x > 0$ and $y = o(x)$ as $x \to \infty$, can be seen as follows: For each $k = 0, 1, \ldots,$ we first have

$$\frac{\widehat{u}_{k+1}(z)}{\widehat{u}_k(z)} = \left(\frac{\alpha_{k+1}}{\alpha_k}\right)^{-z/\beta}. \tag{1.9}$$

Next, by (1.6), we can write

$$\frac{\alpha_{k+1}}{\alpha_k} = \mu_k \exp(\mathrm{i}\,\theta_k), \quad \mu_k > 1, \quad \theta_k \in (-\pi, \pi).$$

Then, we have

$$\begin{aligned}\left|\left(\frac{\alpha_{k+1}}{\alpha_k}\right)^{-z/\beta}\right| &= \exp\Big(-\frac{\log \mu_k}{\beta}\, x + \frac{\theta_k}{\beta}\, y\Big)\\ &\le \exp\Big(-\frac{\log \mu_k}{\beta}\, x + \frac{|\theta_k|}{\beta}\, |y|\Big),\end{aligned}$$

and, since $y = o(x)$ as $x \to \infty$,

$$\begin{aligned}\left|\left(\frac{\alpha_{k+1}}{\alpha_k}\right)^{-z/\beta}\right| &\le \exp\Big(-\frac{\log \mu_k}{\beta}\, x[1 + o(1)]\Big) \quad \text{as } z \to \infty\\ &= o(1) \quad \text{as } z \to \infty.\end{aligned}$$

Here we have made use of the facts that $\log \mu_k > 0$ since $\mu_k > 1$ and that $z \to \infty$ implies $x \to \infty$ when $y = o(x)$ as $x \to \infty$.

We would like to note that the validity of (1.3) for this case has been shown in [6] only for real z.

2. The second case is that with

$$u_k(t) = t^{-\lambda_k} \exp(-\alpha t^{\beta}), \quad k = 0, 1, \dots, \tag{1.10}$$

where α is in general complex, and β and the λ_k are real, such that

$$\Re\alpha > 0; \quad \beta > 0; \quad \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{k\to\infty} \lambda_k = +\infty. \tag{1.11}$$

That $\{u_k(t)\}_{k=0}^{\infty}$ is an asymptotic scale as $t \to \infty$ follows from the fact that, for each $k = 0, 1, \dots,$

$$\frac{u_{k+1}(t)}{u_k(t)} = t^{-(\lambda_{k+1}-\lambda_k)}. \tag{1.12}$$

From this, it is seen immediately that $\{u_k(t)\}_{k=0}^{\infty}$ is an asymptotic scale as $t \to \infty$ with $|\arg t| \leq \pi - \delta < \pi$ in the t-plane, as well as along the $\Re t$-axis. For this case,

$$\widehat{u}_k(z) = \int_0^{\infty} t^{z-1} u_k(t)\, dt = \frac{1}{\beta}\, \alpha^{-(z-\lambda_k)/\beta}\, \Gamma\left(\frac{z-\lambda_k}{\beta}\right). \tag{1.13}$$

Of course, the integral in (1.13) exists for all complex z with $\Re z > \lambda_k$. That $\{\widehat{u}_k(z)\}_{k=0}^{\infty}$ is an asymptotic scale as $z \to \infty$ along any path in the z-plane, such that $|\arg z| \leq \pi - \delta < \pi$, follows from the fact that, for each $k = 0, 1, \dots,$

$$\frac{\widehat{u}_{k+1}(z)}{\widehat{u}_k(z)} \sim \left(\frac{\alpha\beta}{z}\right)^{(\lambda_{k+1}-\lambda_k)/\beta} \quad \text{as } z \to \infty, \quad |\arg z| \leq \pi - \delta < \pi, \tag{1.14}$$

which, in turn, follows from the fact that (see Olver et al. [5, Section 5.11(iii), Eq. 5.11.12]), for fixed a, b,

$$\frac{\Gamma(\zeta + a)}{\Gamma(\zeta + b)} \sim \zeta^{a-b} \quad \text{as } \zeta \to \infty, \quad |\arg \zeta| \leq \pi - \delta < \pi. \tag{1.15}$$

The validity of (1.3) for this case has been shown in [6] only for real z, (i) unconditionally when α is real, and (ii) with some additional analyticity assumptions on $f(t)$ when α is complex.

Before proceeding further, let us introduce some notation that we use throughout the remainder of this paper, and also describe the course of action we take to achieve the proofs of our results.

First, for $c \in (0,1)$ and $\eta > 0$, we let

$$U(c,\eta) = \big\{z = x + \mathrm{i}y :\ x, y \text{ real},\ x > 0,\ |y| \le \eta x^c\big\}. \tag{1.16}$$

Note that, because $0 < c < 1$, all large z in the set $U(c,\eta)$ are contained between the straight lines $y = \pm x$ in the right half of the z-plane, and that the curve $|y| = \eta x^c$, which has the $\Re z$-axis as its axis of symmetry, is the boundary of $U(c,\eta)$. Also note that $z \to \infty$ in the set $U(c,\eta)$ if and only if $x \to \infty$, and that x and y satisfy $y = o(x)$ as $x \to \infty$. In the next sections, we show the validity of (1.3) as $z \to \infty$, $z \in U(c,\eta)$, for some $c \in (0,1)$ and some fixed, but otherwise arbitrary, $\eta > 0$.

Next, let us define

$$r_n(t) = f(t) - \sum_{k=0}^{n-1} A_k u_k(t), \quad n = 0, 1, \dots\ . \tag{1.17}$$

Therefore,

$$\widehat{r}_n(z) = \widehat{f}(z) - \sum_{k=0}^{n-1} A_k \widehat{u}_k(z), \quad n = 0, 1, \dots\ . \tag{1.18}$$

Now, by (1.2), for each fixed n,

$$r_n(t) = O(u_n(t)) \quad \text{as } t \to \infty, \tag{1.19}$$

which means equivalently that there exist positive constants K and T, which may depend on n, such that

$$|r_n(t)| \le K\,|u_n(t)|, \quad t \ge T. \tag{1.20}$$

To prove (1.3), we only need to show that

$$\widehat{r}_n(z) = O(\widehat{u}_n(z)) \quad \text{as } z \to \infty, \quad z \in U(c,\eta), \tag{1.21}$$

or, equivalently, that there exist positive constants L and X, which may depend on n, such that

$$|\widehat{r}_n(z)| \le L|\widehat{u}_n(z)|, \quad z \in U(c,\eta), \quad \Re z \ge X. \tag{1.22}$$

In the next two sections, we treat the two cases described above. In Section 4, we go back to two examples considered in [6] and derive explicit full asymptotic expansions for them. In Section 5, we discuss a way of extending the domains of validity of asymptotic expansions derived in different ways.

2. The Case $u_k(t) = \exp(-\alpha_k t^\beta)$

The main result of this section is an extension of a special case treated in Theorem 2.1 in [6].

Theorem 2.1. *Let $f(t)$ be as in the preceding section, and assume that $f(t)$ satisfies (1.2) with $u_k(t)$ as in (1.5) and (1.6). Then $\widehat{f}(z)$ satisfies*

$$\widehat{f}(z) \sim \sum_{k=0}^{\infty} A_k \widehat{u}_k(z) \quad \text{as } z \to \infty, \quad z \in U(c,\eta), \tag{2.1}$$

where $c \in (0,1)$ and $\eta > 0$ are fixed, but arbitrary otherwise.

*Proof.*We start by observing that, by (1.6), for every integer n, there exists an integer N, $N > n$, such that $\Re\alpha_N > |\alpha_n|$. Then

$$\widehat{r}_n(z) = \sum_{k=n}^{N-1} A_k \widehat{u}_k(z) + \widehat{r}_N(z). \tag{2.2}$$

Clearly,

$$\sum_{k=n}^{N-1} A_k \widehat{u}_k(z) = O(\widehat{u}_n(z)) \quad \text{as } z \to \infty, \quad z \in U(c,\eta). \tag{2.3}$$

Thus, we only need to show that

$$\widehat{r}_N(z) = O(\widehat{u}_n(z)) \quad \text{as } z \to \infty, \quad z \in U(c,\eta). \tag{2.4}$$

By (1.20), we have

$$\begin{aligned} \left|\widehat{r}_N(z)\right| &= \left| \int_0^T t^{z-1}\, r_N(t)\, dt + \int_T^\infty t^{z-1}\, r_N(t)\, dt \right| \\ &\le \int_0^T t^{x-1}\, |r_N(t)|\, dt + \int_T^\infty t^{x-1}\, |r_N(t)|\, dt \\ &\le \int_0^T t^{x-1}\, |r_N(t)|\, dt + K \int_T^\infty t^{x-1}\, |u_N(t)|\, dt. \end{aligned} \tag{2.5}$$

By the integrability conditions imposed on $f(t)$ and by the nature of the $u_k(t)$, it is obvious that $r_N(t)$ is locally integrable for $0 < t < \infty$, and, for some real constant σ, $t^\sigma r_N(t)$ is absolutely integrable in every interval of the form $[0,a]$. Consequently, first

$$\int_0^T t^{x-1}\, |r_N(t)|\, dt = O(T^x) \quad \text{as } z \to \infty, \quad z \in U(c,\eta), \tag{2.6}$$

since $x \to \infty$ when $z \to \infty$ and $z \in U(c, \eta)$. Next, for all large $z \in U(c, \eta)$,

$$\int_T^\infty t^{x-1} \, |u_N(t)| \, dt = \int_T^\infty t^{x-1} \, \exp[-(\Re\alpha_N) t^\beta] \, dt \le \widehat{v}_N(x), \tag{2.7}$$

where

$$\widehat{v}_N(x) = \int_0^\infty t^{x-1} \, \exp[-(\Re\alpha_N) t^\beta] \, dt = \frac{1}{\beta} (\Re\alpha_N)^{-x/\beta} \Gamma\left(\frac{x}{\beta}\right). \tag{2.8}$$

In addition, by (2.5) and the Stirling formula for the Gamma function $\Gamma(x/\beta)$, $\widehat{v}_N(x)$ dominates T^x as $x \to \infty$. As a result, (2.5) reduces to

$$|\widehat{r}_N(z)| \le \widehat{K} |\widehat{v}_N(x)|, \quad z \in U(c, \eta), \quad x \ge X, \tag{2.9}$$

where $\widehat{K}$ and X are some positive constants and $\widehat{K} > K$. By (2.8) and (1.8), and letting $\sigma_n = \arg \alpha_n$, we have

$$\left| \frac{\widehat{v}_N(x)}{\widehat{u}_n(z)} \right| = \left| \frac{\Gamma(x/\beta)}{\Gamma(z/\beta)} \right| \frac{|\alpha_n^{z/\beta}|}{(\Re\alpha_N)^{x/\beta}} = \left| \frac{\Gamma(x/\beta)}{\Gamma(z/\beta)} \right| \left(\frac{|\alpha_n|}{\Re\alpha_N} \right)^{x/\beta} e^{-\sigma_n y/\beta}. \tag{2.10}$$

Now, the Gamma function satisfies (see [5, Section 5.8, Eq. 5.8.3])

$$\begin{aligned} &Q(\rho, \tau) := \left| \frac{\Gamma(\rho)}{\Gamma(\rho + \mathrm{i}\tau)} \right|^2 = \prod_{k=0}^\infty \left[1 + \frac{\tau^2}{(\rho + k)^2} \right], \\ &\rho, \tau \text{ real}, \quad \rho \ne 0, -1, -2, \ldots \; . \end{aligned} \tag{2.11}$$

By the fact that $0 < \log(1 + w) \le w$ for $w > 0$,

$$\log Q(\rho, \tau) = \sum_{k=0}^\infty \log \left[1 + \frac{\tau^2}{(\rho + k)^2} \right] \le \sum_{k=0}^\infty \frac{\tau^2}{(\rho + k)^2} = \tau^2 \zeta(2, \rho), \tag{2.12}$$

where $\sum_{k=0}^\infty (k + a)^{-s} = \zeta(s, a)$, $\Re s > 1$, is the generalized Zeta function (or the Hurwitz function), which also satisfies (see [5, Section 25.11(xii), Eq. 25.11.43])

$$\zeta(s, a) \sim \frac{1}{s-1} a^{1-s} \quad \text{as } a \to \infty.$$

Therefore, there exist positive constants M and R, such that, for $\rho \ge R$,

$$\log Q(\rho, \tau) \le M \frac{\tau^2}{\rho} \quad \Rightarrow \quad Q(\rho, \tau) \le \exp\left(M \frac{\tau^2}{\rho} \right). \tag{2.13}$$

As a result, with $\rho = x/\beta$ and $\tau = y/\beta$ in (2.13), and with

$$\frac{\Re\alpha_N}{|\alpha_n|} = \omega_n > 1, \quad \overline{M} = \frac{M}{2\beta},$$

(2.10) gives the inequality

$$\left|\frac{\widehat{v}_N(x)}{\widehat{u}_n(z)}\right| \leq \exp\left(\left[-\frac{\log\omega_n}{\beta} + \frac{|\sigma_n|}{\beta}\frac{|y|}{x} + \overline{M}\left(\frac{|y|}{x}\right)^2\right]x\right). \tag{2.14}$$

Now, with $z \in U(c,\eta)$ and $0 < c < 1$, we have $\lim_{z\to\infty}(|y|/x) = 0$. Consequently, (2.14) becomes

$$\begin{aligned}\left|\frac{\widehat{v}_N(x)}{\widehat{u}_n(z)}\right| &\leq \exp\left(-\frac{\log\omega_n}{\beta}\,x\,[1+o(1)]\right)\\ &= o(1) \quad \text{as } z\to\infty, \quad z \in U(c,\eta).\end{aligned}$$

Combining this with (2.9), we obtain (2.4). This completes the proof. □

3. The Case $u_k(t) = t^{-\lambda_k}\exp(-\alpha t^\beta)$

Our first result of this section is an extension of Theorem 2.2 in [6].

Theorem 3.1. *Let $f(t)$ be as in the preceding section, and assume that $f(t)$ satisfies (1.2) with $u_k(t)$ as in (1.10) and (1.11), but with real α. Then $\widehat{f}(z)$ satisfies*

$$\widehat{f}(z) \sim \sum_{k=0}^{\infty} A_k\widehat{u}_k(z) \quad \text{as } z\to\infty, \quad z \in U(\tfrac{1}{2},\eta), \tag{3.1}$$

where $\eta > 0$ is fixed, but arbitrary otherwise.

Proof. By (1.20), we have

$$\begin{aligned}\left|\widehat{r}_n(z)\right| &= \left|\int_0^T t^{z-1}\,r_n(t)\,dt + \int_T^\infty t^{z-1}\,r_n(t)\,dt\right|\\ &\leq \int_0^T t^{x-1}\,|r_n(t)|\,dt + \int_T^\infty t^{x-1}\,|r_n(t)|\,dt\\ &\leq \int_0^T t^{x-1}\,|r_n(t)|\,dt + K\int_T^\infty t^{x-1}\,|u_n(t)|\,dt. \end{aligned}\tag{3.2}$$

By the integrability conditions imposed on $f(t)$ and by the nature of the $u_k(t)$, it is obvious that $r_n(t)$ is locally integrable for $0 < t < \infty$, and, for some real constant σ, $t^\sigma r_n(t)$ is absolutely integrable in every interval of the form $[0, a]$. Consequently, first

$$\int_0^T t^{x-1}\,|r_n(t)|\,dt = O(T^x) \quad \text{as } z \to \infty, \quad z \in U(\tfrac{1}{2}, \eta), \tag{3.3}$$

since $x \to \infty$ when $z \to \infty$ and $z \in U(\frac{1}{2}, \eta)$. Next, for all large $z \in U(\frac{1}{2}, \eta)$,

$$\int_T^\infty t^{x-1}\,|u_n(t)|\,dt = \int_T^\infty t^{x-1}\,u_n(t)\,dt \le \int_0^\infty t^{x-1}\,u_n(t)\,dt = \widehat{u}_n(x), \tag{3.4}$$

since $u_k(t)$ are all real and positive for $0 < t < \infty$ when α, β, and the λ_k in (1.10) are all real. In addition, by (1.13) and the Stirling formula for the Gamma function, $\widehat{u}_k(x)$ dominates T^x as $x \to \infty$. As a result, (3.2) reduces to

$$|\widehat{r}_n(z)| \le \widehat{K}|\widehat{u}_n(x)|, \quad z \in U(\tfrac{1}{2}, \eta), \quad x \ge X, \tag{3.5}$$

where $\widehat{K}$ and X are some positive constants and $\widehat{K} > K$. Now, by (1.13) and (2.13), for some constant $\overline{M} > 0$, there holds

$$\begin{aligned} \left|\frac{\widehat{u}_n(x)}{\widehat{u}_n(z)}\right| &= \left|\frac{\Gamma\left(\frac{x-\lambda_n}{\beta}\right)}{\Gamma\left(\frac{z-\lambda_n}{\beta}\right)}\right| = \left[Q\left(\frac{x-\lambda_n}{\beta}, \frac{y}{\beta}\right)\right]^{\frac{1}{2}} \\ &\le \exp\left(\overline{M}\,\frac{y^2}{x}\right) \le \exp\left(\overline{M}\eta^2\right), \quad z \in U(\tfrac{1}{2}, \eta). \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we have

$$|\widehat{r}_n(z)| \le L|\widehat{u}_n(z)|, \quad z \in U(\tfrac{1}{2}, \eta), \quad x \ge X, \tag{3.7}$$

where $L > 0$ is some constant. This completes the proof. $\square$

We now turn to the general case in which α is complex with $\Re\alpha > 0$. This case is covered in Theorem 3.2, which is an extension of Theorem 2.3 in [6]. Theorem 3.2 is obtained by applying Theorem 3.1 to a function derived from $f(t)$ when $f(t)$ is analytic in some sector of the t-plane. For completeness, we provide the proof of this theorem here.

Theorem 3.2. *Let $f(t)$ be as in the preceding section, and assume that $f(t)$ satisfies* (1.2) *with $u_k(t)$ as in* (1.10) *and* (1.11), *with complex α, such that $\Re\alpha > 0$. Denote $\omega = \arg\alpha$, and assume that $|\omega|/\beta < \pi$. Denote also*

$\theta_1 = \min\{0, -\omega/\beta\}$ and $\theta_2 = \max\{0, -\omega/\beta\}$, and let $W = (\theta_1 - \delta, \theta_2 + \delta)$ for some small $\delta > 0$. Assume that, for some $T_0 \geq 0$, the function $f(t)$ is analytic in the set $D = \{t : \ |t| \geq T_0, \ \arg t \in W\}$, and that

$$f(t) \sim \sum_{k=0}^{\infty} A_k u_k(t) \quad \text{as } t \to \infty, \quad t \in D. \tag{3.8}$$

If, in addition,

$$\lim_{R\to\infty} \int_{L(R)} t^{z-1} f(t)\, dt = 0 \quad \text{for all large } z \in U(\tfrac{1}{2}, \eta), \tag{3.9}$$

where

$$L(w) = \{t : \ t = we^{\mathrm{i}\theta}, \ \theta \text{ goes from } 0 \text{ to } -\omega/\beta\}, \quad w > 0,$$

then $\widehat{f}(z)$ satisfies

$$\widehat{f}(z) \sim \sum_{k=0}^{\infty} A_k \widehat{u}_k(z) \quad \text{as } z \to \infty, \quad z \in U(\tfrac{1}{2}, \eta), \tag{3.10}$$

where $\eta > 0$ is fixed, but arbitrary otherwise.

If (3.8) holds uniformly in D, then (3.9) is automatically satisfied.

Proof. By the assumption that $f(t)$ is analytic in the set D, and because $e^{-\alpha t^{\beta}} \to 0$ as $t \to \infty$, $\arg t \in W$, and on account of the assumption in (3.9), we can deform the contour $0 < t < \infty$ in (1.1) and write

$$\widehat{f}(z) = \left(\int_0^{T_0} + \int_{L(T_0)} + \int_C \right) t^{z-1} f(t)\, dt, \tag{3.11}$$

where $C = \{t : \ t = \rho e^{-\mathrm{i}\omega/\beta}, \ \rho \text{ goes from } 0 \text{ to } \infty\}$. Next, as we have already seen, by the integrability conditions imposed on $f(t)$ along the $\Re t$ axis,

$$I_1(z) = \int_0^{T_0} t^{z-1} f(t)\, dt = O(T_0^x) \quad \text{as } z \to \infty, \quad z \in U(\tfrac{1}{2}, \eta). \tag{3.12}$$

Similarly,

$$I_2(z) = \int_{L(T_0)} t^{z-1} f(t)\, dt = O(T_0^x) \quad \text{as } z \to \infty, \quad z \in U(\tfrac{1}{2}, \eta), \tag{3.13}$$

since

$$\left| \int_{L(T_0)} t^{z-1} f(t)\, dt \right| \leq \frac{|\omega|}{\beta} \left(\max_{\theta_1 \leq \theta \leq \theta_2} \left| f\left(T_0 e^{\mathrm{i}\theta}\right) \right| \right) e^{|\omega||y|/\beta} T_0^x \tag{3.14}$$

and $y = o(x)$ as $x \to \infty$, as implied by $z \in U(\frac{1}{2}, \eta)$. Next,

$$
\begin{aligned}
I_3(z) &= \int_C t^{z-1} f(t)\, dt \\
&= e^{-\mathrm{i}\omega z/\beta} \int_{T_0}^{\infty} \rho^{z-1} F(\rho)\, d\rho, \quad F(\rho) = f\big(\rho e^{-\mathrm{i}\,\omega/\beta}\big),
\end{aligned} \tag{3.15}
$$

and, by (3.8), $F(\rho)$ has the asymptotic expansion

$$
F(\rho) \sim \sum_{k=0}^{\infty} A'_k v_k(\rho) \quad \text{as } \rho \to \infty, \tag{3.16}
$$

where

$$
v_k(\rho) = \rho^{-\lambda_k} \exp\big(-|\alpha|\rho^{\beta}\big), \quad A'_k = A_k \exp\left(\frac{\mathrm{i}\,\omega\lambda_k}{\beta}\right). \tag{3.17}
$$

Clearly, the integral $\int_{T_0}^{\infty} \rho^{z-1} F(\rho)\, d\rho$ is the Mellin transform of $H(\rho - T_0)\, F(\rho)$, where $H(u)$ is the Heaviside unit step function.[2] It is easy to see that Theorem 3.1 applies to this Mellin transform, and we have

$$
\int_{T_0}^{\infty} \rho^{z-1} F(\rho)\, d\rho \sim \sum_{k=0}^{\infty} A'_k \widehat{v}_k(z) \quad \text{as } z \to \infty, \quad z \in U(\tfrac{1}{2}, \eta), \tag{3.18}
$$

where, analogously to (1.13),

$$
\widehat{v}_k(z) = \frac{1}{\beta}\, |\alpha|^{-(z-\lambda_k)/\beta}\, \Gamma\left(\frac{z-\lambda_k}{\beta}\right), \quad k = 0, 1, \dots\ . \tag{3.19}
$$

Substituting (3.18) in (3.15), and invoking (3.17) and (3.19), we obtain

$$
I_3(z) \sim \sum_{k=0}^{\infty} A_k \widehat{u}_k(z) \quad \text{as } z \to \infty, \quad z \in U(\tfrac{1}{2}, \eta). \tag{3.20}
$$

Combining (3.12), (3.13), and (3.20) in (3.11), and noting that $I_3(z)$ dominates both $I_1(z)$ and $I_2(z)$, we obtain (3.10).

Let us now assume that (3.8) holds uniformly for $\theta = \arg t \in W$. We want to show that (3.9) is automatically satisfied in this case. To see this, observe that,

[2] Recall that the Heaviside unit step function is defined via

$$
H(u) = \begin{cases} 0 & \text{if } u < 0, \\ 1 & \text{if } u \geq 0. \end{cases}
$$

under this condition, there exist positive constants K and $T > T_0$, independent of t, such that

$$|f(t)| \leq K|u_0(t)|, \quad |t| \geq T, \quad \theta = \arg t \in W. \tag{3.21}$$

Thus, for $R \geq T$,

$$\begin{aligned}\left|\int_{L(R)} t^{z-1} f(t)\,dt\right| &\leq K \int_{L(R)} \left|t^{z-1}\right| |u_0(t)|\,|dt| \\ &\leq KR^{x-\lambda_0} e^{|\omega||y|/\beta} \\ &\quad\times \int_{\theta_1}^{\theta_2} \exp[-|\alpha|R^{\beta}\cos(\omega+\beta\theta)]\,d\theta. \end{aligned} \tag{3.22}$$

Now, for $\theta \in [\theta_1, \theta_2]$, we have $|\omega + \beta\theta| \leq |\omega| < \frac{1}{2}\pi$; therefore, $\cos(\omega+\beta\theta) \geq \cos\omega > 0$. Consequently,

$$\left|\int_{L(R)} t^{z-1} f(t)\,dt\right| \leq K\frac{|\omega|}{\beta} R^{x-\lambda_0} e^{|\omega||y|/\beta} \exp(-|\alpha|R^{\beta}\cos\omega), \tag{3.23}$$

and (3.9) follows by letting $R \to \infty$.

This completes the proof. □

4. Examples

We now demonstrate the application of the results of the preceding sections to two special functions.

Example 4.1. In [6], we derived an asymptotic expansion as $\nu \to \infty$ for $K_\nu(z)$, the modified Bessel function of the second kind of order ν. We did this by using an integral representation that enabled us to express $K_\nu(z)$ as a Mellin transform of some function $f(t)$. Unfortunately, due to the complex nature of $f(t)$, we were not able to get a closed-form expression for the coefficients in the asymptotic expansion of $f(t)$ as $t \to \infty$. We now revisit $K_\nu(z)$, and derive its full asymptotic expansion as $\nu \to \infty$, by using the integral representation

$$K_\nu(z) = \widehat{f}(\nu), \quad f(t) = \frac{1}{2}\exp\left(-\frac{zt}{2} - \frac{z}{2t}\right), \quad \Re z > 0. \tag{4.1}$$

This representation is obtained by letting $\gamma = \delta = \frac{1}{2}z$ in the integral (see Gradshteyn and Ryzhik [2, p. 340, Eq. 9])

$$\int_0^\infty t^{\nu-1}\exp\left(-\gamma t - \frac{\delta}{t}\right)dt = 2\left(\frac{\delta}{\gamma}\right)^{\nu/2} K_\nu\left(2\sqrt{\gamma\delta}\right), \quad \Re\gamma, \Re\delta > 0. \tag{4.2}$$

As $t \to \infty$, $f(t)$ has the (convergent) asymptotic expansion

$$f(t) = \frac{1}{2}\, e^{-zt/2} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}z\right)^k}{k!}\, t^{-k}. \tag{4.3}$$

It is easy to see that $f(t)$ satisfies all the conditions of Theorem 3.2, with $\beta = 1$ and $\lambda_k = k$ there. Therefore,

$$K_\nu(z) \sim \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}z\right)^k}{k!} \frac{\Gamma(\nu - k)}{\left(\frac{1}{2}z\right)^{\nu-k}} \quad \text{as } \nu \to \infty, \quad \nu \in U(\tfrac{1}{2}, \eta), \tag{4.4}$$

for fixed but arbitrary $\eta > 0$. Invoking

$$\frac{\Gamma(\nu - k)}{\Gamma(\nu)} = \frac{1}{\prod_{i=1}^{k} (\nu - i)}, \tag{4.5}$$

(4.4) becomes

$$K_\nu(z) \sim \frac{1}{2} \frac{\Gamma(\nu)}{\left(\frac{1}{2}z\right)^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}z\right)^{2k}}{k!} \frac{1}{\prod_{i=1}^{k} (\nu - i)}$$
$$\text{as } \nu \to \infty, \quad \nu \in U(\tfrac{1}{2}, \eta). \tag{4.6}$$

Now

$$\frac{1}{\prod_{i=1}^{k} (\nu - i)} = \sum_{m=k}^{\infty} S(m,k) \nu^{-m}, \quad |\nu| > k, \tag{4.7}$$

where $S(m,k)$ are the Stirling numbers of the second kind (see [3, Sections 6.1 and 7.4]). Substituting (4.7) in (4.6), and rearranging, we obtain

$$K_\nu(z) \sim \frac{1}{2} \frac{\Gamma(\nu)}{\left(\frac{1}{2}z\right)^\nu} \sum_{m=0}^{\infty} \frac{h_m(z)}{\nu^m} \quad \text{as } \nu \to \infty, \quad \nu \in U(\tfrac{1}{2}, \eta), \tag{4.8}$$

where

$$h_m(z) = \sum_{k=0}^{m} (-1)^k \frac{S(m,k)}{k!} \left(\tfrac{1}{2}z\right)^{2k}, \quad m = 0, 1, \ldots\ . \tag{4.9}$$

Since

$$S(0,0) = 1; \quad S(m,0) = 0, \quad m = 1, 2, \ldots, \tag{4.10}$$

the $h_m(z)$ of (4.9) are actually as in

$$h_0(z) = 1; \quad h_m(z) = \sum_{k=1}^{m} (-1)^k \frac{S(m,k)}{k!} \left(\tfrac{1}{2}z\right)^{2k}, \quad m = 1, 2, \ldots\ . \tag{4.11}$$

Example 4.2. In [6], we also derived the asymptotic expansion as $\nu \to \infty$ of $Y_\nu(z)$, the Bessel function of the second kind of order ν. For this, we used the integral representation (see [5, Section 10.9(i), Eq. 10.9.7])

$$\begin{aligned} Y_\nu(z) =& \frac{1}{\pi}\int_0^\pi \sin(z\sin\theta - \nu\theta)\,d\theta \\ &- \frac{1}{\pi}\int_0^\infty [e^{\nu\tau} + e^{-\nu\tau}\cos(\nu\pi)]\exp(-z\sinh\tau)\,d\tau, \quad \Re z > 0. \end{aligned} \tag{4.12}$$

However, we did not give a closed-form expression for the coefficients in the asymptotic expansion of $Y_\nu(z)$, just as in the preceding example. We do this here.

The integral over θ is $O(\nu^{-1})$ as $\nu \to \infty$, as can be seen by integration parts once. Let us express the integral over τ as the sum of two integrals

$$\begin{aligned} I_1(\nu) &= -\frac{1}{\pi}\int_0^\infty e^{\nu\tau}\exp(-z\sinh\tau)\,d\tau, \\ I_2(\nu) &= -\frac{\cos(\nu\pi)}{\pi}\int_0^\infty e^{-\nu\tau}\exp(-z\sinh\tau)\,d\tau. \end{aligned} \tag{4.13}$$

Using Watson's lemma, it is easy to show that $I_2(\nu) = O(\nu^{-1})$ as $\nu \to \infty$. Let us now make the change of variable $t = e^\tau$ in $I_1(\nu)$. Then

$$I_1(\nu) = \widehat{f}(\nu), \quad f(t) = -\frac{1}{\pi}H(t-1)\exp\left(-\frac{zt}{2} + \frac{z}{2t}\right), \tag{4.14}$$

where $H(u)$ is the Heaviside unit step function, as before. As $t \to \infty$, $f(t)$ has the (convergent) asymptotic expansion

$$f(t) = -\frac{1}{\pi}\,e^{-zt/2}\sum_{k=0}^\infty \frac{(\frac{1}{2}z)^k}{k!}\,t^{-k}. \tag{4.15}$$

Observing the similarity of this $f(t)$ to that in the preceding example, we realize that we can proceed exactly as in the latter, and that the dominant contribution to the asymptotic expansion of $Y_\nu(z)$ comes only from $I_1(\nu)$, the rest of the contributions being negligible. Thus, we obtain

$$Y_\nu(z) \sim -\frac{1}{\pi}\frac{\Gamma(\nu)}{(\frac{1}{2}z)^\nu}\sum_{m=0}^\infty \frac{g_m(z)}{\nu^m} \quad \text{as } \nu \to \infty, \quad \nu \in U(\tfrac{1}{2}, \eta), \tag{4.16}$$

where $g_m(z) = h_m(\mathrm{i}z)$ [$h_m(z)$ as in (4.8) and (4.11)], and hence

$$g_0(z) = 1; \quad g_m(z) = \sum_{k=1}^m \frac{S(m,k)}{k!}\left(\tfrac{1}{2}z\right)^{2k}, \quad m = 1, 2, \ldots\ . \tag{4.17}$$

5. Further Developments

As already mentioned, with the help of Theorems 2.1, 3.1, and 3.2, we can increase the domains of validity of the asymptotic expansions derived in the examples given in [6]. This fact can be used to increase the domains of validity further. We explain how this can be done next via $K_\nu(z)$.

So far, we know that $K_\nu(z)$ has the asymptotic expansion given in (4.8) with (4.9), and that this expansion is valid for $\nu \in U(\frac{1}{2}, \eta)$. In Sidi and Hoggan [7], we also show that the *same* asymptotic expansion is valid for

$$\nu \in V(\epsilon) = \{\nu : \ |\Im\nu| \geq \epsilon, \ |\arg\nu| \leq \tfrac{1}{2}\pi - \delta < \tfrac{1}{2}\pi\}.$$

Since $V(\epsilon) \cap U(\frac{1}{2}, \eta)$ is nonempty and unbounded, and since the asymptotic expansion of a function in terms of a fixed asymptotic scale is unique, we conclude that the asymptotic expansion in (4.8) with (4.9) is actually valid for

$$\nu \in \{\nu : \ |\arg\nu| \leq \tfrac{1}{2}\pi - \delta < \tfrac{1}{2}\pi\} \subset V(\epsilon) \cup U(\tfrac{1}{2}, \eta).$$

This approach can be generalized and formalized as in the following lemma, given in [7]:

Lemma 5.1. *Let D_1 and D_2 be two unbounded sets and let $\widehat{D} = D_1 \cap D_2$ be also unbounded. Let $f(\nu)$ have the following asymptotic expansions:*

$$f(\nu) \sim \sum_{k=0}^{\infty} A_k^{(i)} \phi_k(\nu) \quad \text{as } \nu \to \infty, \quad \nu \in D_i, \quad i = 1, 2,$$

where $\{\phi_k(\nu)\}_{k=0}^{\infty}$ is an asymptotic scale as $\nu \to \infty$ in both D_1 and D_2. Then

$$A_k^{(1)} = A_k^{(2)} = A_k, \quad k = 0, 1, \ldots \ .$$

Consequently, the series $\sum_{k=0}^{\infty} A_k \phi_k(\nu)$ represents $f(\nu)$ asymptotically as $\nu \to \infty$ in $D_1 \cup D_2$.

Proof. Since both series $\sum_{k=0}^{\infty} A_k^{(i)} \phi_k(\nu)$, $i = 1, 2$, represent $f(\nu)$ asymptotically as $\nu \to \infty$ in $\widehat{D}$, they must be the same because the asymptotic expansion of a function in terms of a fixed asymptotic scale is unique (see [4, p. 17, Theorem 7.1]). $\square$

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