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A user-friendly extrapolation method for computing infinite range integrals of products of oscillatory functions

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In this work we propose an extrapolation method, which we call the $mW^{(s)}$-transformation, for the accurate and numerically stable computation of infinite range integrals $I[f] = \int_a^\infty f(x)\,\mathrm{d}x$, $a \geqslant 0$, where the $f(x)$ are products of oscillatory functions, which are ultimately expressible (but do not have to be given explicitly, *a priori*) in the form

$$f(x) = R(x)\exp\left[\hat{\phi}(x)\right]\prod_{j=1}^{s}\left\{g_j^+(x)\exp\left[\mathrm{i}\hat{\theta}(x)\right] + g_j^-(x)\exp\left[-\mathrm{i}\hat{\theta}(x)\right]\right\},$$

where s is an arbitrary integer, $\hat{\phi}(x)$ and $\hat{\theta}(x)$ are real polynomials in x, $\hat{\phi}(x) \equiv 0$ being possible, $\exp[\hat{\phi}(x)]$ is bounded at infinity and $R(x)$ and $g_j^{\pm}(x)$ are smooth functions that have asymptotic expansions of the forms $R(x) \sim \sum_{i=0}^{\infty} b_i x^{\epsilon - i}$ and $g_j^{\pm}(x) \sim \sum_{i=0}^{\infty} c_{ji}^{\pm} x^{\delta_j - i}$ as $x \to \infty$, with arbitrary ϵ and δ_j. We denote the class of such functions by $\widetilde{\mathbf{B}}^{(s)}$. These integrals may converge or diverge and in the case of divergence are defined in some summability sense. The $mW^{(s)}$-transformation we propose here is analogous to the mW-transformation of Sidi (1988, A user-friendly extrapolation method for oscillatory infinite integrals. *Math. Comput.*, **51**, 249–266), which was originally developed for a class of infinite range oscillatory integrals, whose integrands actually belong to a subfamily of $\widetilde{\mathbf{B}}^{(s)}$. The $mW^{(s)}$-transformation determines a two-dimensional array of approximations $A_n^{(j)}$ to $I[f]$. We study some of the convergence properties of the $A_n^{(j)}$. We also provide several numerical examples that illustrate the performance of the method.

Keywords: extrapolation methods; mW-transformation; W-algorithm; infinite range oscillatory integrals; numerical integration; Bessel functions; asymptotic expansions.

1. Introduction

Computation of infinite range integrals of oscillatory functions is an important problem that arises in many different contexts. Several numerical methods have been developed for this purpose, extrapolation methods being the most effective. One class of such integrals, namely, that involving *products* of several oscillatory functions, is particularly interesting and challenging and has been of interest recently. For example, Lucas (1995) and Sidi (2003, Chapter 11, pp. 226–227) have considered integrals of products of two Bessel functions; in Lucas (1995) the two Bessel functions have different arguments, while in Sidi (2003) they have the same argument, and these are two different problems that need different treatments. Integrals of products of an arbitrary number of general oscillatory functions are considered in Sidi (2003, Chapter 11, pp. 236–237), and the approach there has also been used by Van Deun & Cools

(2006, 2008) to develop a method for computing integrals of an arbitrary product of Bessel functions with some special factors; this method is based on asymptotic expansions.

A very effective extrapolation method for computing the integrals above is the D-transformation of Levin & Sidi (1981). If the D-transformation is applied without taking into account the special nature of the problem, however, its computational cost generally increases with the number of oscillatory functions that make up the integrand. By taking into account this special nature these integrals can be evaluated much more economically by the $\bar{D}$-, W- and mW-transformations of Sidi (1980, 1982b, 1988, 1997). For all these methods, and related recent developments, see Sidi (2003, Chapter 11).

The method of extrapolation (for computing integrals involving the product of two Bessel functions) used successfully in Lucas (1995) is the mW-transformation. The effectiveness of this transformation is a result of the two Bessel functions having *different* arguments in Lucas (1995). As defined originally the mW-transformation does not perform well when the arguments of the Bessel functions are the same. This phenomenon is observed also in a recent work by Bailey & Borwein (2011), who apply the mW-transformation (in very high precision) to the integrals

$$\int_1^\infty [J_0(x)]^s \frac{dx}{x}, \quad s = 0, 1, 2, \ldots, \tag{1.1}$$

where $J_0(x)$ is the Bessel function of order zero of the first kind. They report very good numerical results for odd s, and mediocre results for even s. It is a generalization of this special problem case that we wish to address in this work. In the process we will give a rigorous explanation as to why the mW-transformation is effective when s is odd and ineffective when s is even when applied to (1.1). We also show how the mW-transformation can be modified in a simple way and made effective again for even s. We call the resulting extrapolation method the $mW^{(s)}$-transformation since it depends on whether s is even or odd.

In the next section we review the function classes denoted $\mathbf{A}^{(\gamma)}$ and $\mathbf{B}^{(m)}$ that we refer to in the sequel, and we also introduce the function classes $\widetilde{\mathbf{B}}^{(s)}$, $s \geqslant 1$ integers. Members of $\widetilde{\mathbf{B}}^{(s)}$ are products of s functions that oscillate an infinite number of times at infinity and have *asymptotically, as $x \to \infty$, the same phase of oscillation*; specifically, the functions $H_j(x)$ that make up the product are ultimately of the form

$$\hat{\omega}_j^+(x) \exp\left[\mathrm{i}\hat{\theta}(x)\right] + \hat{\omega}_j^-(x) \exp\left[-\mathrm{i}\hat{\theta}(x)\right],$$

where $\hat{\theta}(x)$ is a real polynomial in x (it is the same for every j) and the $\hat{\omega}_j^{\pm}(x)$ are nonoscillatory and smooth as $x \to \infty$.

In Section 3 we give a detailed description of the $mW^{(s)}$-transformation for computing infinite range integrals of functions in $\widetilde{\mathbf{B}}^{(s)}$.

In Section 4 we analyse the integral properties of the functions in $\widetilde{\mathbf{B}}^{(s)}$. This analysis is based completely on Sidi (2003, Section 5.7, pp. 117–120). The following facts transpire from this analysis.

1. If $f \in \widetilde{\mathbf{B}}^{(s)}$, s being an odd integer, then $f(x)$ is a sum of at most $s+1$ oscillatory functions with different phases, and the mW-transformation of Sidi (1988) can be shown to be very effective for computing $\int_a^\infty f(x)\,dx$ in this case.
2. If $f \in \widetilde{\mathbf{B}}^{(s)}$, s being an even integer, then $f(x)$ is a sum of at most s oscillatory functions with different phases, and, in general, one nonoscillatory function, which may be absent in certain cases. It is the presence of the nonoscillatory function that causes the mW-transformation of Sidi (1988) to lose its effectiveness in computing $\int_a^\infty f(x)\,dx$ since the mW-transformation was designed for computing infinite range integrals of oscillatory functions only.

3. We also note that, in the case of divergence, the integrals of the oscillatory functions are defined in the sense of Abel summability (see Sidi, 1987), while the integral of the nonoscillatory function is defined in the sense of Hadamard finite part (see Sidi, 1999).

In view of the study in Section 4, in Section 5, we provide the detailed derivation of the $mW^{(s)}$-transformation. In Section 6 we discuss some of the convergence properties of this transformation in light of the theory presented in Sidi (2003, Chapters 8 and 9). In Section 7 we illustrate the method with convergent and divergent integrals whose integrands are in $\widetilde{\mathbf{B}}^{(s)}$ for various values of s. The appendix to the paper contains an important part of the analysis that leads to the derivation of the $mW^{(s)}$-transformation. As such, it is an integral part of this work.

Finally, the $mW^{(s)}$-transformation is closely related to the $\widetilde{d}^{(m)}$-transformation of the author that is presented in Sidi (2003, Section 6.6, pp. 140–149). We do not go into the details of this relation here.

Before closing, we note that integrands in $\widetilde{\mathbf{B}}^{(s)}$ are quite different from those that are products of oscillatory functions with not necessarily the same phase of oscillation and so are their integral properties and their numerical treatment. The latter have a more complicated structure than the former and involve more analysis and computing. See Sidi (2003, Chapter 11, pp. 236–237) for details. See also Van Deun & Cools (2006, 2008), where integrals of products of Bessel functions with not necessarily the same argument are treated.

2. The function classes $\widetilde{\mathbf{B}}^{(s)}$

We begin with the following definitions (see Sidi, 2003, Chapter 5, Definitions 5.1.1 and 5.1.2).

DEFINITION 2.1 A function $\alpha(x)$ belongs to the set $\mathbf{A}^{(\gamma)}$ if it is infinitely differentiable for all large $x > 0$ and has a Poincaré-type asymptotic expansion of the form

$$\alpha(x) \sim \sum_{i=0}^{\infty} \alpha_i x^{\gamma - i} \quad \text{as } x \to \infty, \tag{2.1}$$

and its derivatives have Poincaré-type asymptotic expansions obtained by differentiating that in (2.1) formally term by term. If, in addition, $\alpha_0 \neq 0$ in (2.1), then $\alpha(x)$ is said to belong to $\mathbf{A}^{(\gamma)}$ strictly. Here γ is complex in general.

Functions in the sets $\mathbf{A}^{(\gamma)}$ have some very useful properties that are listed in Sidi (2003, pp. 96–98). Among these, we note the following, for example:

$$\begin{aligned}
&\alpha \in \mathbf{A}^{(\gamma)} \quad \Rightarrow \quad x^{-\gamma}\alpha(x) \in \mathbf{A}^{(0)},\\
&\alpha \in \mathbf{A}^{(\gamma)}, \quad \beta \in \mathbf{A}^{(\delta)} \quad \Rightarrow \quad \alpha\beta \in \mathbf{A}^{(\gamma+\delta)},\\
&\alpha \in \mathbf{A}^{(\gamma)}, \quad \beta \in \mathbf{A}^{(\delta)} \text{ strictly} \quad \Rightarrow \quad \alpha/\beta \in \mathbf{A}^{(\gamma-\delta)},\\
&\alpha \in \mathbf{A}^{(\gamma+k)} \text{ strictly}, \quad k > 0 \text{ integer}, \quad \beta \in \mathbf{A}^{(\gamma)} \quad \Rightarrow \quad \alpha + \beta \in \mathbf{A}^{(\gamma+k)} \text{ strictly}.
\end{aligned}$$

We advise the reader to familiarize himself with the list given in Sidi (2003, pp. 96–98) as we will be invoking the properties of functions in the classes $\mathbf{A}^{(\gamma)}$ often and without mentioning them.

DEFINITION 2.2 A function $f(x)$ belongs to the class $\mathbf{B}^{(m)}$ if it satisfies a linear homogeneous differential equation of the form

$$f(x) = \sum_{k=1}^{m} p_k(x) f^{(k)}(x); \quad p_k \in \mathbf{A}^{(i_k)}, \quad i_k \text{ integer}, \quad i_k \leqslant k, \quad k = 1, \ldots, m. \tag{2.2}$$

We note that most special functions of applied mathematics that are defined via linear differential equations belong to such classes. For example, all Bessel functions are in $\mathbf{B}^{(2)}$.

Functions in the classes $\mathbf{B}^{(m)}$ have some very interesting and useful properties. For example, in general, if $f \in \mathbf{B}^{(r)}$ and $g \in \mathbf{B}^{(s)}$, then $f + g \in \mathbf{B}^{(m)}$ with $m \leqslant r + s$, while $fg \in \mathbf{B}^{(m)}$ with $m \leqslant rs$. See Levin & Sidi (1981) and Sidi (2003, Chapter 5, pp. 106–111).[1]

We now define the classes of functions we wish to treat in this work.

DEFINITION 2.3 We say that a function $f(x)$ belongs to the class $\widetilde{\mathbf{B}}^{(s)}$ if it can be expressed in the form

$$f(x) = W(x) \prod_{j=1}^{s} H_j(x), \tag{2.3}$$

where

$$W(x) = \exp[\phi(x)]Q(x); \quad \phi \in \mathbf{A}^{(k)}, \quad k \text{ integer}, \quad Q \in \mathbf{A}^{(\epsilon)} \quad \text{strictly},$$
$$\phi(x) = \hat{\phi}(x) + L(x); \quad \hat{\phi}(x) = \sum_{i=0}^{k-1} \lambda_i x^{k-i} \text{ real, with } \lambda_0 < 0 \text{ if } k > 0; \quad L \in \mathbf{A}^{(0)}, \tag{2.4}$$

and

$$H_j(x) = h_j^{+}(x) \exp[\mathrm{i}\theta_j(x)] + h_j^{-}(x) \exp[-\mathrm{i}\theta_j(x)],$$
$$h_j^{\pm} \in \mathbf{A}^{(\delta_j)}, \quad \theta_j \in \mathbf{A}^{(m)}, \quad m > 0 \text{ integer},$$
$$\theta_j(x) = \hat{\theta}(x) + M_j(x); \quad \hat{\theta}(x) = \sum_{i=0}^{m-1} \mu_i x^{m-i} \text{ real, with } \mu_0 > 0; \quad M_j \in \mathbf{A}^{(0)}. \tag{2.5}$$

Note that, when $k \leqslant 0$, we have $\hat{\phi}(x) \equiv 0$; in this case $\exp[\phi(x)] \in \mathbf{A}^{(0)}$ and hence $W \in \mathbf{A}^{(\epsilon)}$. When $k > 0$, $\lim_{x\to\infty} \exp[\phi(x)] = 0$ since $\lambda_0 < 0$.

Remarks.

1. It should be emphasized that $f(x)$ does not have to be given explicitly in the form described in (2.3–2.5). It only has to be *expressible* in that form.
2. Note that $\hat{\theta}(x)$ is the polynomial part of the asymptotic expansion of $\theta_j(x)$ and is the *same* for all j. Similarly, $\hat{\phi}(x)$ is the polynomial part of the asymptotic expansion of $\phi(x)$. The functions $H_j(x)$ determine the oscillatory behaviour of $f(x)$.
3. (a) When $k > 0$, we have $\lim_{x\to\infty} \hat{\phi}(x) = -\infty$; thus, the amplitude of $f(x)$ is modulated by the exponentially decaying factor $W(x)$; $I[f]$ is always convergent in this case. (Note that, if $\lim_{x\to\infty} \hat{\phi}(x) = +\infty$, then $f(x)$ is not integrable in any sense; hence, it is irrelevant from the computational point of view.)

[1] These results are stated as 'heuristics' in Sidi (2003) since they are stated for a relaxed version of the class $\mathbf{B}^{(m)}$ in which the functions $p_k(x)$ in (2.2) are in the classes $\mathbf{A}^{(i_k)}$, where the i_k are integers but are not required to satisfy $i_k \leqslant k$. The conclusions from these heuristics do seem to hold also when the class $\mathbf{B}^{(m)}$ is exactly as in Definition 2.2, however. Nevertheless, to be precise, the statements made in this work concerning the sum and the product of functions in the classes $\mathbf{B}^{(m)}$ are meant to be in the relaxed sense.

(b) If $k \leqslant 0$, then $|f(x)| = \mathcal{O}(x^{\epsilon+\sigma})$ as $x \to \infty$, for some σ to be determined later. Depending on the value of $\Re(\epsilon+\sigma)$, in this case $f(x)$ is integrable at infinity either in the regular sense or in some summability sense, as we describe next.

(i) When s is odd $f(x)$ is integrable in the sense of *Abel summability*.

(ii) When s is even $f(x)$ is the sum of two functions, one of which is integrable in the sense of *Abel summability*, the other (if it is not identically zero) being integrable in the sense of *Hadamard finite part* provided $\epsilon + \sigma \neq -1, 0, 1, 2, \ldots$.

We will discuss the issue of divergent integrals in Section 4.

4. Finally, the class $\widetilde{\mathbf{B}}^{(1)}$ is precisely the class of functions treated in Sidi (1988), for which the mW-transformation was developed.

A general example of integrands in $\widetilde{\mathbf{B}}^{(s)}$, involving Bessel functions, is given next.

EXAMPLE 2.4 Let

$$f(x) = W(x) \prod_{j=1}^{s} \mathcal{C}_{\nu_j}(c_j x),$$

where the c_j are arbitrary constants and

$$\mathcal{C}_{\nu_j}(x) = A_j J_{\nu_j}(x) + B_j Y_{\nu_j}(x),$$

$J_\nu(x)$ and $Y_\nu(x)$ being Bessel functions of order ν, of the first kind and second kind, respectively, and $W(x)$ is as in (2.4). Here A_j and B_j are constants. Since (see Olver *et. al*, 2010, Chapter 10)

$$J_\nu(x) = \phi_{\nu,c}(x)\cos x + \phi_{\nu,s}(x)\sin x, \quad \phi_{\nu,c}, \phi_{\nu,s} \in \mathbf{A}^{\left(-\frac{1}{2}\right)},$$

and $\cos x = \frac{1}{2}(e^{\mathrm{i}x} + e^{-\mathrm{i}x})$ and $\sin x = \frac{1}{2\mathrm{i}}(e^{\mathrm{i}x} - e^{-\mathrm{i}x})$ it follows that $f \in \widetilde{\mathbf{B}}^{(s)}$ with $\delta_j = -\frac{1}{2}$ for all j, when $c_1 = \cdots = c_s$.

From what we stated following Definition 2.2, being a product of s Bessel functions and of $W \in \mathbf{B}^{(1)}$, then $f(x) \in \mathbf{B}^{(m)}$ with $m \leqslant 2^s$. Furthermore, $m = 2^s$ is possible when the c_j are distinct; otherwise, $m < 2^s$. As we show later, when all the c_j are the same, $m \leqslant s+1$, and this is the case we consider in this work.

3. The $mW^{(s)}$-transformation

3.1 *Description of the $mW^{(s)}$-transformation*

We now describe the $mW^{(s)}$-transformation for computing the integral

$$I[f] = \int_a^\infty f(x)\,\mathrm{d}x, \quad a \geqslant 0, \tag{3.1}$$

where $f \in \widetilde{\mathbf{B}}^{(s)}$, as in Definition 2.3, with the notation therein. Here are the steps of this method.

1. Choose x_0 to be the smallest real zero of $\sin\hat{\theta}(x)$ that is greater than a. Thus, x_0 is a real solution of the polynomial equation $\hat{\theta}(x) = q\pi$, where q is the smallest integer possible. Following this,

define x_l to be the largest real root of the polynomial equation $\hat{\theta}(x) = (q + l)\pi$, $l = 1, 2, \ldots$. We can also choose the x_l to be zeros of $\cos\hat{\theta}(x)$. Specifically, we choose x_0 to be a real solution of the polynomial equation $\hat{\theta}(x) = \left(q + \frac{1}{2}\right)\pi$ that is greater than a, where the integer q is the smallest possible. Then we take x_l to be the largest real solution of the polynomial equation $\hat{\theta}(x) = (q + l + \frac{1}{2})\pi$, $l = 1, 2, \ldots$.

As we will see later, whether it is a zero of $\sin\hat{\theta}(x)$ or of $\cos\hat{\theta}(x)$, x_l has a *convergent* expansion of the form

$$x_l = \sum_{i=0}^{\infty} c_i l^{(1-i)/m}, \quad c_0 > 0, \quad \text{for all large } l, \tag{3.2}$$

that is also an *asymptotic expansion* for x_l as $l \to \infty$. However, the method remains just as effective also when x_l has a suitable but not necessarily convergent asymptotic expansion of the form

$$x_l \sim \sum_{i=0}^{\infty} c'_i l^{(1-i)/m} \quad \text{as } l \to \infty, \quad c'_0 > 0. \tag{3.3}$$

For example, if $f(x) = W(x)[J_\nu(x)]^s$, where $J_\nu(x)$ is the Bessel function of the first kind of real order $\nu \geqslant 0$, then we can take the x_l to be the consecutive real zeros or points of extremum of $J_\nu(x)$ with $x_0 > a$. These zeros have asymptotic expansions of the form

$$x_l \sim \pi l + c'_1 + c'_2 l^{-1} + c'_3 l^{-2} + \cdots \quad \text{as } l \to \infty.$$

Choosing the x_l this way may be more beneficial when ν is large.

2. Define

$$F(x) = \int_a^x f(t)\,\mathrm{d}t. \tag{3.4}$$

Compute the finite range integrals $F(x_l)$ and

$$\chi(x_l) = F(x_{l+1}) - F(x_l), \quad l = 0, 1, \ldots.\tag{3.5}$$ [2]

3. • When s is odd set

$$\psi(x_l) = \chi(x_l). \tag{3.6}$$

   • When s is even with k and m as in (2.4) and (2.5), respectively, set

$$\psi(x_l) = \begin{cases} x_l^m \chi(x_l) & \text{if } k \leqslant 0, \\ x_l^{m-k} \chi(x_l) & \text{if } 1 \leqslant k \leqslant m, \\ \chi(x_l) & \text{if } k > m. \end{cases} \tag{3.7}$$

   Note that a more user-friendly choice for even s is

$$\psi(x_l) = x_l^m \chi(x_l), \tag{3.8}$$

[2] Actually, we first set $x_{-1} = a$ and, preferably using a low-order Gaussian quadrature formula, compute $\chi(x_l) = \int_{x_l}^{x_{l+1}} f(t)\,\mathrm{d}t$, $l = -1, 0, 1, \ldots$, to machine accuracy, and then form $F(x_l) = \sum_{i=0}^{l} \chi(x_{i-1})$, $l = 0, 1, \ldots$.

and it will work for *all* three cases in (3.7). With this choice we only have to know $\hat{\theta}(x)$, without having to bother with the rest of $f(x)$.

Note that the choice of $\psi(x_l)$ in (3.6) will not work when s is even and $k < m$. The choice of $\psi(x_l)$ in (3.8), however, will work also when s is odd, although taking $\psi(x_l)$ as in (3.6) is generally more economical for this case in the following sense: if $A_{n_1,1}$ is the approximation $A_{n_1}^{(0)}$ obtained by using $\psi(x_l) = \chi(x_l)$ and $A_{n_2,2}$ is the approximation $A_{n_2}^{(0)}$ obtained by using $\psi(x_l) = x_l^m \chi(x_l)$, and they have the same accuracy, then n_1 is slightly less than n_2, hence, slightly fewer $F(x_l)$'s are needed to compute $A_{n_1,1}$ than $A_{n_2,2}$.

4. Choose an increasing sequence of non-negative integers R_l, $0 \leqslant R_0 < R_1 < \ldots$, and set

$$y_l = x_{R_l}, \quad l = 0, 1, \ldots . \tag{3.9}$$

5. Finally, define the approximations $A_n^{(j)}$ to $I[f]$ via the linear systems of equations

$$F(y_l) = A_n^{(j)} + \psi(y_l) \sum_{i=0}^{n-1} \frac{\bar{\beta}_i}{y_l^i}, \quad l = j, j+1, \ldots, j+n. \tag{3.10}$$

Here the $\bar{\beta}_i$ are additional unknowns that are not of much interest.

Note that, in the case $I[f]$ converges in the regular sense we have $I[f] = \sum_{i=0}^{\infty} \chi(x_{i-1})$. Since the only input to the $mW^{(s)}$-transformation is the sequence $\{\chi(x_l)\}_{l=-1}^{\infty}$ this transformation can be viewed as a *convergence acceleration method* for the sequence of partial sums $F(x_l) = \sum_{i=0}^{l} \chi(x_{i-1})$, $l = 0, 1, \ldots,$ of the infinite series $\sum_{i=0}^{\infty} \chi(x_{i-1})$.

The reader may be wondering about the relation of the $mW^{(s)}$-transformation to the original mW-transformation. Concerning this issue, we note that the mW-transformation of Sidi (1988) is nothing but the $mW^{(1)}$-transformation of this work. This is so because the class $\widetilde{\mathbf{B}}^{(1)}$ is ultimately that for which the mW-transformation was designed, as mentioned also in Remark 4 following Definition 2.3.

3.2 *A closed-form expression for* $A_n^{(j)}$

The following lemma gives a simple closed-form expression for $A_n^{(j)}$ that is useful in the convergence and stability study of the $mW^{(s)}$-transformation.

LEMMA 3.1 The approximation $A_n^{(j)}$ resulting from (3.10) is given by

$$A_n^{(j)} = \frac{\mathcal{D}_n^{(j)}\left\{x^{n-1}F(x)/\psi(x)\right\}}{\mathcal{D}_n^{(j)}\left\{x^{n-1}/\psi(x)\right\}}, \tag{3.11}$$

where $\mathcal{D}_n^{(j)}\{w(x)\}$ is the nth order divided difference of $w(x)$ over the set of points $\{y_j, y_{j+1}, \ldots, y_{j+n}\}$, that is,

$$\mathcal{D}_n^{(j)}\{w(x)\} = \sum_{i=0}^{n} c_{ni}^{(j)} w(y_{j+i}); \quad c_{ni}^{(j)} = \left[\prod_{\substack{r=0\\ r\neq i}}^{n} (y_{j+i} - y_{j+r})\right]^{-1}, \quad i = 0, 1, \ldots, n\,. \tag{3.12}$$

Proof. We start by rewriting the equations in (3.10) in the form

$$\frac{y_l^{n-1}F(y_l)}{\psi(y_l)} = \frac{y_l^{n-1}}{\psi(y_l)}A_n^{(j)} + P(y_l), \quad l = j, j+1, \ldots, j+n; \quad P(x) = \sum_{i=0}^{n-1} \bar{\beta}_i x^{n-i-1}.$$

Multiplying the equation with the index $l = j+i$ by $c_{ni}^{(j)}$, and summing over $i = 0, 1, \ldots, n$, we obtain

$$\mathcal{D}_n^{(j)}\left\{x^{n-1}F(x)/\psi(x)\right\} = \mathcal{D}_n^{(j)}\left\{x^{n-1}/\psi(x)\right\} A_n^{(j)} + \mathcal{D}_n^{(j)}\{P(x)\}.$$

Since $P(x)$ is a polynomial of degree at most $n-1$ we have $\mathcal{D}_n^{(j)}\{P(x)\} = 0$. The result now follows. □

3.3 *Recursive computation via the W-algorithm*

The solution to the system in (3.10) can be achieved recursively by the W-algorithm of Sidi (1982a) (see also Sidi, 2003, Chapter 7) as follows.

1. For $j = 0, 1, \ldots,$ set

$$M_0^{(j)} = \frac{F(y_j)}{\psi(y_j)}, \quad N_0^{(j)} = \frac{1}{\psi(y_j)}.$$

2. For $j = 0, 1, \ldots,$ and $n = 1, 2, \ldots,$ compute

$$M_n^{(j)} = \frac{M_{n-1}^{(j+1)} - M_{n-1}^{(j)}}{y_{j+n}^{-1} - y_j^{-1}}, \quad N_n^{(j)} = \frac{N_{n-1}^{(j+1)} - N_{n-1}^{(j)}}{y_{j+n}^{-1} - y_j^{-1}}.$$

3. For all j and n set

$$A_n^{(j)} = \frac{M_n^{(j)}}{N_n^{(j)}}.$$

Note that this algorithm creates two different two-dimensional tables for the $M_n^{(j)}$ and the $N_n^{(j)}$ that are actually divided difference tables.

3.4 *Stable implementations*

When applying extrapolation methods in finite precision arithmetic, we are sometimes confronted with the problem of numerical instability. This problem exhibits itself as follows: let $\bar{A}_n^{(j)}$ be the numerically computed $A_n^{(j)}$ for all j and n, and suppose that we are computing the sequence $\{\bar{A}_n^{(0)}\}_{n=0}^{\infty}$, for example. If numerical instability is present, then the $\bar{A}_n^{(0)}$ seem to converge to $I[f]$ up to some n, say N, and from that point on, their accuracy deteriorates and eventually is lost completely, even though $A_n^{(0)}$ should be tending to $I[f]$ theoretically. In addition, the accuracy of $\bar{A}_N^{(0)}$, which is the best in the sequence $\{\bar{A}_n^{(0)}\}_{n=0}^{\infty}$, is, generally speaking, significantly lower than the machine accuracy (that is, the maximum accuracy allowed by the finite precision arithmetic being used).

The problem of numerical instability can be treated effectively by suitable choices of the sequence $\{R_l\}_{l=0}^{\infty}$, which we introduced prior to (3.9). Recall that we used the R_l to define the y_l, which we used in the definition of the $mW^{(s)}$-transformation via (3.10). Two such choices for $\{R_l\}$ follow.

1. (i) For all k when s is odd, and (ii) for $k > m$ when s is even choose

$$R_l = l, \quad l = 0, 1, \ldots. \tag{3.13}$$

2. When s is even and $k \leqslant m$ choose

$$R_0 = 0; \quad R_l = \max\{\lfloor \sigma R_{l-1} \rfloor, l\}, \quad l = 1, 2, \ldots, \quad \text{for some fixed } \sigma > 1. \tag{3.14}$$

This choice has been denoted the *geometric progression sampling* (GPS) in Sidi (2003, Chapter 10). (If we choose R_l as in (3.13), the $mW^{(s)}$-transformation suffers from numerical instability and does not achieve high accuracy when s is even and $k \leqslant m$. We will give a more detailed explanation of this point in Section 7.)
Note that $R_l = l$ for $0 \leqslant l \leqslant L$, and $R_l = \lfloor \sigma R_{l-1} \rfloor$ for $l \geqslant L + 1$, where $L = \lceil 2/(\sigma - 1) \rceil$. Therefore,

$$\sigma R_{l-1} - 1 < R_l \leqslant \sigma R_{l-1} \quad \forall\, l > L \quad \Rightarrow \quad \lim_{l \to \infty} R_l / R_{l-1} = \sigma.$$

This means that R_l grows practically like σ^l as l increases. Thus, to determine $A_n^{(0)}$, for example, we need to compute the integrals $\chi(x_i)$, $-1 \leqslant i \leqslant R_n$, a total of $R_n + 2$ integrals, and this number increases *exponentially* like σ^n with n. Therefore, we propose to take $\sigma \in (1, 2)$ to keep the computational cost in check. (In our computations we have preferred to take $\sigma = 1.3$ mostly.)
The expression for L we have given above can be found as follows: first, we have

$$R_L = \max\{\lfloor \sigma(L-1) \rfloor, L\} = L \quad \Rightarrow \quad \lfloor \sigma(L-1) \rfloor \leqslant L,$$

which, recalling that $x - 1 < \lfloor x \rfloor \leqslant x$, gives

$$\sigma(L-1) - 1 < L \quad \Rightarrow \quad L < \frac{\sigma + 1}{\sigma - 1} = \frac{2}{\sigma - 1} + 1.$$

Next by

$$L + 2 \leqslant R_{L+1} = \lfloor \sigma R_L \rfloor = \lfloor \sigma L \rfloor$$

we have

$$L + 2 \leqslant \sigma L \quad \Rightarrow \quad L \geqslant \frac{2}{\sigma - 1}.$$

Combining these two inequalities for L and recalling that $x \leqslant N < x + 1$ for an integer N implies that $N = \lceil x \rceil$, we obtain our expression for L.

4. Properties of functions in $\widetilde{\mathbf{B}}^{(s)}$

4.1 *Algebraic properties*

Let $f \in \widetilde{\mathbf{B}}^{(s)}$ as in Definition 2.3, with the notation therein, and let $I[f]$ be as in (3.1). In order to be able to design effective extrapolation methods for computing $I[f]$ we need to study the asymptotic behaviour of $f(x)$ as $x \to \infty$.

We begin by noting that the functions $h_j^{\pm}(x)\exp[\pm \mathrm{i}\theta_j(x)]$ are all in the class $\mathbf{B}^{(1)}$ by Sidi (2003, Theorem 5.7.2, p. 118). Therefore, $H_j \in \mathbf{B}^{(2)}$ for all j, and hence $\prod_{j=1}^{s} H_j(x) \in \mathbf{B}^{(p)}$, $p \leqslant 2^s$, by Sidi (2003, Heuristic 5.4.1, p. 107). As a matter of fact $p = s+1$, as we show next.

First, by (2.4) and (2.5),

$$\exp[\phi(x)] = v(x)\exp\left[\hat{\phi}(x)\right], \quad v = e^{L} \in \mathbf{A}^{(0)} \quad \text{strictly},$$
$$\exp[\pm \mathrm{i}\theta_j(x)] = u_j^{\pm}(x)\exp\left[\pm \mathrm{i}\hat{\theta}(x)\right], \quad u_j^{\pm} = e^{\pm \mathrm{i}M_j} \in \mathbf{A}^{(0)} \quad \text{strictly}. \tag{4.1}$$

Consequently, we have

$$H_j(x) = \hat{h}_j^{+}(x)\exp\left[\mathrm{i}\hat{\theta}(x)\right] + \hat{h}_j^{-}(x)\exp\left[-\mathrm{i}\hat{\theta}(x)\right], \quad \hat{h}_j^{\pm} = h_j^{\pm}u_j^{\pm} \in \mathbf{A}^{(\delta_j)}. \tag{4.2}$$

Substituting (4.2) in the product $\prod_{j=1}^{s} H_j(x)$, expanding and rearranging we have

$$\prod_{j=1}^{s} H_j(x) = \sum_{\substack{j,r \geqslant 0 \\ j+r=s}} \check{T}_{j,r}(x)\exp\left[\mathrm{i}(j-r)\hat{\theta}(x)\right].$$

Actually, $\check{T}_{j,r}$ is a sum of products of j $\hat{h}_i^{+}$'s and r $\hat{h}_i^{-}$'s, and, as a result, either $\check{T}_{j,r}(x) \equiv 0$ or

$$\check{T}_{j,r} \in \mathbf{A}^{(\sigma_j)} \quad \text{strictly}, \quad \sigma_j = \sum_{k=1}^{m} \delta_k - p_j, \quad p_j \geqslant 0 \ \text{integer}. \tag{4.3}$$

(Thus, for any j and j', $\sigma_j - \sigma_{j'}$ is an integer.) Since $\check{T}_{j,r} = \check{T}_{j,s-j}$ we will denote it by T_j for short. Consequently,

$$\prod_{j=1}^{s} H_j(x) = \sum_{j=0}^{s} T_j(x)\exp\left[\mathrm{i}(-s+2j)\hat{\theta}(x)\right], \quad T_j \in \mathbf{A}^{(\sigma_j)} \quad \text{or} \quad T_j \equiv 0.$$

Thus, the function $f(x)$ can be re-expressed as in

$$f(x) = \sum_{j=0}^{s} f_j(x), \quad f_j(x) = U_j(x)\exp\left[\hat{\phi}(x)\right]\exp\left[\mathrm{i}(-s+2j)\hat{\theta}(x)\right],$$
$$U_j(x) = v(x)Q(x)T_j(x), \quad U_j \in \mathbf{A}^{(\epsilon+\sigma_j)} \quad \text{strictly}, \quad j = 0,1,\ldots,s. \tag{4.4}$$

The next two examples illustrate the different situations that can occur in a simple way.

EXAMPLE 4.1 Consider the case

$$H_j(x) = h^{+}(x)\exp[\mathrm{i}\theta(x)] + h^{-}(x)\exp[-\mathrm{i}\theta(x)] \quad \forall\, j.$$

Assume that $h^{+} \not\equiv 0$ and $h^{-} \not\equiv 0$. Expanding $\prod_{j=1}^{s} H_j(x)$ we obtain

$$\prod_{j=1}^{s} H_j(x) = \sum_{j=0}^{s} \binom{s}{j}[h^{+}(x)]^{j}[h^{-}(x)]^{s-j}\exp[\mathrm{i}(-s+2j)\theta(x)]. \tag{4.5}$$

We thus identify

$$T_j = \binom{s}{j}[h^+]^j[h^-]^{s-j} \not\equiv 0, \quad j = 0, 1, \ldots, s.$$

EXAMPLE 4.2 Consider the case

$$H_1(x) = h^+(x)\exp[\mathrm{i}\theta(x)] + h^-(x)\exp[-\mathrm{i}\theta(x)],$$
$$H_2(x) = h^+(x)\exp[\mathrm{i}\theta(x)] - h^-(x)\exp[-\mathrm{i}\theta(x)].$$

Assume that $h^+ \not\equiv 0$ and $h^- \not\equiv 0$. Expanding $\prod_{j=1}^{2} H_j(x)$ we obtain

$$\prod_{j=1}^{2} H_j(x) = [h^+(x)]^2\exp[2\mathrm{i}\theta(x)] - [h^-(x)]^2\exp[-2\mathrm{i}\theta(x)]. \tag{4.6}$$

We identify

$$T_0 = [h^+]^2, \quad T_1 \equiv 0, \quad T_2 = -[h^-]^2.$$

An example of this is given by $H_1(x) = J_\nu(x)$ and $H_2(x) = Y_\nu(x)$. Since

$$J_\nu(x) = \frac{1}{2}\left[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)\right], \quad Y_\nu(x) = \frac{1}{2\mathrm{i}}\left[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)\right],$$

where $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are Hankel functions of order ν (not to be confused with $H_1(x)$ and $H_2(x)$) we have

$$\prod_{j=1}^{2} H_j(x) = \frac{1}{4\mathrm{i}}\left\{\left[H_\nu^{(1)}(x)\right]^2 - \left[H_\nu^{(2)}(x)\right]^2\right\}.$$

Now, $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are precisely of the form $h^+(x)\exp(\mathrm{i}x)$ and $h^-(x)\exp(-\mathrm{i}x)$, respectively. Thus, $\prod_{j=1}^{2} H_j(x)$ is precisely of the form (4.6), with $\hat{\theta}(x) = x$.

Going back to (4.4), we note that, if $f_j \not\equiv 0$, then $f_j \in \mathbf{B}^{(1)}$ by Sidi (2003, Theorem 5.7.2, p. 118). If $f_j \not\equiv 0$ for all j, then, being the sum of the $s+1$ functions $f_j(x)$ that have different exponential factors, the function $f(x) \in \mathbf{B}^{(s+1)}$. (Before we expressed $f(x)$ as in (4.4) all we were able to say was that $f \in \mathbf{B}^{(p)}$ for some $p \leqslant 2^s$; even for moderate s this is very pessimistic.)

We now analyse the nature of $f(x)$ for all values of s. In the sequel we treat the functions $f_j(x)$ as if $f_j \not\equiv 0$ for all j, but we understand that those $f_j(x)$ that are identically zero are absent from the general picture.

Throughout, we differentiate between two cases.

1. *Odd s:* when s is odd the integers $-s+2j$ in (4.4) assume the values

$$-s, -s+2, \ldots, -3, -1, 1, 3, \ldots, s-2, s.$$

Thus, the functions $f_j(x)$ are all oscillatory since $-s+2j \neq 0$ for any j in this case.

2. *Even s:* when s is even the integers $-s+2j$ in (4.4) assume the values

$$-s, -s+2, \dots, -4, -2, 0, 2, 4, \dots, s-2, s.$$

Note that $-s+2j=0$ for $j=s/2$. Thus,

$$f_{s/2}(x) = U_{s/2}(x)\exp\left[\hat{\phi}(x)\right]. \tag{4.7}$$

Clearly, $f_{s/2}(x)$ is not oscillatory. The rest of the $f_j(x)$ are all oscillatory. As we will see later the case of even s has some surprising aspects and is more difficult to analyse.

4.2 *Integral properties*

Having discovered the algebraic nature of $f(x)$ we now analyse its integral properties. For this we need the integral properties of the $f_j(x)$. Integral properties of the $f_j(x)$ can be studied with the help of Theorem 5.7.3 in Sidi (2003, Chapter 5, p. 119), which we state as Theorem 4.3 next. See also Sidi (2003, Chapter 5, Remarks, p. 120). This theorem concerns the integral properties of functions in the class $\mathbf{B}^{(1)}$.

THEOREM 4.3 Define

$$I[g] = \int_a^\infty g(x)\,\mathrm{d}x \quad \text{and} \quad G(x) = \int_a^x g(t)\,\mathrm{d}t.$$

1. If $g \in \mathbf{A}^{(\gamma)}$ for some $\gamma \neq -1, 0, 1, 2, \dots$, then

$$G(x) = I[g] + xg(x)u(x), \quad u \in \mathbf{A}^{(0)} \quad \text{strictly}.$$

When $\Re\gamma < -1$, $I[g]$ exists in the regular sense, otherwise, it exists in the sense of Hadamard finite part.

2. If $g(x) = \exp[\rho(x)]u(x)$, where $u \in \mathbf{A}^{(\gamma)}$ for arbitrary γ and $\rho \in \mathbf{A}^{(k)}$ for some positive integer k, with $\lim_{x\to\infty}\Re\rho(x) = -\infty$ or $\lim_{x\to\infty}\Re\rho(x)$ finite, then

$$G(x) = I[g] + x^{1-k}g(x)u(x), \quad u \in \mathbf{A}^{(0)} \quad \text{strictly}.$$

When $\lim_{x\to\infty}\Re\rho(x) = -\infty$, $I[g]$ exists in the regular sense for all γ. When $\lim_{x\to\infty}\Re\rho(x)$ is finite but $\lim_{x\to\infty}|\Im\rho(x)| = \infty$, the integral $I[g]$ exists in the regular sense if $\Re\gamma < 0$, and it exists in the sense of Abel summability when $\Re\gamma \geqslant 0$.

Let us now define

$$I[f_j] = \int_a^\infty f_j(x)\,\mathrm{d}x, \quad F_j(x) = \int_a^x f_j(t)\,\mathrm{d}t. \tag{4.8}$$

Then by Theorem 4.3 the following hold.

- For all j when s is odd, and for all $j \neq s/2$ when s is even, there holds

$$F_j(x) = I[f_j] + x^\rho f_j(x)g_j(x), \quad g_j \in \mathbf{A}^{(0)} \quad \text{strictly}, \quad \rho = 1-\max\{m,k\}. \tag{4.9}$$

Invoking (4.4) this can be rewritten as

$$F_j(x) = I[f_j] + x^{\rho+\epsilon+\sigma_j} \exp\left[\hat{\phi}(x)\right] \exp[\mathrm{i}(-s+2j)\hat{\theta}(x)]\hat{g}_j(x),$$
$$\hat{g}_j \in \mathbf{A}^{(0)} \quad \text{strictly}, \quad \rho = 1 - \max\{m,k\}. \tag{4.10}$$

Note that ρ can assume one of the values $0, -1, -2, \ldots$.

- When s is even, for $f_{s/2}$, we have

$$F_{s/2}(x) = I[f_{s/2}] + x^{\rho'} f_{s/2}(x) g_{s/2}(x),$$
$$g_{s/2} \in \mathbf{A}^{(0)} \quad \text{strictly}, \quad \rho' = \begin{cases} 1-k & \text{if } k > 0, \\ 1 & \text{if } k \leqslant 0. \end{cases} \tag{4.11}$$

Invoking (4.4) this can be rewritten as

$$F_{s/2}(x) = I[f_{s/2}] + x^{\rho'+\epsilon+\sigma_{s/2}} \exp\left[\hat{\phi}(x)\right] \hat{g}_{s/2}(x),$$
$$\hat{g}_{s/2} \in \mathbf{A}^{(0)} \quad \text{strictly}, \quad \rho' = \begin{cases} 1-k & \text{if } k > 0, \\ 1 & \text{if } k \leqslant 0. \end{cases} \tag{4.12}$$

Thus, ρ' can assume either the value 1 or one of the values $0, -1, -2, \ldots$.

Important points to realize in connection with the above are as follows.

- When s is odd the integer ρ is the same for all j. When s is even ρ is the same for all $j \neq s/2$, and it may be different from ρ' for $j = s/2$.
- Recall that the σ_j differ from each other by integers. In this respect, we recall the property of the sets $\mathbf{A}^{(\gamma)}$ that if $\alpha \in \mathbf{A}^{(\gamma+k)}$ strictly, k being a positive integer, and $\beta \in \mathbf{A}^{(\gamma)}$, then $\alpha + \beta \in \mathbf{A}^{(\gamma+k)}$ strictly.
- Recall also the following facts.
  - (a) For every j when s is odd, and for every $j \neq s/2$ when s is even, if the integral $\int_a^\infty f_j(x)\,\mathrm{d}x$ is not defined in the regular sense, which can happen only when $k \leqslant 0$, then (4.9) holds with $I[f_j]$ there being defined as the *Abel sum* of $\int_a^\infty f_j(x)\,\mathrm{d}x$, which exists; see Sidi (1987).
  - (b) When s is even, if the integral $\int_a^\infty f_{s/2}(x)\,\mathrm{d}x$ is not defined in the regular sense, which can happen only when $k \leqslant 0$, then (4.11) holds with $I[f_{s/2}]$ there being defined as the *Hadamard finite part* of $\int_a^\infty f_{s/2}(x)\,\mathrm{d}x$, which exists when $\epsilon + \sigma_{s/2} \neq -1, 0, 1, 2, \ldots$; see Sidi (1999).

These observations have useful consequences, as we will see in the next section.

5. Development of the $mW^{(s)}$-transformation

We now want to develop the $mW^{(s)}$-transformation for $I[f]$ with $f \in \widetilde{\mathbf{B}}^{(s)}$ along the lines of the original mW-transformation that was developed in Sidi (1988). We start with the determination of the sequence $\{x_l\}_{l=0}^\infty$.

Let us recall that $x_0 < x_1 < \cdots$ are consecutive zeros of $\sin\hat{\theta}(x)$ or of $\cos\hat{\theta}(x)$ in (a,∞). Thus, for each $l = 0, 1, 2, \ldots$, x_l is the largest real root of the polynomial equation $\hat{\theta}(x) = (q+l)\pi$ or of the polynomial equation $\hat{\theta}(x) = \left(q + l + \frac{1}{2}\right)\pi$, where q is some integer.

As shown in Sidi (1988) and Sidi (2003, Theorem 11.8.4, p. 230), whether it is a zero of $\sin\hat{\theta}(x)$ or of $\cos\hat{\theta}(x)$, x_l has a convergent expansion of the form

$$x_l = \sum_{i=0}^{\infty} c_i l^{(1-i)/m}, \quad c_0 > 0, \quad \text{for all large } l, \tag{5.1}$$

that is also an asymptotic expansion for x_l as $l \to \infty$.

In what follows we take the x_l to be consecutive zeros of $\sin\hat{\theta}(x)$, without loss of generality. Since x_l satisfies the equation $\hat{\theta}(x_l) = (q+l)\pi$ for $l = 0, 1, \ldots$, we have

$$\exp\left[\mathrm{i}(-s+2j)\hat{\theta}(x_l)\right] = \exp[\mathrm{i}(-s+2j)(q+l)\pi] = (-1)^{s(q+l)},$$

from which it is clear that

$$\exp\left[\mathrm{i}(-s+2j)\hat{\theta}(x_l)\right] = \begin{cases} (-1)^{q+l} & \text{if } s \text{ odd,} \\ +1 & \text{if } s \text{ even.} \end{cases} \tag{5.2}$$

This fact has important consequences for the integrals

$$F(x_l) = \int_a^{x_l} f(t)\,\mathrm{d}t, \quad \chi(x_l) = \int_{x_l}^{x_{l+1}} f(t)\,\mathrm{d}t = F(x_{l+1}) - F(x_l), \tag{5.3}$$

to the analysis of which we now turn.

5.1 *Properties of $F(x_l)$*

We now express $F(x_l)$ in a simple way that will enable us to continue our study conveniently.

5.1.1 *For odd s.*

Let us invoke (5.2) in (4.10). We then have, for every j,

$$F_j(x_l) = I[f_j] + (-1)^{q+l} x_l^{\rho+\epsilon+\sigma_j} \exp\left[\hat{\phi}(x_l)\right] \hat{g}_j(x_l),$$
$$\hat{g}_j \in \mathbf{A}^{(0)} \quad \text{strictly}, \quad \rho = 1 - \max\{m, k\}. \tag{5.4}$$

Therefore, by (5.3) and (4.4),

$$F(x_l) = I[f] + (-1)^{q+l} x_l^{\omega} \exp\left[\hat{\phi}(x_l)\right] G(x_l), \quad G \in \mathbf{A}^{(0)} \quad \text{strictly},$$
$$x^{\omega} G(x) = \sum_{j=0}^{s} x^{\rho+\epsilon+\sigma_j} \hat{g}_j(x),$$
$$\omega = \max_j\{\rho + \epsilon + \sigma_j\} - p, \quad p \geqslant 0 \quad \text{an integer}. \tag{5.5}$$

5.1.2 *For even s.*

Let us invoke (5.2) in (4.10). We then have, for every $j \neq s/2$,

$$F_j(x_l) = I[f_j] + x_l^{\rho+\epsilon+\sigma_j} \exp\left[\hat{\phi}(x_l)\right] \hat{g}_j(x_l),$$
$$\hat{g}_j \in \mathbf{A}^{(0)} \quad \text{strictly}, \quad \rho = 1 - \max\{m, k\}. \tag{5.6}$$

For $j = s/2$, from (4.12), we have

$$F_{s/2}(x_l) = I[f_{s/2}] + x_l^{\rho'+\epsilon+\sigma_{s/2}} \exp\left[\hat{\phi}(x_l)\right] \hat{g}_{s/2}(x_l),$$
$$\hat{g}_{s/2} \in \mathbf{A}^{(0)} \quad \text{strictly}, \quad \rho' = \begin{cases} 1-k & \text{if } k > 0, \\ 1 & \text{if } k \leqslant 0. \end{cases} \tag{5.7}$$

Therefore, by (5.3) and (4.4), and recalling that ρ and ρ' are both integers,

$$F(x_l) = I[f] + x_l^{\omega} \exp\left[\hat{\phi}(x_l)\right] G(x_l), \quad G \in \mathbf{A}^{(0)} \quad \text{strictly},$$
$$x^{\omega} G(x) = x^{\rho'+\epsilon+\sigma_{s/2}} \hat{g}_{s/2}(x) + \sum_{\substack{j=0 \\ j \neq s/2}}^{s} x^{\rho+\epsilon+\sigma_j} \hat{g}_j(x),$$
$$\omega = \max\left\{ \max_{j \neq s/2} \{\rho + \epsilon + \sigma_j\}, (\rho' + \epsilon + \sigma_{s/2}) \right\} - p, \quad p \geqslant 0 \quad \text{integer}. \tag{5.8}$$

5.2 *Properties of* $\chi(x_l)$

5.2.1 *For odd s.* When s is odd, by (5.3) and (5.5), for all j, we have

$$\chi(x_l) = (-1)^{q+l+1} \left\{ x_{l+1}^{\omega} \exp\left[\hat{\phi}(x_{l+1})\right] G(x_{l+1}) + x_l^{\omega} \exp\left[\hat{\phi}(x_l)\right] G(x_l) \right\}, \tag{5.9}$$

which we rewrite in the form

$$\chi(x_l) = (-1)^{q+l+1} x_l^{\omega} \exp\left[\hat{\phi}(x_l)\right] G(x_l)[S(x_l) + 1], \tag{5.10}$$

where

$$S(x_l) = R(x_l) \exp\left[\Delta\hat{\phi}(x_l)\right], \tag{5.11}$$

with

$$R(x_l) = \left(\frac{x_{l+1}}{x_l}\right)^{\omega} \left[\frac{G(x_{l+1})}{G(x_l)}\right], \quad \Delta\hat{\phi}(x_l) = \hat{\phi}(x_{l+1}) - \hat{\phi}(x_l). \tag{5.12}$$

By part A.1 of the appendix, $[S(x) + 1] \in \mathbf{A}^{(0)}$ strictly for *all* m and k.
Therefore, we can rewrite (5.10) in the form

$$\chi(x_l) = (-1)^{q+l} x_l^{\omega} \exp\left[\hat{\phi}(x_l)\right] b(x_l), \quad b \in \mathbf{A}^{(0)} \quad \text{strictly}. \tag{5.13}$$

Of course, $b(x) = -G(x)[S(x) + 1]$.

Two interesting conclusions that can be drawn from this analysis are that $\chi(x_l)$ is nonzero for all large l and that the sequence of the $\chi(x_l)$ is ultimately an alternating sequence. That is, $\lim_{l\to\infty} \arg[\chi(x_{l+1})/\chi(x_l)] = \pi$.

5.2.2 *For even s.* When s is even, by (5.3) and (5.8), we have

$$\chi(x_l) = x_{l+1}^{\omega} \exp\left[\hat{\phi}(x_{l+1})\right] G(x_{l+1}) - x_l^{\omega} \exp\left[\hat{\phi}(x_l)\right] G(x_l),$$

which we can rewrite in the form

$$\chi(x_l) = x_l^{\omega} \exp\left[\hat{\phi}(x_l)\right] G(x_l)[S(x_l) - 1], \tag{5.14}$$

where $S(x_l)$, $R(x_l)$ and $\Delta\hat{\phi}(x_l)$ are exactly as in (5.11) and (5.12).

Now, by part A.2 of the appendix, $[S(x) - 1]$ has different properties depending on the values of m and k, and this causes $\chi(x_l)$ to have different properties, as follows.

1. When $k \leqslant 0$,

$$\chi(x_l) = x_l^{\omega-m} \exp\left[\hat{\phi}(x_l)\right] b(x_l) = x_l^{\omega-m} b(x_l), \quad b \in \mathbf{A}^{(0)} \quad \text{strictly}. \tag{5.15}$$

2. When $1 \leqslant k \leqslant m$,

$$\chi(x_l) = x_l^{\omega+k-m} \exp\left[\hat{\phi}(x_l)\right] b(x_l), \quad b \in \mathbf{A}^{(0)} \quad \text{strictly}. \tag{5.16}$$

3. When $k > m$,

$$\chi(x_l) = x_l^{\omega} \exp\left[\hat{\phi}(x_l)\right] b(x_l), \quad b \in \mathbf{A}^{(0)} \quad \text{strictly}. \tag{5.17}$$

Of course, in all three cases, $b(x) = G(x)[S(x) - 1]$.

Two interesting conclusions that can be drawn from this analysis are that $\chi(x_l)$ is nonzero for all large l and that the sequence of the $\chi(x_l)$ is ultimately monotonic. That is, $\lim_{l\to\infty} \arg[\chi(x_{l+1})/\chi(x_l)] = 0$.

5.3 *$F(x_l)$ and $\chi(x_l)$ combined*

5.3.1 *For odd s.* Let us solve (5.13) for the product $(-1)^{q+l} x_l^{\omega} \exp[\hat{\phi}(x_l)]$ and substitute in (5.5). We obtain

$$F(x_l) = I[f] + \psi(x_l)\beta(x_l), \quad \beta(x) = \frac{G(x)}{b(x)} \in \mathbf{A}^{(0)} \quad \text{strictly}, \tag{5.18}$$

where

$$\psi(x_l) = \chi(x_l). \tag{5.19}$$

This is true because both $G(x)$ and $b(x)$ are in $\mathbf{A}^{(0)}$ strictly.

5.3.2 *For even s.* Let us solve each of (5.15–5.17) for the product $x_l^{\omega} \exp[\hat{\phi}(x_l)]$ and substitute in (5.8). We obtain

$$F(x_l) = I[f] + \psi(x_l)\beta(x_l), \quad \beta(x) = \frac{G(x)}{b(x)} \in \mathbf{A}^{(0)} \quad \text{strictly}, \tag{5.20}$$

where

$$\psi(x_l) = \begin{cases} x_l^m \chi(x_l) & \text{if } k \leqslant 0, \\ x_l^{m-k} \chi(x_l) & \text{if } 1 \leqslant k \leqslant m, \\ \chi(x_l) & \text{if } k > m. \end{cases} \tag{5.21}$$

Again, this is true because both $G(x)$ and $b(x)$ are in $\mathbf{A}^{(0)}$ strictly.

5.4 *Derivation of the $mW^{(s)}$-transformation*

Now that we have obtained the relations (5.18) and (5.20), with $\psi(x_l)$ as in (5.19) and (5.21), respectively, we can use these to derive the $mW^{(s)}$-transformation as follows: since $\beta \in \mathbf{A}^{(0)}$ strictly, $\beta(x)$ has an asymptotic expansion of the form

$$\beta(x) \sim \sum_{i=0}^{\infty} \beta_i x^{-i} \quad \text{as } x \to \infty. \tag{5.22}$$

Since $x_l \to \infty$ as $l \to \infty$ the relations in (5.18) and (5.20) can be written also as

$$F(x_l) \sim I[f] + \psi(x_l) \sum_{i=0}^{\infty} \frac{\beta_i}{x_l^i} \quad \text{as } l \to \infty. \tag{5.23}$$

We now apply the formalism of the generalized Richardson extrapolation to this expansion: (i) we truncate the infinite summation in (5.23) at $i = n-1$, (ii) replace the asymptotic equality sign $\sim$ by $=$, (iii) replace β_i by $\bar{\beta}_i$, $0 \leqslant i \leqslant n-1$, and $I[f]$ by $A_n^{(j)}$ and finally (iv) choose an increasing sequence of non-negative integers R_l, $0 \leqslant R_0 < R_1 < \cdots$ and collocate the resulting equality at the points $y_l = x_{R_l}$, $l = j, j+1, \ldots, j+n$. As a result, we end up with the linear system

$$F(y_l) = A_n^{(j)} + \psi(y_l) \sum_{i=0}^{n-1} \frac{\bar{\beta}_i}{y_l^i}, \quad l = j, j+1, \ldots, j+n, \tag{5.24}$$

the unknowns being the $A_n^{(j)}$ and the $\bar{\beta}_i$. (For a formal treatment of generalizations of the Richardson extrapolation, see Sidi, 2003, Chapters 3 and 4, for example.)

As explained in Sidi (2003, Chapter 5, p. 103), for example, we can take

$$\psi(x_l) = x_l^m \chi(x_l) \tag{5.25}$$

for *all* cases with even s. This is more user-friendly than that in (5.21), as it requires us to study the asymptotic behaviour of $\theta_j(x)$ only, without worrying about the rest of $f(x)$. This choice will work also when s is odd, although taking $\psi(x_l)$ as in (5.19) is, in general, more economical when s is odd.

As already mentioned in Section 3 the $A_n^{(j)}$ can be computed in an efficient way via the W-algorithm whose steps are given following (3.10). They can be arranged in a two-dimensional array as shown in Figure 1. In general, the *column sequences* $\{A_n^{(j)}\}_{j=0}^{\infty}$ (n fixed) accelerate convergence, each column being at least as good as the one preceding it. Also, *diagonal sequences* $\{A_n^{(j)}\}_{n=0}^{\infty}$ (j fixed) converge much faster than column sequences.

Recall that we have described two appropriate choices for the integers R_l already in (3.13) and (3.14) in Section 3. We will not repeat them here.

$$\begin{array}{lllll} A_0^{(0)} & & & & \\ A_0^{(1)} & A_1^{(0)} & & & \\ A_0^{(2)} & A_1^{(1)} & A_2^{(0)} & & \\ A_0^{(3)} & A_1^{(2)} & A_2^{(1)} & A_3^{(0)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

FIG. 1.

6. Convergence theory

Judging from the structure of the integrals $F(x_l)$ in (5.18) and (5.20) and from (5.24) that define the $mW^{(s)}$-transformation we realize that this transformation is simply a *generalized Richardson extrapolation process* of the $\mathrm{GREP}^{(1)}$ type. (See Sidi, 1979, 2003, Chapter 4.) The convergence and stability properties of $\mathrm{GREP}^{(1)}$ have been studied by the author in different places. A detailed treatment of these is given in Sidi (2003, Chapters 8 and 9). Now, $F(x)$ and $\psi(x)$, as functions of the *discrete* variable x_l, are exactly of the form discussed in Sidi (2003, Chapters 8 and 9). Consequently, the theory of the latter applies to the $mW^{(s)}$-transformation, and we are able to make definitive statements about the convergence of the $mW^{(s)}$-transformation.

Note that, in this section, $\psi(x_l)$ is exactly as in (5.19) for odd s, and as in (5.21) for even s. As will become clear there is a substantial difference between the convergence results pertaining to odd s and those pertaining to even s.

We begin with the following simple lemma, which follows from Lemma 3.1 and (5.18) and (5.20).

LEMMA 6.1 The error in $A_n^{(j)}$ is given by

$$A_n^{(j)} - I[f] = \frac{\mathcal{D}_n^{(j)}\left\{x^{n-1}\beta(x)\right\}}{\mathcal{D}_n^{(j)}\left\{x^{n-1}/\psi(x)\right\}}, \tag{6.1}$$

where $\beta(x) \in \mathbf{A}^{(0)}$ is the function introduced in (5.18) and (5.20).

Theorem 6.2 concerns the convergence of the $mW^{(s)}$-transformation when s is odd.

THEOREM 6.2 Let $f \in \widetilde{\mathbf{B}}^{(s)}$ as in Definition 2.3, with odd s. Let $A_n^{(j)}$ be defined as in (3.10), with $R_l = l$ as in (3.13). Then the following hold.

1. The column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$ (with fixed n) satisfy

$$A_n^{(j)} - I[f] = \mathcal{O}\left(\psi(x_j)x_j^{-mn-n}\right) \quad \text{as } j \to \infty, \quad \text{if } k \leqslant m, \tag{6.2}$$

$$A_n^{(j)} - I[f] = \mathcal{O}\left(\psi(x_{j+n})x_j^{-mn-n}\right) \quad \text{as } j \to \infty, \quad \text{if } k > m. \tag{6.3}$$

2. The diagonal sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$ (with fixed j), for all m and k, satisfy

$$A_n^{(j)} - I[f] = \mathcal{O}(n^{-p}) \quad \text{as } n \to \infty, \quad \forall\ p > 0. \tag{6.4}$$

Note that (6.2), (6.3) and (6.4) follow from Theorems 9.3.1, 9.3.2 and 9.4.3, respectively, in Sidi (2003, Chapter 9). Note also that (6.2) and (6.3) are different from each other. When $k > m$, $\psi(x_{j+n})$ and $\psi(x_j)$ have entirely different asymptotic behaviour as $j \to \infty$; in fact, $\lim_{j\to\infty} \psi(x_{j+n})/\psi(x_j) = 0$. All of these results can be proved by invoking Lemma A.1 and the asymptotic behaviour of $\psi(x_l)$ as $l \to \infty$ in (6.1) and recalling (3.12).

Clearly, Theorem 6.2 shows that the $mW^{(s)}$-transformation accelerates the convergence of the sequence $\{F(x_j)\}_{j=0}^{\infty}$ to $I[f]$.

The next theorem concerns the convergence of the $mW^{(s)}$-transformation when s is even.

THEOREM 6.3 Let $f \in \widetilde{\mathbf{B}}^{(s)}$ as in Definition 2.3, with even s. Let $A_n^{(j)}$ be defined as in (3.10), where $\lim_{l\to\infty} y_{l+1}/y_l = \kappa$ for some constant $\kappa > 0$. Then the following hold.

1. The column sequences $\{A_n^{(j)}\}_{j=0}^{\infty}$ (with fixed n) satisfy

$$A_n^{(j)} - I[f] = \mathcal{O}\left(\psi(y_j) y_j^{-n}\right) \quad \text{as } j \to \infty, \quad \text{if } k \leqslant 0, \tag{6.5}$$

$$A_n^{(j)} - I[f] = \mathcal{O}\left(\psi(y_j) y_j^{-kn-n}\right) \quad \text{as } j \to \infty, \quad \text{if } k \geqslant 1. \tag{6.6}$$

   The results in (6.5) and (6.6) hold, in particular, when the R_l are as in (3.13) (with $\kappa = 1$) and in (3.14) (with $\kappa = \sigma$). In the case that the R_l are as in (3.13), hence, $y_l = x_l$ for all l, (6.6) for $k > m$ can be improved to read

$$A_n^{(j)} - I[f] = \mathcal{O}\left(\psi(x_{j+n}) x_j^{-mn-n}\right) \quad \text{as } j \to \infty, \quad \text{if } k > m. \tag{6.7}$$

2. When $k \leqslant 0$, if $\epsilon + \sigma_j$ are real and the R_l are chosen as in (3.14), then the diagonal sequences $\{A_n^{(j)}\}_{n=0}^{\infty}$ (with fixed j) satisfy

$$A_n^{(j)} - I[f] = \mathcal{O}\left(e^{-pn}\right) \quad \text{as } n \to \infty, \quad \forall\ p > 0. \tag{6.8}$$

Proof. We proceed through Lemma 6.1. For simplicity, assume that everything is real. (The proof of the complex case can be achieved in the same way, although with some extra effort.) First, by the fact that $\beta(x)$ satisfies (5.22) we have

$$x^{n-1}\beta(x) = N(x) + \Delta(x), \quad N(x) = \sum_{i=0}^{n-1} \beta_i x^{n-i}, \quad \Delta \in \mathbf{A}^{(-1)}.$$

Since $N(x)$ is a polynomial of degree $n-1$ there holds $\mathcal{D}_n^{(j)}\{N(x)\} = 0$. Therefore,

$$\mathcal{D}_n^{(j)}\left\{x^{n-1}\beta(x)\} = \mathcal{D}_n^{(j)}\{\Delta(x)\right\} = \frac{1}{n!}\Delta^{(n)}(\xi_{j,n}) \quad \text{for some } \xi_{j,n} \in (y_j, y_{j+n}).$$

Here $\Delta^{(n)}(x)$ is the nth derivative of $\Delta(x)$. But $\Delta^{(n)} \in \mathbf{A}^{(-1-n)}$. Therefore,

$$\mathcal{D}_n^{(j)}\left\{x^{n-1}\beta(x)\right\} = \mathcal{O}\left(\xi_{j,n}^{-n-1}\right) = \mathcal{O}\left(y_j^{-n-1}\right) \quad \text{as } j \to \infty, \tag{6.9}$$

since $\lim_{j\to\infty}(y_{j+i}/y_j)$ exists for all finite i.

We now turn to the proof of (6.5). When $k \leqslant 0$, $\psi(x) = x^{\omega} b(x)$, $b \in \mathbf{A}^{(0)}$ strictly, by (5.15) and (5.21). Hence, $x^{n-1}/\psi(x) = x^{n-1-\omega}/b(x) \equiv w(x)$, $w \in \mathbf{A}^{(n-1-\omega)}$ strictly. Therefore,

$$\mathcal{D}_n^{(j)}\left\{x^{n-1}/\psi(x)\right\} = \mathcal{D}_n^{(j)}\{w(x)\} = \frac{1}{n!} w^{(n)}(\eta_{j,n}) \quad \text{for some } \eta_{j,n} \in (y_j, y_{j+n}).$$

By the fact that $w^{(n)} \in \mathbf{A}^{(-1-\omega)}$ strictly, we have

$$\mathcal{D}_n^{(j)}\left\{x^{n-1}/\psi(x)\right\} \sim K\eta_{j,n}^{-1-\omega} \sim K y_j^{-1-\omega} \quad \text{as } j \to \infty, \quad K > 0 \text{ some constant.} \tag{6.10}$$

Substituting (6.9) and (6.10) in (6.1) we obtain (6.5).

To prove (6.6) we need to analyse $\mathcal{D}_n^{(j)}\{x^{n-1}/\psi(x)\}$ for $k \geqslant 1$. Now, by (5.16) and (5.17),

$$x^{n-1}/\psi(x) = u(x)\exp[-\hat{\phi}(x)], \quad u(x) \in \mathbf{A}^{(n-1-\omega)} \quad \text{strictly.}$$

Therefore,

$$\mathcal{D}_n^{(j)}\left\{x^{n-1}/\psi(x)\right\} = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{u(x)\exp\left[-\hat{\phi}(x)\right]\right\}\Big|_{x=\eta_{j,n}} \quad \text{for some } \eta_{j,n} \in (y_j, y_{j+n}).$$

It is easy to show, by induction on n, that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{u(x)\exp\left[-\hat{\phi}(x)\right]\right\} = w(x)\exp\left[-\hat{\phi}(x)\right], \quad w \in \mathbf{A}^{(-1-\omega+kn)} \quad \text{strictly.}$$

Therefore,

$$\begin{aligned}\mathcal{D}_n^{(j)}\left\{x^{n-1}/\psi(x)\right\} &= \mathcal{D}_n^{(j)}\left\{w(x)\exp\left[-\hat{\phi}(x)\right]\right\} \\ &\sim L\eta_{j,n}^{-1-\omega+kn}\exp\left[-\hat{\phi}(\eta_{j,n})\right] \quad \text{as } j \to \infty \\ &\sim L y_j^{-1-\omega+kn}\exp\left[-\hat{\phi}(\eta_{j,n})\right] \quad \text{as } j \to \infty, \quad L > 0 \quad \text{some constant.}\end{aligned} \tag{6.11}$$

Substituting (6.9) and (6.11) in (6.1) we obtain

$$A_n^{(j)} - I[f] = \mathcal{O}\left(y_j^{\omega-kn-n}\exp\left[\hat{\phi}(\eta_{j,n})\right]\right) \quad \text{as } j \to \infty.$$

Realizing that $\exp[\hat{\phi}(\eta_{j,n})] \leqslant \exp[\hat{\phi}(y_j)]$ and invoking (5.16) and (5.21) again we finally obtain (6.6).

The result in (6.7) follows from Theorem 9.4.3 in Sidi (2003, Chapter 9), just as that in (6.4) does.

Finally, the result in (6.8) follows from Theorem 8.6.7 in Sidi (2003, Chapter 8). (In this case, it is enough to verify with the help of Lemma A.1 that x_l^{-1} are precisely as needed in Sidi, 2003.) □

We have kept the stability results out, as we do not wish to go into the precise description of the issue of stability here. We encourage the reader to consult the relevant theorems and lemmas in Sidi (2003, Chapters 8 and 9). We only state that the $mW^{(s)}$-transformation is stable when s is odd in every part of Theorem 6.2, and this can be proved rigorously. When $k \leqslant 0$, in part 1 of Theorem 6.3 the $mW^{(s)}$-transformation is unstable with R_l as in (3.13) and it is stable with R_l as in (3.14). When $k \geqslant m$, in part 2 of Theorem 6.3 the $mW^{(s)}$-transformation is stable.

7. Numerical examples

We have applied the $mW^{(s)}$-transformation to many convergent or divergent integrals with integrands in the classes $\widetilde{\mathbf{B}}^{(s)}$. We have done this with different values of s and with every possible type of function in $\widetilde{\mathbf{B}}^{(s)}$. In the case of convergent integrals, in each such application we are able to reach practically machine accuracy. Less than machine accuracy is achieved for divergent integrals, the achievable accuracy depending on the rate at which $|F(x)|$ grows with x. (Note that, when $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent the floating-point errors incurred when computing the $F(x_l)$ tend to infinity as $l \to \infty$, even if the $F(x_l)$ are computed to machine accuracy, and this prevents the computed approximations $A_n^{(j)}$ from achieving machine accuracy.)

All of the integrals considered here are of the form $I[f] = \int_0^\infty f(x)\,\mathrm{d}x$. We have considered two classes of functions.

1. The first class contains functions that can be evaluated in quadruple precision (approximately 35 decimal digits). These are combinations of elementary functions, which are already included in the Fortran language. Such functions enable us to test the $mW^{(s)}$-transformation and the two implementations given in Section 3 to a very high accuracy and draw reliable conclusions from the relevant numerical results.
2. The second class involves functions, such as Bessel functions of different orders, which are provided in double precision (approximately 16 decimal digits), by the IMSL library, for example. Infinite range integrals of such functions have become of some interest lately, as also explained in Section 1.

In all of our examples below, we have implemented the $mW^{(s)}$-transformation by letting

$$\psi(x_l) = \begin{cases} \chi(x_l) & \text{if } s \text{ odd}, \\ x_l^m \chi(x_l) & \text{if } s \text{ even}. \end{cases}$$

We have taken R_l in (3.9) as in (3.13) when s is odd. When s is even we have taken R_l both as in (3.13) and as in (3.14) (GPS) to illustrate the fact that the former gives convergence but is not very stable numerically when $k \leqslant m$, while the second gives convergence and is stable numerically. When using GPS we have taken $\sigma = 1.3$ throughout; with this choice of σ in (3.14) we have $R_l = l$ for $l = 0, 1, \ldots, 7$, and $R_8 = 9$, $R_{12} = 23$, $R_{16} = 62$, $R_{20} = 175$, $R_{24} = 497$ and so on. In our examples we have computed $A_n^{(0)}$, $n = 0, 1, \ldots$, where the $A_n^{(j)}$ are the approximations produced by the $mW^{(s)}$-transformation as in (3.10). Note that $R_n + 2$ is the number of finite range integrals $F(x_l)$ used to obtain $A_n^{(0)}$. Finally, in the tables below, we have defined

$$E_n[f] = \left| A_n^{(0)} - I[f] \right|, \quad n = 0, 1, 2, \ldots.$$

EXAMPLE 7.1 $[k = 0]$ We have applied the $mW^{(s)}$-transformations with integrands $f(x)$ that are of the form

$$f(x) = \frac{\sin^p x \cos^q x}{x^r},$$

where p and q are non-negative integers. Clearly, $s = p + q$ and $\hat{\theta}(x) = x$ for these integrands. Here we present the numerical results we have obtained for two such integrands:

$$f_{1,1}(x) = \frac{\sin^5 x}{x^2}, \quad I[f_{1,1}] = \frac{5}{16}(3\log 3 - \log 5); \quad s = 5,$$

$$f_{1,2}(x) = \frac{\sin^6 x}{x^2}, \quad I[f_{1,2}] = \frac{3}{16}\pi; \quad s = 6.$$

In Table 1 we give the numerical results obtained for the integrals $I[f_{1,1}]$ and $I[f_{1,2}]$. We have used $x_l = \left(l + \frac{1}{2}\right)\pi$.

EXAMPLE 7.2 $[1 \leqslant k < m]$ Here we consider functions of the form

$$f(x;s) = -\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-x}(\cos x^2)^s\right] = e^{-x}\left(\cos x^2\right)^{s-1}\left[\cos x^2 + 2sx\sin x^2\right], \quad I[f] = 1.$$

Of course, $m = 2$ and $k = 1$ for $f(x;s)$, and $f(x;s) \in \widetilde{\mathbf{B}}^{(s)}$. In addition, $\hat{\theta}(x) = x^2$. We have computed $I[f_{2,1}]$ and $I[f_{2,2}]$, with

$$f_{2,1}(x) = f(x;5), \quad f_{2,2}(x) = f(x;6).$$

The numerical results for $I[f_{2,1}]$ and $I[f_{2,2}]$ are given in Table 2. We have used $x_l = \sqrt{(l+1)\pi}$.

EXAMPLE 7.3 $[k = m]$ For this case we consider the functions

$$f_{3,1}(x) = e^{-px}\frac{\sin^3 x}{x}, \quad I[f_{3,1}] = \frac{1}{2}\tan^{-1}\left\{\frac{1}{p}\right\} - \frac{1}{4}\tan^{-1}\left\{\frac{2p}{p^2+3}\right\}; \quad s = 3,$$

TABLE 1 *Errors $E_n[f_{1,1}]$ and $E_n[f_{1,2}]$ for the integrals of Example 7.1*

n	R_n	$E_n[f_{1,1}]$	R_n	$E_n[f_{1,2}]$	R_n	$E_n[f_{1,2}]$
4	4	$2.38D-07$	4	$7.35D-05$	4	$7.35D-05$
8	8	$8.01D-13$	8	$1.33D-08$	9	$1.19D-08$
12	12	$2.00D-17$	12	$3.38D-13$	23	$1.05D-13$
16	16	$4.28D-22$	16	$2.29D-16$	62	$5.72D-19$
20	20	$4.01D-27$	20	$4.18D-21$	175	$1.22D-25$
24	24	$1.55D-31$	24	$6.37D-23$	497	$9.14D-32$
28	28	$1.47D-31$	28	$2.44D-21$		
32	32		32	$1.32D-19$		
36	36		36	$4.33D-17$		
40	40		40	$1.79D-14$		

TABLE 2 *Errors $E_n[f_{2,1}]$ and $E_n[f_{2,2}]$ for the integrals of Example 7.2*

n	R_n	$E_n[f_{2,1}]$	R_n	$E_n[f_{2,2}]$	R_n	$E_n[f_{2,2}]$
4	4	$1.90D-07$	4	$2.74D-06$	4	$2.74D-06$
8	8	$2.74D-13$	8	$6.06D-10$	9	$5.38D-10$
12	12	$7.55D-19$	12	$2.40D-15$	23	$1.13D-15$
16	16	$2.74D-25$	16	$3.95D-18$	62	$2.18D-21$
20	20	$5.93D-31$	20	$2.64D-22$	175	$9.43D-30$
24	24	$8.72D-31$	24	$3.40D-24$	497	$6.01D-31$
28	28	$8.72D-31$	28	$4.77D-22$		
32	32		32	$1.48D-20$		
36	36		36	$3.01D-18$		
40	40		40	$8.55D-16$		

$$f_{3,2}(x) = e^{-px}\frac{\sin^4 x}{x}, \quad I[f_{3,2}] = \frac{1}{8}\log\left\{\frac{p^2+4}{p^2}\right\} + \frac{1}{16}\log\left\{\frac{(p^2+4)^2}{p^2(p^2+16)}\right\}; \quad s = 4.$$

In addition, $\hat{\theta}(x) = x$. We have chosen $p = 0.1$ for our computations. The numerical results for $I[f_{3,1}]$ and $I[f_{3,2}]$ are given in Table 3. We have used $x_l = (l+1)\pi$.

The next examples involve products of Bessel functions.

EXAMPLE 7.4 We now consider the functions

$$f(x; s) = [J_0(x)]^{s-1}J_1(x), \quad I[f] = \frac{1}{s}.$$

This can be verified by invoking the fact that $J_0'(x) = -J_1(x)$. Of course, for these integrals, $m = 1$ and $k = 0$ and $f(x; s) \in \widetilde{\mathbf{B}}^{(s)}$. In addition, $\hat{\theta}(x) = x$. We have computed $I[f_{4,1}]$ and $I[f_{4,2}]$, with

$$f_{4,1}(x) = f(x; 9), \quad f_{4,2}(x) = f(x; 10).$$

The numerical results for $I[f_{4,1}]$ and $I[f_{4,2}]$ are given in Table 4. We have used $x_l = \left(l + \frac{1}{2}\right)\pi$.

TABLE 3 *Errors $E_n[f_{3,1}]$ and $E_n[f_{3,2}]$ for the integrals of Example 7.3*

n	R_n	$E_n[f_{3,1}]$	R_n	$E_n[f_{3,2}]$	R_n	$E_n[f_{3,2}]$
4	4	$1.68D-07$	4	$9.47D-04$	4	$9.47D-04$
8	8	$1.01D-13$	8	$1.55D-06$	9	$1.31D-06$
12	12	$1.79D-18$	12	$2.44D-09$	23	$1.17D-10$
16	16	$1.19D-23$	16	$3.84D-12$	62	$3.15D-18$
20	20	$4.57D-29$	20	$6.03D-15$	175	$1.93D-34$
24	24	$3.85D-34$	24	$9.49D-18$	497	$5.78D-34$
28	28	$0.00D+00$	28	$1.49D-20$		
32	32		32	$2.35D-23$		
36	36		36	$6.35D-25$		
40	40		40	$4.38D-24$		

TABLE 4 *Errors $E_n[f_{4,1}]$ and $E_n[f_{4,2}]$ for the integrals of Example 7.4*

n	R_n	$E_n[f_{4,1}]$	R_n	$E_n[f_{4,2}]$	R_n	$E_n[f_{4,2}]$
4	4	$4.41D-12$	4	$1.48D-12$	4	$1.48D-12$
8	8	$5.55D-17$	8	$1.39D-17$	9	$2.36D-16$
12	12	$8.33D-17$	12	$1.95D-14$	23	$1.39D-17$
16	16	$1.11D-16$	16	$4.34D-13$	62	$9.71D-17$
20	20	$9.71D-17$	20	$4.15D-12$	175	$4.16D-17$
24	24	$9.71D-17$	24	$4.85D-09$	497	$5.55D-17$
28	28	$1.39D-16$	28	$2.70D-07$		

EXAMPLE 7.5 We now consider

$$f_{5,1}(x) = x[J_0(x)]^3, \quad I[f_{5,1}] = \frac{2}{\pi\sqrt{3}}; \quad s = 3,$$
$$f_{5,2}(x) = x^2[J_0(x)]^2 J_1(x), \quad I[f_{5,2}] = \frac{4}{3\pi\sqrt{3}}; \quad s = 3.$$

Here $m = 1$, $k = 0$ and $\hat{\theta}(x) = x$. Of these integrals $I[f_{5,1}]$ is convergent, while $I[f_{5,2}]$ is divergent but defined in the sense of Abel summability. Actually, $I[f_{5,2}]$ can be obtained using $I[f_{5,1}]$ as follows. Integrating by parts we obtain

$$I[f_{5,2}] = \int_0^\infty x^2[J_0(x)]^2 J_1(x)\,\mathrm{d}x = -\frac{1}{3}x^2[J_0(x)]^3|_0^\infty + \frac{2}{3}\int_0^\infty x[J_0(x)]^3\,\mathrm{d}x$$
$$= \frac{2}{3}\int_0^\infty x[J_0(x)]^3\,\mathrm{d}x = \frac{2}{3}I[f_{5,1}]$$

since $x^2[J_0(x)]^3$ vanishes at $x = 0$ and it behaves like $x^{1/2}(A\,e^{\mathrm{i}x} + B\,e^{-\mathrm{i}x})^3 + o(1)$ as $x \to \infty$, and thus makes no contribution to the Abel sum of $I[f_{5,2}]$. The numerical results for $I[f_{5,1}]$ and $I[f_{5,2}]$ are given in Table 5. We have used $x_l = \left(l + \frac{1}{2}\right)\pi$.

EXAMPLE 7.6 Next we consider

$$f(x;\lambda) = x^{-\lambda}[J_0(x)]^2, \quad I[f] = \frac{\Gamma(\lambda)\Gamma\left(\frac{1-\lambda}{2}\right)}{2^\lambda\Gamma\left(\frac{1+\lambda}{2}\right)}, \quad 0 < \Re\lambda < 1; \quad s = 2.$$

Again, $m = 1$ and $k = 0$ and $\hat{\theta}(x) = x$. The constraints $\Re\lambda < 1$ and $\Re\lambda > 0$ guarantee that $f(x)$ is integrable at $x = 0$ and $x = \infty$, respectively. For $\Re\lambda < 0$, $f(x)$ is integrable partly in the sense of Abel summability and partly in the sense of Hadamard finite part, as explained at the end of Section 4. In fact, $I[f]$ can be continued analytically to the whole λ-plane except $\lambda = -2i$ and $\lambda = 2i + 1$, $i = 0, 1, \ldots,$ where it has simple poles. Since, in our method, we need to compute the finite range integrals $F(x) = \int_0^x f(x)\,\mathrm{d}x$, we cannot work with those values of λ for which $\Re\lambda \geqslant 1$. We can, however, work with those for which $\Re\lambda \leqslant 0$. In this example, we consider

$$f_{6,1}(x) = f\left(x; -\tfrac{1}{2}\right), \quad f_{6,2}(x) = f\left(x; -\tfrac{3}{2}\right).$$

TABLE 5 *Errors $E_n[f_{5,1}]$ and $E_n[f_{5,2}]$ for the integrals of Example 7.5*

n	R_n	$E_n[f_{5,1}]$	$E_n[f_{5,2}]$
4	4	$3.97D-06$	$2.01D-05$
8	8	$2.57D-11$	$1.11D-10$
12	12	$3.89D-16$	$7.49D-16$
16	16	$7.22D-16$	$2.75D-15$
20	20	$6.11D-16$	$7.91D-15$
24	24	$8.33D-16$	$1.12D-14$

We have used $x_l = \left(l + \frac{1}{2}\right)\pi$. The numerical results for $I[f_{6,1}]$ and $I[f_{6,2}]$ are given in Table 6. This example shows that the $mW^{(s)}$-transformation is an effective tool for analytic continuation as well.

The only case not tested by example so far is that with $k > m$. The next example does this testing with an integrand in $\widetilde{\mathbf{B}}^{(2)}$ involving Bessel functions.

EXAMPLE 7.7 [$k > m$] We apply the $mW^{(s)}$-transformation to the integral

$$f_7(x) = xe^{-px^2/2}[J_0(x)]^2, \quad I[f_7] = \frac{1}{p}\exp(-1/p)I_0(1/p); \quad s = 2.$$

Here $I_0(x)$ is the modified Bessel function of order zero of the first kind. Note that $2 = k > m = 1$ for $f_7(x)$. For our computations we have chosen $p = 0.02$. We have applied the $mW^{(s)}$-transformation with R_l as in both (3.13) and (3.14). We have also used $x_l = \left(l + \frac{1}{2}\right)\pi$. The numerical results for $I[f_7]$ are given in Table 7. Note that, in this case, both choices of the R_l are very effective. This should be contrasted with the cases in which s is even and $k \leqslant m$.

EXAMPLE 7.8 Before ending this section we would like to study the case given in Example 4.2, as this case is somewhat unusual. Clearly, the integral of $f(x)$ can be computed by the $mW^{(2)}$-transformation precisely as described earlier. It can also be computed at much smaller cost by taking into account the special structure of $f(x)$, however. Recall that $f(x)$ has the form

$$f(x) = U^+(x)\exp\left[2\mathrm{i}\widehat{\theta}(x)\right] + U^-(x)\exp\left[-2\mathrm{i}\widehat{\theta}(x)\right],$$

where $U^{\pm}$ have the same structure as the U_j in (4.4). Thus, $f(x)$ is purely oscillatory. Actually, $f(x) \in \widetilde{\mathbf{B}}^{(1)}$, with the function $\widehat{\theta}(x)$ in (4.4) replaced by $2\widehat{\theta}(x)$, and its integral can be computed with high precision and stably by using the $mW^{(1)}$-transformation, despite the fact that $s = 2$ (even) in this case to begin with. For this we choose the x_l as the roots of the polynomial equations $2\widehat{\theta}(x) = (q + l)\pi$ or $2\widehat{\theta}(x) = (q + l + \frac{1}{2})\pi$, $l = 0, 1, \ldots$, where q is an integer for which x_0 is the smallest zero greater than

TABLE 6 *Errors $E_n[f_{6,1}]$ and $E_n[f_{6,2}]$ for the integrals of Example 7.6*

n	R_n	$E_n[f_{6,1}]$	R_n	$E_n[f_{6,1}]$	R_n	$E_n[f_{6,2}]$	R_n	$E_n[f_{6,2}]$
4	4	$1.86D-03$	4	$1.86D-03$	4	$1.10D-01$	4	$1.10D-01$
8	8	$1.41D-08$	9	$1.51D-08$	8	$3.99D-04$	9	$3.79D-04$
12	12	$2.74D-07$	23	$1.50D-09$	12	$7.63D-06$	23	$1.04D-07$
16	16	$6.82D-05$	62	$3.90D-09$	16	$4.83D-03$	62	$5.33D-07$
20	20	$1.62D-02$	175	$3.11D-10$	20	$1.66D+00$	175	$3.51D-07$
24	24	$6.02D-01$	497	$1.46D-08$	24	$4.98D+01$	497	$1.59D-05$

TABLE 7 *Errors $E_n[f_7]$ for the integrals of Example 7.7*

n	R_n	$E_n[f_7]$	R_n	$E_n[f_7]$
4	4	$4.46D-03$	4	$4.46D-03$
8	8	$2.38D-08$	9	$9.72D-09$
12	12	$5.42D-14$	23	$0.00D+00$
16	16	$1.33D-15$	62	$1.78D-15$

a of $\sin[2\widehat{\theta}(x)]$ and of $\cos[2\widehat{\theta}(x)]$, respectively. Now we proceed by choosing $R_l = l$, instead of $R_l = \max\{\max\lfloor\sigma R_{l-1}\rfloor, l\}$ with $R_0 = 0$. This applies to functions of the form $f(x) = W(x)J_\nu(x)Y_\nu)(x)$, as discussed in Example 4.2.

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Appendix A

In this appendix we give the complete asymptotic analysis of $R(x_l)$ and $\Delta\hat{\phi}(x_l)$ as $l \to \infty$, and deduce the asymptotic behaviour of $S(x_l)$ that is required in Section 5. Since everything depends on the asymptotic behaviour of x_l as $l \to \infty$ we review this topic and its consequences first. Before doing that we recall that x_l is the largest real solution of the polynomial equations $\hat{\theta}(x) = (q+l)\pi$ or $\hat{\theta}(x) = (q+l+\frac{1}{2})\pi$, where $\hat{\theta}(x) = \sum_{i=0}^{m-1} \mu_i x^{m-i}$, $\mu_0 > 0$.

In the sequel, $T(l)$ will denote *generically* any function of l that has a convergent expansion for all large l or an asymptotic expansion as $l \to \infty$, of the form $\sum_{i=0}^{\infty} \tau_i l^{-i/m}$, with $\tau_0 = 1$. Let us denote the class of such functions by $\mathcal{T}$. It is easy to show that if $T_1(l)$ and $T_2(l)$ are two such functions in $\mathcal{T}$, then

$T_3(l) \equiv [T_1(l)]^p[T_2(l)]^q$ is also in $\mathcal{T}$, for any p and q, whether integer or not. In the sequel, we express this generically in the form $[T(l)]^p[T(l)]^q = T(l)$.

The following lemma is Lemma 3.4 in Sidi (1988).

LEMMA A.1 As a function of l, and for sufficiently large l, x_l has the convergent expansion

$$x_l = \sum_{i=0}^{\infty} a_i l^{(1-i)/m} = a_0 l^{1/m} T(l), \quad a_0 = \left(\frac{\pi}{\mu_0}\right)^{1/m} > 0. \tag{A.1}$$

Thus, we also have

$$x_{l+1} = \sum_{i=0}^{\infty} a_i' l^{(1-i)/m}; \quad a_i' = a_i, \quad 0 \leqslant i \leqslant m-1, \quad a_m' = a_m + \frac{a_0}{m}, \tag{A.2}$$

$$x_{l+1}^p - x_l^p = \tilde{a}_0^{(p)} l^{-1+p/m} T(l), \quad \tilde{a}_0^{(p)} = \frac{p}{m} a_0^p \quad \forall\, p, \tag{A.3}$$

$$\frac{x_{l+1}}{x_l} = 1 + \frac{1}{m} l^{-1} T(l) \quad \Rightarrow \quad \left(\frac{x_{l+1}}{x_l}\right)^{\omega} = 1 + \frac{\omega}{m} l^{-1} T(l) \quad \forall\, \omega. \tag{A.4}$$

First, note that the convergent series in (A.1) represents x_l asymptotically as $l \to \infty$ as well. As mentioned earlier we can also choose the sequence $\{x_l\}$ such that x_l has a not necessarily convergent asymptotic expansion of the form

$$x_l \sim \sum_{i=0}^{\infty} a_i l^{(1-i)/m} \quad \text{as } l \to \infty \quad \Rightarrow \quad x_l = a_0 T(l). \tag{A.5}$$

Whether x_l satisfies (A.1) or (A.5), $x_l \sim a_0 l^{1/m}$ as $l \to \infty$. Therefore, a function $P(l)$ that has a convergent or asymptotic expansion (as $l \to \infty$) of the form $l^{\kappa/m} \sum_{i=0}^{\infty} d_i l^{-i/m}$, where $d_0 \neq 0$ and κ is an integer, also has a convergent or asymptotic expansion (as $x_l \to \infty$) of the form $x_l^{\kappa} \sum_{i=0}^{\infty} e_i x_l^{-i}$, where $e_0 = d_0/a_0^{\kappa} \neq 0$. This can be seen easily in the case that the x_l are solutions of the polynomial equations $\hat{\theta}(x) = (q+l)\pi$ or $\hat{\theta}(x) = \left(q + l + \frac{1}{2}\right)\pi$; in these cases, l is a polynomial in x_l and substituting this polynomial in the series $l^{\kappa/m} \sum_{i=0}^{\infty} d_i l^{-i/m}$ and re-expanding in negative powers of x_l (since $x_l \to \infty$ as $l \to \infty$) we obtain the expansion $x_l^{\kappa} \sum_{i=0}^{\infty} e_i x_l^{-i}$. This last point will be of help in reaching our final conclusions about $S(x_l)$.

Making use of Lemma A.1, we can now go on to analyse the quantities $R(x_l)$, $\Delta\hat{\phi}(x_k)$ and $S(x_l)$ as $l \to \infty$.

Asymptotics of $R(x_l)$ Since $G(x)$ is strictly in $\mathbf{A}^{(0)}$ it satisfies $G(x) \sim \sum_{i=0}^{\infty} c_i x^{-i}$ as $x \to \infty$, $c_0 \neq 0$. Therefore, $G(x_l) = c_0 T(l)$. Consequently, if c_r is the first nonzero c_i following c_0, then, by (A.3)

$$G(x_{l+1}) - G(x_l) \sim \sum_{i=0}^{\infty} c_i \left(x_{l+1}^{-i} - x_l^{-i}\right) = -\frac{r c_r}{m a_0^r} l^{-1-r/m} T(l), \quad r \geqslant 1,$$

and hence

$$\frac{G(x_{l+1})}{G(x_l)} = \frac{G(x_{l+1}) - G(x_l)}{G(x_l)} + 1 = 1 - \frac{r c_r}{m c_0 a_0^r} l^{-1-r/m} T(l). \tag{A.6}$$

Combining (A.4) and (A.6) in (5.12) we obtain

$$R(x_l) = \left[1 + \frac{\omega}{m} l^{-1} T(l)\right]\left[1 - \frac{r c_r}{m c_0 a_0^r} l^{-1-r/m} T(l)\right]$$
$$= 1 + \frac{\omega}{m} l^{-1} T(l). \tag{A.7}$$

Asymptotics of $\exp\left[\Delta\hat{\phi}(x_l)\right]$ To begin there are two cases to consider.

1. When $k \leqslant 0$ we have that $\hat{\phi}(x) \equiv 0$, and therefore, $\Delta\hat{\phi}(x_l) = 0$ and

$$\exp\left[\Delta\hat{\phi}(x_l)\right] \equiv 1 \quad \forall\, l. \tag{A.8}$$

2. When $k > 0$, by (A.3), we have

$$\Delta\hat{\phi}(x_l) = \sum_{i=0}^{k-1} \lambda_i \left(x_{l+1}^{k-i} - x_l^{k-i}\right) = \sum_{i=0}^{k-1} \lambda_i \frac{k-i}{m} a_0^{k-i} l^{-1+(k-i)/m} T(l).$$

Consequently,

$$\Delta\hat{\phi}(x_l) = C l^{-1+k/m} T(l), \quad C = \frac{k}{m}\lambda_0 a_0^k < 0 \quad \text{since } \lambda_0 < 0. \tag{A.9}$$

The behaviour of $\exp[\Delta\hat{\phi}(x_l)]$ varies depending on whether $1 \leqslant k < m$ or $k = m$ or $k > m$.

(a) When $1 \leqslant k < m$, $\lim_{l\to\infty} \Delta\hat{\phi}(x_l) = 0$. As a result,

$$\exp\left[\Delta\hat{\phi}(x_l)\right] = 1 + C l^{-1+k/m} T(l). \tag{A.10}$$

(b) When $k = m$, (A.9) becomes

$$\Delta\hat{\phi}(x_l) = C T(l) = C + D l^{-i/m} T(l) \quad \text{for some } i \geqslant 1 \text{ and } D \neq 0.$$

Since $\lim_{l\to\infty} l^{-i/m} T(l) = 0$ there holds

$$\exp[D l^{-i/m} T(l)] = 1 + D l^{-i/m} T(l).$$

Therefore, with C as in (A.9),

$$\exp\left[\Delta\hat{\phi}(x_l)\right] = e^C \exp\left[D l^{-i/m} T(l)\right] = e^C \left[1 + D l^{-i/m} T(l)\right]$$
$$\text{for some } D \neq 0 \text{ and integer } i \geqslant 1. \tag{A.11}$$

(c) When $k > m$ we have $\lim_{l\to\infty} \Delta\hat{\phi}(x_l) = -\infty$ since $C < 0$ by (A.9). As a result, $\lim_{l\to\infty} \exp[\Delta\hat{\phi}(x_l)] = 0$, which implies that $\exp[\Delta\hat{\phi}(x_l)]$ has an empty asymptotic expansion, meaning that it tends to zero faster than any negative power of l as $l \to \infty$, that is,

$$\exp\left[\Delta\hat{\phi}(x_l)\right] = \mathcal{O}(l^{-p}) \quad \text{as } l \to \infty, \quad \forall\, p > 0. \tag{A.12}$$

Asymptotics of $S(x_l)$ Combining the above we have the following results for $S(x_l)$.

1. When $k \leqslant 0$,

$$S(x_l) = R(x_l) = 1 + \frac{\omega}{m} l^{-1} T(l). \tag{A.13}$$

2. When $1 \leqslant k < m$,

$$S(x_l) = \left[1 + \frac{\omega}{m} l^{-1} T(l)\right]\left[1 + C l^{-1+k/m} T(l)\right] = 1 + C l^{-1+k/m} T(l). \tag{A.14}$$

3. When $k = m$ there exists a constant $M \neq 0$ and an integer $\kappa \geqslant \min\{i, m\}$ such that

$$S(x_l) = e^C \left[1 + \frac{\omega}{m} l^{-1} T(l)\right]\left[1 + D l^{-i/m} T(l)\right] = e^C + M l^{-\kappa/m} T(l). \tag{A.15}$$

4. When $k > m$,

$$S(x_l) = \mathcal{O}(l^{-p}) \quad \text{as } l \to \infty, \quad \forall\, p > 0. \tag{A.16}$$

In our treatment of the quantities $[S(x_l) \pm 1]$ below we also recall the fact that a function $P(l)$ that has a convergent or asymptotic expansion (as $l \to \infty$) of the form $l^{\kappa/m} \sum_{i=0}^{\infty} d_i l^{-i/m}$, where $d_0 \neq 0$ and κ is an integer, also has a convergent or asymptotic expansion (as $x_l \to \infty$) of the form $x_l^{\kappa} \sum_{i=0}^{\infty} e_i x_l^{-i}$, where $e_0 = d_0/a_0^{\kappa} \neq 0$.

A.1 $[S(x_l) + 1]$ *when s is odd*

1. When $k \leqslant 0$,

$$S(x_l) + 1 = 2 + \frac{\omega}{m} l^{-1} T(l) \quad \Rightarrow \quad [S(x) + 1] \in \mathbf{A}^{(0)} \quad \text{strictly}. \tag{A.17}$$

2. When $1 \leqslant k < m$,

$$S(x_l) + 1 = 2 + C l^{-1+k/m} T(l) \quad \Rightarrow \quad [S(x) + 1] \in \mathbf{A}^{(0)} \quad \text{strictly}. \tag{A.18}$$

3. When $k = m$,

$$S(x_l) + 1 = \left(e^C + 1\right) + M l^{-\kappa/m} T(l) \quad \Rightarrow \quad [S(x) + 1] \in \mathbf{A}^{(0)} \quad \text{strictly}. \tag{A.19}$$

4. When $k > m$,

$$S(x_l) + 1 = 1 + \mathcal{O}(l^{-p}) \quad \text{as } l \to \infty, \quad \forall\, p > 0 \quad \Rightarrow \quad [S(x) + 1] \in \mathbf{A}^{(0)} \quad \text{strictly}. \tag{A.20}$$

A.2 $[S(x_l) - 1]$ *when s is even*

1. When $k \leqslant 0$,

$$S(x_l) - 1 = \frac{\omega}{m} l^{-1} T(l) \quad \Rightarrow \quad [S(x) - 1] \in \mathbf{A}^{(-m)} \quad \text{strictly}. \tag{A.21}$$

2. When $1 \leqslant k < m$,

$$S(x_l) - 1 = Cl^{-1+k/m} T(l) \quad \Rightarrow \quad [S(x) - 1] \in \mathbf{A}^{(k-m)} \quad \text{strictly.} \tag{A.22}$$

3. When $k = m$, because $C < 0$ and hence $e^C \neq 1$,

$$S(x_l) - 1 = \left(e^C - 1\right) + Ml^{-\kappa/m} T(l) \quad \Rightarrow \quad [S(x) - 1] \in \mathbf{A}^{(0)} \quad \text{strictly.} \tag{A.23}$$

4. When $k > m$,

$$S(x_l) - 1 = -1 + \mathcal{O}(l^{-p}) \quad \text{as } l \to \infty, \quad \forall\, p > 0 \quad \Rightarrow \quad [S(x) - 1] \in \mathbf{A}^{(0)} \quad \text{strictly.} \tag{A.24}$$