J. Math. Anal. Appl. 397 (2013) 91–108

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Polynomials biorthogonal to dilations of measures, and their asymptotics[☆]

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ARTICLE INFO

Article history:
Received 15 February 2011
Available online 20 July 2012
Submitted by D. Khavinson

Keywords:
Biorthogonal
Asymptotics
Polynomials

ABSTRACT

We analyze polynomials P_n that are biorthogonal to dilates of a positive measure μ, supported on $(0, \infty)$:

$$\int_0^\infty P_n(x)\, d\mu\left(\sigma_{n,j} x\right) = 0, \quad 1 \le j \le n.$$

We establish representations for P_n in terms of the *associated dilation polynomial*

$$R_n(y) = \prod_{j=1}^{n} \left(y + 1/\sigma_{n,j}\right).$$

In the case where

$$d\mu(t) = t^\alpha e^{-t^\beta} dt \quad \text{on } (0, \infty),$$

we show that strong asymptotics for R_n in the complex plane, as $n \to \infty$, lead to strong asymptotics for P_n, via the method of steepest descent.

1. Introduction

Let $\alpha > -1$ and $\left\{\sigma_{n,j}\right\}_{j=1}^n$ be distinct exponents in $(0, \infty)$. Then we may determine a monic polynomial P_n of degree n, determined by the biorthogonality conditions

$$\int_0^\infty x^\alpha P_n(x) e^{-\sigma_{n,j} x} dx = 0, \quad 1 \le j \le n.$$

In developing methods for convergence acceleration, and numerical integration of singular integrands, the second author introduced some classes of polynomials of this type [1–5]. These include

(I) the polynomials $P_n = \Lambda_n^{(\alpha,\Delta)}$, for which $\sigma_{n,j} = j + \Delta,\ 1 \le j \le n$;
(II) the polynomials $P_n = G_n^{(\alpha)}$, for which $\left\{\sigma_{n,j}^{-1}\right\}_{j=1}^n$ are the zeros of the Sidi polynomials $D_n^{(0,0)}$;
(III) the polynomials $P_n = M_n^{(\alpha)}$, for which $\left\{\sigma_{n,j}^{-1}\right\}_{j=1}^n$ are the zeros of the Legendre polynomials scaled to $(0, 1)$.

We explored these polynomials, and their zero distribution, in an earlier paper [6].

☆ Research supported by NSF grant DMS1001182 and US–Israel BSF grant 2008399.
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doi:10.1016/j.jmaa.2012.06.043

In this paper, we consider the more general case when P_n is determined by orthogonality to dilates of a positive measure μ. Thus we assume that μ is a positive measure supported on the real line, with all moments

$$\mu_j = \int_0^\infty x^j d\mu(x), \quad j = 0, 1, 2, \ldots, \tag{1.1}$$

finite. We assume that for $n \geq 1$, we are given distinct positive numbers $\left\{\sigma_{n,j}\right\}_{j=1}^n$, and determine a monic polynomial P_n of degree n by the conditions

$$\int_0^\infty P_n(x) d\mu\left(\sigma_{n,j} x\right) = 0, \quad 1 \leq j \leq n. \tag{1.2}$$

Equivalently,

$$\int_0^\infty P_n\left(\frac{t}{\sigma_{n,j}}\right) d\mu\,(t) = 0, \quad 1 \leq j \leq n. \tag{1.3}$$

As in [6], P_n is closely related to the polynomial

$$R_n(y) = \prod_{j=1}^n \left(y + \sigma_{n,j}^{-1}\right) = \sum_{j=0}^n r_{n,j} y^j. \tag{1.4}$$

We call R_n the *dilation polynomial associated with* P_n. The following simple proposition establishes the relationship between P_n and R_n:

Theorem 1.1. *Let μ be a positive measure on $(0, \infty)$ with infinitely many points in its support, and finite moments $\left\{\mu_j\right\}$. Let $\left\{\sigma_{n,j}\right\}_{j=1}^n$ be distinct positive numbers. Let P_n be a monic polynomial of degree n, determined by the orthogonality relations (1.2), and let R_n be given by (1.4). Then P_n exists, is unique, and*

(I)

$$P_n(x) = (-1)^n \sum_{j=0}^n r_{n,j} \frac{\mu_n}{\mu_j} (-x)^j, \tag{1.5}$$

while

$$(-1)^n R_n(-y) = \frac{1}{\mu_n} \int_0^\infty P_n(ty) d\mu(t). \tag{1.6}$$

(II) *There exists $r > 0$ such that*

$$P_n(x) = \mu_n \frac{(-1)^n}{2\pi i} \int_{|t|=r} R_n\left(-\frac{x}{t}\right) G\,(t)\, \frac{dt}{t} \tag{1.7}$$

where

$$G(t) = \sum_{j=0}^\infty \frac{t^j}{\mu_j}. \tag{1.8}$$

(III) *Write $\sigma_j = \sigma_{n,j},\ 1 \leq j \leq n$. Then*

$$\frac{P_n(x)}{\mu_n} = \frac{\det \begin{bmatrix} 1 & \sigma_1^{-1} & \sigma_1^{-2} & \cdots & \sigma_1^{-n} \\ 1 & \sigma_2^{-1} & \sigma_2^{-2} & \cdots & \sigma_2^{-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_n^{-1} & \sigma_n^{-2} & \cdots & \sigma_n^{-n} \\ \dfrac{1}{\mu_0} & \dfrac{x}{\mu_1} & \dfrac{x^2}{\mu_2} & \cdots & \dfrac{x^n}{\mu_n} \end{bmatrix}}{\det \begin{bmatrix} 1 & \sigma_1^{-1} & \sigma_1^{-2} & \cdots & \sigma_1^{-n+1} \\ 1 & \sigma_2^{-1} & \sigma_2^{-2} & \cdots & \sigma_2^{-n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_n^{-1} & \sigma_n^{-2} & \cdots & \sigma_n^{-n+1} \end{bmatrix}}. \tag{1.9}$$

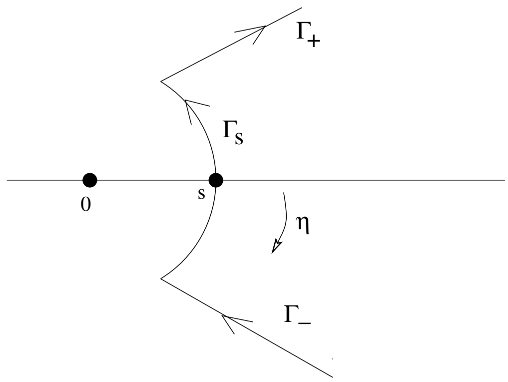

Fig. 1. The contour Γ.

(IV) *If μ has the form*

$$d\mu(t) = t^{\alpha} e^{-t^{\beta}} dt, \quad t \in (0, \infty),$$

where $\alpha > -1$, $\beta > 0$, then P_n has n simple zeros in $(0, \infty)$.

We shall prove this simple result in Section 2. In the special case where $d\mu(t) = t^{\alpha} e^{-t} dt$, it is a combination of Proposition 1.1 and Theorem 1.5 in [6]. There we denoted R_n by Q_n. In the general case, parts of Proposition 1.1 overlap with results of Brezinski [7], Iserles et al. [8,9] on biorthogonal polynomials in a more general setting. Algebraic and asymptotic properties of related biorthogonal polynomials have been explored in [10–13].

The main focus of this paper is the asymptotic behavior of $\{P_n\}$. These are based on a simple new contour integral representation of P_n:

Theorem 1.2. *Let $\beta \geq 1, \alpha > -1$, and $d\mu(t) = t^{\alpha} e^{-t^{\beta}} dt$, $t \in (0, \infty)$. Let $\left\{\sigma_{n,j}\right\}_{j=1}^{n}$ be distinct positive numbers. Let P_n be a monic polynomial of degree n, determined by the orthogonality relations (1.2), and R_n be given by (1.4). Let*

$$\frac{\pi}{2\beta} < \eta < \frac{\pi}{\beta}, \tag{1.10}$$

$s > 0$, and let Γ be the contour consisting of the rays $\Gamma_+ = \left\{re^{i\eta} : r \geq s\right\}$, $\Gamma_- = \left\{re^{-i\eta} : r \geq s\right\}$, and the circular arc $\Gamma_s = \left\{se^{i\theta} : |\theta| \leq \eta\right\}$ (see Fig. 1). Assume that Γ is traversed in such a way that Γ_s is traversed anticlockwise. Then for all complex z,

$$P_n(z) = \frac{\beta^2 (-1)^n \mu_n}{2\pi i} \int_{\Gamma} e^{t^{\beta}} t^{\beta-\alpha-2} R_n\left(-\frac{z}{t}\right) dt. \tag{1.11}$$

Let ν_n denote the zero counting function for R_n, so that

$$\nu_n [a, b] = \frac{1}{n} (\text{Number of zeros of } R_n \text{ in } [a, b]). \tag{1.12}$$

Equivalently,

$$\nu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{-1/\sigma_{n,j}},$$

where δ_a denotes a unit mass at a. In [6, Theorem 1.2, p. 347], we showed that for $\beta = 1$, the zero counting measures of $P_n(-4nx)$ converge weakly as $n \to \infty$, iff ν_n converges weakly as $n \to \infty$. Moreover, we related the limiting zero counting distributions of R_n and P_n. Note that

$$R_n(z) = \exp\left(n \int \log(z - t)\, d\nu_n(t)\right),$$

with the usual branch of the log, at least for $z \notin (-\infty, 0)$. In this paper, we establish a stronger form of asymptotic. We shall make some assumptions about the behavior of R_n as $n \to \infty$:

Hypotheses on R_n.

(I) For all j, n,

$$\sigma_{n,j} \in [1, \infty). \tag{1.13}$$

(II) There is a probability distribution ν on $[-1, 0]$, a sequence $\{\lambda_n\}$ of non-negative numbers, and functions D and E such that as $n \to \infty$,

$$R_n(z) = \exp\left(n \int \log(z-t)\, d\nu(t) + \lambda_n D(z) + E(z) + o(1)\right), \tag{1.14}$$

uniformly for z in compact subsets of $\mathbb{C} \setminus [-1, 0]$.

(III) For each $\varepsilon > 0$, as $n \to \infty$,

$$\lambda_n = o\left(n^{\varepsilon}\right). \tag{1.15}$$

(IV) $\exp(D)$ and $\exp(E)$ are analytic in $\mathbb{C} \setminus [-1, 0]$.

We need the scaled monic polynomials

$$Q_n(z) = \left(-\frac{\mu_{n-1}}{\mu_n}\right)^n P_n\left(-\frac{\mu_n}{\mu_{n-1}} z\right). \tag{1.16}$$

For $u \in \mathbb{C} \setminus (-\infty, 0]$, and $\frac{z}{u} \notin (-\infty, 0]$, we let

$$\Psi(z, u) = \int_{-1}^{0} \log\left(\frac{z}{u} - t\right) d\nu(t) + \frac{u^{\beta} - 1}{\beta}. \tag{1.17}$$

The critical points of $\Psi(z, \cdot)$ determine the main part of the asymptotic for Q_n. In applying steepest descent, the second derivative of Ψ plays a role. Accordingly, we need, that when $\frac{\partial \Psi(z,u)}{\partial u} = 0$,

$$\begin{aligned} B(z, u) &= u^2 \frac{\partial^2 \Psi(z, u)}{\partial u^2} \\ &= -\int_{-1}^{0} \frac{uzt}{(z - tu)^2} d\nu(t) + \beta u^{\beta}. \end{aligned} \tag{1.18}$$

The following function is useful in describing the asymptotics:

$$H(t) = \sqrt{\beta}\, t^{\beta - \alpha - 1} \exp\left(\frac{t^{\beta} - 1}{\beta}\left[\alpha + \frac{1 - \beta}{2}\right]\right). \tag{1.19}$$

Following is our asymptotic result.

Theorem 1.3. *Assume that $\beta \geq 1$ and $\alpha > -1$. Let $d\mu(t) = t^{\alpha} \exp\left(-t^{\beta}\right) dt$ on $(0, \infty)$ and assume that for $n \geq 1$, we are given distinct exponents $\left\{\sigma_{n,j}\right\}_{j=1}^{n}$. Let $\{P_n\}$ denote the corresponding monic biorthogonal polynomials, and $\{R_n\}$ the associated exponent polynomials. Assume the hypotheses* (I)–(IV) *above.*

(a) *For* $\operatorname{Re} z \geq 0,\ z \neq 0$, *there is a unique solution $u = \psi(z)$ of the equation*

$$u^{\beta} = \int_{-1}^{0} \frac{z}{z - ut} d\nu(t) \tag{1.20}$$

with $|\arg \psi(z)| < \frac{\pi}{2\beta}$ and $|\psi(z)| \leq 1$. Here ψ is an analytic function of z.

(b) *Uniformly for z in compact sets of* $\operatorname{Re} z \geq 0,\ z \neq 0$,

$$Q_n(z) = \frac{H(\psi(z))}{\sqrt{B(z, \psi(z))}} \exp\left(n\Psi(z, \psi(z)) + \lambda_n D\left(\frac{z}{\psi(z)}\right) + E\left(\frac{z}{\psi(z)}\right)\right)(1 + o(1)). \tag{1.21}$$

Remarks.

(a) Let ω_n denote the zero counting function of Q_n, so that

$$d\omega_n(t) = \frac{1}{n} \sum_{x: Q_n(x) = 0} \delta_x.$$

It has support in $(-\infty, 0)$. By taking absolute values and nth roots in the asymptotic above, we see that, at least for $\operatorname{Re}(z) \geq 0$,

$$\lim_{n \to \infty} \int \log|z - t|\, d\omega_n(t) = \int_{-1}^{0} \log\left|\frac{z}{\psi(z)} - t\right| d\nu(t) + \operatorname{Re}\left(\frac{\psi(z)^{\beta} - 1}{\beta}\right).$$

It follows from this limit, that ω_n converges weakly to a distribution ω that satisfies, at least for $\operatorname{Re} z \geq 0$,

$$\int \log |z-t|\, d\omega(t) = \int_{-1}^{0} \log \left| \frac{z}{\psi(z)} - t \right| d\nu(t) + \operatorname{Re}\left(\frac{\psi(z)^{\beta}-1}{\beta} \right).$$

(b) Since $\frac{\mu_n}{\mu_{n-1}} = \left(\frac{n}{\beta}\right)^{1/\beta}(1+o(1))$, and because the convergence is locally uniform in z, it follows that

$$Q_n^{\#}(z) = \left(-\frac{\mu_n}{\mu_{n-1}}\right)^n P_n\left(-\left(\frac{n}{\beta}\right)^{1/\beta} z\right),$$

has the same asymptotic representation as Q_n.

(c) We expect that the same asymptotic holds more generally for $z \in \mathbb{C} \setminus (-\infty, 0]$. We could not do this, because we could not show there is a unique critical point when $\operatorname{Re} z < 0$

We illustrate Theorem 1.3 with two examples from [6].

Example 1. Consider the polynomials P_n determined by the conditions

$$\int_0^{\infty} P_n(x) e^{-(j+\Delta)x^{\beta}} x^{\alpha} dx = 0, \quad 1 \leq j \leq n.$$

In the case $\beta = 1$, these polynomials were considered in [6]. The associated dilation polynomials are

$$R_n(x) = \prod_{j=1}^{n} \left(x + (j+\Delta)^{-1}\right).$$

Let

$$F(z) = \log(1+z) - z, \quad z \notin (-\infty, -1).$$

We see that

$$\frac{R_n(z)}{z^n} = \exp\left(\frac{1}{z} \sum_{j=1}^{n} \frac{1}{j+\Delta} + \sum_{j=1}^{n} F\left(\frac{1}{(j+\Delta)z}\right) \right). \tag{1.22}$$

Here

$$\begin{aligned} \sum_{j=1}^{n} \frac{1}{j+\Delta} &= \sum_{j=1}^{n} \frac{1}{j} - \Delta \sum_{j=1}^{n} \frac{1}{j(j+\Delta)} \\ &= \log n + \gamma - \Delta \sum_{j=1}^{\infty} \frac{1}{j(j+\Delta)} + o(1), \end{aligned}$$

where γ is Euler's constant. Let us set

$$\begin{aligned} &\lambda_n = \log n; \\ &D(z) = \frac{1}{z}; \\ &E(z) = \frac{1}{z}\left[\gamma - \Delta \sum_{j=1}^{\infty} \frac{1}{j(j+\Delta)}\right] + \sum_{j=1}^{\infty} F\left(\frac{1}{(j+\Delta)z}\right). \end{aligned}$$

As $F(v) = O\left(v^2\right), v \to 0$, the second series in the last line converges, uniformly for z in compact subsets of $\mathbb{C} \setminus (-\infty, 0]$. Also let

$$\nu = \delta_0,$$

a unit mass at 0. Then we see from (1.22) that

$$\begin{aligned} R_n(z) &= \exp\left(n \log z + \frac{\log n}{z} + E(z) + o(1) \right) \\ &= \exp\left(n \int \log(z-t)\, d\nu(t) + \lambda_n D(z) + E(z) + o(1) \right). \end{aligned}$$

Thus (1.14) holds. Moreover, Eq. (1.20) for $u = \psi(z)$ becomes

$$u^{\beta} = 1.$$

Thus

$$\psi(z) = 1 \quad \text{for all } z,$$

while from (1.17),

$$\Psi(z, \psi(z)) = \log z,$$

and from (1.18),

$$B(z, \psi(z)) = \beta.$$

Moreover,

$$H(\psi(z)) = \sqrt{\beta}.$$

So the asymptotic (1.21) for Q_n becomes,

$$Q_n(z) = z^n n^{\frac{1}{z}} e^{E(z)} (1 + o(1)),$$

at least for $\operatorname{Re} z \geq 0,\ z \neq 0$. The limiting zero distribution ω satisfies

$$\int \log|z - t|\, d\omega(t) = \log|z|,$$

suggesting that $\omega = \delta_0$. For $\beta = 1$, this was proved in [6].

Example 2. Let R_n denote the Legendre polynomial for $[-1, 0]$. By translating asymptotics for the Legendre polynomials [14, p. 194; p. 63] from $[-1, 1]$ to $[-1, 0]$, we see that uniformly for z in compact subsets of $\mathbb{C} \setminus [-1, 0]$,

$$R_n(z) = \phi(z)^n \exp(E(z)) (1 + o(1)),$$

where

$$\phi(z) = \frac{1}{4}\left(2z + 1 + \sqrt{(2z+1)^2 - 1}\right),$$

and

$$\exp(E(z)) = \sqrt{2}\left((2z+1)^2 - 1\right)^{-1/4} \phi(z)^{1/2}.$$

It is well known that 4ϕ is the conformal map of $\mathbb{C} \setminus [-1, 0]$ onto the exterior of the unit ball, and

$$\phi(z) = \exp\left(\int_{-1}^{0} \log(z - t) \frac{dt}{\pi\sqrt{t(1+t)}}\right),$$

so

$$d\nu(t) = \frac{dt}{\pi\sqrt{t(1+t)}}, \quad t \in (-1, 0).$$

Moreover, $u = \psi(z)$ is the root of

$$u^{\beta} = \frac{z}{u}\frac{\phi'}{\phi}\left(\frac{z}{u}\right) = \frac{2z}{u\sqrt{\left(2\frac{z}{u} + 1\right)^2 - 1}}.$$

We see then that (1.21) becomes

$$Q_n(z) = \frac{H(\psi(z))}{\sqrt{B(z, \psi(z))}} \exp\left(n\left[\log\phi\left(\frac{z}{\psi(z)}\right) + \frac{\psi(z)^{\beta} - 1}{\beta}\right] + E\left(\frac{z}{\psi(z)}\right) + o(1)\right).$$

Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t and polynomials P of degree at most n. We write $C = C(\lambda), C \neq C(\lambda)$ to indicate dependence on, or independence of, a parameter λ. The same symbol does not necessarily denote the same constant in different occurrences. We denote the polynomials of degree $\leq n$ by $\mathcal{P}_n$.

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2. In Section 4, we present some preliminaries for the proof of Theorem 1.3. In Section 5, we analyze the critical points of Ψ. We prove Theorem 1.3 in Section 6.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Throughout the proof, we write $\sigma_j = \sigma_{n,j},\ 1 \le j \le n$. We first prove the existence and uniqueness of

$$P_n(x) = \sum_{j=0}^{n} p_j x^j.$$

The defining relations (1.3) are easily rewritten as the linear system

$$\begin{bmatrix} 1 & \sigma_1^{-1} & \sigma_1^{-2} & \cdots & \sigma_1^{-n+1} \\ 1 & \sigma_2^{-1} & \sigma_2^{-2} & \cdots & \sigma_2^{-n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_n^{-1} & \sigma_n^{-2} & \cdots & \sigma_n^{-n+1} \end{bmatrix} \begin{bmatrix} p_0\mu_0 \\ p_1\mu_1 \\ \vdots \\ p_{n-1}\mu_{n-1} \end{bmatrix} = -\mu_n \begin{bmatrix} \sigma_1^{-n} \\ \sigma_2^{-n} \\ \vdots \\ \sigma_n^{-n} \end{bmatrix}, \tag{2.1}$$

recall that $p_n = 1$. The matrix on the left-hand side is a Vandermonde matrix, and all the $\{\sigma_{n,j}\}$ are distinct, so the matrix is non-singular. Hence the system has a unique solution for $\{p_j\}_{j=0}^{n-1}$.

(I) We see that

$$\int_0^\infty \left(\sum_{j=0}^{n} r_{n,j} \frac{\mu_n}{\mu_j} (-ty)^j \right) d\mu(t) = \mu_n R_n(-y),$$

and by definition of R_n, this vanishes when $y = \sigma_k^{-1}$, $1 \le k \le n$. Then uniqueness of P_n gives (1.5). The last equation also gives (1.6).

(II) Since μ has infinitely many points in its support in $[0, \infty)$, necessarily,

$$s = \liminf_{j\to\infty} \mu_j^{1/j} > 0.$$

Then G of (1.8) has radius of convergence $s > 0$. For $r < s$, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{|t|=r} \frac{R_n\left(-\frac{x}{t}\right)}{t} G(t)dt &= \sum_{j=0}^{n} r_{n,j} (-x)^j \frac{1}{2\pi i} \int_{|t|=r} \frac{G(t)}{t^{j+1}} dt \\ &= \sum_{j=0}^{n} r_{n,j} (-x)^j \mu_j^{-1} = \frac{(-1)^n}{\mu_n} P_n(x). \end{aligned}$$

(III) This follows by applying Cramer's rule to solve (2.1) for $\{p_j\}_{j=0}^{n-1}$. Alternatively, one can compute the integral

$$\int_0^\infty P_n\left(\frac{x}{\sigma_k}\right) d\mu(x)$$

by integrating the determinant in (1.9), and then observing that one obtains two identical rows.

(IV) By a substitution, we can recast the orthogonality relations (1.2) as

$$\int_0^\infty P_n\left(t^{1/\beta}\right) t^{\frac{\alpha+1}{\beta}-1} e^{-\sigma_k^\beta t} dt = 0, \quad 1 \le k \le n. \tag{2.2}$$

If P_n has less than n sign changes in $(0, \infty)$, then so also does $P_n\left(t^{1/\beta}\right)$. Since all $\{\sigma_k\}_{k=1}^n$ are distinct, $\left\{e^{-\sigma_k^\beta t}\right\}_{k=1}^n$ is a Chebyshev system, so we can find a linear combination $h(t)$ that has sign changes at precisely the less than n sign changes of $P_n\left(t^{1/\beta}\right)$ in $(0, \infty)$. Now (2.2) gives

$$0 = \int_0^\infty P_n\left(t^{1/\beta}\right) t^{\frac{\alpha+1}{\beta}-1} h(t)dt.$$

This is impossible, as the integrand has one sign in $(0, \infty)$ except at the sign changes of $P_n\left(t^{1/\beta}\right)$. $\square$

3. Proof of Theorem 1.2

In the sequel $\beta \ge 1$ and $\alpha > -1$, while $d\mu(t) = t^\alpha e^{-t^\beta} dt$. The moments are

$$\mu_n = \int_0^\infty t^{n+\alpha} e^{-t^\beta} dt = \frac{1}{\beta} \Gamma\left(\frac{n+\alpha+1}{\beta}\right). \tag{3.1}$$

It is possible to derive (1.11) using the results in Djrbashian's book [15], but it is actually easier to start with the standard Hankel loop integral for the reciprocal of the gamma function. Let $\rho > 0$. Then [16, p. 13], for $a > 0$,

$$\frac{1}{\Gamma(a)} = \frac{1}{2\pi i}\int_{\mathcal{C}} e^t t^{-a} dt.$$

Here $\mathcal{C}$ is a contour that starts at the lower edge of the negative real axis, cut from $-\infty$ to $-\rho$, then traverses the circle $t = \rho e^{i\theta}, -\pi < \theta < \pi$ anticlockwise, and then traverses the upper edge of the negative real axis, cut from $-\rho$ to $-\infty$.

In particular,

$$\frac{1}{\mu_j} = \frac{\beta}{\Gamma\left(\frac{j+\alpha+1}{\beta}\right)} = \frac{\beta}{2\pi i}\int_{\mathcal{C}} e^t t^{-\frac{j+\alpha+1}{\beta}} dt.$$

Then (1.5) gives

$$\begin{aligned} P_n(x) &= \frac{\beta(-1)^n \mu_n}{2\pi i}\int_{\mathcal{C}} e^t t^{-\frac{\alpha+1}{\beta}} \sum_{j=0}^{n} r_{n,j}\left(-\frac{x}{t^{1/\beta}}\right)^j dt \\ &= \frac{\beta(-1)^n \mu_n}{2\pi i}\int_{\mathcal{C}} e^t t^{-\frac{\alpha+1}{\beta}} R_n\left(-\frac{x}{t^{1/\beta}}\right) dt. \end{aligned}$$

We make the substitution $t = u^\beta$ in the last integral, giving

$$P_n(x) = \frac{\beta^2(-1)^n \mu_n}{2\pi i}\int_{\mathcal{C}_1} e^{u^\beta} u^{\beta-\alpha-1} R_n\left(-\frac{x}{u}\right) du.$$

Here $\mathcal{C}_1$ is the image of the contour $\mathcal{C}$ under the map $u = t^{1/\beta}$. Thus $\mathcal{C}_1$ consists of a circular arc $t = \rho^{1/\beta} e^{i\theta}, -\frac{\pi}{\beta} < \theta < \frac{\pi}{\beta}$, and rays from $\rho^{1/\beta} e^{\pm i\frac{\pi}{\beta}}$ to ∞, in the left-half plane. Now the integrand is analytic in $\mathbb{C} \setminus (-\infty, 0]$ as a function of u, while if $u = re^{i\theta}$,

$$\left|e^{u^\beta}\right| = e^{r^\beta \cos\beta\theta}.$$

This decays rapidly to 0 as $r \to \infty$, as long as $\theta \in \left(\frac{\pi}{2\beta}, \frac{\pi}{\beta}\right)$. It follows that we can deform the contour $\mathcal{C}_1$ into the contour Γ described in the statement of Theorem 1.2. $\square$

For Q_n of (1.16), we deduce:

Lemma 3.1.

$$Q_n(z) = \frac{\delta_n}{2\pi i}\int_\Gamma e^{\left(t\frac{\mu_n}{\mu_{n-1}}\right)^\beta} t^{\beta-\alpha-2} R_n\left(\frac{z}{t}\right) dt, \tag{3.2}$$

where

$$\delta_n = \beta^2\left(\frac{\mu_{n-1}}{\mu_n}\right)^{n-\beta+\alpha+1} \mu_n. \tag{3.3}$$

Proof. We make the substitutions $z \to -\frac{\mu_n}{\mu_{n-1}} z$ and $t \to t\frac{\mu_n}{\mu_{n-1}}$ and $s \to s/\frac{\mu_n}{\mu_{n-1}}$ in (1.11). Then by multiplying by $\left(-\frac{\mu_{n-1}}{\mu_n}\right)^n$, we obtain

$$Q_n(z) = \left[\left(\frac{\mu_{n-1}}{\mu_n}\right)^n \mu_n\right] \frac{\beta^2}{2\pi i}\int_\Gamma e^{\left(t\frac{\mu_n}{\mu_{n-1}}\right)^\beta} \left(t\frac{\mu_n}{\mu_{n-1}}\right)^{\beta-\alpha-2} R_n\left(\frac{z}{t}\right) \frac{\mu_n}{\mu_{n-1}} dt. \quad \square$$

4. Preliminary results

In the sequel, we let

$$\Omega_n(z) = \exp\left(\lambda_n D(z) + E(z)\right). \tag{4.1}$$

Recall that H was defined at (1.19).

Lemma 4.1.

(a)

$$\frac{\mu_n}{\mu_{n-1}} = \left(\frac{n}{\beta}\right)^{1/\beta} \left(1 + \frac{1}{n}\frac{1}{\beta}\left[\alpha + \frac{1-\beta}{2}\right] + O\left(n^{-2}\right)\right); \tag{4.2}$$

$$\left(\frac{\mu_n}{\mu_{n-1}}\right)^{\beta} = \frac{n}{\beta} + \frac{1}{\beta}\left[\alpha + \frac{1-\beta}{2}\right] + O\left(\frac{1}{n}\right). \tag{4.3}$$

(b) *Let δ_n be defined by (3.3). Then*

$$\delta_n = \sqrt{2\pi\beta n} e^{-\frac{n}{\beta}} \exp\left(-\frac{1}{\beta}\left[\alpha + \frac{1-\beta}{2}\right]\right)(1 + o(1)). \tag{4.4}$$

(c) *Uniformly for t in compact subsets of $\mathbb{C}\setminus(-\infty, 0]$,*

$$\delta_n e^{\left(t\frac{\mu_n}{\mu_{n-1}}\right)^{\beta}} t^{\beta-\alpha-1} = \sqrt{2\pi n}\exp\left(\frac{n\left(t^{\beta}-1\right)}{\beta} + o(1)\right) H(t). \tag{4.5}$$

Proof.

(a) Using (3.1) and the asymptotic [17, p. 257, 6.1.47]

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)} = 1 + \frac{(a-b)(a+b-1)}{2z} + O\left(\frac{1}{z^2}\right),$$

we see that

$$\frac{\mu_n}{\mu_{n-1}} = \left(\frac{n}{\beta}\right)^{1/\beta}\left(1 + \frac{\frac{2\alpha+1}{\beta} - 1}{2n} + O\left(n^{-2}\right)\right),$$

$$\left(\frac{\mu_n}{\mu_{n-1}}\right)^{\beta} = \frac{n}{\beta} + \frac{1}{\beta}\left[\alpha + \frac{1-\beta}{2}\right] + O\left(\frac{1}{n}\right).$$

(b) This follows from (4.3), and Stirling's formula. We omit the lengthy, but straightforward, calculation.

(c) By (a) and (b) of the lemma,

$$\begin{aligned}\delta_n e^{\left(t\frac{\mu_n}{\mu_{n-1}}\right)^{\beta}} t^{\beta-\alpha-1} &= \sqrt{2\pi\beta n} t^{\beta-\alpha-1}\exp\left(t^{\beta}\left[\frac{n}{\beta} + \frac{1}{\beta}\left[\alpha + \frac{1-\beta}{2}\right] + O\left(\frac{1}{n}\right)\right]\right.\\ &\qquad \left. - \frac{1}{\beta}\left[\alpha + \frac{1-\beta}{2}\right] - \frac{n}{\beta} + o(1)\right)\\ &= \sqrt{2\pi n}\exp\left(\frac{n\left(t^{\beta}-1\right)}{\beta} + o(1)\right)H(t). \quad \square\end{aligned}$$

Recall that $\Psi(z, u)$ was defined by (1.17), while Ω_n was defined at (4.1). Moreover, the contours $\Gamma_s, \Gamma_-, \Gamma_+$ were defined in Theorem 1.2. We now write Q_n as a sum of three terms, of which the first will be the main one.

Lemma 4.2. *Let $z \in \mathbb{C}\setminus[-1, 0]$, $m \geq 1$, and $s > 0$. Let $R \geq |z|$, and B_R denote the ball center 0, radius R. Then*

$$Q_n(z) = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3, \tag{4.6}$$

where

$$\mathrm{I}_1 = \frac{\sqrt{2\pi n}}{2\pi i}\int_{\Gamma_s} e^{n\Psi(z,t)}\Omega_n\left(\frac{z}{t}\right)H(t)(1 + o(1))\frac{dt}{t}; \tag{4.7}$$

$$\mathrm{I}_2 = \frac{\sqrt{2\pi n}}{2\pi i}\int_{(\Gamma_+\cup\Gamma_-)\cap B_R} e^{n\Psi(z,t)}\Omega_n\left(\frac{z}{t}\right)H(t)(1 + o(1))\frac{dt}{t}; \tag{4.8}$$

$$|\mathrm{I}_3| \leq C_1\sqrt{n}\left(2/e^{1/\beta}\right)^n e^{-C_2 nR^{\beta}}. \tag{4.9}$$

The estimates hold uniformly for z in compact sets of $\mathbb{C}\setminus[-1, 0]$. C_1 and C_2 are independent of n and R.

Proof. We use the integral formula in Lemma 3.1:

$$Q_n(z) = \frac{\delta_n}{2\pi i} \int_{\Gamma_s \cup \Gamma_+ \cup \Gamma_-} e^{\left(t \frac{\mu_n}{\mu_{n-1}}\right)^\beta} t^{\beta-\alpha-2} R_n\left(\frac{z}{t}\right) dt.$$

Here by our hypothesis (1.14) on R_n, and by Lemma 4.1,

$$\delta_n R_n\left(\frac{z}{t}\right) e^{\left(t \frac{\mu_n}{\mu_{n-1}}\right)^\beta} t^{\beta-\alpha-1} = \sqrt{2\pi n} \exp\left(n\Psi\left(z,t\right) + o(1)\right) \Omega_n\left(\frac{z}{t}\right) H(t),$$

uniformly for t in compact subsets of $|t| \geq s$, and z in a compact set. Then (4.6) follows, with

$$I_3 = \frac{\delta_n}{2\pi i} \int_{(\Gamma_+ \cup \Gamma_-)\setminus B_R} e^{\left(t \frac{\mu_n}{\mu_{n-1}}\right)^\beta} t^{\beta-\alpha-2} R_n\left(\frac{z}{t}\right) dt.$$

Here as all zeros of R_n lie in $[-1, 0]$, for $|t| \geq R \geq |z|$, we have

$$\left|R_n\left(\frac{z}{t}\right)\right| \leq \left(1 + \frac{|z|}{|t|}\right)^n \leq 2^n.$$

Then

$$\begin{aligned} |I_3| &\leq C\delta_n 2^n \int_R^\infty e^{-\left(\frac{\mu_n}{\mu_{n-1}} r\right)^\beta |\cos(\beta\eta)|} r^{\beta-\alpha-2} dr \\ &\leq C_1 \delta_n 2^n e^{-C_2 n R^\beta}, \end{aligned}$$

by (4.2) and some straightforward estimation. Here C_1 and C_2 do depend on η, but not on R or n, and may be taken to be the same for z in a compact set. Finally apply (4.4) for δ_n. □

5. The critical points of Ψ

In order to apply the method of steepest descent, we need to study the critical points of Ψ. We prove the following.

Theorem 5.1. *For each $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$, $z \neq 0$, there exists a unique $u = \psi(z)$ such that $|\arg(\psi(z))| < \frac{\pi}{2\beta}$; $|\psi(z)| < 1$, and*

$$\frac{\partial}{\partial u}\Psi(z, u) = 0. \tag{5.1}$$

ψ is an analytic function of z. Moreover, if $\mathrm{Im}(z) \geq 0$, $\arg(\psi(z)) \in [0, \frac{\pi}{2\beta})$.

Observe from (1.17) that

$$\frac{\partial}{\partial u}\Psi(z, u) = \frac{1}{u}\left[-\int_{-1}^0 \frac{1}{1 - t\frac{u}{z}} d\nu(t) + u^\beta\right]. \tag{5.2}$$

We shall make the substitution

$$v = \frac{1}{z}$$

and analyze

$$F(v, u) = -\int_{-1}^0 \frac{1}{1 - tuv} d\nu(t) + u^\beta. \tag{5.3}$$

We shall prove the result by first showing that for $\mathrm{Re}(v) \geq 0 > \mathrm{Im}(v)$, $F(v, \cdot)$ maps the boundary of the sector of the unit ball,

$$D_+ = \left\{re^{i\theta} : r \in (0, 1), 0 < \theta < \frac{\pi}{2\beta}\right\}, \tag{5.4}$$

onto a curve enclosing 0. We let $\gamma = \partial D_+$ and $F \circ \gamma$ denote the image set $\{F(v, u) : u \in \gamma\}$, for a given fixed v satisfying $\mathrm{Re}(v) \geq 0 > \mathrm{Im}(v)$. Observe that for such v, $F(v, \cdot)$ is a single valued analytic function for $u \in \mathbb{C} \setminus (-\infty, 0]$. Indeed, as $\arg(v) \in [-\frac{\pi}{2}, 0)$, while for $u \in \gamma$, $\arg(u) \in \left[0, \frac{\pi}{2\beta}\right] \subseteq \left[0, \frac{\pi}{2}\right]$, we have $\arg(uv) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then for $t \in [-1, 0]$,

$$\mathrm{Re}(1 - tuv) = 1 + |t|\,\mathrm{Re}(uv) \neq 0.$$

It follows that $F \circ \gamma$ is a piecewise smooth closed curve. We define the half open quadrants as follows:

$$\mathcal{Q}_1 = \{w : \mathrm{Re}(w) \geq 0, \mathrm{Im}\,(w) > 0\};$$
$$\mathcal{Q}_2 = \{w : \mathrm{Re}(w) \leq 0, \mathrm{Im}\,(w) > 0\};$$
$$\mathcal{Q}_3 = \{w : \mathrm{Re}(w) \leq 0, \mathrm{Im}\,(w) < 0\};$$
$$\mathcal{Q}_4 = \{w : \mathrm{Re}(w) \geq 0, \mathrm{Im}\,(w) < 0\}.$$

The interior of $\mathcal{Q}_j$ is $\mathcal{Q}_j^0$. We also let

$$D_- = \left\{re^{i\theta} : r \in (0,1), 0 > \theta > -\frac{\pi}{2\beta}\right\}. \tag{5.5}$$

Our main lemma is the following.

Lemma 5.2.

(a) *Let $v \in \mathcal{Q}_4$. Then there exists a unique $\varphi(v) \in D_+ \cup \{1\}$ such that*

$$F(v, \varphi(v)) = 0. \tag{5.6}$$

Moreover, φ is an analytic function of $v \in \mathcal{Q}_4$.

(b) *Let $v \in \mathcal{Q}_1$. Then there exists a unique $\varphi(v) \in D_- \cup \{1\}$ such that (5.6) holds. Moreover, φ is an analytic function of $v \in \mathcal{Q}_1$.*

Proof.

(a) Let us first deal with the "trivial" case where ν is a unit mass at 0. Then we see that

$$F(v, u) = -1 + u^\beta,$$

and we can choose $u = 1$, that is $u = \varphi(v) = 1$ for all v. In the sequel, we assume that ν is not a unit mass at 0. □

We let $\gamma_1 = [0, 1]$; $\gamma_2 = \left\{e^{i\theta} : \theta \in \left[0, \frac{\pi}{2\beta}\right]\right\}$; $\gamma_3 = \left\{xe^{i\frac{\pi}{2\beta}} : x \in [0,1]\right\}$ denote the three arcs of γ. Write

$$v = \rho e^{i\sigma} \quad \text{and} \quad u = re^{i\theta},$$

where $\sigma \in [-\frac{\pi}{2}, 0)$ and $\theta \in \left[0, \frac{\pi}{2\beta}\right]$. Now if $u \neq 0$, then $r > 0$, so $\mathrm{Re}(uv) = \rho r \cos(\theta + \sigma) \geq 0$, with strict inequality unless $\sigma = -\frac{\pi}{2}$ and $\theta = 0$. In that exceptional case, $\mathrm{Im}(uv) = \rho r \sin\left(-\frac{\pi}{2}\right) \neq 0$. Then

$$\begin{aligned}\left|\int_{-1}^{0} \frac{1}{1 - tuv} d\nu(t)\right| &\leq \int_{-1}^{0} \frac{1}{|1 - tuv|} d\nu\,(t) \\ &\leq \int_{-1}^{0} \frac{1}{\left[(1 + |t|\,\mathrm{Re}(uv))^2 + (|t|\,\mathrm{Im}(uv))^2\right]^{1/2}} d\nu(t) < 1,\end{aligned} \tag{5.7}$$

as ν is not a unit mass at 0. We shall use (5.7) repeatedly. Next, we consider the location of the curves $F(v, \gamma_j)$, $j = 1, 2, 3$.

Step 1 $F(v, \gamma_1)$.

Now

$$F(v, 0) = -\int_{-1}^{0} d\nu(t) = -1.$$

Moreover, by (5.7),

$$\mathrm{Re}\, F(v, 1) = 1 - \mathrm{Re} \int_{-1}^{0} \frac{d\nu\,(t)}{1 - tv} > 0.$$

Also, for $u \in (0, 1]$,

$$\mathrm{Im}\, F(v, u) = \int_{-1}^{0} \frac{u\,|t|\,\mathrm{Im}(v)}{|1 + |t|\,uv|^2} d\nu(t) < 0.$$

It follows that $F(v, \gamma_1) = \{F(v, u) : u \in [0, 1]\}$ is a path in the lower half-plane, starting at $F(v, 0) = -1$ and ending at a point $F(v, 1)$ in $\mathcal{Q}_4^0$.

Step 2 $F(v, \gamma_2)$.

Here, using (5.7),

$$\left|F\left(v, e^{i\theta}\right)\right| = \left|e^{i\beta\theta} - \int_{-1}^{0} \frac{d\nu(t)}{1-tuv}\right| \\ \geq 1 - \left|\int_{-1}^{0} \frac{d\nu(t)}{1-tuv}\right| > \varepsilon > 0,$$

where ε is independent of $\theta \in \left[0, \frac{\pi}{2\beta}\right]$. As $e^{i\beta\theta}$ lies on the unit circle and is in $\mathcal{Q}_1$, it follows that $F\left(v, e^{i\theta}\right)$ is a point in the quadrants $\mathcal{Q}_1$, $\mathcal{Q}_2$, or $\mathcal{Q}_4$, with modulus at least ε. In particular, it cannot lie in $\mathcal{Q}_3$.

Moreover,

$$F\left(v, e^{i\frac{\pi}{2\beta}}\right) = i - \int_{-1}^{0} \frac{d\nu\,(t)}{1 - e^{i\frac{\pi}{2\beta}} vt}$$

has

$$\operatorname{Im} F\left(v, e^{i\frac{\pi}{2\beta}}\right) \geq 1 - \left|\int_{-1}^{0} \frac{d\nu(t)}{1 - e^{i\frac{\pi}{2\beta}} vt}\right| > 0;$$

$$\operatorname{Re} F\left(v, e^{i\frac{\pi}{2\beta}}\right) = -\int_{-1}^{0} \frac{1 + |t|\,\rho\cos\left(\frac{\pi}{2\beta} + \sigma\right)}{\left|1 - e^{i\frac{\pi}{2\beta}} vt\right|^2} d\nu(t) < 0.$$

Thus $F(v, \gamma_2)$ is a path from $F\,(v, 1)$ in $\mathcal{Q}_4^0$ to $F\left(v, e^{i\frac{\pi}{2\beta}}\right)$ in $\mathcal{Q}_2^0$, that does not intersect $\mathcal{Q}_3$, nor the ball center 0, radius ε.

Step 3 $F(v, \gamma_3)$.

Here for $u = xe^{i\frac{\pi}{2\beta}} \in \gamma_3,\ x \in [0, 1]$, we have

$$F\,(v, u) = x^{\beta} i - \int_{-1}^{0} \frac{d\nu(t)}{1 - xe^{i\frac{\pi}{2\beta}} vt}$$

so

$$\operatorname{Re} F\,(v, u) = -\int_{-1}^{0} \frac{1 + |t|\,x\rho\cos\left(\frac{\pi}{2\beta} + \sigma\right)}{\left|1 - xe^{i\frac{\pi}{2\beta}} vt\right|^2} d\nu(t) < 0.$$

Thus $F\,(v, u)$ traces a path in the left-half plane from $F\left(v, e^{i\frac{\pi}{2\beta}}\right)$ in $\mathcal{Q}_2^0$ to $F\,(v, 0) = -1$.

In summary, as we traverse γ counterclockwise, $F\,(v, u)$ traces a path

(i) through the lower half-plane, starting at $F\,(v, 0) = -1$ and ending at a point $F\,(v, 1)$ in $\mathcal{Q}_4^0$;

(ii) then from $F\,(v, 1)$ to $F\left(v, e^{i\frac{\pi}{2\beta}}\right)$ in $\mathcal{Q}_2^0$, not intersecting $\mathcal{Q}_3$, nor the ball center 0, radius ε;

(iii) then in the open left-half plane, from $F\left(v, e^{i\frac{\pi}{2\beta}}\right)$ to $F\,(v, 0) = -1$.

It follows that $F(v, \gamma)$ encloses 0 in its interior, so that the winding number about 0 is at least 1. It cannot be more than 1—otherwise we would obtain contradictions to (i), (ii), or (iii). So

$$\frac{1}{2\pi i} \int_{F(v,\gamma)} \frac{dt}{t} = 1.$$

The substitution $t = F\,(v, u)$ leads to

$$\frac{1}{2\pi i} \int_{\gamma} \frac{\frac{\partial}{\partial u} F\,(v, u)}{F\,(v, u)} du = 1.$$

Since $F\,(v, \cdot)$ is analytic inside γ, and continuous on γ, it follows that $F\,(v, \cdot)$ has a simple zero inside γ. That is, there is a unique $u = \varphi(v)$ satisfying (5.6). The uniqueness of the solution, and the local mapping theorem (or implicit function theorem) then yield local, and hence global, analyticity of φ.

(b) This follows from the identity

$$F\,(\bar{v}, \bar{u}) = \overline{F\,(v, u)}. \quad \square$$

Proof of Theorem 5.1. Thus far, we have shown that for $v \in \mathcal{Q}_1 \cup \mathcal{Q}_4$, there exists a unique $\varphi(v)$ satisfying (5.6). We must still treat the case where $v \in (0, \infty)$. For such v, we see that

$$F(v, 0) = -1; \qquad F(v, 1) = 1 - \int_{-1}^{0} \frac{d\nu(t)}{1 + |t|\, v} \geq 0.$$

We have strict inequality in the second inequality unless ν is a unit mass at 0. It follows that $F(v, u) = 0$ has a root $u \in (0, 1]$. Moreover,

$$\frac{\partial F(v, u)}{\partial u} = \beta u^{\beta - 1} + \int_{-1}^{0} \frac{v\,|t|}{(1 - uvt)^2} d\nu(t) > 0,$$

so the root u of $F(v, u) = 0$ is unique. Note that as $F(v, u)$ will be non-real for non-real u, the root will also be unique when u ranges over $\overline{D_+ \cup D_-}$.

Now recall that we set $v = \frac{1}{z}$. Since v maps the open right-half z-plane conformally onto itself, we can set

$$\psi(z) = \varphi(v) = \varphi\left(\frac{1}{z}\right),$$

and obtain a unique root $u = \psi(z)$ of the equation

$$\frac{\partial \Psi(u, z)}{\partial u}_{|u = \psi(z)} = 0.$$

The analyticity of ψ follows from that of φ. Here if $\mathrm{Im}\, z \geq 0$, $\mathrm{Im}\, v \leq 0$, so $\arg \psi(z) = \arg \varphi(v) \in [0, \frac{\pi}{2\beta})$, by Lemma 5.2(a). □

6. Proof of Theorem 1.3

We can prove the asymptotic for $\mathrm{Im}\, z \geq 0$, since P_n has real zeros. Thus we assume that

$$z = \rho e^{i\sigma}, \quad \text{with } \sigma \in \left[0, \frac{\pi}{2}\right] \text{ and } \rho > 0.$$

Then, writing

$$\psi(z) = s e^{i\theta_0},$$

we have $s \in (0, 1]$, and

$$\theta_0 \in \left[0, \frac{\pi}{2\beta}\right).$$

Recall that in Lemma 4.2, we split Q_n as a sum of three terms. The main contribution will come from I_1. We now further divide

$$\mathrm{I}_1 = \mathrm{I}_{11} + \mathrm{I}_{12} + \mathrm{I}_{13}, \tag{6.1}$$

where

$$\mathrm{I}_{11} = \frac{\sqrt{2\pi n}}{2\pi} \int_{\theta_0 - n^{-\frac{1}{3}-\varepsilon}}^{\theta_0 + n^{-\frac{1}{3}-\varepsilon}} e^{n\Psi(z, se^{i\theta})} \Omega_n\left(\frac{z}{se^{i\theta}}\right) H\left(se^{i\theta}\right)(1 + o(1))\, d\theta; \tag{6.2}$$

$$\mathrm{I}_{12} = \frac{\sqrt{2\pi n}}{2\pi} \int_{\mathcal{J} \setminus \left[\theta_0 - n^{-\frac{1}{3}-\varepsilon}, \theta_0 + n^{-\frac{1}{3}-\varepsilon}\right]} e^{n\Psi(z, se^{i\theta})} \Omega_n\left(\frac{z}{se^{i\theta}}\right) H\left(se^{i\theta}\right)(1 + o(1))\, d\theta; \tag{6.3}$$

$$\mathrm{I}_{13} = \frac{\sqrt{2\pi n}}{2\pi} \int_{[-\eta, \eta] \setminus \mathcal{J}} e^{n\Psi(z, se^{i\theta})} \Omega_n\left(\frac{z}{se^{i\theta}}\right) H\left(se^{i\theta}\right)(1 + o(1))\, d\theta. \tag{6.4}$$

Here

$$0 < \varepsilon < \frac{1}{6}$$

and

$$\mathcal{J} = \left[\max\left\{-\frac{\pi}{2\beta}, \sigma - \frac{\pi}{2}\right\}, \eta\right]. \tag{6.5}$$

The parameter η satisfies (1.10), but will be fixed below to be close enough to $\frac{\pi}{2\beta}$. We start with the central term.

Lemma 6.1. *Let* $0 < \varepsilon < \frac{1}{6}$. *Then*

$$\mathrm{I}_{11} = \frac{H(\psi(z))}{\sqrt{B(z, \psi(z))}} e^{n\Psi(z,\psi(z))} \Omega_n\left(\frac{z}{\psi(z)}\right)(1 + o(1)). \tag{6.6}$$

Proof. Let us fix $z = \rho e^{i\sigma}$ and abbreviate $\Psi(u) = \Psi(z, u)$ in this proof. Recall that $\Psi'\left(se^{i\theta_0}\right) = 0$. By a Taylor series expansion, for $|\theta - \theta_0| \le n^{-1/3-\varepsilon}$,

$$\begin{aligned} n\Psi\left(se^{i\theta}\right) &= n\Psi\left(se^{i\theta_0}\right) + \frac{1}{2}\left(se^{i\theta} - se^{i\theta_0}\right)^2 n\Psi''\left(se^{i\theta_0}\right) + O\left(n\left|se^{i\theta} - se^{i\theta_0}\right|^3\right) \\ &= n\Psi\left(se^{i\theta_0}\right) + \frac{1}{2}\left(se^{i\theta} - se^{i\theta_0}\right)^2 n\Psi''\left(se^{i\theta_0}\right) + O\left(n^{-3\varepsilon}\right). \end{aligned} \tag{6.7}$$

The order term may be taken uniform for z in compact sets. Now let

$$\begin{aligned} f(\theta) &= \mathrm{Re}\,\Psi\left(se^{i\theta}\right) \\ &= \int_{-1}^{0} \log\left|\frac{z}{se^{i\theta}} - t\right| d\nu(t) + \frac{s^\beta \cos\beta\theta - 1}{\beta}. \end{aligned} \tag{6.8}$$

We see that

$$f'(\theta) = \frac{\rho}{s}\sin(\sigma - \theta)\int_{-1}^{0} \frac{|t|}{\left|\frac{z}{se^{i\theta}} - t\right|^2} d\nu(t) - s^\beta \sin\beta\theta;$$

$$f''(\theta) = -\frac{\rho}{s}\cos(\sigma - \theta)\int_{-1}^{0} \frac{|t|}{\left|\frac{z}{se^{i\theta}} - t\right|^2} d\nu(t) - 2\left(\frac{\rho}{s}\sin(\sigma - \theta)\right)^2 \int_{-1}^{0} \frac{|t|^2}{\left|\frac{z}{se^{i\theta}} - t\right|^4} d\nu(t) - \beta s^\beta \cos\beta\theta.$$

Then

$$f''(\theta) < 0 \quad \text{for } |\theta| \le \frac{\pi}{2\beta} \text{ and } |\sigma - \theta| \le \frac{\pi}{2}. \tag{6.9}$$

In particular, as $0 \le \theta_0 < \frac{\pi}{2\beta}$, we have

$$f''(\theta_0) < 0. \tag{6.10}$$

Note too that

$$\begin{aligned} f'(\theta) &= -\mathrm{Im}\left(\Psi'\left(se^{i\theta}\right)se^{i\theta}\right); \\ f''(\theta) &= -\mathrm{Re}\left(\Psi''\left(se^{i\theta}\right)\left(se^{i\theta}\right)^2 + \Psi'\left(se^{i\theta}\right)se^{i\theta}\right) \end{aligned}$$

so in particular, recalling that $\psi(z) = se^{i\theta_0}$,

$$0 > f''(\theta_0) = -\mathrm{Re}\left(\Psi''\left(se^{i\theta_0}\right)\left(se^{i\theta_0}\right)^2\right) = -\mathrm{Re}\left(B(z, \psi(z))\right). \tag{6.11}$$

Here $B(z, u)$ is given by (1.18). For $|\theta - \theta_0| \le n^{-\frac{1}{3}-\varepsilon}$, we have

$$\begin{aligned} \frac{1}{2}\left(se^{i\theta} - se^{i\theta_0}\right)^2 n\Psi''\left(se^{i\theta_0}\right) &= -\frac{n}{2}\left(se^{i\theta_0}\right)^2 \Psi''\left(se^{i\theta_0}\right)(\theta - \theta_0)^2 + O\left(n|\theta - \theta_0|^3\right) \\ &= -\frac{n}{2}B(z, \psi(z))(\theta - \theta_0)^2 + O\left(n^{-3\varepsilon}\right). \end{aligned} \tag{6.12}$$

Next, recalling that Ω_n was defined at (4.1),

$$\begin{aligned} \Omega_n\left(\frac{z}{se^{i\theta}}\right) &= \exp\left(\lambda_n D\left(\frac{z}{se^{i\theta}}\right) + E\left(\frac{z}{se^{i\theta}}\right)\right) \\ &= \exp\left(\lambda_n D\left(\frac{z}{se^{i\theta_0}}\right) + O\left(\lambda_n n^{-\frac{1}{3}-\varepsilon}\right) + E\left(\frac{z}{se^{i\theta_0}}\right) + O\left(n^{-\frac{1}{3}-\varepsilon}\right)\right) \\ &= \Omega_n\left(\frac{z}{se^{i\theta_0}}\right)(1 + O(n^{-1/3})). \end{aligned}$$

Moreover, recalling the form (1.19) of H,

$$H\left(se^{i\theta}\right) = H\left(se^{i\theta_0}\right)\left(1 + O\left(n^{-1/3}\right)\right).$$

Combining the last two relations, together with (6.7) and (6.12), we see that

$$\int_{\theta_0 - n^{-\frac{1}{3}-\varepsilon}}^{\theta_0 + n^{-\frac{1}{3}-\varepsilon}} e^{n\Psi\left(z, se^{i\theta}\right)} \Omega_n\left(\frac{z}{se^{i\theta}}\right) H\left(se^{i\theta}\right) d\theta$$
$$= e^{n\Psi(z,\psi(z))} \Omega_n\left(\frac{z}{\psi(z)}\right) H(\psi(z)) \int_{\theta_0 - n^{-\frac{1}{3}-\varepsilon}}^{\theta_0 + n^{-\frac{1}{3}-\varepsilon}} e^{-\frac{n}{2}B(z,\psi(z))(\theta-\theta_0)^2} (1 + o(1))\, d\theta.$$

The crucial point here is that $\text{Re}\, B(z, \psi(z)) > 0$, so that by rotating the line segment,

$$\int_{\theta_0 - n^{-\frac{1}{3}-\varepsilon}}^{\theta_0 + n^{-\frac{1}{3}-\varepsilon}} e^{-\frac{n}{2}B(z,\psi(z))(\theta-\theta_0)^2} (1 + o(1))\, d\theta = \sqrt{\frac{2}{nB(z, \psi(z))}} \int_{-\infty}^{\infty} e^{-y^2} dy + o\left(\frac{1}{\sqrt{n}}\right)$$
$$= \sqrt{\frac{2\pi}{nB(z, \psi(z))}} (1 + o(1)).$$

Then we obtain (6.6). □

Next, we estimate the tail in the main integral, using the notation defined in the previous proof:

Lemma 6.2.

$$|I_{12}| \le C_1 e^{n\text{Re}\,\Psi(z,\psi(z))} \exp\left(-C_2 n^{1/3-2\varepsilon}\right). \tag{6.13}$$

Proof. We use the notation of the previous proof. We see that for some constant C_1 independent of n, θ,

$$\left|\Omega_n\left(\frac{z}{se^{i\theta}}\right)\right| \le \exp(C_1\lambda_n).$$

Then from (6.3),

$$|I_{12}| \le C_2\sqrt{n}\exp(C_1\lambda_n) \int_{\mathcal{J}\setminus\left[\theta_0 - n^{-\frac{1}{3}-\varepsilon}, \theta_0 + n^{-\frac{1}{3}-\varepsilon}\right]} e^{nf(\theta)} d\theta.$$

Here, from (6.9), $f'(\theta)$ is decreasing for $|\theta| \le \frac{\pi}{2\beta}$ and $|\sigma - \theta| \le \frac{\pi}{2}$, so $f'(\theta)$ is decreasing in the interval $\left[\max\left\{-\frac{\pi}{2\beta}, \sigma - \frac{\pi}{2}\right\}, \frac{\pi}{2\beta}\right]$. (Recall that $\sigma \in \left[0, \frac{\pi}{2}\right]$.) Note that if $\eta > \frac{\pi}{2\beta}$ is close enough to $\frac{\pi}{2\beta}$, we have

$$f(\eta) < f\left(\theta_0 + \frac{1}{2}\left(\frac{\pi}{2\beta} - \theta_0\right)\right),$$

for the right-hand side is greater than $f\left(\frac{\pi}{2\beta}\right)$. It is also less than $f(\theta_0)$. Choose such an η. For $\theta \in \left[\theta_0 + n^{-\frac{1}{3}-\varepsilon}, \eta\right]$, we have from (6.7), at least for large enough n,

$$\begin{aligned} nf(\theta) &\le nf\left(\theta_0 + n^{-\frac{1}{3}-\varepsilon}\right) \\ &\le nf(\theta_0) - \frac{n}{2}\text{Re}\left(B(\psi(z))\right)\left(n^{-\frac{1}{3}-\varepsilon}\right)^2 + O\left(n^{-3\varepsilon}\right) \\ &\le nf(\theta_0) - \frac{\text{Re}\left(B(\psi(z))\right)}{2} n^{1/3-2\varepsilon} \end{aligned}$$

provided $\varepsilon < \frac{1}{6}$, as we assumed. A similar estimate holds for $\theta \in \left[\max\left\{-\frac{\pi}{2\beta}, \sigma - \frac{\pi}{2}\right\}, \theta_0 - n^{-\frac{1}{3}-\varepsilon}\right]$. Then

$$|I_{12}| \le C_2\sqrt{n}\exp(C_1\lambda_n)\, e^{n\text{Re}\,\Psi(z,\psi(z))} \exp\left(-\frac{\text{Re}\left(B(\psi(z))\right)}{2} n^{1/3-2\varepsilon}\right).$$

Because of our growth assumption (1.15) on λ_n, the stated estimate follows. □

Now we deal with I_{13} given by (6.4):

Lemma 6.3.

$$|I_{13}| \le C_1 e^{n\mathrm{Re}\ \Psi(z,\psi(z))} \exp\left(-C_2 n^{1/3-2\varepsilon}\right). \tag{6.14}$$

Proof. Let us first assume that ν is not a unit mass at 0. Observe that $[-\eta, \eta] \setminus \mathcal{J} \subset (-\infty, 0)$. For $\theta \in (0, \eta]$, a simple calculation shows that

$$\begin{aligned} f(\theta) - f(-\theta) &= \int_{-1}^{0} \log \left| \frac{\frac{\rho}{s} e^{i(\sigma-\theta)} - t}{\frac{\rho}{s} e^{i(\sigma+\theta)} - t} \right| d\nu(t) \\ &= \frac{1}{2} \int_{-1}^{0} \log \left(1 + \frac{4\frac{\rho}{s} |t| (\sin\sigma)(\sin\theta)}{\left|\frac{\rho}{s} e^{i(\sigma+\theta)} - t\right|^2} \right) d\nu(t) \\ &> 0. \end{aligned}$$

Since $[-\eta, \eta] \setminus \mathcal{J}$ omits 0, we see there exists $\Delta > 0$ such that for $\theta \ge 0$ with $-\theta \in [-\eta, \eta] \setminus \mathcal{J}$,

$$f(\theta) - f(-\theta) \ge \Delta.$$

Then straightforward estimation gives

$$\begin{aligned} |I_{13}| &\le C_1 \sqrt{n} e^{C_2 \lambda_n - n\Delta} \int_0^{\eta} e^{nf(\theta)} d\theta \\ &\le C_1 \sqrt{n} e^{C_2 \lambda_n - n\Delta} \int_0^{\eta} e^{n\mathrm{Re}\ \Psi\left(z, se^{i\theta}\right)} d\theta \end{aligned}$$

and (6.14) follows in a stronger form, from the previous lemmas. Finally, if $\nu = \delta_0$, a unit mass at 0, then

$$\Psi(z, u) = \log \frac{z}{u} + \frac{u^{\beta} - 1}{\beta},$$

so

$$\frac{\partial \Psi(z, u)}{\partial u} = -\frac{1}{u} + u^{\beta-1}$$

and this vanishes when $u = 1$. So in this case $u = \psi(z) = 1$, and $\theta_0 = 0$. We then have symmetry of the integrals about the real line, and I_{13} can be estimated by a constant multiple of $|I_{12}|$. □

Lemma 6.4. *For some $C_1, C_2 > 0$, I_2 of (4.8) admits the estimate*

$$|I_2| \le C_1 e^{n\mathrm{Re}\ \Psi(z,\psi(z))} \exp(-C_2 n). \tag{6.15}$$

Proof. We split I_2 of (4.8) into integrals over the upper and lower half rays:

$$I_{2\pm} = \frac{\sqrt{2\pi n}}{2\pi i} \int_{\Gamma_\pm \cap B_R} e^{n\Psi(z,t)} \Omega_n\left(\frac{z}{t}\right) H(t)(1 + o(1)) \frac{dt}{t}. \tag{6.16}$$

Let us suppose first that

$$|\sigma - \eta| \le \frac{\pi}{2}. \tag{6.17}$$

(We shall discuss this condition later.) Then for $r \ge s$, $\frac{z}{re^{i\eta}} = \frac{\rho}{r} e^{i(\sigma-\eta)}$ lies in the right-half plane, so that $\left|\frac{z}{re^{i\eta}} - t\right|$ decreases as r increases for $t \in [-1, 0]$. Hence

$$\begin{aligned} \int_{-1}^{0} \log \left| \frac{z}{re^{i\eta}} - t \right| d\nu(t) &\le \int_{-1}^{0} \log \left| \frac{z}{se^{i\eta}} - t \right| d\nu(t) \\ &= \mathrm{Re}\ \Psi\left(z, se^{i\eta}\right) - \frac{s^{\beta} \cos\beta\eta - 1}{\beta}, \end{aligned}$$

so

$$\mathrm{Re}\ \Psi\left(z, re^{i\eta}\right) \le \mathrm{Re}\ \Psi\left(z, se^{i\eta}\right) + \left(r^{\beta} - s^{\beta}\right) \frac{\cos\beta\eta}{\beta}.$$

Thus

$$
\begin{aligned}
|\mathrm{I}_{2+}| &\le \frac{\sqrt{2\pi n}}{2\pi} \int_{\Gamma_+} \left| e^{n\Psi(z,t)} \Omega_n\left(\frac{z}{t}\right) H(t) \right| (1+o(1)) \frac{|dt|}{|t|} \\
&\le \frac{\sqrt{2\pi n}}{2\pi} \left| e^{n\Psi\left(z,se^{i\eta}\right)} \right| \int_s^\infty e^{-n\left(r^\beta - s^\beta\right)\frac{|\cos\beta\eta|}{\beta}} \left| \Omega_n\left(\frac{z}{re^{i\eta}}\right) \right| \left| H\left(re^{i\eta}\right) \right| dr \\
&\le C_1 \left| e^{n\Psi\left(z,se^{i\eta}\right)} \right| n^{-1/2} e^{C_2\lambda_n}.
\end{aligned}
$$

Now, as we saw in the proof of Lemma 6.2,

$$
\begin{aligned}
a &= \mathrm{Re}\,\Psi\left(z, se^{i\eta}\right) - \mathrm{Re}\,\Psi\left(z, se^{i\theta_0}\right) \\
&= f(\eta) - f(\theta_0) < 0,
\end{aligned}
$$

so

$$
|\mathrm{I}_{2+}| \le C_3 e^{-nC_4} \left| e^{n\Psi(z,\psi(z))} \right|. \tag{6.18}
$$

Next, we attend to the condition (6.17). Since $\sigma \in \left[0, \frac{\pi}{2}\right]$,

$$
-\eta \le \sigma - \eta \le \frac{\pi}{2} - \eta < \frac{\pi}{2}.
$$

As our only restrictions on η are $\frac{\pi}{2\beta} < \eta < \frac{\pi}{\beta}$, and $\beta \ge 1$, we can choose η to satisfy (6.17) unless $\beta = 1$ and $\sigma = 0$, that is unless z lies in $(0, \infty)$. We now attend to this case. We use the fact that in this case $s = \psi(z)$ also is in $(0, \infty)$ (as was shown in the proof of Theorem 5.1). We have, for $r \ge s$,

$$
\begin{aligned}
\mathrm{Re}\,\Psi\left(z, re^{i\eta}\right) &= \int_{-1}^0 \log\left| \frac{z}{re^{i\eta}} + |t| \right| d\nu(t) + \frac{r^\beta \cos\beta\eta - 1}{\beta} \\
&\le \int_{-1}^0 \log\left| \frac{z}{r} + |t| \right| d\nu(t) + \frac{r^\beta \cos\beta\eta - 1}{\beta} \\
&\le \int_{-1}^0 \log\left| \frac{z}{s} + |t| \right| d\nu(t) + \frac{r^\beta \cos\beta\eta - 1}{\beta} \\
&= \Psi(z, s) + \frac{r^\beta \cos\beta\eta - s^\beta}{\beta} \\
&= \Psi(z, \psi(z)) - \frac{r^\beta |\cos\beta\eta| + s^\beta}{\beta}.
\end{aligned}
$$

We can now proceed much as above, to obtain (6.15). □

Proof of Theorem 1.3. From Lemmas 4.2 and 6.1 to 6.4,

$$
\begin{aligned}
Q_n(z) &= \mathrm{I}_{11} + \mathrm{I}_{12} + \mathrm{I}_{13} + \mathrm{I}_2 + \mathrm{I}_3 \\
&= \frac{H(\psi(z))}{\sqrt{B(z, \psi(z))}} e^{n\Psi(z,\psi(z))} \Omega_n\left(\frac{z}{\psi(z)}\right)(1+o(1)) + O\left(e^{n\mathrm{Re}\,\Psi(z,\psi(z))} e^{-C_2 n^{1/3-2\varepsilon}}\right) + O\left(2^n e^{-C_2 nR^\beta}\right).
\end{aligned}
$$

Recall here that $0 < \varepsilon < \frac{1}{6}$. Now R may be chosen so large that

$$
2^n e^{-C_2 nR^\beta} \le e^{n\mathrm{Re}\,\Psi(z,\psi(z)) - n}.
$$

Then the result follows. □

References

[1] A. Sidi, Numerical quadrature and non-linear sequence transformations: unified rules for efficient computation of integrals with algebraic and logarithmic endpoint singularities, Math. Comp. 35 (1980) 851–874.
[2] A. Sidi, Numerical quadrature for some infinite range integrals, Math. Comp. 38 (1982) 127–142.
[3] A. Sidi, Problems 5–8, in: H. Brass, G. Hämmerlin (Eds.), Numerical Integration III, Birkhäuser, Berlin, 1988, pp. 321–325.
[4] A. Sidi, Practical Extrapolation Methods, Cambridge University Press, Cambridge, 2003.
[5] A. Sidi, D.S. Lubinsky, Biorthogonal polynomials and numerical integration formulas for infinite intervals, J. Numer. Anal. Ind. Appl. Math. 2 (2007) 209–226.
[6] D.S. Lubinsky, A. Sidi, Zero distribution of composite polynomials and polynomials biorthogonal to exponentials, Constr. Approx. 28 (2008) 343–371.
[7] C. Brezinski, Biorthogonality and its Applications to Numerical Analysis, Marcel Dekker, New York, 1992.
[8] A. Iserles, S.P. Norsett, Bi-orthogonality and zeros of transformed polynomials, J. Comput. Appl. Math. 19 (1987) 39–45.
[9] A. Iserles, S.P. Norsett, E.B. Saff, On transformations and zeros of polynomials, Rocky Mountain J. Math. 21 (1991) 331–357.

[10] D.S. Lubinsky, A. Sidi, Strong asymptotics for polynomials biorthogonal to powers of $\log x$, Analysis 14 (1994) 341–379.
[11] D.S. Lubinsky, I. Soran, Weights whose biorthogonal polynomials admit a Rodrigues formula, J. Math. Anal. Appl. 324 (2006) 805–819.
[12] D.S. Lubinsky, H. Stahl, Some explicit biorthogonal polynomials, in: C.K. Chui, M. Neamtu, L.L. Schumaker (Eds.), Approximation Theory XI, Nashboro Press, Brentwood, TN, 2005, pp. 279–285.
[13] A. Sidi, D.S. Lubinsky, On the zeros of some polynomials that arise in numerical quadrature and convergence acceleration, SIAM J. Numer. Anal. 20 (1983) 589–598.
[14] G. Szegő, Orthogonal Polynomials, fourth ed., in: American Mathematical Society Colloquium Publications, vol. 23, American Math. Soc., Providence, 1975.
[15] M.M. Djrbashian, Harmonic Analysis and Boundary Value Problems in the Complex Domain, Birkhäuser, Basel, 1993.
[16] A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, Vol. 1, McGraw Hill, New York, 1953.
[17] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1970.