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Euler–Maclaurin Expansions for Integrals with Arbitrary Algebraic-Logarithmic Endpoint Singularities

Avram Sidi

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Abstract In this paper, we provide the Euler–Maclaurin expansions for (offset) trapezoidal rule approximations of the finite-range integrals $I[f]=\int_a^b f(x)\,dx$, where $f\in C^\infty(a,b)$ but can have general algebraic-logarithmic singularities at one or both endpoints. These integrals may exist either as ordinary integrals or as Hadamard finite part integrals. We assume that $f(x)$ has asymptotic expansions of the general forms

$$f(x)\sim \widehat{P}\big(\log(x-a)\big)(x-a)^{-1}+\sum_{s=0}^{\infty}P_s\big(\log(x-a)\big)(x-a)^{\gamma_s}\quad \text{as } x\to a+,$$

$$f(x)\sim \widehat{Q}\big(\log(b-x)\big)(b-x)^{-1}+\sum_{s=0}^{\infty}Q_s\big(\log(b-x)\big)(b-x)^{\delta_s}\quad \text{as } x\to b-,$$

where $\widehat{P}(y)$, $P_s(y)$ and $\widehat{Q}(y)$, $Q_s(y)$ are polynomials in y. The γ_s and δ_s are distinct, complex in general, and different from -1. They also satisfy

$$\Re\gamma_0\le\Re\gamma_1\le\cdots,\qquad \lim_{s\to\infty}\Re\gamma_s=+\infty;$$

$$\Re\delta_0\le\Re\delta_1\le\cdots,\qquad \lim_{s\to\infty}\Re\delta_s=+\infty.$$

The results we obtain in this work extend the results of a recent paper [A. Sidi, *Numer. Math.* 98:371–387, 2004], which pertain to the cases in which $\widehat{P}(y)\equiv 0$ and $\widehat{Q}(y)\equiv 0$. They are expressed in very simple terms based only on the asymptotic expansions of $f(x)$ as $x\to a+$ and $x\to b-$. The results we obtain in this work gen-

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A. Sidi (✉)
Computer Science Department, Technion–Israel Institute of Technology, Haifa 32000, Israel
e-mail: asidi@cs.technion.ac.il

eralize, and include as special cases, all those that exist in the literature. Let $D_\omega = \frac{d}{d\omega}$, $h = (b-a)/n$, where n is a positive integer, and define $\check{T}_n[f] = h\sum_{i=1}^{n-1} f(a+ih)$. Then with $\widehat{P}(y) = \sum_{i=0}^{\hat{p}} \hat{c}_i y^i$ and $\widehat{Q}(y) = \sum_{i=0}^{\hat{q}} \hat{d}_i y^i$, one of these results reads

$$\begin{aligned}\check{T}_n[f] \sim I[f] &+ \sum_{i=0}^{\hat{p}} \left[\sum_{r=i}^{\hat{p}} \binom{r}{i} \hat{c}_r \sigma_{r-i}\right] (\log h)^i - \sum_{i=0}^{\hat{p}} \hat{c}_i \frac{(\log h)^{i+1}}{i+1} \\ &+ \sum_{s=0}^{\infty} P_s(D_{\gamma_s})\big[\zeta(-\gamma_s)\, h^{\gamma_s+1}\big] + \sum_{s=0}^{\infty} Q_s(D_{\delta_s})\big[\zeta(-\delta_s)\, h^{\delta_s+1}\big] \\ &+ \sum_{i=0}^{\hat{q}} \left[\sum_{r=i}^{\hat{q}} \binom{r}{i} \hat{d}_r \sigma_{r-i}\right] (\log h)^i - \sum_{i=0}^{\hat{q}} \hat{d}_i \frac{(\log h)^{i+1}}{i+1} \quad \text{as } h \to 0,\end{aligned}$$

where $\zeta(z)$ is the Riemann Zeta function and σ_i are Stieltjes constants defined via $\sigma_i = \lim_{n\to\infty}[\sum_{k=1}^{n} \frac{(\log k)^i}{k} - \frac{(\log n)^{i+1}}{i+1}]$, $i = 0, 1, \ldots$.

Keywords Euler–Maclaurin expansions · Asymptotic expansions · Trapezoidal rule · Endpoint singularities · Algebraic singularities · Logarithmic singularities · Hadamard finite part · Zeta function · Stieltjes constants

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1 Introduction

Euler–Maclaurin (E–M) expansions for trapezoidal rule approximations of finite-range integrals $\int_a^b f(x)\,dx$, and their various generalizations in the presence of possible algebraic and/or logarithmic endpoint singularities, are of interest in many different contexts. For example, they play an important role in the design of methods for the efficient numerical evaluation of such integrals.

In this work, we derive E–M expansions for trapezoidal rule approximations to $\int_a^b f(x)\,dx$, in the presence of arbitrary algebraic-logarithmic endpoint singularities. Specifically, we assume that $f(x)$ has the following properties:

1. $f \in C^\infty(a,b)$ and has the asymptotic expansions

$$\begin{aligned} f(x) &\sim \widehat{P}\big(\log(x-a)\big)(x-a)^{-1} + \sum_{s=0}^{\infty} P_s\big(\log(x-a)\big)\,(x-a)^{\gamma_s} \quad \text{as } x \to a+, \\ f(x) &\sim \widehat{Q}\big(\log(b-x)\big)(b-x)^{-1} + \sum_{s=0}^{\infty} Q_s\big(\log(b-x)\big)\,(b-x)^{\delta_s} \quad \text{as } x \to b-, \end{aligned} \tag{1.1}$$

where $\widehat{P}(y)$, $P_s(y)$ and $\widehat{Q}(y)$, $Q_s(y)$ are polynomials in y given as in

$$\begin{aligned} \widehat{P}(y) &= \sum_{i=0}^{\hat{p}} \hat{c}_i y^i, \qquad & P_s(y) &= \sum_{i=0}^{p_s} c_{si} y^i, \\ \widehat{Q}(y) &= \sum_{i=0}^{\hat{q}} \hat{d}_i y^i, \qquad & Q_s(y) &= \sum_{i=0}^{q_s} d_{si} y^i, \end{aligned} \tag{1.2}$$

and γ_s and δ_s are distinct and, in general, complex, and satisfy

$$\begin{aligned} &\gamma_s \neq -1 \quad \forall s; \qquad \Re\gamma_0 \leq \Re\gamma_1 \leq \Re\gamma_2 \leq \cdots; \qquad \lim_{s\to\infty} \Re\gamma_s = +\infty, \\ &\delta_s \neq -1 \quad \forall s; \qquad \Re\delta_0 \leq \Re\delta_1 \leq \Re\delta_2 \leq \cdots; \qquad \lim_{s\to\infty} \Re\delta_s = +\infty. \end{aligned} \tag{1.3}$$

Here, $\Re z$ stands for the real part of z.[1]

As an example, consider the case

$$f(x) = \bigl[\log(x-a)\bigr]^i (x-a)^{-p} g_a(x) = \bigl[\log(b-x)\bigr]^j (b-x)^{-q} g_b(x),$$

where p and q are positive integers and $g_a \in C^\infty[a,b)$ and $g_b \in C^\infty(a,b]$. If $g_a(x)$ and $g_b(x)$ have full Taylor series about $x=a$ and $x=b$, respectively, then the γ_s and the δ_s are, respectively,

$$\begin{aligned} &-p, -p+1, \ldots, -3, -2, 0, 1, 2, \ldots, \quad \text{and} \\ &-q, -q+1, \ldots, -3, -2, 0, 1, 2, \ldots, \end{aligned}$$

and we have

$$\widehat{P}(y) = \frac{g_a^{(p-1)}(a)}{(p-1)!} y^i \quad \text{and} \quad \widehat{Q}(y) = (-1)^{q-1} \frac{g_b^{(q-1)}(b)}{(q-1)!} y^j.$$

2. If $\hat{p} = \deg(\widehat{P})$, $p_s = \deg(P_s)$, $\hat{q} = \deg(\widehat{Q})$, and $q_s = \deg(Q_s)$ for each s, then the γ_s and δ_s are ordered such that

$$p_s \geq p_{s+1} \quad \text{if } \Re\gamma_{s+1} = \Re\gamma_s; \qquad q_s \geq q_{s+1} \quad \text{if } \Re\delta_{s+1} = \Re\delta_s. \tag{1.4}$$

[1] We can write the expansions in (1.1) in the "simpler" form

$$f(x) \sim \sum_{s=0}^{\infty} P_s\bigl(\log(x-a)\bigr)(x-a)^{\gamma_s} \quad \text{as } x \to a+, \qquad f(x) \sim \sum_{s=0}^{\infty} Q_s\bigl(\log(b-x)\bigr)(b-x)^{\delta_s} \quad \text{as } x \to b-,$$

allowing one of the γ_s and/or one of the δ_s to be equal to -1. However, this complicates the statements of our results. Therefore, we have chosen to separate these two exponents as in (1.1).

3. By (1.1), we mean that, for every integer $r = 0, 1, \ldots,$

$$\begin{aligned} f(x) - \left[\widehat{P}\big(\log(x-a)\big)(x-a)^{-1} + \sum_{s=0}^{r-1} P_s\big(\log(x-a)\big)\,(x-a)^{\gamma_s} \right] \\ = O\big(P_r\big(\log(x-a)\big)\,(x-a)^{\gamma_r}\big) \quad \text{as } x \to a+, \\ f(x) - \left[\widehat{Q}\big(\log(b-x)\big)(b-x)^{-1} + \sum_{s=0}^{r-1} Q_s\big(\log(b-x)\big)\,(b-x)^{\delta_s} \right] \\ = O\big(Q_r\big(\log(b-x)\big)\,(b-x)^{\delta_r}\big) \quad \text{as } x \to b-. \end{aligned} \tag{1.5}$$

This is consistent with (1.3) and (1.4). Note that when $r = 0$, (1.5) is valid, since the respective summations there are now empty (zero).

4. For each $k = 1, 2, \ldots,$ the kth derivative of $f(x)$ also has asymptotic expansions as $x \to a+$ and $x \to b-$ that are obtained by differentiating those in (1.1) term by term.

The following facts are consequences of (1.3):

(i) There are only finitely many γ_s having the same real parts, and only finitely many δ_s having the same real parts; consequently, $\Re\gamma_s < \Re\gamma_{s+1}$ and $\Re\delta_{s'} < \Re\delta_{s'+1}$ for infinitely many values of the indices s and s'.

(ii) The sequences $\{(x-a)^{\gamma_s}\}_{s=0}^{\infty}$ and $\{(b-x)^{\delta_s}\}_{s=0}^{\infty}$ are asymptotic scales as $x \to a+$ and $x \to b-$, respectively, in the following sense: For each $s = 0, 1, \ldots,$

$$\lim_{x\to a+} \left| \frac{(x-a)^{\gamma_{s+1}}}{(x-a)^{\gamma_s}} \right| = \begin{cases} 1 & \text{if } \Re\gamma_s = \Re\gamma_{s+1}, \\ 0 & \text{if } \Re\gamma_s < \Re\gamma_{s+1}, \end{cases}$$

$$\lim_{x\to b-} \left| \frac{(b-x)^{\delta_{s+1}}}{(b-x)^{\delta_s}} \right| = \begin{cases} 1 & \text{if } \Re\delta_s = \Re\delta_{s+1}, \\ 0 & \text{if } \Re\delta_s < \Re\delta_{s+1}. \end{cases}$$

(iii) The integral $\int_a^b f(x)\,dx$ exists in the ordinary sense only if $\widehat{P}(y) \equiv 0$, $\widehat{Q}(y) \equiv 0$, and $\Re\gamma_0 > -1$, $\Re\delta_0 > -1$. Otherwise, it exists in the sense of *Hadamard finite part* (HFP).[2] The latter is defined as follows: Let the integers μ and ν be such that

$$\Re\gamma_\mu > -1, \qquad \Re\delta_\nu > -1. \tag{1.6}$$

[2]The usual notation for Hadamard finite part (HFP) integrals is $⨍_a^b f(x)\,dx$. For simplicity, in this work, we use $\int_a^b f(x)\,dx$ to denote both ordinary and Hadamard finite part integrals. For the definition and properties of Hadamard finite part integrals, see Davis and Rabinowitz [3, pp. 11–14], for example. These integrals have some of the usual properties of regular integrals and some properties that are quite unusual. For example, they are invariant under a linear transformation of the variable of integration that is of the form $y = \alpha \pm x$, but they are not necessarily invariant under a transformation of the form $y = \alpha + \beta x$, $|\beta| \neq 1$.

Define also

$$\phi_\mu(x) := f(x) - \left[\widehat{P}\big(\log(x-a)\big)(x-a)^{-1} + \sum_{s=0}^{\mu-1} P_s\big(\log(x-a)\big)\,(x-a)^{\gamma_s} \right],$$

$$\psi_\nu(x) := f(x) - \left[\widehat{Q}\big(\log(b-x)\big)(b-x)^{-1} + \sum_{s=0}^{\nu-1} Q_s\big(\log(b-x)\big)\,(b-x)^{\delta_s} \right]. \tag{1.7}$$

Then, for arbitrary $t \in (a,b)$,

$$\begin{aligned}\int_a^b f(x)\,dx = {} & \sum_{i=0}^{\hat{p}} \hat{c}_i \frac{[\log(t-a)]^{i+1}}{i+1} + \sum_{s=0}^{\mu-1}\sum_{i=0}^{p_s} c_{si} \frac{d^i}{d\gamma_s^i} \frac{(t-a)^{\gamma_s+1}}{\gamma_s+1} \\ & + \int_a^t \phi_\mu(x)\,dx + \sum_{i=0}^{\hat{q}} \hat{d}_i \frac{[\log(b-t)]^{i+1}}{i+1} \\ & + \sum_{s=0}^{\nu-1}\sum_{i=0}^{q_s} d_{si} \frac{d^i}{d\delta_s^i} \frac{(b-t)^{\delta_s+1}}{\delta_s+1} + \int_t^b \psi_\nu(x)\,dx. \end{aligned} \tag{1.8}$$

Here the integrals of $\phi_\mu(x)$ and $\psi_\nu(x)$ exist in the ordinary sense, as is clear from the way we have chosen μ and ν in (1.6).

The earlier literature on E–M expansions for finite-range integrals $\int_a^b f(x)\,dx$ concerns either the case (i) $f \in C^\infty[a,b]$, or the case (ii) $f(x) = (x-a)^\gamma \times [\log(x-a)]^p g_a(x) = (b-x)^\delta [\log(b-x)]^q g_b(x)$, where p and q are nonnegative integers and $g_a \in C^\infty[a,b)$ and $g_b \in C^\infty(a,b]$. The case (i) is treated in many books on numerical analysis; see, for example, Atkinson [1], Davis and Rabinowitz [3], Ralston and Rabinowitz [12], Steffensen [20], or Stoer and Bulirsch [21]. The case (ii), with $\delta = 0$ and $q = 0$, was first treated in two papers by Navot, namely, in [8] with $\Re\gamma > -1$ and $p = 0$, and in [9] with $\Re\gamma > -1$ and $p = 1$; the treatment of [9] can be extended easily to arbitrary $p \geq 1$ using the technique described there. [Actually, Navot's results can easily be used to treat the case (ii) as well.] The case (ii) was later considered by Lyness and Ninham [6] using a different method involving generalized functions. Navot's results were later applied by Sidi and Israeli [19] to derive E–M expansions and quadrature methods of high accuracy for periodic singular and weakly singular Fredholm integral equations. (For a brief survey of the relevant expansions, see also Sidi [13, Appendix D].)

Subsequently, in a paper by Ninham [10], Navot's expansions were shown to hold also for the case in which $\Re\gamma \leq -1$ and/or $\Re\delta \leq -1$, such that γ and δ are different from $-1, -2, \ldots$; in this case, $\int_a^b f(x)\,dx$ is defined as an HFP integral. Finally, the remaining case in which γ or δ or both are negative integers has recently been dealt with by Lyness [5] and by Monegato and Lyness [7]. The technique used in [7] unifies the treatments of the various expansions; it is based on an approach introduced by Verlinden [23] that employs the Mellin transform.

Lately, E–M expansions associated with functions $f(x)$ described as in (1.1)–(1.5), but with $\widehat{P}(y) \equiv 0$ and $\widehat{Q}(y) \equiv 0$, have been considered in Sidi [14].[3] These generalized E–M expansions can be utilized to improve the performance of the trapezoidal rule approximations when the latter are preceded by an appropriate variable transformation that may be singular at the endpoints. This approach was suggested by the author in the papers Sidi [15, 16], and [17], in which some novel singular variable transformations are constructed and used in a way that "optimizes" the accuracy of the trapezoidal rule approximations to finite-range regular integrals with possibly algebraic endpoint singularities. The accuracies that can be achieved by this approach are remarkable.

Yet in another recent work by Sidi [18], E–M expansions are derived in the presence of functions $f(x)$ satisfying (1.1)–(1.5), with the limitation that $\widehat{P}(y)$, $P_s(y)$ and $\widehat{Q}(y)$, $Q_s(y)$ are *constant* polynomials. An interesting problem precisely with this feature has been considered in a recent paper by Brauchart, Hardin, and Saff [2], and this problem was tackled also in [18] with the help of the new technique developed there.

By allowing $\widehat{P}(y)$ and/or $\widehat{Q}(y)$ to be arbitrary nonzero polynomials, the present work thus completes the treatments of [14] and [18]. Furthermore, because we have allowed *arbitrary* algebraic-logarithmic endpoint singularities, our class of functions, as characterized via (1.1)–(1.5), contains, but is not contained in, the totality of all the previous classes. Thus, all of the classes of functions $f(x)$ treated in the earlier literature are subclasses of our general class here.

In the next section, we state the main results of this work. In Sect. 3, we provide some useful technical preliminaries. Finally, in Sect. 4, we provide the proofs of the main results. Our results have the pleasant feature that they are expressed in extremely simple terms based only on the asymptotic expansions in (1.1).

Before closing this section, we note that we have assumed that $f \in C^\infty(a,b)$ only for the sake of simplifying the presentation. We can assume that $f \in C^m(a,b)$ for some finite m, and obtain the appropriate E–M expansion for this case in the same way we obtain the E–M expansion for $f \in C^\infty(a,b)$. The method of proof applies to this case without any changes.

2 Main Results

Throughout the remainder of the paper, we use the notation

$$I[f] := \int_a^b f(x)\,dx, \tag{2.1}$$

[3] Even though the results of [14] are correct under the condition that $\gamma_s, \delta_s \neq -1$, they were stated with the unnecessarily stringent condition that $\gamma_s, \delta_s \neq -1, -2, -3, \dots,$ due to an unfortunate oversight. In this work, we correct this blunder.

whether $\int_a^b f(x)\,dx$ exists as an ordinary integral or as an HFP integral, and

$$\widetilde{T}_n[f;\theta] := h\sum_{i=0}^{n-1} f(a+ih+\theta h); \quad h=\frac{b-a}{n},\ n=1,2,\dots. \tag{2.2}$$

Here $\widetilde{T}_n[f;\theta]$ is the *offset trapezoidal rule* approximation to $I[f]$, and $\theta\in[0,1]$. Because $f\in C^\infty(a,b)$, $\widetilde{T}_n[f;\theta]$ with $\theta\in(0,1)$ is well defined. Note that $\widetilde{T}_n[f;\frac{1}{2}]$ is simply the midpoint rule approximation to $I[f]$. We also use the notation

$$\check{T}_n[f] := h\sum_{i=1}^{n-1} f(a+ih), \qquad T_n[f] := \check{T}_n[f] + \frac{h}{2}\big[f(a)+f(b)\big]. \tag{2.3}$$

By the fact that $f\in C^\infty(a,b)$, $\check{T}_n[f]$ is always well defined just as $\widetilde{T}_n[f;\theta]$ with $0<\theta<1$. Note that $\check{T}_n[f]$ is analogous to (but not the same as) $\widetilde{T}_n[f;1]$. In addition, provided $f(a)$ and $f(b)$ exist, which is the case, for example, when $f\in C[a,b]$, $T_n[f]$ is the ordinary trapezoidal rule approximation to $I[f]$.

In our results below, $\zeta(z,\theta)$, the *generalized Zeta function* (or the *Hurwitz Zeta function*), plays an important role. $\zeta(z,\theta)$ is defined by the convergent Dirichlet series $\sum_{k=0}^\infty 1/(k+\theta)^z$ for $\Re z>1$ and continued analytically to the whole complex z-plane, with the exception of $z=1$, where it has a simple pole with residue 1. For $\theta=1$, $\zeta(z,1)$ is simply $\zeta(z)$, the *Riemann Zeta function.*

We also make use of the *Bernoulli polynomials* $B_j(\theta)$ and the *Bernoulli numbers* B_j. In particular, we make use of the following:

$$\begin{gathered} B_j(1-\theta) = (-1)^j B_j(\theta), \quad j=0,1,\dots \\ B_j(0)=B_j, \quad j\ge 0; \qquad B_1(1)=-B_1; \qquad B_j(1)=B_j, \quad j\ge 0,\ j\ne 1, \\ B_0=1, \qquad B_1=-\frac{1}{2}; \qquad B_{2j+1}=0, \qquad B_{2j}\ne 0, \quad j=1,2,\dots, \\ B_{2j+1}\left(\frac{1}{2}\right)=0, \qquad B_{2j}\left(\frac{1}{2}\right)\ne 0, \quad j=0,1,\dots. \end{gathered} \tag{2.4}$$

For the properties of the Zeta functions, see Titchmarsh [22] or Olver et al. [11, Chap. 25], for example. For the Bernoulli polynomials and numbers, see [11, Chap. 24], for example. For a brief summary of these topics, see also Sidi [13, Appendices D and E].

Finally, we also make use of the *Stieltjes constants* $\sigma_s(\theta)$ that are defined as in

$$\sigma_s(\theta) := \lim_{N\to\infty}\left(\sum_{k=0}^{N-1}\frac{[\log(k+\theta)]^s}{k+\theta} - \frac{(\log N)^{s+1}}{s+1}\right), \quad s=0,1,\dots. \tag{2.5}$$

We set $\sigma_s\equiv\sigma_s(1)$, which, of course, are defined via

$$\sigma_s := \lim_{N\to\infty}\left(\sum_{k=1}^{N}\frac{(\log k)^s}{k} - \frac{(\log N)^{s+1}}{s+1}\right), \quad s=0,1,\dots. \tag{2.6}$$

It is known that

$$\sigma_0(\theta) = -\psi(\theta), \quad \sigma_0 = C, \tag{2.7}$$

where $\psi(z) = \frac{d}{dz}\log\Gamma(z)$ is the *Psi function* and $C = 0.577\ldots$ is *Euler's constant*. For Stieltjes constants, see Ivić [4], for example.[4]

The following theorem gives our main results:

Theorem 2.1 *Let $f(x)$ be as in Sect. 1 [in particular, as in (1.1)–(1.5)], with the notation therein. Set $D_\omega = \frac{d}{d\omega}$. For an arbitrary polynomial $W(y) = \sum_{i=0}^{k} e_i y^i$ and an arbitrary function u that is differentiable sufficiently often as a function of the parameter ω, define also*

$$W(D_\omega)u := \sum_{i=0}^{k} e_i \left[D_\omega^i u\right] = \sum_{i=0}^{k} e_i \frac{d^i u}{d\omega^i}.$$

Then the following are true:

(1) *For $0 < \theta < 1$, as $h \to 0$, $\widetilde{T}_n[f;\theta]$ has the asymptotic expansion*

$$\begin{aligned}\widetilde{T}_n[f;\theta] \sim I[f] &+ \sum_{i=0}^{\hat{p}}\left[\sum_{k=i}^{\hat{p}}\binom{k}{i}\hat{c}_k\sigma_{k-i}(\theta)\right](\log h)^i - \sum_{i=0}^{\hat{p}}\hat{c}_i\frac{(\log h)^{i+1}}{i+1} \\ &+ \sum_{s=0}^{\infty}P_s(D_{\gamma_s})\left[\zeta(-\gamma_s,\theta)\,h^{\gamma_s+1}\right] + \sum_{s=0}^{\infty}Q_s(D_{\delta_s})\left[\zeta(-\delta_s,1-\theta)\,h^{\delta_s+1}\right] \\ &+ \sum_{i=0}^{\hat{q}}\left[\sum_{k=i}^{\hat{q}}\binom{k}{i}\hat{d}_k\sigma_{k-i}(1-\theta)\right](\log h)^i - \sum_{i=0}^{\hat{q}}\hat{d}_i\frac{(\log h)^{i+1}}{i+1}.\end{aligned} \tag{2.8}$$

(2) *As $h \to 0$, $\check{T}_n[f]$ has the asymptotic expansion*

$$\begin{aligned}\check{T}_n[f] \sim I[f] &+ \sum_{i=0}^{\hat{p}}\left[\sum_{k=i}^{\hat{p}}\binom{k}{i}\hat{c}_k\sigma_{k-i}\right](\log h)^i - \sum_{i=0}^{\hat{p}}\hat{c}_i\frac{(\log h)^{i+1}}{i+1} \\ &+ \sum_{s=0}^{\infty}P_s(D_{\gamma_s})\left[\zeta(-\gamma_s)\,h^{\gamma_s+1}\right] + \sum_{s=0}^{\infty}Q_s(D_{\delta_s})\left[\zeta(-\delta_s)\,h^{\delta_s+1}\right] \\ &+ \sum_{i=0}^{\hat{q}}\left[\sum_{k=i}^{\hat{q}}\binom{k}{i}\hat{d}_k\sigma_{k-i}\right](\log h)^i - \sum_{i=0}^{\hat{q}}\hat{d}_i\frac{(\log h)^{i+1}}{i+1}.\end{aligned} \tag{2.9}$$

[4]The usual notation used for Stieltjes constants defined via (2.5)–(2.7) is $\gamma_s(\theta)$ and γ_s. We have changed the notation to $\sigma_s(\theta)$ and σ_s, respectively, to avoid confusion with our γ_s in (1.1). These constants appear also in the Laurent expansions of $\zeta(z,\theta)$ and $\zeta(z)$ as in

$$\zeta(z,\theta) = \frac{1}{z-1} + \sum_{j=0}^{\infty}(-1)^j\frac{\sigma_j(\theta)}{j!}(z-1)^j, \qquad \zeta(z) = \frac{1}{z-1} + \sum_{j=0}^{\infty}(-1)^j\frac{\sigma_j}{j!}(z-1)^j.$$

In (2.8) *and* (2.9), $I[f]$ *is defined in the sense of HFP as described via* (1.6)–(1.8).

Remark To see the explicit form of the expansions in Theorem 2.1, we first note

$$D^i_\omega\big[\zeta(-\omega,\theta)\,h^{\omega+1}\big]=h^{\omega+1}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}\zeta^{(i-j)}(-\omega,\theta)(\log h)^j,$$

where $\zeta^{(k)}(z,\theta)$ is the kth derivative of $\zeta(z,\theta)$ with respect to z. Using this, it can be seen that, for example,

$$P_s(D_{\gamma_s})\big[\zeta(-\gamma_s,\theta)\,h^{\gamma_s+1}\big]=h^{\gamma_s+1}\sum_{j=0}^{p_s}w_{sj}(\log h)^j,$$

where

$$w_{sj}=\sum_{i=j}^{p_s}(-1)^{i-j}\binom{i}{j}c_{si}\,\zeta^{(i-j)}(-\gamma_s,\theta),\qquad j=0,1,\dots,p_s.$$

From this and from (1.3) and (1.4), we see that (2.8) and (2.9) are genuine asymptotic expansions.

3 Preliminaries

3.1 E–M Expansion for $\sum_{j=0}^{N-1}[\log(j+\theta)]^s/(j+\theta)$

We begin by stating the classical result on the E–M expansion for sums. For a proof of this result, we refer the reader to Steffensen [20].

Theorem 3.1 *Let* $F(y)\in C^m[J,\infty)$, *where* J *is an integer, and let* $\theta\in[0,1]$ *be fixed. Then for any integer* $N>J$,

$$\sum_{j=J}^{N-1}F(j+\theta)=\int_J^N F(y)\,dy+\sum_{k=1}^{m}\frac{B_k(\theta)}{k!}\big[F^{(k-1)}(N)-F^{(k-1)}(J)\big]+R_m(N;\theta),$$

with the remainder term $R_m(N;\theta)$ *given by*

$$R_m(N;\theta)=-\int_J^N F^{(m)}(y)\frac{\bar{B}_m(\theta-y)}{m!}\,dy,$$

where $\bar{B}_k(x)$ *is the* periodic Bernoullian function *that is the* 1-*periodic extension of the Bernoulli polynomial* $B_k(x)$, *given as* $\bar{B}_k(x)=B_k(x-\lfloor x\rfloor)$.

We use Theorem 3.1 to prove the next result.

Theorem 3.2 *Let*

$$F_s(y) = \frac{(\log y)^s}{y}, \quad s = 0, 1, 2, \dots. \tag{3.1}$$

(1) *For* $0 < \theta \le 1$ *and with* $m > 0$, *as* $N \to \infty$,

$$\sum_{j=0}^{N-1} F_s(j+\theta) = \sigma_s(\theta) + \int_1^N F_s(y)\,dy + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} F_s^{(k-1)}(N) + O\bigl(N^{-m}(\log N)^s\bigr). \tag{3.2}$$

(2) *For* $\theta = 1$ *and with* $m > 0$, *as* $N \to \infty$,

$$\sum_{j=1}^{N-1} F_s(j) + \frac{1}{2} F_s(N) = \sigma_s + \int_1^N F_s(y)\,dy + \sum_{k=2}^{m} \frac{B_k}{k!} F_s^{(k-1)}(N) + O\bigl(N^{-m}(\log N)^s\bigr). \tag{3.3}$$

[*Note that, in this case, the summation on k actually contains only the terms with even k since, by* (2.4), $B_{2s+1} = 0$ *for* $s = 1, 2, \dots$.]

Observe that, for all $s = 0, 1, \dots,$

$$\int_1^N F_s(y)\,dy = \frac{(\log N)^{s+1}}{s+1}. \tag{3.4}$$

Proof By Theorem 3.1, with $F(y) = F_s(y)$ and $J = 1$ there, we first have

$$\sum_{j=0}^{N-1} F_s(j+\theta) = F_s(\theta) + \int_1^N F_s(y)\,dy + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} \bigl[F_s^{(k-1)}(N) - F_s^{(k-1)}(1)\bigr] + R_m(N;\theta), \tag{3.5}$$

where

$$\sum_{j=0}^{N-1} F_s(j+\theta) = \sum_{j=0}^{N-1} \frac{[\log(j+\theta)]^s}{j+\theta}, \qquad R_m(N;\theta) = -\int_1^N F_s^{(m)}(y) \frac{\bar{B}_m(\theta - y)}{m!}\,dy. \tag{3.6}$$

Noting that

$$F_s^{(m)}(y) = \frac{1}{y^{m+1}} \sum_{k=0}^{\min\{m,s\}} \alpha_{smk} (\log y)^{s-k} \tag{3.7}$$

for some constants α_{smk}, and that $|\bar{B}_m(x)|$ is uniformly bounded for all real x, we have that $R_m(\infty;\theta) = \lim_{N\to\infty} R_m(N;\theta)$ exists for every $m \geq 1$. This fact and (3.6) enable us to rewrite (3.5) in the form

$$\sum_{j=0}^{N-1} \frac{[\log(j+\theta)]^s}{j+\theta} = \frac{(\log N)^{s+1}}{s+1} + S'_m(\theta) + S''_m(N;\theta), \tag{3.8}$$

where

$$S'_m(\theta) = F_s(\theta) - \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} F_s^{(k-1)}(1) + R_m(\infty;\theta),$$

and

$$S''_m(N;\theta) = \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} F_s^{(k-1)}(N) + \widetilde{R}_m(N;\theta), \tag{3.9}$$

with

$$\widetilde{R}_m(N;\theta) = \int_N^\infty F_s^{(m)}(y) \frac{\bar{B}_m(\theta - y)}{m!}\, dy.$$

Now, by (3.7), we have

$$F_s^{(k-1)}(N) = O\big(N^{-k}(\log N)^s\big), \qquad \widetilde{R}_m(N;\theta) = O\big(N^{-m}(\log N)^s\big) \quad \text{as } N \to \infty. \tag{3.10}$$

Therefore, $\lim_{N\to\infty} S''_m(N;\theta) = 0$, which, together with (2.5), implies that

$$S'_m(\theta) = \lim_{N\to\infty} \left[\sum_{j=0}^{N-1} \frac{[\log(j+\theta)]^s}{j+\theta} - \frac{(\log N)^{s+1}}{s+1} \right] = \sigma_s(\theta) \quad \text{independently of } m. \tag{3.11}$$

Substituting (3.11) and (3.9) in (3.8), and invoking (3.10), we obtain (3.2).

To prove (3.3), we let $\theta = 1$ in (3.2), and make use of the fact that $B_k(1) = B_k$ for $k \geq 2$ and $B_1(1) = -B_1 = \frac{1}{2}$. □

3.2 E–M Expansions for $\int_a^t [\log(x-a)]^i (x-a)^\omega\, dx$ and $\int_t^b [\log(b-x)]^i (b-x)^\omega\, dx$, $a < t < b$

Throughout the remainder of this work, we will be using the following notation:

Analogously to (2.1), for arbitrary α, β, we set

$$I^{(\alpha,\beta)}[g] = \int_\alpha^\beta g(x)\, dx. \tag{3.12}$$

We define two integers $r, \bar{r}$ as in

$$r = \left\lfloor \frac{n}{2} \right\rfloor, \qquad \bar{r} = n - r = \left\lfloor \frac{n+1}{2} \right\rfloor, \tag{3.13}$$

and let

$$t = a + rh, \quad \bar{t} = a + \bar{r}h \quad \Rightarrow \quad b - t = \bar{t} - a. \tag{3.14}$$

Clearly, $r, \bar{r}, t, \bar{t}$ are all functions of n, and satisfy the asymptotic equalities

$$r \sim \bar{r} \sim n/2, \qquad t - a \sim b - t \sim (b-a)/2, \qquad t \sim \bar{t} \sim (a+b)/2 \quad \text{as } n \to \infty. \tag{3.15}$$

Next, analogously to (2.2) and (2.3), we define

$$\widetilde{T}_r^{(a,t)}[g;\theta] = h \sum_{j=0}^{r-1} g(a + jh + \theta h), \qquad \widetilde{T}_{\bar{r}}^{(t,b)}[g;\theta] = h \sum_{j=r}^{n-1} g(a + jh + \theta h), \tag{3.16}$$

and

$$\mathring{T}_r^{(a,t)}[g] := h \sum_{j=1}^{r-1} g(a + jh) + \frac{h}{2} g(t), \qquad \overset{*}{T}{}_{\bar{r}}^{(t,b)}[g] := \frac{h}{2} g(t) + h \sum_{j=r+1}^{n-1} g(a + jh). \tag{3.17}$$

It is easy to see that $\widetilde{T}_r^{(a,t)}[g;\theta]$ and $\widetilde{T}_{\bar{r}}^{(t,b)}[g;\theta]$ are the offset trapezoidal rule approximations to the integrals $I^{(a,t)}[g]$ and $I^{(t,b)}[g]$, respectively. Obviously,

$$\widetilde{T}_r^{(a,t)}[g;\theta] + \widetilde{T}_{\bar{r}}^{(t,b)}[g;\theta] = \widetilde{T}_n[g;\theta]. \tag{3.18}$$

Similarly,

$$\mathring{T}_r^{(a,t)}[g] + \overset{*}{T}{}_{\bar{r}}^{(t,b)}[g] = \check{T}_n[g]. \tag{3.19}$$

Let us now define

$$\begin{aligned}
u_{\omega,i}(x) &:= \left[\log(x-a)\right]^i (x-a)^{\omega} \quad \forall w \in \mathbb{C} \\
v_{\omega,i}(x) &:= \left[\log(b-x)\right]^i (b-x)^{\omega} \quad \forall \omega \in \mathbb{C}, \\
\hat{u}_i(x) &:= u_{-1,i}(x) = \left[\log(x-a)\right]^i (x-a)^{-1}, \\
\hat{v}_i(x) &:= v_{-1,i}(x) = \left[\log(b-x)\right]^i (b-x)^{-1}.
\end{aligned} \tag{3.20}$$

Then we have

$$\begin{aligned}
I^{(a,t)}[u_{\omega,i}] &= \int_a^t u_{\omega,i}(x)\,dx = \frac{d^i}{d\omega^i} \frac{(t-a)^{\omega+1}}{\omega+1}, \quad \text{if } \omega \neq -1, \\
I^{(a,t)}[\hat{u}_i] &= \int_a^t \hat{u}_i(x)\,dx = \frac{[\log(t-a)]^{i+1}}{i+1},
\end{aligned} \tag{3.21}$$

$$
\begin{aligned}
I^{(t,b)}[v_{\omega,i}] &= \int_t^b v_{\omega,i}(x)\,dx = \frac{d^i}{d\omega^i}\frac{(b-t)^{\omega+1}}{\omega+1}, \quad \text{if } \omega \neq -1,\\
I^{(t,b)}[\hat{v}_i] &= \int_t^b \hat{u}_i(x)\,dx = \frac{[\log(b-t)]^{i+1}}{i+1}.
\end{aligned}
\tag{3.22}
$$

The results concerning $I^{(a,t)}[u_{\omega,i}]$ and $I^{(t,b)}[v_{\omega,i}]$ when $\omega \neq -1$ follow from the fact that

$$
u_{\omega,i} = \frac{d^i}{d\omega^i}u_\omega, \qquad v_{\omega,i} = \frac{d^i}{d\omega^i}v_\omega. \tag{3.23}
$$

The integrals $I^{(a,t)}[u_{\omega,i}]$ and $I^{(t,b)}[v_{\omega,i}]$ exist in the ordinary sense only when $\Re\omega > -1$; otherwise, they exist in the sense of HFP. The integrals $I^{(a,t)}[\hat{u}_i]$ and $I^{(t,b)}[\hat{v}_i]$ exist only in the sense of HFP. Clearly, by (3.14), (3.21), and (3.22),

$$
I^{(t,b)}[v_{\omega,i}] = I^{(a,\bar{t})}[u_{\omega,i}] \quad \text{if } \omega \neq -1, \qquad I^{(t,b)}[\hat{v}_i] = I^{(a,\bar{t})}[\hat{u}_i]. \tag{3.24}
$$

In addition, by (3.14), (3.16), and (3.17),

$$
\widetilde{T}_{\bar{r}}^{(t,b)}[v_{\omega,i};\theta] = \widetilde{T}_{\bar{r}}^{(a,\bar{t})}[u_{\omega,i};1-\theta], \qquad \widetilde{T}_{\bar{r}}^{(t,b)}[\hat{v}_i;\theta] = \widetilde{T}_{\bar{r}}^{(a,\bar{t})}[\hat{u}_i;1-\theta], \tag{3.25}
$$

and

$$
\overset{*}{T}{}_{\bar{r}}^{(t,b)}[v_{\omega,i}] = \overset{\circ}{T}{}_{\bar{r}}^{(a,\bar{t})}[u_{\omega,i}], \qquad \overset{*}{T}{}_{\bar{r}}^{(t,b)}[\hat{v}_i] = \overset{\circ}{T}{}_{\bar{r}}^{(a,\bar{t})}[\hat{u}_i]. \tag{3.26}
$$

These facts enable us to reduce the amount of work we need to do for the proofs considerably.

The following theorem, concerning the E–M expansions of $\widetilde{T}_r^{(a,t)}[u_{\omega,i};\theta]$ when $\theta \in (0,1]$ and of $\widetilde{T}_{\bar{r}}^{(t,b)}[v_{\omega,i};\theta]$ when $\theta \in [0,1)$, will be used in our proof of Theorem 2.1.

Theorem 3.3 *Set $D_\omega = \frac{d}{d\omega}$.*

(1) *We have the following E–M expansions for $\widetilde{T}_r^{(a,t)}[u_{\omega,i};\theta]$ when $\theta \in (0,1]$ and for $\widetilde{T}_{\bar{r}}^{(t,b)}[v_{\omega,i};\theta]$ when $\theta \in [0,1)$:*

(1-a) *For $\omega \neq -1$ and with $m > \Re\omega + 1$, as $h \to 0$,*

$$
\begin{aligned}
\widetilde{T}_r^{(a,t)}[u_{\omega,i};\theta] &= I^{(a,t)}[u_{\omega,i}] + D_\omega^i\big[\zeta(-\omega,\theta)\,h^{\omega+1}\big]\\
&\quad + \sum_{k=1}^m \frac{B_k(\theta)}{k!}u_{\omega,i}^{(k-1)}(t)\,h^k + O\big(h^m\big); \quad 0 < \theta \le 1,
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
\widetilde{T}_{\bar{r}}^{(t,b)}[v_{\omega,i};\theta] &= I^{(t,b)}[v_{\omega,i}] + D_\omega^i\big[\zeta(-\omega,1-\theta)\,h^{\omega+1}\big]\\
&\quad - \sum_{k=1}^m \frac{B_k(\theta)}{k!}v_{\omega,i}^{(k-1)}(t)\,h^k + O\big(h^m\big); \quad 0 \le \theta < 1.
\end{aligned}
\tag{3.28}
$$

(1-b) *With* $m > 0$, *as* $h \to 0$,

$$\widetilde{T}_r^{(a,t)}[\hat{u}_i;\theta] = I^{(a,t)}[\hat{u}_i] + \left[\sum_{s=0}^{i}\binom{i}{s}\sigma_s(\theta)(\log h)^{i-s} - \frac{(\log h)^{i+1}}{i+1}\right] + \sum_{k=1}^{m}\frac{B_k(\theta)}{k!}\hat{u}_i^{(k-1)}(t)\,h^k + O\big(h^m(\log h)^i\big);\quad 0<\theta\le 1, \tag{3.29}$$

$$\widetilde{T}_{\bar{r}}^{(t,b)}[\hat{v}_i;\theta] = I^{(t,b)}[\hat{v}_i] + \left[\sum_{s=0}^{i}\binom{i}{s}\sigma_s(1-\theta)(\log h)^{i-s} - \frac{(\log h)^{i+1}}{i+1}\right] - \sum_{k=1}^{m}\frac{B_k(\theta)}{k!}\hat{v}_i^{(k-1)}(t)\,h^k + O\big(h^m(\log h)^i\big);\quad 0\le\theta< 1. \tag{3.30}$$

(2) *Letting* $\theta = 1$ *in* (3.27) *and* (3.29), *and* $\theta = 0$ *in* (3.28) *and* (3.30), *we obtain the following E–M expansions*:

(2-a) *For* $\omega \neq -1$ *and with* $m > \Re\omega + 1$, *as* $h \to 0$,

$$\overset{\circ}{T}{}_r^{(a,t)}[u_{\omega,i}] = I^{(a,t)}[u_{\omega,i}] + D_\omega^i\big[\zeta(-\omega)\,h^{\omega+1}\big] + \sum_{k=2}^{m}\frac{B_k}{k!}u_{\omega,i}^{(k-1)}(t)\,h^k + O\big(h^m\big), \tag{3.31}$$

$$\overset{*}{T}{}_{\bar{r}}^{(t,b)}[v_{\omega,i}] = I^{(t,b)}[v_{\omega,i}] + D_\omega^i\big[\zeta(-\omega)\,h^{\omega+1}\big] - \sum_{k=2}^{m}\frac{B_k}{k!}v_{\omega,i}^{(k-1)}(t)\,h^k + O\big(h^m\big). \tag{3.32}$$

(2-b) *With* $m > 0$, *as* $h \to 0$,

$$\overset{\circ}{T}{}_r^{(a,t)}[\hat{u}_i] = I^{(a,t)}[\hat{u}_i] + \left[\sum_{s=0}^{i}\binom{i}{s}\sigma_s\cdot(\log h)^{i-s} - \frac{(\log h)^{i+1}}{i+1}\right] + \sum_{k=2}^{m}\frac{B_k}{k!}\hat{u}_i^{(k-1)}(t)\,h^k + O\big(h^m(\log h)^i\big), \tag{3.33}$$

$$\overset{*}{T}{}_{\bar{r}}^{(t,b)}[\hat{v}_i] = I^{(t,b)}[\hat{v}_i] + \left[\sum_{s=0}^{i}\binom{i}{s}\sigma_s\cdot(\log h)^{i-s} - \frac{(\log h)^{i+1}}{i+1}\right] - \sum_{k=2}^{m}\frac{B_k}{k!}\hat{v}_i^{(k-1)}(t)\,h^k + O\big(h^m(\log h)^i\big). \tag{3.34}$$

The $O(h^m)$ and $O(h^m(\log h)^i)$ terms in the results above are all uniformly valid in t. [Note that, by (2.4), the summations on k in (3.31)–(3.34) actually contain only the terms with even k, since $B_{2s+1}=0$ for $s=1,2,\ldots$.]

Remark Note that the summations involving the $B_k(\theta)$ in each of the four parts of this theorem are the same in form, except that they have opposite signs. This fact plays an important role in the proofs of our main results.

Proof The proofs of the results pertaining to $u_{\omega,i}$ and $v_{\omega,i}$ with $\omega \neq -1$ can be found in Sidi [14]. They are obtained by repeated application of a powerful device suggested and used by Navot [9], which is based on the observation in (3.23).

We now turn to the proofs of the results concerning $\hat{u}_i$ and $\hat{v}_i$. It is sufficient to give the details for $\widetilde{T}_r^{(a,t)}[\hat{u}_i;\theta]$. Making use of the fact that $\log[(j+\theta)h]=\log(j+\theta)+\log h$, we start by observing that

$$\widetilde{T}_r^{(a,t)}[\hat{u}_i;\theta]=\sum_{j=0}^{r-1}\frac{(\log[(j+\theta)h])^i}{j+\theta}=\sum_{s=0}^{i}\binom{i}{s}(\log h)^{i-s}\sum_{j=0}^{r-1}F_s(j+\theta), \tag{3.35}$$

with $F_s(y)$ as in Theorem 3.2. Invoking (3.2) in (3.35) and rearranging, we obtain

$$\begin{aligned}\widetilde{T}_r^{(a,t)}[\hat{u}_i;\theta]&=\sum_{s=0}^{i}\binom{i}{s}\sigma_s(\theta)(\log h)^{i-s}+\int_1^r\sum_{s=0}^{i}\binom{i}{s}(\log h)^{i-s}F_s(y)\,dy\\&\quad+\sum_{k=1}^{m}\frac{B_k(\theta)}{k!}\sum_{s=0}^{i}\binom{i}{s}(\log h)^{i-s}F_s^{(k-1)}(r)\\&\quad+\sum_{s=0}^{i}\binom{i}{s}(\log h)^{i-s}O\big(r^{-m}(\log r)^s\big)\quad\text{as } r\to\infty.\end{aligned} \tag{3.36}$$

We can now complete the proof of (3.29) (i) by observing that, by (3.1) and (3.4),

$$\sum_{s=0}^{i}\binom{i}{s}(\log h)^{i-s}F_s(y)=\frac{[\log(ht)]^i}{y}=hF_i(hy),$$

$$\sum_{s=0}^{i}\binom{i}{s}(\log h)^{i-s}F_s^{(k)}(y)=h\frac{d^k}{dt^k}F_i(hy)=h^{k+1}F_i^{(k)}(hy),$$

$$\begin{aligned}\int_1^r\sum_{s=0}^{i}\binom{i}{s}(\log h)^{i-s}F_s(y)\,dy&=h\int_1^r F_i(hy)\,dy\\&=\int_h^{rh}F_i(\tau)\,d\tau=I^{(a,t)}[\hat{u}_i]-\frac{(\log h)^{i+1}}{i+1},\end{aligned}$$

and

$$F_i(x-a)=\hat{u}_i(x) \quad\Rightarrow\quad F_i^{(k)}(x-a)=\hat{u}_i^{(k)}(x)$$
$$\Rightarrow\quad F_i^{(k)}(rh)=F_i^{(k)}(t-a)=\hat{u}_i^{(k)}(t),$$

where we have recalled that $rh=t-a$, and (ii) by noting that

$$(\log h)^{i-s}r^{-m}(\log r)^s \sim Kh^m(\log h)^i \quad \text{as } h\to 0, \quad |K|=\left[\frac{1}{2}(b-a)\right]^{-m}>0,$$

since $r=(t-a)/h\sim \frac{1}{2}(b-a)h^{-1}$ as $n\to\infty$.

To prove (3.30), we start by replacing θ and r in (3.29) by $1-\theta$ and $\bar{t}$, respectively. To complete the proof, we recall that $\widetilde{T}_{\bar{r}}^{(t,b)}[\hat{v}_i;\theta]=\widetilde{T}_{\bar{r}}^{(a,\bar{t})}[\hat{u}_i;1-\theta]$ by (3.25), $I^{(t,b)}[\hat{v}_i]=I^{(a,\bar{t})}[\hat{u}_i]$ by (3.24), and $B_k(1-\theta)=(-1)^kB_k(\theta)$ by (2.4), and observe that

$$\hat{u}_i^{(k)}(\bar{t})=(-1)^k\hat{v}_i^{(k)}(t) \quad\Rightarrow\quad B_k(1-\theta)\hat{u}_i^{(k-1)}(\bar{t})=-B_k(\theta)\hat{v}_i^{(k-1)}(t).$$

The proof of the remaining parts is similar to that of part (2) of Theorem 3.2, and we leave it to the reader. □

Remark Before closing, we would like to emphasize that the remainder terms present throughout the statement of Theorem 3.3 all depend on t. Since t changes with n (hence with h), we might be led to believe that these terms cannot be bounded independently of t after all. Actually, they can be bounded by some constant multiples of h^m and $h^m(\log h)^i$ independently of t, because t remains in a small neighborhood of $x=\frac{1}{2}(a+b)$ by (3.15) and hence the intervals $[a,t]$ and $[t,b]$ are asymptotically of fixed and nonzero length $\frac{1}{2}(b-a)$ as $n\to\infty$ (equivalently, as $h\to 0$).

4 Proof of Theorem 2.1

We now turn to the proof of Theorem 2.1. We carry out the proof of (2.8) for the case $0<\theta<1$ only. The proof of (2.9) is almost identical; we give a brief sketch of it, leaving the details to the reader.

We begin by stating the classical result on the E–M expansion for the trapezoidal rule. For a proof of this result, we refer the reader to Steffensen [20].

Theorem 4.1 *Let $g\in C^m[\alpha,\beta]$, where $[\alpha,\beta]$ is a finite interval, and define $I[g]=\int_\alpha^\beta g(x)\,dx$. Let $h=(\beta-\alpha)/N$, where $N=1,2,\ldots,$ and*

$$\widetilde{T}_N[g;\theta]=h\sum_{j=0}^{N-1}g(\alpha+jh+\theta h), \qquad T_N[g]=h\sum_{j=0}^{N-1}g(\alpha+jh)+\frac{h}{2}\big[g(\alpha)+g(\beta)\big].$$

(1) *For all* $\theta \in [0,1]$,

$$\widetilde{T}_N[g;\theta] = I[g] + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!}\left[g^{(k-1)}(\beta) - g^{(k-1)}(\alpha)\right]h^k + U_m(h;\theta),$$

where the remainder term $U_m(h;\theta)$ *is given by*

$$U_m(h;\theta) = -h^m \int_\alpha^\beta g^{(m)}(x) \frac{\bar{B}_m(\theta - N\frac{x-\alpha}{\beta-\alpha})}{m!}\,dx = O\big(h^m\big) \quad \textit{as } h \to 0.$$

As before, $\bar{B}_k(x)$ *is the* periodic Bernoullian function *that is the* 1*-periodic extension of the Bernoulli polynomial* $B_k(x)$.

(2) *For the case* $\theta = 1$, *the result of part* (1) *can be rewritten as*

$$T_N[g] = I[g] + \sum_{\substack{k=2\\ k\text{even}}}^{m} \frac{B_k}{k!}\left[g^{(k-1)}(\beta) - g^{(k-1)}(\alpha)\right]h^k + O\big(h^m\big) \quad \textit{as } h \to 0.$$

For $\mu \geq 0$ and $\nu \geq 0$ arbitrary integers, and with $u_{\omega,i}(x)$ and $v_{\omega,i}(x)$ defined as in (3.20), we split the function $f(x)$ as in

$$\begin{aligned} f(x) &= \sum_{i=0}^{\hat{p}} \hat{c}_i \hat{u}_i(x) + \sum_{s=0}^{\mu-1}\sum_{i=0}^{p_s} c_{si}\, u_{\gamma_s,i}(x) + \phi_\mu(x),\\ f(x) &= \sum_{i=0}^{\hat{q}} \hat{d}_i \hat{v}_i(x) + \sum_{s=0}^{\nu-1}\sum_{i=0}^{q_s} d_{si}\, v_{\delta_s,i}(x) + \psi_\nu(x). \end{aligned} \tag{4.1}$$

Note that $\phi_\mu(x)$ and $\psi_\nu(x)$ are exactly as in (1.7) and (1.8). Clearly, $\phi_\mu \in C^\infty(a,b)$ and $\psi_\nu \in C^\infty(a,b)$, and they have the asymptotic expansions

$$\phi_\mu(x) \sim \sum_{s=\mu}^{\infty}\sum_{i=0}^{p_s} c_{si}\, u_{\gamma_s,i}(x) \quad \text{as } x \to a+,$$

$$\psi_\nu(x) \sim \sum_{s=\nu}^{\infty}\sum_{i=0}^{q_s} d_{si}\, v_{\delta_s,i}(x) \quad \text{as } x \to b-,$$

which, by our assumptions on $f^{(k)}(x)$, are termwise differentiable infinitely many times. Thus,

$$\begin{aligned} \phi_\mu^{(k)}(x) &= O\big((x-a)^{\gamma_\mu - k}\big[\log(x-a)\big]^{p_\mu}\big) \quad \text{as } x \to a+,\ k = 0,1,2,\ldots,\\ \psi_\nu^{(k)}(x) &= O\big((b-x)^{\delta_\nu - k}\big[\log(b-x)\big]^{q_\nu}\big) \quad \text{as } x \to b-,\ k = 0,1,2,\ldots. \end{aligned} \tag{4.2}$$

Let m be an arbitrary positive integer, and let μ and ν be the smallest integers for which

$$\gamma_\mu > m, \qquad \delta_\nu > m. \tag{4.3}$$

Because $\lim_{s\to\infty} \Re\gamma_s = +\infty$ and $\lim_{s\to\infty} \Re\delta_s = +\infty$, such μ and ν exist and are unique. Then, by (4.2) and (4.3), for every $t \in (a,b)$, there hold

$$\begin{aligned} &\phi_\mu \in C^m[a,t]; \qquad \phi_\mu^{(k)}(a) = 0, \quad k = 0,1,\dots,m-1,\\ &\psi_\nu \in C^m[t,b]; \qquad \psi_\nu^{(k)}(b) = 0, \quad k = 0,1,\dots,m-1. \end{aligned} \tag{4.4}$$

We now split the integral $I[f] = \int_a^b f(x)\,dx$ as in

$$I[f] = I^{(a,t)}[f] + I^{(t,b)}[f], \tag{4.5}$$

where

$$I^{(a,t)}[f] := \int_a^t f(x)\,dx, \qquad I^{(t,b)}[f] := \int_t^b f(x)\,dx. \tag{4.6}$$

We also split the offset trapezoidal rule $\widetilde{T}_n[f;\theta]$ as in

$$\widetilde{T}_n[f;\theta] = \widetilde{T}_r^{(a,t)}[f;\theta] + \widetilde{T}_{\bar{r}}^{(t,b)}[f;\theta], \tag{4.7}$$

where

$$\widetilde{T}_r^{(a,t)}[f;\theta] := h\sum_{i=0}^{r-1} f(a+ih+\theta h), \qquad \widetilde{T}_{\bar{r}}^{(t,b)}[f;\theta] := h\sum_{i=r}^{n-1} f(a+ih+\theta h). \tag{4.8}$$

4.1 E–M Expansions for $\widetilde{T}_r^{(a,t)}[f;\theta]$ and $\widetilde{T}_{\bar{r}}^{(t,b)}[f;\theta]$

We first give a detailed derivation of the E–M expansion associated with $\widetilde{T}_r^{(a,t)}[f;\theta]$.

In view of the splittings in (4.1), we first have

$$\widetilde{T}_r^{(a,t)}[f;\theta] = \sum_{i=0}^{\hat{p}} \hat{c}_i \widetilde{T}_r^{(a,t)}[\hat{u}_i;\theta] + \sum_{s=0}^{\mu-1}\sum_{i=0}^{p_s} c_{si}\widetilde{T}_r^{(a,t)}[u_{\gamma_s,i};\theta] + \widetilde{T}_r^{(a,t)}[\phi_\mu;\theta]. \tag{4.9}$$

By parts (1-a) and (1-b) of Theorem 3.3 on the $\hat{u}_i$ and $u_{\gamma_s,i}$, and by part (1) of Theorem 4.1 on ϕ_μ, and letting

$$\widehat{G}_i(\log h) = \sum_{s=0}^{i}\binom{i}{s}\sigma_s(\theta)(\log h)^{i-s} - \frac{(\log h)^{i+1}}{i+1}, \tag{4.10}$$

(4.9) becomes

$$\begin{aligned}
&\widetilde{T}_r^{(a,t)}[f;\theta] \\
&\quad = \sum_{i=0}^{\hat{p}} \hat{c}_i \left\{ I^{(a,t)}[\hat{u}_i] + \widehat{G}_i(\log h) + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} \hat{u}_i^{(k-1)}(t) + O\big(h^m (\log h)^i\big) \right\} \\
&\qquad + \sum_{s=0}^{\mu-1} \sum_{i=0}^{p_s} c_{si} \left\{ I^{(a,t)}[u_{\gamma_s,i}] + D^i_{\gamma_s}\big[\zeta(-\gamma_s,\theta)\, h^{\gamma_s+1}\big] \right. \\
&\qquad \left. + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} u_{\gamma_s,i}^{(k-1)}(t)\, h^k + O\big(h^m\big) \right\} \\
&\qquad + \left\{ I^{(a,t)}[\phi_\mu] + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} \big[\phi_\mu^{(k-1)}(t) - \phi_\mu^{(k-1)}(a)\big] h^k + O\big(h^m\big) \right\} \quad \text{as } h \to 0.
\end{aligned} \tag{4.11}$$

Invoking the fact that $\phi_\mu^{(k-1)}(a) = 0$ for $k = 1, \ldots, m$, which follows from (4.4), and rearranging, (4.11) becomes

$$\begin{aligned}
\widetilde{T}_r^{(a,t)}[f;\theta] &= I^{(a,t)}\left[\sum_{i=0}^{\hat{p}} \hat{c}_i \hat{u}_i + \sum_{s=0}^{\mu-1} \sum_{i=0}^{p_s} c_{si} u_{\gamma_s,i} + \phi_\mu \right] \\
&\quad + \sum_{i=0}^{\hat{p}} \hat{c}_i \widehat{G}_i(\log h) + \sum_{s=0}^{\mu-1} \sum_{i=0}^{p_s} c_{si} D^i_{\gamma_s}\big[\zeta(-\gamma_s,\theta)\, h^{\gamma_s+1}\big] \\
&\quad + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} \left[\sum_{i=0}^{\hat{p}} \hat{c}_i \hat{u}_i^{(k-1)}(t) + \sum_{s=0}^{\mu-1} \sum_{i=0}^{p_s} c_{si} u_{\gamma_s,i}^{(k-1)}(t) + \phi_\mu^{(k-1)}(t) \right] h^k \\
&\quad + O\big(h^m (\log h)^{\hat{p}}\big) \quad \text{as } h \to 0.
\end{aligned} \tag{4.12}$$

Invoking (4.1), we finally obtain

$$\begin{aligned}
\widetilde{T}_r^{(a,t)}[f;\theta] &= I^{(a,t)}[f] + \sum_{i=0}^{\hat{p}} \hat{c}_i \widehat{G}_i(\log h) + \sum_{s=0}^{\mu-1} \sum_{i=0}^{p_s} c_{si} D^i_{\gamma_s}\big[\zeta(-\gamma_s,\theta)\, h^{\gamma_s+1}\big] \\
&\quad + \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} f^{(k-1)}(t) h^k + O\big(h^m (\log h)^{\hat{p}}\big) \quad \text{as } h \to 0.
\end{aligned} \tag{4.13}$$

We now employ the E–M expansion of $\widetilde{T}_r^{(a,t)}[f;\theta]$ given in (4.13) in conjunction with (3.22), (3.24), and (3.25) to write down the E–M expansion associated with

$\widetilde{T}_{\tilde{r}}^{(t,b)}[f;\theta]$ without further effort. Letting

$$\widehat{H}_i(\log h) = \sum_{s=0}^{i} \binom{i}{s} \sigma_s(1-\theta)(\log h)^{i-s} - \frac{(\log h)^{i+1}}{i+1}, \tag{4.14}$$

we obtain, analogously to (4.13),

$$\begin{aligned}\widetilde{T}_{\tilde{r}}^{(t,b)}[f;\theta] = I^{(t,b)}[f] &+ \sum_{i=0}^{\hat{q}} \hat{d}_i \widehat{H}_i(\log h) + \sum_{s=0}^{\nu-1}\sum_{i=0}^{q_s} d_{si} D_{\delta_s}^i \big[\zeta(-\delta_s, 1-\theta)\, h^{\delta_s+1}\big] \\ &- \sum_{k=1}^{m} \frac{B_k(\theta)}{k!} f^{(k-1)}(t) h^k + O\big(h^m (\log h)^{\hat{q}}\big) \quad \text{as } h \to 0.\end{aligned} \tag{4.15}$$

4.2 Completion of Proof for $\widetilde{T}_n[f;\theta]$

Substituting (4.13) and (4.15) in (4.7), and recalling also (4.5), we obtain

$$\begin{aligned}\widetilde{T}_n[f;\theta] = I[f] &+ \sum_{i=0}^{\hat{p}} \hat{c}_i \widehat{G}_i(\log h) + \sum_{s=0}^{\mu-1}\sum_{i=0}^{p_s} c_{si} D_{\gamma_s}^i \big[\zeta(-\gamma_s, \theta)\, h^{\gamma_s+1}\big] \\ &+ \sum_{i=0}^{\hat{q}} \hat{d}_i \widehat{H}_i(\log h) + \sum_{s=0}^{\nu-1}\sum_{i=0}^{q_s} d_{si} D_{\delta_s}^i \big[\zeta(-\delta_s, 1-\theta)\, h^{\delta_s+1}\big] \\ &+ O\big(h^m (\log h)^L\big) \quad \text{as } h \to 0;\ L = \max\{\hat{p}, \hat{q}\}.\end{aligned} \tag{4.16}$$

Note that there is no contribution to the expansion in (4.16) from $x = t$. This is a consequence of the fact that the summations involving the $B_k(\theta)$ in (4.13) and (4.15) have opposite signs and hence cancel each other. In addition, by (4.10) and (4.14),

$$\begin{aligned}\sum_{i=0}^{\hat{p}} \hat{c}_i \widehat{G}_i(\log h) &= \sum_{i=0}^{\hat{p}} \left[\sum_{r=i}^{\hat{p}} \binom{r}{i} \hat{c}_r \sigma_{r-i}(\theta)\right] (\log h)^i - \sum_{i=0}^{\hat{p}} \hat{c}_i \frac{(\log h)^{i+1}}{i+1}, \\ \sum_{i=0}^{\hat{q}} \hat{d}_i \widehat{H}_i(\log h) &= \sum_{i=0}^{\hat{q}} \left[\sum_{r=i}^{\hat{q}} \binom{r}{i} \hat{d}_r \sigma_{r-i}(1-\theta)\right] (\log h)^i - \sum_{i=0}^{\hat{q}} \hat{d}_i \frac{(\log h)^{i+1}}{i+1},\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\sum_{i=0}^{p_s} c_{si} D_{\gamma_s}^i \big[\zeta(-\gamma_s, \theta)\, h^{\gamma_s+1}\big] &= P_s(D_{\gamma_s})\big[\zeta(-\gamma_s, \theta)\, h^{\gamma_s+1}\big], \\ \sum_{i=0}^{q_s} d_{si} D_{\delta_s}^i \big[\zeta(-\delta_s, 1-\theta)\, h^{\delta_s+1}\big] &= Q_s(D_{\delta_s})\big[\zeta(-\delta_s, 1-\theta)\, h^{\delta_s+1}\big].\end{aligned} \tag{4.18}$$

The result in (2.8) now follows by substituting (4.17) and (4.18) in (4.16) and by recalling that m is an arbitrary integer and that $m, \mu, \nu \to \infty$ simultaneously. □

4.3 Sketch of Proof for $\check{T}_n[f]$

The proof of (2.9) can be carried out similarly. This time, we split $\check{T}_n[f]$ as in

$$\check{T}_n[f] = \mathring{T}_r^{(a,t)}[f] + \overset{*}{T}_{\bar{r}}^{(t,b)}[f],$$

and compute the E–M expansions for $\mathring{T}_r^{(a,t)}[f]$ and $\overset{*}{T}_{\bar{r}}^{(t,b)}[f]$. To do this, we make use of parts (2-a) and (2-b) of Theorem 3.3 on the $\hat{u}_i$ and $u_{\gamma_s,i}$, and part (2) of Theorem 4.1 on ϕ_μ and ψ_ν, realizing that $T_r^{(a,t)}[\phi_\mu]$ and $T_{\bar{r}}^{(t,b)}[\psi_\mu]$, the trapezoidal rule approximations for the integrals $I^{(a,t)}[\phi_\mu]$ and $I^{(t,b)}[\psi_\mu]$, respectively, satisfy

$$T_r^{(a,t)}[\phi_\mu] = \mathring{T}_r^{(a,t)}[\phi_\mu], \qquad T_{\bar{r}}^{(t,b)}[\psi_\mu] = \overset{*}{T}_n^{(t,b)}[\psi_\nu]$$

because $\phi_\mu(a) = 0$ and $\psi_\nu(b) = 0$ by (4.4), and continue as above. We leave the details to the reader.

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