J Sci Comput (2013) 54:145–176
DOI 10.1007/s10915-012-9610-y

Compact Numerical Quadrature Formulas for Hypersingular Integrals and Integral Equations

Avram Sidi

Received: 8 March 2012 / Revised: 11 May 2012 / Accepted: 14 May 2012 / Published online: 7 June 2012

Abstract In the first part of this work, we derive compact numerical quadrature formulas for finite-range integrals $I[f]=\int_a^b f(x)\,dx$, where $f(x)=g(x)|x-t|^{\beta}$, β being real. Depending on the value of β, these integrals are defined either in the regular sense or in the sense of Hadamard finite part. Assuming that $g\in C^{\infty}[a,b]$, or $g\in C^{\infty}(a,b)$ but can have arbitrary algebraic singularities at $x=a$ and/or $x=b$, and letting $h=(b-a)/n$, n an integer, we derive asymptotic expansions for $T^{*}_n[f]=h\sum_{1\leq j\leq n-1,\ x_j\neq t} f(x_j)$, where $x_j=a+jh$ and $t\in\{x_1,\ldots,x_{n-1}\}$. These asymptotic expansions are based on some recent generalizations of the Euler–Maclaurin expansion due to the author (A. Sidi, Euler–Maclaurin expansions for integrals with arbitrary algebraic endpoint singularities, in Math. Comput., 2012), and are used to construct our quadrature formulas, whose accuracies are then increased at will by applying to them the Richardson extrapolation process. We pay particular attention to the case in which $\beta=-2$ and $f(x)$ is T-periodic with $T=b-a$ and $f\in C^{\infty}(-\infty,\infty)\setminus\{t+kT\}_{k=-\infty}^{\infty}$, which arises in the context of periodic hypersingular integral equations. For this case, we propose the remarkably simple and compact quadrature formula $\widehat{Q}_n[f]=h\sum_{j=1}^{n} f(t+jh-h/2)-\pi^2 g(t)h^{-1}$, and show that $\widehat{Q}_n[f]-I[f]=O(h^{\mu})$ as $h\to 0\ \forall\mu>0$, and that it is exact for a class of singular integrals involving trigonometric polynomials of degree at most n. We show how $\widehat{Q}_n[f]$ can be used for solving hypersingular integral equations in an efficient manner. In the second part of this work, we derive the Euler–Maclaurin expansion for integrals $I[f]=\int_a^b f(x)dx$, where $f(x)=g(x)(x-t)^{\beta}$, with $g(x)$ as before and $\beta=-1,-3,-5,\ldots$, from which suitable quadrature formulas can be obtained. We revisit the case of $\beta=-1$, for which the known quadrature formula $\widetilde{Q}_n[f]=h\sum_{j=1}^{n} f(t+jh-h/2)$ satisfies $\widetilde{Q}_n[f]-I[f]=O(h^{\mu})$ as $h\to 0\ \forall\mu>0$, when $f(x)$ is T-periodic with $T=b-a$ and $f\in C^{\infty}(-\infty,\infty)\setminus\{t+kT\}_{k=-\infty}^{\infty}$. We show that this formula too is exact for a class of singular integrals involving trigonometric polynomials of degree at most $n-1$. We provide numerical examples involving periodic integrands that confirm the theoretical results.

This paper is dedicated to the memory of my dear friend Professor Moshe Israeli.

A. Sidi (✉)
Computer Science Department, Technion—Israel Institute of Technology, Haifa 32000, Israel
e-mail: asidi@cs.technion.ac.il
url: http://www.cs.technion.ac.il/~asidi/

Keywords Hadamard finite part · Hypersingular integrals · Hypersingular integral equations · Numerical quadrature · Trapezoidal rule · Midpoint rule · Euler–Maclaurin expansions · Asymptotic expansions · Richardson extrapolation

1 Introduction and Background

Let the function $g(x)$ be in $C^\infty(a,b)$ with possible arbitrary algebraic singularities at $x=a$ and $x=b$. In particular, we assume that $g(x)$ has the following asymptotic expansions

$$\begin{aligned} g(x) &\sim \sum_{s=0}^{\infty} c_s (x-a)^{\gamma_s} \quad \text{as } x\to a+, \\ g(x) &\sim \sum_{s=0}^{\infty} d_s (b-x)^{\delta_s} \quad \text{as } x\to b-, \end{aligned} \tag{1.1}$$

where c_s and d_s are some constants, γ_s and δ_s are distinct and, in general, complex, and satisfy

$$\begin{aligned} &\gamma_s \neq -1 \quad \forall s; \qquad \operatorname{Re}\gamma_0 \le \operatorname{Re}\gamma_1 \le \operatorname{Re}\gamma_2 \le \cdots; \qquad \lim_{s\to\infty} \operatorname{Re}\gamma_s = +\infty, \\ &\delta_s \neq -1 \quad \forall s; \qquad \operatorname{Re}\delta_0 \le \operatorname{Re}\delta_1 \le \operatorname{Re}\delta_2 \le \cdots; \qquad \lim_{s\to\infty} \operatorname{Re}\delta_s = +\infty. \end{aligned} \tag{1.2}$$

In the first part of this work, we shall be concerned mainly with the derivation of numerical quadrature formulas for computing convergent or divergent integrals of the form

$$I[f] = \int_a^b f(x)\,dx, \qquad f(x) = g(x)|x-t|^\beta, \ \beta \text{ real}; \ a<t<b. \tag{1.3}$$

Here x is the variable while t is a fixed parameter.

It is clear that (i) when $g(x)$ is not integrable at $x=a$ and/or $x=b$, or (ii) when $\beta\le -1$, such integrals do not exist in the regular sense. They do exist in the sense of Hadamard finite part (HFP), however, and we let $I[f]$ denote the HFP of $\int_a^b f(x)\,dx$ in these cases.[1]

A case of special interest is that with $\beta=-2$, which arises in connection with so called *hypersingular* integral equations that are of the form

$$\lambda\phi(t) + \int_a^b K(t,x)\phi(x)\,dx = u(t), \qquad t\in(a,b), \ \lambda \text{ scalar}, \tag{1.4}$$

where $K(t,x)$ is of the form

$$K(t,x) = \frac{H(t,x)}{(x-t)^2}, \qquad H\in C^\infty\big([a,b]\times[a,b]\big). \tag{1.5}$$

[1] The usual notation for integrals defined in the sense of the Hadamard finite part (HFP) is $⨍_a^b f(x)\,dx$. In this work, we denote them by $\int_a^b f(x)\,dx$, as in (1.3), for simplicity. For the definition and properties of Hadamard finite part integrals, see Davis and Rabinowitz [2], Evans [3], or Kythe and Schäferkotter [6], for example. These integrals have most of the properties of regular integrals and some properties that are quite unusual. For example, they are invariant with respect to translation, but they are not necessarily invariant under a scaling of the variable of integration.

An important example of (1.4)–(1.5) is that in which (i) $K(t,x)$ is periodic in both x and t, with period $T = b - a$, and, as a function of x, it is infinitely differentiable in $(-\infty, \infty) \setminus \{t + kT\}_{k=-\infty}^{\infty}$, and (ii) $u(x)$ and the solution $\phi(x)$ are both T-periodic and infinitely differentiable in $(-\infty, \infty)$. For hypersingular integral equations, see Ladopoulos [7] or Lifanov, Poltavskii, and Vainikko [10], for example.

In this work, we are first concerned with the development of some compact numerical quadrature formulas for efficient computation of the integrals $I[f]$ in (1.3), whether these exist in the regular sense or in the sense of HFP. The formulas we derive are actually what we shall call "corrected" trapezoidal rules, and they are obtained via the approach developed in the paper by Sidi and Israeli [22] (see also Sidi [17]) in conjunction with some novel generalizations and extensions of the Euler–Maclaurin (E–M) expansions that were obtained by the author recently in the paper Sidi [20].[2] We also note that the integrals treated in [22] are those with $\beta > -1$ in (1.3), as well as integrals of the form $\int_a^b g(x)|x-t|^\beta \log|x-t|\,dx$ with $\beta > -1$ and Cauchy principal value (CPV) integrals[3] of the form $\int_a^b g(x)/(x-t)\,dx$, and their application to so called weakly singular and singular integral equations .

The derivation of the appropriate E–M expansions for $I[f]$, whether defined in the regular sense or in the sense of HFP, will be the subject of the next section. These expansions form the basis for the development of the corrected trapezoidal rules for any β.

Following the derivation of Sect. 2 for general algebraic endpoint singularities and general β, in Sect. 3, we concentrate on the special case of no endpoint singularities and $\beta = -2$. In the process, we derive quadrature formulas that are remarkably simple and compact and seem to be new. We also show how the accuracy of these formulas can be increased arbitrarily via the Richardson extrapolation process. For the Richardson extrapolation process and related subjects, see Stoer and Bulirsch [23] or Sidi [18].

In Sect. 4, we turn to the periodic hypersingular case (of $\beta = -2$) discussed above and show that the new quadrature formulas will have very good performance for integrals of periodic hypersingular functions. We actually show that they produce very high accuracy (called "spectral" accuracy in the literature) for such integrals. These formulas can be used very easily in the numerical solution of (periodic) hypersingular integral equations with very high accuracy. One of these quadrature formulas is

$$\widehat{Q}_n[f] = h \sum_{j=1}^{n} f(t + jh - h/2) - \pi^2 g(t) h^{-1}, \quad h = \frac{b-a}{n}.$$

In Sect. 5, we show that $\widehat{Q}_n[f]$ is exact for some periodic hypersingular integrals involving a family of trigonometric polynomials. The details of the proof are included in Appendix A. In Sect. 6, we prove that the convergence of $\widehat{Q}_n[f]$ is actually of exponential accuracy when the integrands are analytic in a strip of the complex plane that includes the real axis.

In Sect. 7, we turn to the numerical solution of hypersingular integral equations (the case $\beta = -2$), both periodic and nonperiodic. In Sect. 8, we illustrate the accuracy of our quadrature formulas with a suitable numerical example and confirm the theory of the preceding sections.

[2] The paper [22] makes use of the generalizations of the E–M expansions due to Navot [14] and [15] and it treats the convergent cases of $\operatorname{Re}\beta > -1$ in (1.3). The generalized E–M expansions of [20], however, help to treat in a simple way all of the divergent cases resulting from $\operatorname{Re}\beta \le -1$ as well.

[3] The usual notation for integrals defined in the sense of Cauchy principal value (CPV) is $⨍_a^b f(x)\,dx$. In this work, we denote them by $\int_a^b f(x)\,dx$ for simplicity. For the definition and properties of Cauchy principal value integrals, see [2, 3], or [6], for example.

In the second part of this work, we consider the integrals

$$I[f] = \int_a^b f(x)\,dx; \quad f(x) = g(x)(x-t)^\beta,\ a < t < b,\ \beta = -1, -3, -5, \dots, \tag{1.6}$$

which are not included in (1.3). Of these integrals, the ones with $\beta = -1$ are defined in the sense of CPV and have received much attention. The ones with $\beta = -3$ have also been investigated, but much less than the case $\beta = -1$. In Sect. 9, we extend the approach of Sect. 2 to obtain the E–M expansion of these integrals and to obtain suitable numerical quadrature formulas from them. In Sect. 10, we investigate the exactness properties of the quadrature formula developed in [22], namely,

$$\widetilde{Q}_n[f] = h\sum_{j=1}^{n} f(t + jh - h/2), \quad h = \frac{b-a}{n},$$

for $\beta = -1$; we show that $\widetilde{Q}_n[f]$, analogously to $\widehat{Q}_n[f]$, is exact for some periodic CPV integrals involving a family of trigonometric polynomials. The details of the proof are included in Appendix B. In Sect. 11, we illustrate the use of the quadrature method $\widetilde{Q}_n[f]$ for $\beta = -1$ with a numerical example.

Throughout this work, we allow the functions $g(x)$ to have arbitrary algebraic endpoint singularities as in (1.1). Using the results of [21], we can also allow them to have arbitrary algebraic-logarithmic endpoint singularities without any extra theoretical effort. This complicates the technical details, however. We leave the treatment of this case to another publication.

The numerical treatment of singular integrals of the form $\int_a^b g(x)(x-t)^\beta dx$, with $\beta = -1$ and $\beta = -2$, and of the corresponding integral equations, has been the subject of intensive research. The papers by Monegato [13], Mastroianni and Occorsio [11], Mastronardi and Occorsio [12], and Capobianco, Mastroianni, and Russo [1], to name a few, deal with the cases in which $g(x)$ is of the form $g(x) = w(x)u(x)$, where $w(x)$ is some admissible weight function for the interval $[a, b]$ (mostly the Jacobi weight function), and $u \in C^p[a, b]$ for some integer p, such that $\int_a^b g(x)(x-t)^\beta dx$ exists in the sense of HFP. (More references on this subject are provided in the bibliographies of these papers. For the earlier literature, see also Davis and Rabinowitz [2, pp. 182–190].) The quadrature methods developed and/or analyzed in these works are based on polynomial approximations of the functions $u(x)$ over the whole interval $[a, b]$, and hence are entirely different from those we consider in this work.

Since our work is based on the trapezoidal rule approximation of singular integrals, here we will recall some of the recent works whose methods are based on applications of the generalized E-M expansions in conjunction with certain representations of hypersingular integrals. The paper by Huang, Wang, and Zhu [5], which is one of these, approaches the problem [with the restriction $-2 \le \beta < -1$ in (1.3)] by following [22], which is based on the generalizations of the E–M expansions by Navot [14]. As such, [5] is closest to the present work. The papers by Wu and Sun [25], and Wu, Dai, and Zhang [24] take similar approaches. Our work here is also along the lines of [22], but is based on the most recent developments in E–M expansions of Sidi [20] that are valid for all HFP integrals with algebraic endpoint singularities. As a consequence of these new E–M expansions, our quadrature formulas are *compact* in that they consist of trapezoidal like rules with very simple "correction" terms. They are thus simpler than those that already exist in the literature. For a large bibliography on the subject, we refer the reader to [5], for example.

Before proceeding to the next sections, we would like to recall some of the properties of the Riemann Zeta function $\zeta(z)$ and the Bernoulli numbers B_k and the connection between them for future reference:

$$\begin{aligned} &B_0 = 1, \qquad B_1 = -\frac{1}{2}; \qquad B_{2k+1} = 0, \qquad B_{2k} \neq 0, \quad k = 1, 2, \ldots, \\ &\zeta(0) = -\frac{1}{2}; \qquad \zeta(-2k) = 0, \qquad \zeta(1-2k) = -\frac{B_{2k}}{2k} \neq 0, \quad k = 1, 2, \ldots. \end{aligned} \tag{1.7}$$

2 Euler–Maclaurin Expansions and Quadrature Formulas for $\int_a^b g(x)|x-t|^\beta$

Our starting point is Theorem 2.3 in Sidi [20] that concerns the generalization and extension of the Euler–Maclaurin expansion to integrands with arbitrary algebraic endpoint singularities. We state it as Theorem 2.1 next.

Theorem 2.1 *Let $u \in C^\infty(a,b)$, and assume that $u(x)$ has the asymptotic expansions*

$$\begin{aligned} u(x) &\sim K(x-a)^{-1} + \sum_{s=0}^{\infty} c_s\,(x-a)^{\gamma_s'} \quad \text{as } x \to a+, \\ u(x) &\sim L(b-x)^{-1} + \sum_{s=0}^{\infty} d_s\,(b-x)^{\delta_s'} \quad \text{as } x \to b-, \end{aligned} \tag{2.1}$$

where the γ_s' and δ_s' are distinct complex numbers that satisfy

$$\begin{aligned} &\gamma_s' \neq -1 \quad \forall s; \qquad \operatorname{Re}\gamma_0' \le \operatorname{Re}\gamma_1' \le \operatorname{Re}\gamma_2' \le \cdots; \qquad \lim_{s\to\infty} \operatorname{Re}\gamma_s' = +\infty, \\ &\delta_s' \neq -1 \quad \forall s; \qquad \operatorname{Re}\delta_0' \le \operatorname{Re}\delta_1' \le \operatorname{Re}\delta_2' \le \cdots; \qquad \lim_{s\to\infty} \operatorname{Re}\delta_s' = +\infty. \end{aligned} \tag{2.2}$$

Assume furthermore that, for each positive integer k, $u^{(k)}(x)$ has asymptotic expansions as $x \to a+$ and $x \to b-$ that are obtained by differentiating those of $u(x)$ term by term k times. Let also $h = (b-a)/n$ for $n = 1, 2, \ldots$. Then, as $h \to 0$,

$$\begin{aligned} h\sum_{j=1}^{n-1} u(a+jh) \sim \int_a^b u(x)\,dx + K(C - \log h) + \sum_{\substack{s=0 \\ \gamma_s' \notin \{2,4,6,\ldots\}}}^{\infty} c_s\,\zeta\left(-\gamma_s'\right) h^{\gamma_s'+1} \\ + L(C - \log h) + \sum_{\substack{s=0 \\ \delta_s' \notin \{2,4,6,\ldots\}}}^{\infty} d_s\,\zeta\left(-\delta_s'\right) h^{\delta_s'+1}, \end{aligned} \tag{2.3}$$

where $C = 0.577\ldots$ is Euler's constant and $\zeta(z)$ is the Riemann Zeta function.[4]

[4] We can write the expansions in (2.1) in the "simpler" form

$$u(x) \sim \sum_{s=0}^{\infty} c_s\,(x-a)^{\gamma_s'} \quad \text{as } x \to a+, \qquad u(x) \sim \sum_{s=0}^{\infty} d_s\,(b-x)^{\delta_s'} \quad \text{as } x \to b-,$$

It is clear from Theorem 2.1 that the positive even powers of $(x-a)$ and $(b-x)$, if present in the asymptotic expansions of $u(x)$ as $x\to a+$ and $x\to b-$, do not contribute to the asymptotic expansion of $h\sum_{j=1}^{n-1}u(a+jh)$ as $h\to 0$, the reason being that $\zeta(-2k)=0$ for $k=1,2,\dots$, by (1.7).

In addition, if γ'_p is the first of the γ'_s that is different from $2,4,6,\dots$, and if δ'_q is the first of the δ'_s that is different from $2,4,6,\dots$, then we have the useful observation that

$$\left[h\sum_{j=1}^{n-1}u(a+jh)-(K+L)(C-\log h)\right]-\int_a^b u(x)\,dx=O\big(h^{\sigma+1}\big)\quad\text{as } h\to 0;$$
$$\sigma=\min\big\{\operatorname{Re}\gamma'_p,\operatorname{Re}\delta'_q\big\}.$$

Remark Note that the paper Sidi [20] concerns only arbitrary algebraic endpoint singularities. The treatment of algebraic-logarithmic endpoint singularities is the subject of Sidi [19] and [21].

Theorem 2.2 *Let $f(x)$ and $g(x)$ be exactly as in (1.3) and (1.1)–(1.2), respectively, with the notation therein. Let also $\{n_k\}_{k=0}^{\infty}$ be a sequence of positive integers, $n_0<n_1<n_2<\cdots$, and let $h_k=(b-a)/n_k$. Let t be such that $t\in\{a+jh_k\}_{j=1}^{n_k-1}$ for every $k=0,1,\dots$. (This is guaranteed if each n_k is an integer multiple of n_0 and $t\in\{a+jh_0\}_{j=1}^{n_0-1}$.) Let $n\in\{n_k\}_{k=0}^{\infty}$ and let $h=(b-a)/n$ and $x_j=a+jh$, $j=0,1,\dots,n$, and define*

$$T^*_n[f]=h\sum_{\substack{j=1\\x_j\neq t}}^{n-1}f(x_j).\tag{2.4}$$

Define also

$$\begin{aligned}C_{si}(t;\beta)&=(-1)^i\binom{\beta}{i}c_s(t-a)^{\beta-i},\quad s,i=0,1,\dots,\\ D_{si}(t;\beta)&=(-1)^i\binom{\beta}{i}d_s(b-t)^{\beta-i},\quad s,i=0,1,\dots.\end{aligned}\tag{2.5}$$

Then the following are true:

1. *For $\beta\neq -1,-2,\dots,$*

$$\begin{aligned}T^*_n[f]\sim I[f]&+\sum_{s=0}^{\infty}\sum_{\substack{i=0\\ \gamma_s+i\notin\{2,4,6,\dots\}}}^{\infty}C_{si}(t;\beta)\zeta(-\gamma_s-i)h^{\gamma_s+i+1}\\&+\sum_{s=0}^{\infty}\sum_{\substack{i=0\\ \delta_s+i\notin\{2,4,6,\dots\}}}^{\infty}D_{si}(t;\beta)\zeta(-\delta_s-i)h^{\delta_s+i+1}\\&+2\sum_{i=0}^{\infty}\frac{g^{(2i)}(t)}{(2i)!}\zeta(-\beta-2i)h^{\beta+2i+1}\quad\textit{as } h\to 0.\end{aligned}\tag{2.6}$$

ordering the γ'_s and the δ'_s as in (2.2), and allowing now one of the γ'_s and/or one of the δ'_s to be equal to -1. However, this complicates the statements of our results. Therefore, we have chosen to separate these two exponents as in (2.1).

2. *For* $\beta = -m$, $m = 1, 2, \ldots,$

$$
\begin{aligned}
T_n^*[f] \sim I[f] &+ \sum_{s=0}^{\infty} \sum_{\substack{i=0 \\ \gamma_s+i \notin \{2,4,6,\ldots\}}}^{\infty} C_{si}(t; -m)\zeta(-\gamma_s - i)h^{\gamma_s+i+1} \\
&+ \sum_{s=0}^{\infty} \sum_{\substack{i=0 \\ \delta_s+i \notin \{2,4,6,\ldots\}}}^{\infty} D_{si}(t; -m)\zeta(-\delta_s - i)h^{\delta_s+i+1} \\
&+ 2 \sum_{\substack{i=0 \\ 2i \neq m-1}}^{\infty} \frac{g^{(2i)}(t)}{(2i)!}\zeta(m - 2i)h^{-m+2i+1} \\
&+ \left[1 - (-1)^m\right]\frac{g^{(m-1)}(t)}{(m-1)!}(C - \log h) \quad \text{as } h \to 0. \qquad (2.7)
\end{aligned}
$$

Here, by "$h \to 0$*" we mean "*$h \to 0$, $h \in \{h_k\}_{k=0}^{\infty}$*".*

Proof We start by writing

$$
\int_a^b f(x)\,dx = \int_a^t f(x)\,dx + \int_t^b f(x)\,dx.
$$

We next apply Theorem 2.1 to the sums $h\sum_{j\ge 1, x_j<t} f(x_j)$ and $h\sum_{j\le n-1, x_j>t} f(x_j)$ with $\int_a^t f(x)\,dx$ and $\int_t^b f(x)\,dx$, respectively. For this, we need the asymptotic expansions of $f(x)$ as $x \to a+$, as $x \to b-$, and as $x \to t\pm$. This is the only thing that is needed since, by our assumption that $g \in C^\infty(a,b)$ and by the fact that $a < t < b$, it is clear that $f \in C^\infty(a,t)$ and $f \in C^\infty(t,b)$.

Expanding $f(x)$ at $x = t\pm$, we have

$$
\begin{aligned}
f(x) &\sim \sum_{i=0}^{\infty} \frac{g^{(i)}(t)}{i!}(x-t)^{\beta+i} \quad \text{as } x \to t+, \\
f(x) &\sim \sum_{i=0}^{\infty} (-1)^i \frac{g^{(i)}(t)}{i!}(t-x)^{\beta+i} \quad \text{as } x \to t-.
\end{aligned}
$$

In addition, by expanding $|x-t|^\beta$ and invoking (1.1), it is easy to see that

$$
\begin{aligned}
f(x) &\sim \sum_{s=0}^{\infty}\sum_{i=0}^{\infty} C_{si}(t;\beta)(x-a)^{\gamma_s+i} \quad \text{as } x \to a+, \\
f(x) &\sim \sum_{s=0}^{\infty}\sum_{i=0}^{\infty} D_{si}(t;\beta)(b-x)^{\delta_s+i} \quad \text{as } x \to b-.
\end{aligned}
$$

Applying now Theorem 2.1 to the sums $h\sum_{j\ge 1, x_j<t} f(x_j)$ and $h\sum_{j\le n-1, x_j>t} f(x_j)$, and summing the resulting expansions, we obtain the asymptotic expansions in (2.6) and (2.7). □

Remarks

1. In the sequel, we let $f(x)$ and $g(x)$ be exactly as in (1.3) and (1.1)–(1.2), respectively, with the notation therein. In addition, by "$h \to 0$" we shall always mean "$h \to 0$, $h \in \{h_k\}_{k=0}^{\infty}$", the h_k being as defined in the statement of Theorem 2.2.
2. In case $g \in C^\infty[a,b]$, $f(x)$ is infinitely differentiable at $x = a+$ and $x = b-$ since $a < t < b$. Consequently, the asymptotic expansions of $f(x)$ as $x \to a+$ and $x \to b-$ are simply its Taylor series at $x = a+$ and $x = b-$, respectively, namely,
$$f(x) \sim \sum_{s=0}^{\infty} \frac{f^{(s)}(a)}{s!}(x-a)^s \quad \text{as } x \to a+,$$
$$f(x) \sim \sum_{s=0}^{\infty} (-1)^s \frac{f^{(s)}(b)}{s!}(b-x)^s \quad \text{as } x \to b-.$$
Thus, by (1.7), the sum of the first two summations in both (2.6) and (2.7) is simply
$$-\frac{h}{2}\big[f(a)+f(b)\big] + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}\big[f^{(2i-1)}(b) - f^{(2i-1)}(a)\big]h^{2i}. \tag{2.8}$$
3. In case $g(x) = (x-a)^\gamma v_a(x) = (b-x)^\delta v_b(x)$, where $v_a \in C^\infty[a,b)$ and $v_b \in C^\infty(a,b]$, we have
$$\gamma_s = \gamma + s, \qquad \delta_s = \delta + s, \quad s = 0, 1, \ldots.$$
Consequently, after proper rearrangement of the powers $\gamma_s + i + 1$ and $\delta_s + i + 1$, the sum of the first two summations in both (2.6) and (2.7) is simply
$$\sum_{\substack{s=0\\ \gamma+s\notin\{2,4,6,\ldots\}}}^{\infty} A_s(t;\beta)\zeta(-\gamma-s)h^{\gamma+s+1} + \sum_{\substack{s=0\\ \delta+s\notin\{2,4,6,\ldots\}}}^{\infty} B_s(t;\beta)\zeta(-\delta-s)h^{\delta+s+1},$$
where
$$A_s(t;\beta) = \sum_{i=0}^{s} C_{i,s-i}(t;\beta), \qquad B_s(t;\beta) = \sum_{i=0}^{s} D_{i,s-i}(t;\beta), \quad s = 0, 1, \ldots.$$
Such cases arise, for example, by introducing a weight function with algebraic end point singularities into the integral $\int_a^b f(x)\,dx$. A frequently treated example is $g(x) = v(x)(1-x^2)^\lambda$, $v(x)$ being a nice function on $[-1,1]$.
4. From the result in (2.7), it is seen that in case $\beta = -m$, $m = 1, 2, \ldots$, the asymptotic expansion of $T_n^*[f]$ takes on different forms depending on whether m is even or odd.
   (a) When m is even, let $m = 2r$, $r = 1, 2, \ldots$. Then, the $(C - \log h)$ terms disappear. In addition, by the fact that $\zeta(-2k) = 0$, for $k = 1, 2, \ldots$, all the terms involving $\zeta(2r - 2i)$ with $i > r$ disappear. As a result, we have
$$\begin{aligned} T_n^*[f] \sim I[f] &+ \sum_{s=0}^{\infty} \sum_{\substack{i=0\\ \gamma_s+i\notin\{2,4,6,\ldots\}}}^{\infty} C_{si}(t;-2r)\zeta(-\gamma_s-i)h^{\gamma_s+i+1} \\ &+ \sum_{s=0}^{\infty} \sum_{\substack{i=0\\ \delta_s+i\notin\{2,4,6,\ldots\}}}^{\infty} D_{si}(t;-2r)\zeta(-\delta_s-i)h^{\delta_s+i+1} \\ &+ 2\sum_{i=0}^{r} \frac{g^{(2i)}(t)}{(2i)!}\zeta(2r-2i)h^{-2r+2i+1} \quad \text{as } h \to 0. \end{aligned} \tag{2.9}$$

(b) When m is odd, let $m = 2r + 1$, $r = 0, 1, \ldots$. Then we have

$$T_n^*[f] \sim I[f] + \sum_{s=0}^{\infty} \sum_{\substack{i=0 \\ \gamma_s + i \notin \{2,4,6,\ldots\}}}^{\infty} C_{si}(t; -2r-1)\zeta(-\gamma_s - i)h^{\gamma_s+i+1}$$

$$+ \sum_{s=0}^{\infty} \sum_{\substack{i=0 \\ \delta_s + i \notin \{2,4,6,\ldots\}}}^{\infty} D_{si}(t; -2r-1)\zeta(-\delta_s - i)h^{\delta_s+i+1}$$

$$+ 2\sum_{\substack{i=0 \\ i \neq r}}^{\infty} \frac{g^{(2i)}(t)}{(2i)!}\zeta(2r+1-2i)h^{-2r+2i}$$

$$+ 2\frac{g^{(2r)}(t)}{(2r)!}(C - \log h) \quad \text{as } h \to 0. \tag{2.10}$$

With the asymptotic expansions of $T_n^*[f]$ available, we can derive various compact numerical quadrature formulas to approximate $I[f]$. This can be done by subtracting some of the lowest order terms in the asymptotic expansions, or suitable approximations to them, from $T_n^*[f]$. In the next section, we show how this is achieved for $\beta = -2$, a case that has been the subject of quite a few publications in the past. The treatment of the rest of the cases is quite similar.

3 Compact Quadrature Formulas for the Special Case $\boldsymbol{\beta = -2}$ and $\boldsymbol{g \in C^\infty[a, b]}$

When $\beta = -2$, and $g \in C^\infty[a, b]$, by Remarks 2 and 4 of the preceding section, (2.9) assumes the form

$$T_n^*[f] \sim I[f] + 2\zeta(2)g(t)h^{-1} + \zeta(0)g''(t)h - \frac{h}{2}\big[f(a) + f(b)\big]$$

$$+ \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}\big[f^{(2i-1)}(b) - f^{(2i-1)}(a)\big]h^{2i} \quad \text{as } h \to 0. \tag{3.1}$$

Using the fact that $\zeta(0) = -1/2$ and $\zeta(2) = \pi^2/6$, and defining

$$Q_n[f] = \left(T_n^*[f] + \frac{h}{2}\big[f(a) + f(b)\big]\right) - \frac{\pi^2}{3}g(t)h^{-1},$$

$$= h\left[\frac{1}{2}f(a) + \sum_{\substack{j=1 \\ x_j \neq t}}^{n-1} f(x_j) + \frac{1}{2}f(b)\right] - \frac{\pi^2}{3}g(t)h^{-1}, \tag{3.2}$$

we rewrite (3.1) as in

$$Q_n[f] \sim I[f] - \frac{1}{2}g''(t)h + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}\big[f^{(2i-1)}(b) - f^{(2i-1)}(a)\big]h^{2i} \quad \text{as } h \to 0. \tag{3.3}$$

Here we have made the natural assumption that $g(t)$ is readily available so that $Q_n[f]$ can be computed. We can use this result to devise quadrature formulas for $I[f]$ and also to accelerate their convergence via the Richardson extrapolation process, as in the next subsections.

3.1 Quadrature Formula when $g''(t)$ Is Available

If $g''(t)$ is available, then we define

$$\bar{Q}_n[f] = Q_n[f] + \frac{1}{2}g''(t)h \tag{3.4}$$

as an approximation to $I[f]$ with error expansion in the powers $h^2, h^4, h^6, \ldots$, given as in

$$\bar{Q}_n[f] \sim I[f] + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}\left[f^{(2i-1)}(b) - f^{(2i-1)}(a)\right]h^{2i} \quad \text{as } h \to 0. \tag{3.5}$$

Consequently, we can apply the Richardson extrapolation process to the sequence $\bar{Q}_{n_k}[f]$, with the n_k as in the statement of Theorem 2.2.

3.2 Quadrature Formulas when $g''(t)$ Is Not Available

When $g''(t)$ is not available, we have a few options.

1. We can take $Q_n[f]$ given in (3.2) as our approximation to $I[f]$. Note that the asymptotic expansion of $Q_n[f]$ given in (3.3) involves the powers $h^1, h^2, h^4, h^6, \ldots$. Thus, we can improve the convergence of $Q_n[f]$ by applying the Richardson extrapolation process to the sequence $Q_{n_k}[f]$, with the n_k as in the statement of Theorem 2.2.
2. We can approximate $g''(t)$ by a central difference as in

$$g''(t) \approx \frac{g(t+h) - 2g(t) + g(t-h)}{h^2},$$

and replace $g''(t)$ in the formula for $\bar{Q}_n[f, t]$ given in (3.5), to define a new approximation to $I[f]$, namely,

$$\bar{Q}'_n[f] = Q_n[f] + \frac{g(t+h) - 2g(t) + g(t-h)}{2h}. \tag{3.6}$$

Note that $t+h$ and $t-h$, just as t, belong to the set $\{x_0, x_1, \ldots, x_n\}$, so that no extra evaluation of $g(x)$ is necessary here. By the fact that

$$\frac{g(t+h) - 2g(t) + g(t-h)}{h^2} \sim g''(t) + 2\sum_{i=2}^{\infty} \frac{g^{(2i)}(t)}{(2i)!}h^{2i-2} \quad \text{as } h \to 0,$$

and by (3.5), there holds

$$\bar{Q}'_n[f] \sim I[f] + \sum_{i=1}^{\infty} w_i(t)h^{i+1} \quad \text{as } h \to 0. \tag{3.7}$$

Here,

$$\begin{aligned} w_{2i-1}(t) &= \frac{B_{2i}}{(2i)!}\left[f^{(2i-1)}(b) - f^{(2i-1)}(a)\right], \quad i = 1, 2, \ldots, \\ w_{2i}(t) &= 2\frac{g^{(2i+2)}(t)}{(2i+2)!}, \quad i = 1, 2, \ldots. \end{aligned} \tag{3.8}$$

Consequently, we can improve the convergence of $\bar{Q}'_n[f]$ by applying the Richardson extrapolation to the sequence $\bar{Q}'_{n_k}[f]$, with the n_k as in the statement of Theorem 2.2.

Note that the asymptotic expansion of the error in $\bar{Q}'_n[f]$ contains *all* the powers h^i, $i = 2, 3, \ldots$, and this should be contrasted with the asymptotic expansion of the error in $\bar{Q}_n[f]$, which contains only the *even* powers h^{2i}, $i = 1, 2, \ldots$. This implies that both of the errors $\bar{Q}_n[f] - I[f]$ and $\bar{Q}'_n[f] - I[f]$ are asymptotically equal to $\frac{B_2}{2!}[f'(b) - f'(a)]h^2$ as $h \to 0$. When applying the Richardson extrapolation process, however, the cost incurred for $\bar{Q}'_n[f]$ is larger than that incurred for $\bar{Q}_n[f]$ since we have more (twice as many) powers to eliminate by extrapolation in the former case.

3. Finally, we can also approximate $I[f]$ by combining $Q_n[f]$ with $Q_{2n}[f]$ as follows:

$$\widehat{Q}_n[f] = 2Q_{2n}[f] - Q_n[f]. \tag{3.9}$$

This amounts to performing one step of the Richardson extrapolation process on the asymptotic expansion of $Q_n[f]$ given in (3.3) to eliminate the first power of h and hence the need to know, or approximate, $g''(t)$. This is precisely what was done in [22] in the treatment of CPV integrals. (Observe that t belongs to both $\{a + jh\}_{j=1}^{n-1}$ and $\{a + jh/2\}_{j=1}^{2n-1}$, the sets of abscissas for $Q_n[f]$ and $Q_{2n}[f]$, respectively.) Since the step size associated with $Q_{2n}[f]$ is $h/2$, we have

$$\widehat{Q}_n[f] = h\sum_{j=1}^{n} f(a + jh - h/2) - \pi^2 g(t)h^{-1}, \tag{3.10}$$

that is, $\widehat{Q}_n[f]$ is the sum of $h\sum_{j=1}^{n} f(a + jh - h/2)$, which is simply the midpoint rule with step size h, and the correction term $-\pi^2 g(t)h^{-1}$. Note also that even though t is in the set $\{a + jh/2\}_{j=1}^{2n-1}$ *as well as* the set $\{a + jh\}_{j=1}^{n-1}$, it is *not* in the set $\{a + jh - h/2\}_{j=1}^{n}$. As already mentioned, this procedure eliminates the term $-\frac{1}{2}g''(t)h$ from the asymptotic expansion of the error in $Q_n[f]$ in (3.3), and we have the asymptotic expansion in the powers $h^2, h^4, h^6, \ldots$, given as in

$$\widehat{Q}_n[f] \sim I[f] + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}\left(2^{1-2i} - 1\right)\left[f^{(2i-1)}(b) - f^{(2i-1)}(a)\right]h^{2i} \quad \text{as } h \to 0. \tag{3.11}$$

Choosing $n_k = 2^k n_0$, $k = 1, 2, \ldots$, in this case, we can apply the Richardson extrapolation process to the sequence $\{\widehat{Q}_{n_k}[f]\}_{k=0}^{\infty}$.

4 Quadrature Formulas for $\beta = -2$ when $f(x)$ is Periodic

In the preceding section, we assumed that $\beta = -2$ and $g \in C^\infty[a, b]$, which means that $f \in C^\infty[a, b] \setminus \{t\}$. We now assume, in addition, that $f(x)$ is a T-periodic function of x for $x \in \mathbb{R}_t$, and that $f \in C^\infty(\mathbb{R}_t)$, where

$$T = b - a \quad \text{and} \quad \mathbb{R}_t = \mathbb{R} \setminus \{t + kT\}_{k=-\infty}^{\infty}, \quad \text{with } \mathbb{R} = (-\infty, \infty). \tag{4.1}$$

In this case,

$$I[f] = \int_{a'}^{a'+T} f(x)\,dx, \quad a' \text{ arbitrary}.$$

Thus, for each n, we can cause any given t to belong to a set $\{a' + jh\}_{j=1}^{n-1}$ by choosing a' suitably. Starting with this observation, and following [22, pp. 213–214, Remarks], we can express the quadrature formulas $\bar{Q}_n[f]$ and $\widehat{Q}_n[f]$ also in the simple forms

$$\begin{aligned} \bar{Q}_n[f] &= h\sum_{j=1}^{n-1} f(t+jh) - \frac{\pi^2}{3}g(t)h^{-1} + \frac{1}{2}g''(t)h, \\ \widehat{Q}_n[f] &= h\sum_{j=1}^{n} f(t+jh-h/2) - \pi^2 g(t)h^{-1}. \end{aligned} \tag{4.2}$$

For this periodic case, we have $f^{(i)}(a) = f^{(i)}(b)$, $i = 0, 1, \ldots$. Therefore, the summations

$$\sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}\big[f^{(2i-1)}(b) - f^{(2i-1)}(a)\big]h^{2i}$$

and

$$\sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}\big(2^{1-2i}-1\big)\big[f^{(2i-1)}(b) - f^{(2i-1)}(a)\big]h^{2i}$$

in (3.5) and (3.11), respectively, are now empty. This results in the following theorem:

Theorem 4.1 *Let $f(x)$ be as in the first paragraph of this subsection and let $\bar{Q}_n[f]$ and $\widehat{Q}_n[f]$ be as defined in (4.2). Then*

$$\bar{Q}_n[f] - I[f] = O\big(h^{\mu}\big) \quad \text{and} \quad \widehat{Q}_n[f] - I[f] = O\big(h^{\mu}\big) \quad \text{as } h \to 0\ \forall \mu > 0. \tag{4.3}$$

In words, the errors in $\bar{Q}_n[f]$ and $\widehat{Q}_n[f]$ tend to zero as $h \to 0$ faster than any positive power of h. In the nomenclature of the common literature, the quadrature formulas $\bar{Q}_n[f]$ and $\widehat{Q}_n[f]$ have "spectral" accuracy. Thus, the quadrature formulas $\bar{Q}_n[f]$ and $\widehat{Q}_n[f]$ become excellent methods for computing $I[f]$ when $f(x)$ is T-periodic and infinitely smooth on $\mathbb{R}_t$, with $\mathbb{R}_t$ as defined in (4.1).

What we have done with the formulas $\bar{Q}_n[f]$ and $\widehat{Q}_n[f]$ cannot be done with the formula $\bar{Q}'_n[f]$ because the error expansion for $\bar{Q}'_n[f]$ is *not* empty (it contains the powers $h^3, h^5, h^7, \ldots$) as can be seen from (3.7) and (3.8).

4.1 Another Expression for $\widehat{Q}_n[f]$

If $f(x)$ is T-periodic, by letting $u(x) = f(x)\sin^2\frac{\pi(x-t)}{T}$, it may be more convenient to express $f(x)$ in the form

$$f(x) = \frac{u(x)}{\sin^2\frac{\pi(x-t)}{T}}. \tag{4.4}$$

Note that the function $1/\sin^2[\pi(x-t)/T]$ is T-periodic as well, and is in $C^{\infty}(\mathbb{R}_t)$. This implies that $u(x)$ is T-periodic too, and is in $C^{\infty}(\mathbb{R})$. Here, $\mathbb{R}$ and $\mathbb{R}_t$ are as in (4.1). Then,

$$g(x) = \frac{(x-t)^2}{\sin^2\frac{\pi(x-t)}{T}}u(x),$$

and $g(t)$, which is needed for the quadrature formula $\widehat{Q}_n[f]$, is obtained from $g(t) = \lim_{x\to t} g(x)$, which gives

$$g(t) = \frac{T^2}{\pi^2}u(t).$$

Thus, the quadrature formula $\widehat{Q}_n[f]$ assumes the form

$$\widehat{Q}_n[f] = h\sum_{j=1}^{n} f(t + jh - h/2) - T^2u(t)h^{-1}. \tag{4.5}$$

In the next section, we will show that this formula can be reduced as in

$$\widehat{Q}_n[f] = h\sum_{j=1}^{n} \frac{u(t + jh - h/2) - u(t)}{\sin^2[(2j-1)\frac{\pi}{2n}]}. \tag{4.6}$$

5 Exactness Property of $\widehat{Q}_n[f]$ and Consequences

5.1 Exactness Property of $\widehat{Q}_n[f]$

It is well known that the trapezoidal rule

$$T_n[u] = h\left[\frac{1}{2}u(a) + \sum_{j=1}^{n-1} u(a + jh) + \frac{1}{2}u(b)\right]; \quad h = \frac{b-a}{n},\ n \text{ integer},$$

is exact for *regular* integrals $\int_a^b u(x)\,dx$, when $u(x)$ is a trigonometric polynomial of degree at most $n-1$ with period $T = b - a$. That is,

$$T_n[u] = \int_a^b u(x)\,dx, \quad u(x) = \sum_{m=-(n-1)}^{n-1} c_m e^{\mathrm{i}2m\pi x/T}.$$

Interestingly, the quadrature formula $\widehat{Q}_n[f]$ for *hypersingular* integrals $I[f]$, with $f(x) = g(x)/(x-t)^2$ and $g \in C^\infty(a, b)$, has an analogous exactness property. Specifically, we have the following theorem:

Theorem 5.1 *Let*

$$f_m(x) = \frac{e^{\mathrm{i}2m\pi x/T}}{\sin^2 \frac{\pi(x-t)}{T}}, \quad m \text{ integer}. \tag{5.1}$$

[Note that $f_m(x)$ is T-periodic with a hypersingularity of the form $(x-t)^{-2}$ at $x = t$.] Then the following are true:

1. *The integral $I[f_m] = \int_a^b f_m(x)\,dx$, $b - a = T$, satisfies*

$$I[f_m] = -2T|m|e^{\mathrm{i}2m\pi t/T}, \quad m = 0, \pm1, \pm2, \ldots. \tag{5.2}$$

2. *The quadrature formula* $\widehat{Q}_n[f_m]$ *for* $I[f_m]$ *satisfies*

$$\widehat{Q}_n[f_m] = I[f_m] = -2T|m|e^{\mathrm{i}2m\pi t/T}, \quad m = 0, \pm 1, \pm 2, \dots, \pm n, \tag{5.3}$$

while for arbitrary m, *we have*

$$\widehat{Q}_n[f_m] = T\big[(-1)^k(n-2r) - n\big]e^{\mathrm{i}2m\pi t/T}, \tag{5.4}$$

where k *and* r *are unique integers*, $k \ge 0$ *and* $0 \le r \le n-1$, *such that* $|m| = kn + r$, *in which case*,

$$\widehat{Q}_n[f_m] - I[f_m] = T\big\{\big[(-1)^k - 1\big](n-2r) + 2kn\big\}e^{\mathrm{i}2m\pi t/T}. \tag{5.5}$$

3. *If* $f(x)$ *is of the form*

$$f(x) = \frac{u(x)}{\sin^2\frac{\pi(x-t)}{T}}, \quad u(x) = \sum_{m=-n}^{n} c_m e^{\mathrm{i}2m\pi x/T}, \tag{5.6}$$

then the quadrature formula $\widehat{Q}_n[f]$ *for* $I[f] = \int_a^b f(x)\,dx$, $b - a = T$, *satisfies*

$$\widehat{Q}_n[f] = I[f]. \tag{5.7}$$

The proof of Theorem 5.1 is provided in Appendix A. The result of part 1 of this theorem is actually known and can be found in Lifanov and Poltavskii [9]. Nevertheless, we have provided an independent proof of it.

5.2 Consequences of Theorem 5.1

With $f(x)$ as in (4.4), we can show that the hypersingular integral $\int_a^b f(x)\,dx$ can be expressed as a CPV integral. This is done in Theorem 5.2.

Theorem 5.2 *Let us express* $f(x)$ (*whether periodic or not*) *as in*

$$f(x) = \frac{u(x)}{\sin^2\frac{\pi(x-t)}{T}}, \quad a < t < b;\ T = b - a. \tag{5.8}$$

Then

$$I[f] = \int_a^b f(x)\,dx = \int_a^b \frac{u(x) - u(t)}{\sin^2\frac{\pi(x-t)}{T}}\,dx, \tag{5.9}$$

which is a CPV integral.

Proof We start by expressing $I[f]$ in the form

$$I[f] = \int_a^b \frac{u(x) - u(t)}{\sin^2\frac{\pi(x-t)}{T}}\,dx + u(t)\int_a^b \frac{1}{\sin^2\frac{\pi(x-t)}{T}}\,dx.$$

Next, by the fact that $\sin^2[\pi(x-t)/T]$ is T-periodic, and by (5.2) with $m = 0$, we have

$$\int_a^b \frac{1}{\sin^2\frac{\pi(x-t)}{T}}\,dx = I[f_0] = 0.$$

The result in (5.9) now follows. That the integral in (5.9) is a CPV integral follows from the fact that

$$\frac{u(x)-u(t)}{\sin^2\frac{\pi(x-t)}{T}} \sim \frac{(T/\pi)^2u'(t)}{x-t} \quad \text{as } x\to t.$$

This completes the proof. □

We have a similar result that concerns $\widehat{Q}_n[f]$:

Theorem 5.3 *Let $f(x)$ and $I[f]$ be as in Theorem 5.2, and let $h=T/n$. Then, provided $t\in\{a+jh\}_{j=1}^{n-1}$, we have*

$$\widehat{Q}_n[f]=h\sum_{j=1}^{n}\frac{u(a+jh-h/2)-u(t)}{\sin^2\frac{\pi(a+jh-h/2-t)}{T}}. \tag{5.10}$$

If $f(x)$ is T-periodic, then

$$\widehat{Q}_n[f]=h\sum_{j=1}^{n}\frac{u(t+jh-h/2)-u(t)}{\sin^2[(2j-1)\frac{\pi}{2n}]}. \tag{5.11}$$

Proof By (4.5),

$$\widehat{Q}_n[f]=h\sum_{j=1}^{n}\frac{u(a+jh-h/2)}{\sin^2\frac{\pi(a+jh-h/2-t)}{T}}-T^2u(t)h^{-1}.$$

Hence

$$\widehat{Q}_n[f]=h\sum_{j=1}^{n}\frac{u(a+jh-h/2)-u(t)}{\sin^2\frac{\pi(a+jh-h/2-t)}{T}}+K_nu(t),$$

$$K_n=h\sum_{j=1}^{n}\frac{1}{\sin^2\frac{\pi(a+jh-h/2-t)}{T}}-T^2h^{-1}. \tag{5.12}$$

By the fact that $\sin^2[\pi(x-t)/T]$ is T-periodic, and $t=a+ih$ for some $i\in\{1,\dots,n-1\}$, we have that $K_n=\widehat{Q}_n[f_0]$, with $f_m(x)$ as in (5.1). But, by (5.3), $\widehat{Q}_n[f_0]=I[f_0]=0$, and this implies that $K_n=0$. Substituting this in (5.12), we obtain (5.10). The result in (5.11) follows from (5.10) by invoking the fact that $f(x)$ is T-periodic. □

The next theorem concerns the error in $\widehat{Q}_n[f]$ in terms of the Fourier series of the function $u(x)$ in (5.8) of Theorem 5.2.

Theorem 5.4 *Let $f(x)$ and $u(x)$ be as in Theorem 5.2, and let $u(x)$ have the Fourier series*

$$u(x)=\sum_{m=-\infty}^{\infty}c_me^{\mathrm{i}2m\pi x/T};\qquad c_m=\frac{1}{T}\int_a^b e^{-\mathrm{i}2m\pi x/T}u(x)\,dx. \tag{5.13}$$

Then, with k and r as in Theorem 5.1,

$$\widehat{Q}_n[f]-I[f]=T\sum_{m=n+1}^{\infty}\left\{\left[(-1)^k-1\right](n-2r)+2kn\right\}\left[c_m e^{\mathrm{i}2m\pi t/T}+c_{-m}e^{-\mathrm{i}2m\pi t/T}\right]. \quad (5.14)$$

Consequently,

$$\left|\widehat{Q}_n[f]-I[f]\right|\le 2T\sum_{m=n+1}^{\infty}(m+n)\left(|c_m|+|c_{-m}|\right). \quad (5.15)$$

Proof The proof of (5.14) is achieved by invoking Theorem 5.1. The proof of (5.15) is achieved by noting that $kn\le m$ and $|n-2r|\le n$ since $0\le r\le n-1$ and $|m|\ge kn$ for $|m|\ge n$. □

6 Error in $\widehat{Q}_n[f]$ for Analytic and Periodic $f(x)$

We now go back to the periodic integrand functions $f(x)$ discussed in Sect. 4. There, we assumed that $f(x)$ is infinitely differentiable on $(-\infty,\infty)$ excluding the points of singularity $t+kT$, $k=0,\pm1,\pm2,\dots$, where $T=b-a$. Concerning this case, we recall Theorem 4.1 that states that the error $\widehat{Q}_n[f]-I[f]$ tends to zero faster as $h\to 0$ (equivalently, as $n\to\infty$) than every positive power of h. In this section we prove that the error tends to zero exponentially in n as $n\to\infty$ if $f(z)$, as a function of the complex variable z, is also analytic in a strip containing the $\mathrm{Re}\,z$ axis, with the exception of the points $t+kT$, $k=0,\pm1,\pm2,\dots$. This is the subject of Theorem 6.2 below.

In proving Theorem 6.2, we will make use of Theorem 6.1 that follows next. This theorem was stated and proved as Theorem 9 in Sidi and Israeli [22].

Theorem 6.1 *Let $G(z)$ be T-periodic and analytic in the strip $|\mathrm{Im}\,z|\le\sigma$ for some $\sigma>0$, with simple poles at the points $t+kT$, $k=0,\pm1,\pm2,\dots$. Let $I[G]=\int_a^b G(x)\,dx$, $b-a=T$.*[5] *Next, let $h=(b-a)/n$ and*

$$\widetilde{Q}_n[G]=h\sum_{j=1}^{n}G(t+jh-h/2).$$

Then

$$\left|\widetilde{Q}_n[G]-I[G]\right|\le 2TM\left(\sigma'\right)\frac{\exp(-2\pi n\sigma'/T)}{1-\exp(-2\pi n\sigma'/T)},\quad \forall\sigma'<\sigma,$$

where

$$M(\tau)=\max\left\{\max_{x\in\mathbb{R}}\left|G_e(x+\mathrm{i}\tau)\right|,\max_{x\in\mathbb{R}}\left|G_e(x-\mathrm{i}\tau)\right|\right\},$$

and

$$G_e(\xi)=\frac{1}{2}\left[G(t+\xi)+G(t-\xi)\right].$$

[*Note that $G_e(z)$ is analytic throughout the strip $|\mathrm{Im}\,z|\le\sigma$.*]

[5] In this case, $I[G]$ is simply the Cauchy principal value of $\int_a^b G(x)\,dx$.

Theorem 6.2 *Let $T = b - a$ and let $f(z)$ be T-periodic and analytic in the strip $|\mathrm{Im}\, z| \le \sigma$ for some $\sigma > 0$, with double poles at the points $t + kT$, $k = 0, \pm 1, \pm 2, \ldots$. Express $f(z)$ as in*

$$f(z) = \frac{u(z)}{\sin^2 \frac{\pi(z-t)}{T}}, \tag{6.1}$$

and define the function $G(z)$ as in

$$G(z) = \frac{u(z) - u(t)}{\sin^2 \frac{\pi(z-t)}{T}}, \tag{6.2}$$

and let

$$G_e(\xi) = \frac{1}{2}\big[G(t+\xi) + G(t-\xi)\big]. \tag{6.3}$$

Then

$$\big|\widehat{Q}_n[f] - I[f]\big| \le 2TM\big(\sigma'\big)\frac{\exp(-2\pi n\sigma'/T)}{1 - \exp(-2\pi n\sigma'/T)}, \quad \forall \sigma' < \sigma, \tag{6.4}$$

where

$$M(\tau) = \max\Big\{\max_{x\in\mathbb{R}} \big|G_e(x + \mathrm{i}\tau)\big|, \max_{x\in\mathbb{R}} \big|G_e(x - \mathrm{i}\tau)\big|\Big\}. \tag{6.5}$$

Proof We start by noting that $1/\sin^2[\pi(z-t)/T]$, just as $f(z)$, is T-periodic and meromorphic in the strip $|\mathrm{Im}\, z| \le \sigma$ with double poles at the points $t + kT$, $k = 0, \pm 1, \pm 2, \ldots$. Consequently, $u(z)$ is analytic and T-periodic throughout the strip $|\mathrm{Im}\, z| \le \sigma$. Consequently, $G(z)$ is T-periodic and meromorphic in the strip $|\mathrm{Im}\, z| \le \sigma$ with simple poles at the points $t + kT$, $k = 0, \pm 1, \pm 2, \ldots$.

Now, by Theorem 5.2, we have that $I[f] = I[G]$. In Theorem 5.3, we showed that $\widehat{Q}_n[f]$ ultimately has the form shown in (5.11). It is easy to verify from this that $\widehat{Q}_n[f] = \widetilde{Q}_n[G]$ defined in Theorem 6.1. Consequently, $\widehat{Q}_n[f] - I[f] = \widetilde{Q}_n[G] - I[G]$. Therefore, Theorem 6.1 applies, and the result follows. □

7 Application of $\widehat{Q}_n[f]$ to Hypersingular Integral Equations

We now consider the application of the quadrature formula $\widehat{Q}_n[f]$ to the numerical solution of hypersingular integral equations of the form

$$\lambda\phi(t) + \int_a^b K(t,x)\phi(x)\,dx = w(t), \quad t \in (a,b),\ \lambda \text{ scalar}, \tag{7.1}$$

such that,

$$K(t,x) = \frac{H(t,x)}{(x-t)^2}, \quad H \in C^\infty\big([a,b]\times[a,b]\big). \tag{7.2}$$

In some cases, additional conditions are imposed on the solution to ensure uniqueness, which we will skip below. The important thing here is to see how the quadrature formula $\widehat{Q}_n$ is being used in the context of integral equations.

7.1 Application to Periodic Hypersingular Integral Equations

In the periodic case, with T, $\mathbb{R}$ and $\mathbb{R}_t$ as in (4.1), we assume that

1. (a) $K(t,x)$ is T-periodic in both x and t, and is in $C^\infty(\mathbb{R}_t)$ as a function of x.
   (b) As a function of x, $K(t,x)$ has double poles at the points $x=t+kT$, $k=0,\pm1,\pm2,\dots$.
2. $w(x)$ and the solution $\phi(x)$ are both T-periodic in x and in $C^\infty(\mathbb{R})$. (That $\phi\in C^\infty(\mathbb{R})$ can be argued heuristically as was done in [22, Introduction].)

Following [22, Sect. 4], for a given integer n, let $h=h_{2n}=T/(2n)$, and $x_j=a+jh$, $j=1,\dots,2n$, and let t be any one of the x_j. Let us approximate the integral $\int_a^b K(t,x)\phi(x)\,dx$ by the rule $\widehat{Q}_n$ with step size $2h$ (and not by $\widehat{Q}_{2n}$ with step size h), namely,

$$\widehat{Q}_n\big[K(t,\cdot)\phi\big]=2h\sum_{j=1}^{n}K(t,t+2jh-h)\phi(t+2jh-h)-\pi^2H(t,t)\phi(t)(2h)^{-1}.$$

Finally, letting $t=x_i$ and noting that $t+2jh-h=x_{i+2j-1}$, and replacing the $\phi(t+2jh-h)$ by corresponding approximations $\widetilde{\phi}_{i+2j-1}$, and recalling that everything here is T-periodic, [for example, $\phi(x_{j+2n})=\phi(x_j+T)=\phi(x_j)$ for all j, and the same holds true for $K(t,x)$ and $w(x)$], we write down the following set of equations for the $\widetilde{\phi}_j$:

$$\lambda\widetilde{\phi}_i+2h\sum_{j=1}^{2n}\epsilon_{ij}K(x_i,x_j)\widetilde{\phi}_j-\pi^2H(x_i,x_i)\widetilde{\phi}_i(2h)^{-1}=w(x_i),\quad i=1,\dots,2n,\tag{7.3}$$

where

$$\epsilon_{ij}=\begin{cases}1 & \text{if } |i-j| \text{ odd},\\ 0 & \text{if } |i-j| \text{ even}.\end{cases}\tag{7.4}$$

Note that we do not need to know $H(t,x)$ for all t,x. It is enough to know $H(t,t)$, which can be obtained from $K(t,x)$ via $H(t,t)=\lim_{x\to t}[(x-t)^2K(t,x)]$. The linear equations in (7.3) can be rewritten in the form

$$\begin{aligned}&\sum_{j=1}^{2n}\widetilde{K}_{ij}\widetilde{\phi}_j=w(x_i),\quad i=1,\dots,2n,\\ &\widetilde{K}_{ij}=2h\epsilon_{ij}K(x_i,x_j)+\big[\lambda-\pi^2H(x_i,x_i)(2h)^{-1}\big]\delta_{ij},\quad \forall i,j.\end{aligned}\tag{7.5}$$

Here δ_{ij} stands for the Kronecker delta.

7.2 Application to Nonperiodic Hypersingular Integral Equations

Direct use of the quadrature formula $\widehat{Q}_n$ in the solution of nonperiodic hypersingular integral equations in (7.1) will not yield good approximations, because (i) the solution $\phi(x)$ is usually singular at the endpoints with a rather complicated singularity structure; as a result, the integrand $K(t,x)\phi(x)$ has the same characteristics at the endpoints, and (ii) the integrand $K(t,x)\phi(x)$ has a double pole singularity at $x=t$ since $K(t,x)$ does. Thus, the error in $\widehat{Q}_n[K(t,\cdot)\phi]$ is at best $O(h^2)$ as $h\to0$, contributed by the point of singularity $x=t$.

We can increase the accuracy if we apply the quadrature formula $\widehat{Q}_n$ after transforming the variable of integration x in (7.1) suitably, provided $K(t,x)\phi(x)$ is integrable through $x=a$ and $x=b$ in the regular sense. A brief description of the procedure follows:

Let $x=\psi(\xi)$, be the variable transformation alluded to, with the properties

$$\psi\colon [0,1]\to[a,b]; \qquad \psi(0)=a, \qquad \psi(1)=b; \qquad \psi'(\xi)>0 \quad \text{for } 0<\xi<1,$$

and

$$\psi^{(k)}(0)=0, \qquad \psi^{(k)}(1)=0, \quad k=1,\dots,r, \text{ for some } r.$$

Since $\psi(\xi)$ is increasing for $0<\xi<1$, we have $t=\psi(\tau)$ for every $t\in[a,b]$. Of course, τ is unique. Then (7.1) becomes

$$\lambda\Phi(\tau)+\int_0^1 \mathcal{K}(\tau,\xi)\Phi(\xi)\,d\xi=\mathcal{W}(\tau), \qquad \tau\in(0,1), \tag{7.6}$$

where

$$\Phi(\xi)=\phi\bigl(\psi(\xi)\bigr), \qquad \mathcal{W}(\xi)=w\bigl(\psi(\xi)\bigr), \qquad \mathcal{K}(\tau,\xi)=K\bigl(\psi(\tau),\psi(\xi)\bigr)\psi'(\xi). \tag{7.7}$$

By taking r sufficiently large, we can make many of the derivatives of the integrand $\mathcal{K}(\tau,\xi)\Phi(\xi)$ vanish at $\xi=0$ and $\xi=1$, and this makes the 1-periodic extension of $\mathcal{K}(\tau,\xi)\Phi(\xi)$ as a function of ξ quite smooth, except at the points $\xi=\tau+k$, $k=0,\pm1,\pm2,\dots$. (This is the reason why such variable transformations are also called "periodizing" transformations.) Thus, we can apply the quadrature formula $\widehat{Q}_n$ to the transformed integral equation in (7.6) exactly in the same way we have described for the periodic case in the preceding subsection. We leave the details to the reader.

8 A Numerical Example with $\widehat{Q}_n[f]$

As mentioned earlier, because we know the form of the asymptotic expansions of the errors in the quadrature formulas $\bar{Q}_n[f]$, $\bar{Q}'_n[f]$, and $\widehat{Q}_n[f]$, we can apply the Richardson extrapolation process to these formulas and obtain approximations to $I[f]$ that have increasingly high accuracy. As this is a very well known subject, we will not give numerical examples to illustrate it. Instead, we will concentrate on the use of $\widehat{Q}_n[f]$ in the context of periodic integrands since this is of interest when solving periodic hypersingular integral equations and produces spectral accuracy.

Let us consider the integral $I[f]$, where

$$I[f]=\int_{-\pi}^{\pi} f(x)\,dx, \qquad f(x)=\frac{u(x)}{\sin^2\frac{x-t}{2}}, \tag{8.1}$$

with

$$u(x)=\sum_{m=0}^{\infty}\eta^m\cos mx=\frac{1-\eta\cos x}{1-2\eta\cos x+\eta^2}, \qquad \eta \text{ real}, \ |\eta|<1, \tag{8.2}$$

which follows from

$$u(x)=\operatorname{Re}\sum_{m=0}^{\infty}\eta^m e^{\mathrm{i}mx}=\operatorname{Re}\frac{1}{1-\eta e^{\mathrm{i}x}}.$$

Table 1 Numerical results for the integral in (8.1)–(8.3) with $t=1$ throughout. Here $E_n(\eta=c)=|\widehat{Q}_n[f]-I[f]|$ for $\eta=c$

n	E_n $(\eta=0.1)$	E_n $(\eta=0.2)$	E_n $(\eta=0.3)$	E_n $(\eta=0.4)$	E_n $(\eta=0.5)$
10	5.04D–11	2.30D–07	3.23D–05	1.05D–03	1.50D–02
20	1.91D–20	5.21D–14	3.15D–10	1.49D–07	1.69D–05
30	2.68D–30	6.54D–21	2.00D–15	1.47D–11	1.33D–08
40	4.54D–32	5.78D–28	8.80D–21	9.43D–16	5.76D–12
50	5.07D–32	1.33D–32	1.76D–26	4.40D–21	3.27D–15
60	4.79D–32	1.14D–32	1.68D–31	9.60D–24	1.09D–17
70	1.04D–31	3.42D–32	7.48D–33	1.74D–27	1.47D–20
80	9.23D–32	4.98D–32	6.92D–32	2.78D–31	1.37D–23
90	1.36D–31	2.70D–33	1.06D–31	6.34D–32	8.44D–27
100	1.61D–31	1.66D–32	3.10D–32	3.49D–32	7.53D–31

By (5.2), we have

$$I[f]=-4\pi\sum_{m=0}^{\infty}m\eta^m\cos mt=-4\pi\eta\frac{\partial}{\partial\eta}u(t)=-4\pi\eta\frac{(1+\eta^2)\cos t-2\eta}{(1-2\eta\cos t+\eta^2)^2}. \tag{8.3}$$

For this $u(x)$, the function $f(z)$ is meromorphic in the strip $|\operatorname{Im} z|<\sigma=\log\eta^{-1}$ with double poles at the points $z=t+2k\pi$, $k=0,\pm1,\pm2,\dots$.

We have applied $\widehat{Q}_n[f]$ with $t=1$ and $\eta=0.1(0.1)0.5$. The results of this computation, using quadruple precision arithmetic (approximately 35 decimal digits) are given in Table 1.

By Theorem 6.2, $E_n=|\widehat{Q}_n[f]-I[f]|=O(\eta^n)$ as $n\to\infty$. As a result, we should have, $E_{n+k}/E_n\approx\eta^k$, and this can be seen from the results in Table 1.

9 Euler–Maclaurin Expansions and Quadrature Formulas for $\int_a^b g(x)(x-t)^\beta dx$, $\beta=-1,-3,-5,\dots$

We now turn to the treatment of the integrals

$$I[f]=\int_a^b f(x)\,dx;\quad f(x)=g(x)(x-t)^\beta,\ \beta=-1,-3,-5,\dots, \tag{9.1}$$

where $g(x)$ is exactly as described in the Introduction via (1.1) and (1.2). (As mentioned earlier, these integrals are not contained in those of (1.3).) Then we have the following result, which can be obtained using the technique of proof of Theorem 2.2.

Theorem 9.1 *Let $g(x)$ be exactly as in (1.1)–(1.2) with the notation therein, and let $f(x)$ be as in (9.1). Write $\beta=-(2r+1)$, $r=0,1,\dots,$ for convenience. Let also $\{n_k\}_{k=0}^{\infty}$ be a sequence of positive integers, $n_0<n_1<n_2<\cdots$, and let $h_k=(b-a)/n_k$. Let t be such that $t\in\{a+jh_k\}_{j=1}^{n_k-1}$ for every $k=0,1,\dots$. (This is guaranteed if each n_k is an integer multiple of n_0, $k=1,2,\dots,$ and $t\in\{a+jh_0\}_{j=1}^{n_0-1}$.) Let $n\in\{n_k\}_{k=0}^{\infty}$ and let $h=(b-a)/n$*

and $x_j = a + jh$, $j = 0, 1, \dots, n$, *and define*

$$T_n^*[f] = h \sum_{\substack{j=1 \\ x_j \neq t}}^{n-1} f(x_j). \tag{9.2}$$

Define also

$$\begin{aligned} C_{si}(t;\beta) &= (-1)^{i+1} \binom{\beta}{i} c_s (t-a)^{\beta-i}, \quad s, i = 0, 1, \dots, \\ D_{si}(t;\beta) &= (-1)^{i} \binom{\beta}{i} d_s (b-t)^{\beta-i}, \quad s, i = 0, 1, \dots. \end{aligned} \tag{9.3}$$

Then

$$\begin{aligned} T_n^*[f] \sim I[f] &+ \sum_{s=0}^{\infty} \sum_{\substack{i=0 \\ \gamma_s + i \notin \{2,4,6,\dots\}}}^{\infty} C_{si}(t;\beta) \zeta(-\gamma_s - i) h^{\gamma_s+i+1} \\ &+ \sum_{s=0}^{\infty} \sum_{\substack{i=0 \\ \delta_s + i \notin \{2,4,6,\dots\}}}^{\infty} D_{si}(t;\beta) \zeta(-\delta_s - i) h^{\delta_s+i+1} \\ &+ 2 \sum_{i=0}^{r} \frac{g^{(2i+1)}(t)}{(2i+1)!} \zeta(2r-2i) h^{-2r+2i+1} \quad \text{as } h \to 0. \end{aligned} \tag{9.4}$$

As before, by "$h \to 0$" we mean "$h \to 0$, $h \in \{h_k\}_{k=0}^{\infty}$".

Proof The proof is carried out again by writing $I[f] = \int_a^t f(x)\,dx + \int_t^b f(x)\,dx$ and proceeding exactly as in the proof of Theorem 2.2. Note that each of the two integrals $\int_a^t f(x)\,dx$ and $\int_t^b f(x)\,dx$ is defined in the sense of HFP, and the corresponding E–M expansions have a $(C - \log h)$ term in them contributed by the point of singularity $x = t$. Upon summing the two expansions, these terms cancel each other out, hence the sum given in (9.4) contains no terms involving $(C - \log h)$. We leave the details to the reader. □

Note that Remarks 2 and 3 following the proof of Theorem 2.2 apply to the expansion in (9.4) with no changes. In particular, when $g \in C^\infty[a,b]$, after some simple manipulation, the expansion in (9.4) becomes

$$\begin{aligned} Q_n[f] \sim I[f] &+ 2 \sum_{i=0}^{r} \frac{g^{(2i+1)}(t)}{(2i+1)!} \zeta(2r-2i) h^{-2r+2i+1} \\ &+ \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} \big[f^{(2i-1)}(b) - f^{(2i-1)}(a) \big] h^{2i} \quad \text{as } h \to 0, \end{aligned} \tag{9.5}$$

where

$$Q_n[f] = T_n^*[f] + \frac{h}{2}\big[f(a) + f(b)\big] = h\left[\frac{1}{2} f(a) + \sum_{\substack{j=1 \\ x_j \neq t}}^{n-1} f(x_j) + \frac{1}{2} f(b)\right]. \tag{9.6}$$

Note that the asymptotic expansion of the error $Q_n[f]-I[f]$ involves only the derivatives of $f(x)$ and $g(x)$.

Upon invoking $\zeta(0)=-1/2$ and $\zeta(2)=\pi^2/6$, for the two important cases of $\beta=-1$ (hence $r=0$) and $\beta=-3$ (hence $r=1$), (9.5) becomes

$$\begin{aligned} Q_n[f] &\sim I[f]-g'(t)h \\ &\quad+\sum_{i=1}^{\infty}\frac{B_{2i}}{(2i)!}\big[f^{(2i-1)}(b)-f^{(2i-1)}(a)\big]h^{2i} \quad \text{as } h\to 0,\ (\beta=-1), \end{aligned} \tag{9.7}$$

which is precisely what was obtained in [22] (in a much different and lengthy way) for the CPV integral $\int_a^b g(x)/(x-t)\,dx$, $a<t<b$, and

$$\begin{aligned} Q_n[f] &\sim I[f]+\frac{\pi^2}{3}g'(t)h^{-1}-\frac{1}{6}g'''(t)h \\ &\quad+\sum_{i=1}^{\infty}\frac{B_{2i}}{(2i)!}\big[f^{(2i-1)}(b)-f^{(2i-1)}(a)\big]h^{2i} \quad \text{as } h\to 0,\ (\beta=-3). \end{aligned} \tag{9.8}$$

9.1 The Case $\beta=-1$

As is seen from (9.7), when $\beta=-1$, hence $r=0$, the E–M expansion of $Q_n[f]$ for the CPV integral $\int_a^b g(x)/(x-t)\,dx$, $a<t<b$, contains the term $-g'(t)h$. When $g'(t)$ is known, we can use it to define

$$Q_n'[f]=Q_n[f]+g'(t)h=h\left[\frac{1}{2}f(a)+\sum_{\substack{j=1\\ x_j\neq t}}^{n-1}f(x_j)+\frac{1}{2}f(b)\right]+g'(t)h$$

as our numerical quadrature formula. Otherwise, we can replace $g'(t)$ by a (preferably central) differentiation formula of suitable accuracy that is at our disposal.

In case we do not know $g'(t)$, and we do not wish to bother with it, the term $-g'(t)h$ in the asymptotic expansion of (9.7) can be eliminated again by approximating $I[f]$ via a linear combination of $Q_n[f]$ and $Q_{2n}[f]$ as in

$$\widetilde{Q}_n[f]=2Q_{2n}[f]-Q_n[f]. \tag{9.9}$$

(This is precisely what was done in [22] in the treatment of CPV integrals. We have already used this approach in our treatment of the case $\beta=-2$ in Sect. 3.) Since the step size associated with $Q_{2n}[f]$ is $h/2$, we have

$$\widetilde{Q}_n[f]=h\sum_{j=1}^{n}f(a+jh-h/2), \tag{9.10}$$

that is, $\widetilde{Q}_n[f]$ is simply the midpoint rule with step size $h=(b-a)/n$. As a result, the asymptotic expansion of $\widetilde{Q}_n[f]-I[f]$ contains only the powers $h^2, h^4, h^6, \dots$, and is given as in

$$\widetilde{Q}_n[f]\sim I[f]+\sum_{i=1}^{\infty}\frac{B_{2i}}{(2i)!}\big(2^{1-2i}-1\big)\big[f^{(2i-1)}(b)-f^{(2i-1)}(a)\big]h^{2i} \quad \text{as } h\to 0. \tag{9.11}$$

If, in addition, $f(x)$ is a T-periodic function of $x \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R}_t)$, where $T = b - a$, $\mathbb{R} = (-\infty, \infty)$, and $\mathbb{R}_t = \mathbb{R} \setminus \{t + kT\}_{-\infty}^{\infty}$, then (9.11) becomes

$$\widetilde{Q}_n[f] - I[f] = O\big(h^\mu\big) \quad \forall \mu > 0. \tag{9.12}$$

All this is already in [22], although we have shown that the E–M expansion for $\beta = -1$ can be obtained by a technique that is much simpler than the one used in [22]. The exactness result of the next section pertaining to the case $\beta = -1$ seems to be new, however.

10 Exactness Property of $\widetilde{Q}_n[f]$ for $\beta = -1$

The quadrature formula $\widetilde{Q}_n[f]$ for CPV integrals $I[f] = \int_a^b g(x)/(x-t)\,dx$ has an exactness property analogous to that of $\widehat{Q}_n[f]$ for hypersingular integrals $I[f] = g(x)/(x-t)^2\,dx$ described in Theorem 5.1. Specifically, we have the following result that involves the Hilbert kernel:

Theorem 10.1 *Let*

$$f_m(x) = \cot\frac{\pi(x-t)}{T}\, e^{\mathrm{i}2m\pi x/T}, \quad m \text{ integer}. \tag{10.1}$$

[*Note that $f_m(x)$ is T-periodic with a Cauchy singularity of the form $(x-t)^{-1}$ at $x = t$.*] *Then the following are true with the convention that* $\operatorname{sgn}(0) = 0$ *throughout*:

1. *The integral $I[f_m] = \int_a^b f_m(x)\,dx$, $b - a = T$, satisfies*

$$I[f_m] = \mathrm{i}T\operatorname{sgn}(m)\,e^{\mathrm{i}2m\pi t/T}, \quad m = 0, \pm1, \pm2, \ldots. \tag{10.2}$$

2. *The quadrature formula $\widetilde{Q}_n[f_m]$ for $I[f_m]$ satisfies*

$$\widetilde{Q}_n[f_m] = I[f_m] = \mathrm{i}T\operatorname{sgn}(m)\,e^{\mathrm{i}2m\pi t/T}, \quad m = 0, \pm1, \ldots, \pm(n-1), \tag{10.3}$$

   while for arbitrary m, we have

$$\widetilde{Q}_n[f_m] = \mathrm{i}T\operatorname{sgn}(m)V_{m,n}\,e^{\mathrm{i}2m\pi t/T}, \tag{10.4}$$

   with $V_{-m,n} = V_{m,n}$, and

$$V_{m,n} = \begin{cases} 0 & \text{if } m = 0, \\ 1 & \text{if } |m| = 1, \ldots, n-1, \end{cases} \quad \text{and} \quad V_{\pm m,n} = V_{kn+r,n} = (-1)^k V_{r,n}, \tag{10.5}$$

   where k and r are unique integers, $k \ge 0$ and $0 \le r \le n-1$, such that $|m| = kn + r$, in which case,

$$\widetilde{Q}_n[f_m] - I[f_m] = \mathrm{i}T\operatorname{sign}(m)(V_{m,n} - 1)e^{\mathrm{i}2m\pi t/T}. \tag{10.6}$$

3. *If $f(x)$ is of the form*

$$f(x) = \cot\frac{\pi(x-t)}{T}\,u(x), \quad u(x) = \sum_{m=-(n-1)}^{n-1} c_m e^{\mathrm{i}2m\pi x/T}, \tag{10.7}$$

then the quadrature formula $\widetilde{Q}_n[f]$ *for* $I[f]=\int_a^b f(x)\,dx$, $b-a=T$, *satisfies*

$$\widetilde{Q}_n[f]=I[f]. \tag{10.8}$$

Note that $V_{\pm kn,n}=0$, for every integer k, by (10.5).

The proof of Theorem 10.1 is provided in Appendix B. The result of part 1 of this theorem is actually known and can be found in Lifanov [8], for example. Nevertheless, we have provided an independent proof of it.

The next theorem concerns the error in $\widehat{Q}_n[f]$ in terms of the Fourier series of the function $u(x)$ in (5.8) of Theorem 5.2.

Theorem 10.2 *Let*

$$f(x)=\cot\frac{\pi(x-t)}{T}\,u(x),$$

and let $u(x)$ *have the Fourier series*

$$u(x)=\sum_{m=-\infty}^{\infty} c_m e^{\mathrm{i}2m\pi x/T}; \quad c_m=\frac{1}{T}\int_a^b e^{-\mathrm{i}2m\pi x/T}u(x)\,dx. \tag{10.9}$$

Then, with $V_{m,n}$ *as in Theorem* 10.1,

$$\widetilde{Q}_n[f]-I[f]=\mathrm{i}\,T\sum_{m=n}^{\infty}(V_{m,n}-1)\big[c_m e^{\mathrm{i}2m\pi t/T}-c_{-m}e^{-\mathrm{i}2m\pi t/T}\big]. \tag{10.10}$$

Consequently,

$$\big|\widetilde{Q}_n[f]-I[f]\big|\le 2T\sum_{m=n}^{\infty}\big(|c_m|+|c_{-m}|\big). \tag{10.11}$$

Proof The proof of (10.10) is achieved by invoking Theorem 10.1. The proof of (10.11) is achieved by noting that $|V_{m,n}|\le 1$ for all m and n. □

11 A Numerical Example with $\widetilde{Q}_n[f]$ for $\beta=-1$

In this section we concentrate on the use of $\widetilde{Q}_n[f]$ in the context of periodic integrands since this is of interest when solving periodic singular integral equations involving the Cauchy kernel and produces spectral accuracy.

Let us consider the integral $I[f]$, where

$$I[f]=\int_{-\pi}^{\pi} f(x)\,dx, \quad f(x)=\cot\frac{x-t}{2}\,u(x), \tag{11.1}$$

with

$$u(x)=\sum_{m=0}^{\infty}\eta^m\cos mx=\frac{1-\eta\cos x}{1+\eta^2-2\eta\cos x}, \quad \eta \text{ real}, \ |\eta|<1. \tag{11.2}$$

Table 2 Numerical results for the integral in (8.1)–(8.3) with $t=1$ throughout. Here $E_n(\eta=c)=|\widetilde{Q}_n[f]-I[f]|$ for $\eta=c$

n	E_n $(\eta=0.1)$	E_n $(\eta=0.2)$	E_n $(\eta=0.3)$	E_n $(\eta=0.4)$	E_n $(\eta=0.5)$
10	4.74D–10	6.28D–07	4.45D–05	9.25D–04	9.64D–03
20	6.77D–20	8.11D–14	2.97D–10	9.89D–08	8.68D–06
30	6.63D–30	7.34D–21	1.39D–15	7.23D–12	5.04D–09
40	2.34D–32	4.12D–28	3.40D–21	1.85D–16	1.91D–14
50	1.18D–32	1.44D–32	1.46D–26	4.70D–20	4.84D–15
60	1.24D–32	1.71D–32	2.43D–31	1.03D–23	7.91D–18
70	3.85D–34	3.85D–34	1.54D–33	1.30D–27	8.35D–21
80	3.03D–32	3.47D–32	3.81D–32	1.53D–31	6.14D–24
90	1.06D–31	1.07D–31	1.09D–31	1.10D–31	2.10D–27
100	8.62D–32	8.72D–32	8.47D–32	9.48D–32	2.50D–30

Thus, by the fact that

$$u(x)=\mathrm{Re}\sum_{m=0}^{\infty}\eta^m e^{\mathrm{i}mx}=\mathrm{Re}\frac{1}{1-\eta e^{\mathrm{i}x}}$$

and by (10.2),

$$I[f]=\mathrm{Re}\left(2\mathrm{i}\pi\sum_{m=1}^{\infty}\eta^m e^{\mathrm{i}mt}\right)=-2\pi\left[\mathrm{Im}\left(-1+\sum_{m=0}^{\infty}\eta^m e^{\mathrm{i}mt}\right)\right]=-2\pi\left(\mathrm{Im}\,\frac{1}{1-\eta e^{\mathrm{i}t}}\right).$$

Therefore,

$$I[f]=-2\pi\,\frac{\eta\sin t}{1+\eta^2-2\eta\cos t}.$$

For this $u(x)$, the function $f(z)$ is meromorphic in the strip $|\mathrm{Im}\,z|<\sigma=\log\eta^{-1}$ with simple poles at the points $z=t+2k\pi$, $k=0,\pm1,\pm2,\ldots.$

We have applied $\widetilde{Q}_n[f]$ with $t=1$ and $\eta=0.1(0.1)0.5$. The results of this computation, using quadruple precision arithmetic (approximately 35 decimal digits) are given in Table 2.

By Theorem 6.1, $E_n=|\widetilde{Q}_n[f]-I[f]|=O(\eta^n)$ as $n\to\infty$. As a result, we should have, $E_{n+k}/E_n\approx\eta^k$, and this can be seen from the results in Table 2.

Acknowledgement This research was supported in part by the United States–Israel Binational Science Foundation grant no. 2008399.

Appendix A: Proof of Theorem 5.1

Proof of Part 1 We start by noting that

$$I[f_m]=\left[\int_a^b\frac{e^{\mathrm{i}2m\pi(x-t)/T}}{\sin^2\frac{\pi(x-t)}{T}}\,dx\right]e^{\mathrm{i}2m\pi t/T}.$$

Making the variable transformation $y=2\pi(x-t)/T$ in the integral inside the square brackets, and using the fact that the integrand is T-periodic, we obtain

$$I[f_m]=\frac{T}{2\pi}L_m e^{\mathrm{i}2m\pi t/T};\qquad L_m=\int_{-\pi}^{\pi}\frac{e^{\mathrm{i}my}}{\sin^2(\frac{1}{2}y)}\,dy. \tag{A.1}$$

Next,

$$L_m=\int_{-\pi}^{\pi}\frac{\cos(my)}{\sin^2(\frac{1}{2}y)}\,dy+\mathrm{i}\int_{-\pi}^{\pi}\frac{\sin(my)}{\sin^2(\frac{1}{2}y)}\,dy,$$

and since $\int_{-\pi}^{\pi}\frac{\sin(my)}{\sin^2(\frac{1}{2}y)}\,dy=0$ due to its integrand being odd, it follows that

$$L_m=\int_{-\pi}^{\pi}\frac{\cos(my)}{\sin^2(\frac{1}{2}y)}\,dy\quad\Rightarrow\quad L_{-m}=L_m. \tag{A.2}$$

Thus, it suffices to treat only the case $m=0,1,\dots$. Now,

$$\begin{aligned}\cos\big[(m+1)y\big]+\cos\big[(m-1)y\big]&=2\cos y\cos(my)\\&=2\left[1-2\sin^2\left(\frac{1}{2}y\right)\right]\cos(my),\end{aligned}$$

which, upon dividing both sides by $\sin^2(\frac{1}{2}y)$, gives the identity

$$\frac{\cos[(m+1)y]}{\sin^2(\frac{1}{2}y)}-2\frac{\cos(my)}{\sin^2(\frac{1}{2}y)}+\frac{\cos[(m-1)y]}{\sin^2(\frac{1}{2}y)}=-4\cos(my). \tag{A.3}$$

Integrating both sides of this identity over $[-\pi,\pi]$, and invoking (A.2), we obtain

$$L_{m+1}-2L_m+L_{m-1}=-4\int_{-\pi}^{\pi}\cos(my)\,dy. \tag{A.4}$$

Upon setting $m=0$, and invoking the fact that $L_{-1}=L_1$, (A.4) gives

$$2L_1-2L_0=-8\pi\quad\Rightarrow\quad L_1=L_0-4\pi, \tag{A.5}$$

while for $m\geq 1$, because $\int_{-\pi}^{\pi}\cos(my)\,dy=0$, (A.4) gives the recurrence relation

$$L_{m+1}-2L_m+L_{m-1}=0,\qquad m=1,2,\dots,$$

whose general solution is of the form $L_m=Am+B$. We can determine A and B by invoking the values of L_0 and L_1. Now, applying to L_0 the known result

$$\int_a^b\frac{g(x)}{(x-t)^2}\,dx=\lim_{\epsilon\to 0}\left[\int_a^{t-\epsilon}\frac{g(x)}{(x-t)^2}\,dx+\int_{t+\epsilon}^b\frac{g(x)}{(x-t)^2}\,dx-2\frac{g(t)}{\epsilon}\right],\quad a<t<b,$$

we obtain

$$L_0=\int_{-\pi}^{\pi}\csc^2\left(\frac{1}{2}y\right)dy=\lim_{\epsilon\to 0}\big[4\cot(\epsilon/2)-8/\epsilon\big]=0.$$

As for L_1, by (A.5), we have

$$L_1 = L_0 - 4\pi = -4\pi.$$

Consequently, $A = -4\pi$ and $B = 0$. Hence, also by the fact that $L_{-m} = L_m$,

$$L_m = -4\pi|m|, \quad m = 0, \pm 1, \pm 2, \dots. \tag{A.6}$$

Substituting this in (A.1), we obtain (5.2).

Proof of Part 2 To prove (5.3) and (5.4), we proceed similarly. First, by (4.5),

$$\widehat{Q}_n[f_m] = \frac{T}{n}\big(W_{m,n} - n^2\big)e^{\mathrm{i}2m\pi t/T}, \tag{A.7}$$

$$W_{m,n} = \sum_{j=1}^{n} \frac{e^{\mathrm{i}my_j}}{\sin^2(\frac{1}{2}y_j)}, \quad y_j = (2j-1)\frac{\pi}{n}, \ j = 1, \dots, n. \tag{A.8}$$

Next,

$$W_{m,n} = \sum_{j=1}^{n} \frac{\cos(my_j)}{\sin^2(\frac{1}{2}y_j)} + \mathrm{i}\sum_{j=1}^{n} \frac{\sin(my_j)}{\sin^2(\frac{1}{2}y_j)}.$$

By the fact that

$$\frac{\sin(my_{n-j+1})}{\sin^2(\frac{1}{2}y_{n-j+1})} = -\frac{\sin(my_j)}{\sin^2(\frac{1}{2}y_j)}, \quad j = 1, \dots, n,$$

and since $\sum_{j=1}^{n} = \sum_{j=1}^{n} w_{n-j+1}$, we have $\sum_{j=1}^{n} \frac{\sin(my_j)}{\sin^2(\frac{1}{2}y_j)} = 0$. Consequently,

$$W_{m,n} = \sum_{j=1}^{n} \frac{\cos(my_j)}{\sin^2(\frac{1}{2}y_j)} \quad \Rightarrow \quad W_{-m,n} = W_{m,n}. \tag{A.9}$$

Thus, it suffices to treat only the case $m = 0, 1, \dots$.

Next, for every $m \ge 0$, there exist unique integers $k, r \ge 0$, $r \le n-1$, such that $m = kn + r$. (Thus, $k = 0$ and $r = 0$ for $m = 0$, while $k = 1$ and $r = 0$ for $m = n$.) By the fact that

$$\cos\big[(kn+r)y_j\big] = \cos\big[k(2j-1)\pi + ry_j\big] = (-1)^k\cos(ry_j), \quad j = 1, \dots, n,$$

we realize that

$$W_{m,n} = W_{kn+r,n} = (-1)^k W_{r,n}. \tag{A.10}$$

Thus, we need to concern ourselves only with $0 \le m \le n-1$. (Note that this also implies that $|W_{m,n}| \le \max_{0\le i\le n-1}|W_{i,n}|$ for every m, hence $\{W_{m,n}\}_{m=-\infty}^{\infty}$ is a bounded sequence for fixed n.)

Setting $y = y_j$ in (A.3) and summing over j from 1 to n, and invoking (A.9), we obtain

$$W_{m+1,n} - 2W_{m,n} + W_{m-1,n} = -4\sum_{j=1}^{n}\cos(my_j) = -4\left[\operatorname{Re}\sum_{j=1}^{n} e^{\mathrm{i}my_j}\right]. \tag{A.11}$$

Upon setting $m=0$ in (A.11), and invoking the fact that $W_{-1,0}=W_{1,0}$, we first obtain

$$2W_{1,n}-2W_{0,n}=-4n \quad \Rightarrow \quad W_{1,n}=W_{0,n}-2n. \tag{A.12}$$

Let us now consider $1\le m\le n-1$ in (A.11). For this, we need to compute $\sum_{j=1}^{n}\cos(my_j)$. Invoking (A.8), we have

$$\sum_{j=1}^{n}e^{\mathrm{i}my_j}=e^{\mathrm{i}m\pi/n}\sum_{j=0}^{n-1}\left(e^{\mathrm{i}2m\pi/n}\right)^j=e^{\mathrm{i}m\pi/n}\frac{1-\left(e^{\mathrm{i}2m\pi/n}\right)^n}{1-e^{\mathrm{i}2m\pi/n}}=0,$$

since $0<2m\pi/n<2\pi$ for $1\le m\le n-1$. As a result, $\sum_{j=1}^{n}\cos(my_j)=0$, and (A.11) becomes

$$W_{m+1,n}-2W_{m,n}+W_{m-1,n}=0, \quad m=1,2,\dots,n-1, \tag{A.13}$$

which enables us to compute $W_{m,n}$ for $2\le m\le n$ when $W_{0,n}$ and $W_{1,n}$ are known. The general solution of (A.13) is of the form $W_{m,n}=Am+B$ for some constants A and B, which we determine by invoking two initial values, namely, those of $W_{0,n}$ and $W_{1,n}$. To compute $W_{0,n}$, let

$$\theta_j=\frac{1}{2}y_j \quad \text{and} \quad \xi_j=\cos\theta_j, \quad j=1,\dots,n, \tag{A.14}$$

and note that $\xi_1,\dots,\xi_n$ are the zeros of $T_n(\xi)$, the nth Chebyshev polynomial of the first kind. (For Chebyshev polynomials and their properties, see Rivlin [16], for example.) By (A.9) and (A.14),

$$W_{0,n}=\sum_{j=1}^{n}\frac{1}{\sin^2\theta_j}=\sum_{j=1}^{n}\frac{1}{1-\cos^2\theta_j}=\sum_{j=1}^{n}\frac{1}{1-\xi_j^2}=\frac{1}{2}\left[\sum_{j=1}^{n}\frac{1}{1-\xi_j}+\sum_{j=1}^{n}\frac{1}{1+\xi_j}\right],$$

and since $\xi_{n-j+1}=-\xi_j$, $j=1,\dots,n$, we have

$$\sum_{j=1}^{n}\frac{1}{1-\xi_j}=\sum_{j=1}^{n}\frac{1}{1+\xi_j} \quad \Rightarrow \quad W_{0,n}=\sum_{j=1}^{n}\frac{1}{1-\xi_j}.$$

Now, since ξ_j are the zeros of $T_n(\xi)$, we have the identity

$$\frac{T_n'(\xi)}{T_n(\xi)}=\sum_{j=1}^{n}\frac{1}{\xi-\xi_j}.$$

Letting $\xi=1$ in this identity, and making use of the facts that $T_n(1)=1$, $T_n'(1)=n^2$, we have

$$W_{0,n}=\sum_{j=1}^{n}\frac{1}{1-\xi_j}=\frac{T_n'(1)}{T_n(1)}=n^2.$$

As for $W_{1,n}$, using (A.12), we have

$$W_{1,n}=W_{0,n}-2n=n^2-2n.$$

Consequently, $A = -2n$ and $B = n^2$, and hence

$$W_{m,n} = n^2 - 2mn, \quad m = 0, 1, \dots, n.$$

Recalling also the fact that $W_{-m,n} = W_{m,n}$, we replace m by $|m|$ everywhere, thus obtaining

$$W_{m,n} = n^2 - 2|m|n, \quad m = 0, \pm 1, \dots, \pm n. \tag{A.15}$$

Finally, substituting (A.15) in (A.7), we obtain (5.3).

The result in (5.4) is obtained by substituting (A.10), with $W_{r,n} = n^2 - 2rn$, in (A.8). The result in (5.5) is obtained by subtracting (5.2) from (5.4).

Proof of Part 3 This follows by invoking Part 2 of the theorem in

$$\widehat{Q}_n[f] - I[f] = \sum_{m=-n}^{n} c_m \bigl(\widehat{Q}_n[f_m] - I[f_m]\bigr).$$

□

Appendix B: Proof of Theorem 10.1

Proof of Part 1 We start by noting that

$$I[f_m] = \left[\int_a^b \cot\frac{\pi(x-t)}{T}\, e^{\mathrm{i}2m\pi(x-t)/T}\, dx\right] e^{\mathrm{i}2m\pi t/T}.$$

Making the variable transformation $y = 2\pi(x-t)/T$ in the integral inside the square brackets, and using the fact that the integrand is T-periodic, we obtain

$$I[f_m] = \frac{T}{2\pi} K_m e^{\mathrm{i}2m\pi t/T}; \quad K_m = \int_{-\pi}^{\pi} \cot\left(\frac{1}{2}y\right) e^{\mathrm{i}my}\, dy. \tag{B.1}$$

Next,

$$K_m = \int_{-\pi}^{\pi} \cot\left(\frac{1}{2}y\right) \cos(my)\, dy + \mathrm{i}\int_{-\pi}^{\pi} \cot\left(\frac{1}{2}y\right) \sin(my)\, dy,$$

and since $\int_{-\pi}^{\pi} \cot(\frac{1}{2}y)\cos(my)\, dy = 0$ due to its integrand being odd, it follows that

$$K_m = \mathrm{i}\int_{-\pi}^{\pi} \cot\left(\frac{1}{2}y\right) \sin(my)\, dy \quad \Rightarrow \quad K_0 = 0, \qquad K_{-m} = -K_m. \tag{B.2}$$

Thus, it suffices to treat only the case $m = 1, 2, \dots$. For $m \geq 1$, the integral in (B.2) representing K_m is regular. By making the variable transformation $z = \frac{1}{2}y$ in this integral, and invoking Gradshteyn and Ryzhik [4, p. 366, formula 3.612(7)], we obtain

$$K_m = 4\mathrm{i}\int_0^{\pi/2} \cos z\, \frac{\sin(2mz)}{\sin z}\, dz = 2\pi\mathrm{i}, \quad m = 1, 2, \dots. \tag{B.3}$$

Substituting this in (B.1), we obtain (10.2), with $\mathrm{sgn}(0) = 0$.

Proof of Part 2 By (9.10), we have first

$$\widetilde{Q}_n[f_m] = \frac{T}{n}\, U_{m,n}\, e^{\mathrm{i}\,2m\pi t/T},$$

$$U_{m,n} = \sum_{j=1}^{n} \cot\left(\frac{1}{2} y_j\right) e^{\mathrm{i} m y_j}, \quad y_j = (2j-1)\frac{\pi}{n},\ j = 1, \dots, n. \tag{B.4}$$

Next,

$$U_{m,n} = \sum_{j=1}^{n} \cot\left(\frac{1}{2} y_j\right) \cos(m y_j) + \mathrm{i} \sum_{j=1}^{n} \cot\left(\frac{1}{2} y_j\right) \sin(m y_j).$$

By the fact that

$$\cot\left(\frac{1}{2} y_{n-j+1}\right) \cos(m y_{n-j+1}) = -\cot\left(\frac{1}{2} y_j\right) \cos(m y_j), \quad j = 1, \dots, n,$$

and since $\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} w_{n-j+1}$, we have $\sum_{j=1}^{n} \cot(\frac{1}{2} y_j) \cos(m y_j) = 0$. Consequently,

$$U_{m,n} = \mathrm{i} \sum_{j=1}^{n} \cot\left(\frac{1}{2} y_j\right) \sin(m y_j) \quad \Rightarrow \quad U_{0,n} = 0, \qquad U_{-m,n} = -U_{m,n}. \tag{B.5}$$

Thus, it suffices to treat only the case $m = 1, 2, \dots$.

Next, for every $m \geq 0$, there exist unique integers $k, r \geq 0$, $r \leq n-1$, such that $m = kn + r$. (Thus, $k = 0$ and $r = 0$ for $m = 0$, while $k = 1$ and $r = 0$ for $m = n$, and, therefore, $U_{\pm kn,n} = 0$ for $k = 0, 1, \dots$.) By the fact that

$$\sin\big[(kn + r) y_j\big] = \sin\big[k(2j-1)\pi + r y_j\big] = (-1)^k \sin(r y_j), \quad j = 1, \dots, n,$$

we realize that

$$U_{m,n} = U_{kn+r,n} = (-1)^k U_{r,n}. \tag{B.6}$$

Thus, we need to concern ourselves only with $1 \leq m \leq n-1$. (Note that this also implies that $|U_{m,n}| \leq \max_{0 \leq i \leq n-1} |U_{i,n}|$ for every m, hence $\{U_{m,n}\}_{m=-\infty}^{\infty}$ is a bounded sequence for fixed n.)

Now,

$$\sin\left(my \pm \frac{1}{2} y\right) = \sin(my) \cos\left(\frac{1}{2} y\right) \pm \cos(my) \sin\left(\frac{1}{2} y\right). \tag{B.7}$$

Dividing both sides of this identity by $\sin(\frac{1}{2} y)$, we obtain

$$\frac{\sin(my \pm \frac{1}{2} y)}{\sin(\frac{1}{2} y)} = \cot\left(\frac{1}{2} y\right) \sin(my) \pm \cos(my). \tag{B.8}$$

Let us now set $y = y_j$ in this identity and sum over j from 1 to n. Invoking (B.5), we obtain

$$\sum_{j=1}^{n} \frac{\sin(m y_j \pm \frac{1}{2} y_j)}{\sin(\frac{1}{2} y_j)} = -\mathrm{i} U_{m,n} \pm \sum_{j=1}^{n} \cos(m y_j). \tag{B.9}$$

Note that the y_j in (B.4) are the same as the y_j in (A.8), and hence $\sum_{j=1}^{n}\cos(my_j)=0$ for $1\le m\le n-1$, as shown in the proof of Theorem 5.1 given in Appendix A. Consequently, (B.9) becomes

$$D_{m+1}=-\mathrm{i}U_{m,n}=D_m,\quad 1\le m\le n-1;\qquad D_k=\sum_{j=1}^{n}\frac{\sin(ky_j-\frac{1}{2}y_j)}{\sin(\frac{1}{2}y_j)}. \tag{B.10}$$

Therefore, $D_n=D_{n-1}=\cdots=D_2=D_1=n$, so that

$$U_{m,n}=\mathrm{i}n,\quad m=1,\ldots,n-1.$$

Recalling also (B.5), we have

$$U_{m,n}=\mathrm{i}\,\mathrm{sgn}(m)n,\quad m=0,\pm1,\ldots,\pm(n-1);\qquad \mathrm{sign}(0)=0. \tag{B.11}$$

Substituting (B.11) in (B.4), we obtain (10.3). Combining (B.11) with (B.5) and (B.6), we obtain (10.4) with (10.5). The result in (10.6) is obtained by subtracting (10.2) from (10.4).

Proof of Part 3 This follows by invoking Part 2 of the theorem in

$$\widehat{Q}_n[f]-I[f]=\sum_{m=-(n-1)}^{n-1}c_m\big(\widehat{Q}_n[f_m]-I[f_m]\big).$$

□

References

1. Capobianco, M.R., Mastroianni, G., Russo, M.G.: Pointwise and uniform approximation of the finite Hilbert transform. In: Approximation and Optimization, vol. I, Cluj-Napoca, 1996, pp. 45–66. Transilvania, Cluj-Napoca (1997)
2. Davis, P.J., Rabinowitz, P.: Methods of Numerical Integration, 2nd edn. Academic Press, New York (1984)
3. Evans, G.: Practical Numerical Integration. Wiley, New York (1993)
4. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products. Academic Press, New York (1980). 4th printing (1983)
5. Huang, J., Wang, Z., Zhu, R.: Asymptotic error expansions for hypersingular integrals. Adv. Comput. Math. (2011). Published online: September 30. doi:10.1007/s10444-011-9236-x
6. Kythe, P.K., Schäferkotter, M.R.: Handbook of Computational Methods for Integration. Chapman & Hall/CRC Press, New York (2005)
7. Ladopoulos, E.G.: Singular Integral Equations: Linear and Non-linear Theory and Its Applications in Science and Engineering. Springer, Berlin (2000)
8. Lifanov, I.K.: An instance of numerical solution of integral equations with singularities in the periodic case. Differ. Equ. **42**(9), 1134–1342 (2006). Translated from Differ. Uravn. **42**, 1263–1271 (2006)
9. Lifanov, I.K., Poltavskii, L.N.: On the numerical solution of hypersingular and singular integral equations on the circle. Differ. Equ. **39**(8), 1175–1197 (2003). Translated from Differ. Uravn. **39**, 1115–1136 (2003)
10. Lifanov, I.K., Poltavskii, L.N., Vainikko, G.M.: Hypersingular Integral Equations and Their Applications. CRC Press, New York (2004)
11. Mastroanni, G., Occorsio, D.: Interlacing properties of the zeros of the orthogonal polynomials and approximation of the Hilbert transform. Comput. Math. Appl. **30**, 155–168 (1995)
12. Mastronardi, N., Occorsio, D.: Some numerical algorithms to evaluate Hadamard finite-part integrals. J. Comput. Appl. Math. **70**, 75–93 (1996)
13. Monegato, G.: Numerical evaluation of hypersingular integrals. J. Comput. Appl. Math. **50**, 9–31 (1994)
14. Navot, I.: An extension of the Euler–Maclaurin summation formula to functions with a branch singularity. J. Math. Phys. **40**, 271–276 (1961)

15. Navot, I.: A further extension of the Euler–Maclaurin summation formula. J. Math. Phys. **41**, 155–163 (1962)
16. Rivlin, T.J.: Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory, 2nd edn. Wiley, New York (1990)
17. Sidi, A.: Comparison of some numerical quadrature formulas for weakly singular periodic Fredholm integral equations. Computing **43**, 159–170 (1989)
18. Sidi, A.: Practical Extrapolation Methods: Theory and Applications. Cambridge Monographs on Applied and Computational Mathematics, vol. 10. Cambridge University Press, Cambridge (2003)
19. Sidi, A.: Euler–Maclaurin expansions for integrals with endpoint singularities: a new perspective. Numer. Math. **98**, 371–387 (2004)
20. Sidi, A.: Euler–Maclaurin expansions for integrals with arbitrary algebraic endpoint singularities. Math. Comput. (2012). Published online: April 10. doi:10.1090/S0025-5718-2012-02597-x
21. Sidi, A.: Euler–Maclaurin expansions for integrals with arbitrary algebraic-logarithmic endpoint singularities. Constr. Approx. (2011). Published online: September 28. doi:10.1007/s00365-011-9140-0
22. Sidi, A., Israeli, M.: Quadrature methods for periodic singular and weakly singular Fredholm integral equations. J. Sci. Comput. **3**, 201–231 (1988). Originally appeared as Technion Computer Science Dept. Technical Report No. 384 (1985) and also as ICASE Report No. 86-50 (1986)
23. Stoer, J., Bulirsch, R.: Introduction to Numerical Analysis, 3rd edn. Springer, New York (2002)
24. Wu, J., Dai, Z., Zhang, X.: The superconvergence of the composite midpoint rule for the finite-part integral. J. Comput. Appl. Math. **233**, 1954–1968 (2010)
25. Wu, J., Sun, W.: The superconvergence of Newton–Cotes rules for the Hadamard finite-part integral on an interval. Numer. Math. **109**, 143–165 (2008)